On Equivariant Block Triangularization of Matrix Cocyles

Christopher Bose, Joseph Horan, Anthony Quas

September 24, 2014

Abstract

The Multiplicative Ergodic Theorem shows that a real matrix cocycle is block diagonalizable over the real numbers, given mild hypotheses; that is, the cocycle is cohomologous to one in which the matrices are supported on blocks corresponding to the Lyapunov exponents. It is shown that it is not always possible to block *triangularize* the cocycle, even over the complex numbers.

1 Introduction

The Multiplicative Ergodic Theorem has a rich history of generalizations and variations on a theme; there are versions of the theorem in many different situations (see Froyland, Lloyd, Quas [3] for a brief survey). The original proof of the theorem, involving a cocycle into invertible matrices over an invertible ergodic map, is due to Oseledets [4] (in Russian); an updated proof can be found in Barreira and Pesin [1]. One of the key steps of part of the proof is to construct a related *triangular* cocycle over an extended base space, thereby reducing the computation to the simpler case of triangular cocycles. Other parts of the theorem are obtained, however, using the original (non-triangularized) cocycle. Therefore, it is natural to ask if in general, the cocycle may be replaced with a cohomologous triangular cocycle, thus refining the MET and also simplifying the proof. In this paper, we show that this is not possible. Specifically, we construct an invertible, measurable cocycle over an invertible, measure-preserving, ergodic base that cannot admit a cohomologous triangularization.

In Section 2, we detail the setting and formulate the question. In Section 3, we construct a particular cocycle and prove that it is not cohomologous to a triangular cocycle, in the sense of definition 2.3.

2 Preliminaries

Definition 2.1 (Invertible Matrix Cocycle)**.** Let $(X, \mathcal{B}, \mu, \sigma)$ be a dynamical system, where σ is an invertible measure-preserving transformation, and $A_0 : X \to GL_d(\mathbb{R})$ be measurable. Then we define the *cocycle* $A : X \times \mathbb{Z} \to GL_d(\mathbb{R})$ by:

$$\begin{aligned} A(x,1) &= A_0(x), \\ A(x,0) &= I, \\ A(x,n) &= A(\sigma^{n-1}(x),1)\cdots A(\sigma(x),1)A(x,1), \\ A(x,-n) &= A(\sigma^{-n}(x),1)^{-1}\cdots A(\sigma^{-1}(x),1)^{-1}, \end{aligned}$$

for all $n \in \mathbb{Z},\ n > 0$. One easily checks that $A(x, n+m) = A(\sigma^n(x), m)A(x,n)$, for all $n, m \in \mathbb{Z}$. A_0 is the generator for the cocycle, and we often abuse notation by using the same letter for the cocycle and its generator.

The MET, in its classical form, yields a measurable equivariant decomposition of $\mathbb{R}^n$. We state an equivalent formulation of the MET, in the case of invertible matrix cocycles, which is about block diagonalization of the matrices; Theorem 6.3.2 in [1] outlines this statement. Following the theorem, we shall implicitly assume the existence of the base dynamics over which a cocycle is defined.

Theorem 2.1 (Equivalent Formulation of MET for Invertible Matrix Cocycles)**.** *Let $(X, \mathcal{B}, \mu, \sigma)$ be a dynamical system, and $A : X \times \mathbb{Z} \to GL_d(\mathbb{R})$ be a measurable cocycle. Suppose that σ is ergodic, and that A satisfies*

$$\int_X \log^+(\|A(x,1)\|)\ d\mu < \infty, \quad \int_X \log^+(\|A(x,1)^{-1}\|)\ d\mu < \infty.$$

Then there exists $k \in \mathbb{N} = \{1, 2, \dots\}$, $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq -\infty$, positive integers $m_1, m_2, \dots, m_k$ such that $m_1 + \cdots + m_k = d$, a measurable function $C : X \to GL_d(\mathbb{R})$, and a σ-invariant set of full measure, $\tilde{X} \subset X$ with the following properties:

1. ***Equivariance:*** *For all $x \in \tilde{X}$, we have that*
$$C(\sigma(x))^{-1} A(x,1) C(x)$$
is block diagonal with block sizes $(m_1, \cdots, m_k)$;
2. ***Bilateral Growth:*** *Given $x \in \tilde{X}$ and $i \in \{1, \dots, k\}$, for all non-zero v in the columnspace of the ith block, we have:*
$$\frac{1}{n} \log(\|A(x,n)v\|) \underset{n\to\infty}{\longrightarrow} \lambda_i, \ \frac{1}{n} \log(\|A(x,n)v\|) \underset{n\to-\infty}{\longrightarrow} \lambda_i.$$

To see the equivalence between this formulation of the MET and the classical statement of the theorem, first notice that given the matrices $C(x)$, we may measurably construct the equivariant subspaces by taking $\text{span}_{\mathbb{R}}\{C(x)e_{M_i+1}, \dots, C(x)e_{M_i+m_i}\}$, where $i = 1, \dots, k$ and $M_i = m_1 + \cdots + m_{i-1}$, with $M_1 = 0$. Conversely, given the subspaces $V_i(x),\ i = 1, \dots, k$ in the classical MET, one may measurably choose basis vectors for each of them; call them v_i^j, for $j = 1, \dots, m_i$. Then construct the matrix $C(x)$ whose columns are the v_i^j, in order, to obtain the block diagonalization of A.

Definition 2.2 (Block Diagonalization)**.** We say that a measurable cocycle $A(x)$ can be put into *block diagonal* form if there exists a measurable family of matrices $C : X \to GL_d(\mathbb{R})$ such that $C(\sigma(x))^{-1}A(x,1)C(x)$ is block diagonal, and satisfying the equivariance and growth conditions as in Theorem 2.1.

Definition 2.3 (Block Upper-Triangularization)**.** We say that a measurable cocycle $A(x)$ can be put into *block upper-triangular* form over the field $\mathbb{F}$ if there exists a measurable family of matrices $C : X \to GL_d(\mathbb{F})$ such that $C(\sigma(x))^{-1}A(x,1)C(x)$ is block diagonal, satisfying the equivariance and growth conditions as in Theorem 2.1, and that each of the blocks is upper-triangular.

Remark 1. We wish to consider trying to block upper-triangularize a cocycle A. We must decide over which field to do this. Consider the case of a cocycle where the base is the one-point space. Then the cocycle consists of powers of a single matrix, and we have that the cocycle is block upper-triangularizable if and only if the matrix is conjugate to a matrix in block upper-triangular form. It is well known that this cannot always be accomplished over the real numbers; for example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is not conjugate to a real upper-triangular matrix. For this reason, we shall consider performing block upper-triangularization over $\mathbb{C}$.

Given these definitions, we may now state precisely the question which we wish to ask:
Question: Given a cocycle A satisfying the MET (Theorem 2.1), can we necessarily always block upper-triangularize A, in the sense of Definition 2.3?

We now proceed to answer this question in the negative, via the following example.

3 A non-triangularizable cocycle

Consider the following invertible matrix cocycle. Let $X = [0,1)$ be the additive 1-torus, with Borel σ-algebra and Lebesgue measure λ. Let $\eta \in [0,1)$ be irrational, and define $\sigma : X \to X,\ \sigma(x) = x + \eta$. Let $\alpha \in [0,1)$

be irrational, and define the cocycle $A : X \times \mathbb{Z} \to GL_d(\mathbb{R})$ by specifying its generator:

$$A(x,1) = \begin{cases} \begin{bmatrix} \cos(\pi\alpha) & -\sin(\pi\alpha) \\ \sin(\pi\alpha) & \cos(\pi\alpha) \end{bmatrix} & x \in [0, 1-\eta), \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & x \in [1-\eta, 1). \end{cases}$$

We shall, in the remainder of the paper, prove the following.

Theorem 3.1. *The cocycle A cannot be block upper-triangularized over $\mathbb{C}$.*

A is a measurable cocycle of invertible matrices, as its generator $A(\cdot, 1)$ is measurable, and

$$\log(\|A(x,1)v\|) = 0$$

for all non-zero $v \in \mathbb{R}^2$ and $x \in X$, as $A(\cdot, 1)$ is an isometry, thus the integrability condition is satisfied. Finally, σ is ergodic (see [6, Theorem 1.8]), and therefore the MET applies, so that we obtain a measurable block diagonalization of the cocycle. Importantly, we see that we can determine in which space a vector resides by calculating its Lyapunov exponent. Observe that for all $x \in X, 0 \neq v \in \mathbb{R}^2$, we have:

$$\frac{1}{n}\log(\|A(x,n)v\|) = \frac{1}{n}\log(\|v\|) \underset{n\to\infty}{\longrightarrow} 0,$$

and likewise for $n \to -\infty$, since $A(x, n)$ is an isometry for all x and for all n. Therefore, the only possible Lyapunov exponent is $\lambda_1 = 0$. This forces the decomposition to be simply $V_1(x) = V_1 = \mathbb{R}^2$. Thus the MET has told us nothing new about the situation.

Suppose now, for contradiction, that the cocycle A may be upper-triangularized. This implies that we may find a measurable equivariant family of 1-D complex subspaces of $\mathbb{C}^2$. To see this, note that if $C(\sigma(x))^{-1}A(x,1)C(x)$ is upper-triangular, then the family of subspaces $V(x) = \mathrm{span}_{\mathbb{C}}\{C(x)e_1\}$, for $e_1 = (1, 0)$, is equivariant:

$$A(x,1)C(x)e_1 = C(\sigma(x))C(\sigma(x))^{-1}A(x,1)C(x)e_1 = \delta C(\sigma(x))e_1,$$

for some non-zero complex number δ, and so $A(x, 1)V(x) = V(\sigma(x))$. This property says that the graph of the function $x \to V(x)$ from X to $\mathrm{Gr}_1(\mathbb{C}^2)$, as a subset of $X \times \mathrm{Gr}_1(\mathbb{C}^2)$, is invariant under the action of $\sigma \times A$.

Given a basis $\{v_1, v_2\}$ of $\mathbb{C}^2$, there is a measurable bijection between the Grassmannian $\mathrm{Gr}_1(\mathbb{C}^2)$ (equipped with Borel subspace σ-algebra) and the extended complex plane $\bar{\mathbb{C}}$ (equipped with the usual σ-algebra), given by:

$$\mathrm{span}_{\mathbb{C}}\{v_1 + zv_2\} \longleftrightarrow z, \ \mathrm{span}_{\mathbb{C}}\{v_2\} \longleftrightarrow \infty.$$

We therefore consider functions from X into $\bar{\mathbb{C}}$ instead of into the Grassmannian, and moreover we may choose an appropriate basis, so that the cocycle action is simplified. Observe that by writing sine and cosine as linear combinations of complex exponentials, we may write, for $x \in [0, 1-\eta)$:

$$A(x,1) = \begin{bmatrix} \cos(\pi\alpha) & -\sin(\pi\alpha) \\ \sin(\pi\alpha) & \cos(\pi\alpha) \end{bmatrix} = \frac{e^{\pi i\alpha}}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} + \frac{e^{-\pi i\alpha}}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}.$$

We observe that:

$$A(x,1)\left(a\begin{bmatrix} 1 \\ i \end{bmatrix} + b\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) = e^{-\pi i\alpha}a\begin{bmatrix} 1 \\ i \end{bmatrix} + e^{\pi i\alpha}b\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Then for $x \in [1-\eta, 1)$, we have

$$A(x,1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and observe that:

$$A(x,1)\left(a\begin{bmatrix} 1 \\ i \end{bmatrix} + b\begin{bmatrix} 1 \\ -i \end{bmatrix}\right) = b\begin{bmatrix} 1 \\ i \end{bmatrix} + a\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

We therefore choose the basis

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Define skew product dynamics on the space $X \times \mathrm{Gr}_1(\mathbb{C}^2)$ given by $(x, W) \mapsto (\sigma(x), A(x,1)W)$. We write this map in the previously introduced coordinates, on the space $Y = X \times \bar{\mathbb{C}}$. For $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$:

$$T(x,z) = \begin{cases} (\sigma(x), e^{2\pi\alpha i} z) & x \in [0, 1-\eta), \\ (\sigma(x), \frac{1}{z}) & x \in [1-\eta, 1), \end{cases}$$

while for $z = 0$ or $z = \infty$, we have:

$$T(x,0) = \begin{cases} (\sigma(x), 0) & x \in [0, 1-\eta) \\ (\sigma(x), \infty) & x \in [1-\eta, 1) \end{cases},$$

$$T(x,\infty) = \begin{cases} (\sigma(x), \infty) & x \in [0, 1-\eta) \\ (\sigma(x), 0) & x \in [1-\eta, 1) \end{cases}.$$

For $x \in [0, 1-\eta)$, T rotates the plane (and thus circles centered at the origin, including the points 0 and ∞) by $2\pi\alpha$, and for $x \in [1-\eta, 1)$, T performs an inversion, swapping the circles of radius r and radius $\frac{1}{r}$ and flipping in the real axis, with 0 corresponding to ∞. T is clearly measurable and invertible.

Now, the function $x \to V(x)$ associated to the one-dimensional equivariant subspaces now gives rise to a measurable function $f : X \to \bar{\mathbb{C}}$ with the property that its graph (as a subset of $X \times \bar{\mathbb{C}}$) is T-invariant. We shall now prove that there is no measurable function f which has a T-invariant graph.

Recall that we defined equivariance, and so invariance of the graph, as an almost everywhere statement. We obtain a pointwise statement by removing a T-invariant null set from the dynamics space. From the MET, we obtain a null set of points N in X on which the equivariance condition fails. The union of the kth images of this null set under the map σ is exactly the collection of orbits of these points, and thus is the set of points in X for which equivariance fails at some time. The invertibility of σ says that $N \times \bar{\mathbb{C}}$ is a T-invariant set, and it is a null set. In the sequel, we continue to write Y in place of $Y \setminus N \times \bar{\mathbb{C}}$, and we speak only of invariance of the graph.

Consider now the graph of f, which is now pointwise invariant under T. Notice that each orbit of T are confined to unions of two circles (which may be $\{0\}$ or $\{\infty\}$), one outside of the unit circle and one inside, or to the unit circle itself. Let $k : X \to \mathbb{R}$, $k(x) = \min\{|f(x)|, \frac{1}{|f(x)|}\}$. We have, by invariance of the graph under T:

$$\begin{aligned} k(\sigma(x)) &= \min\{|f(\sigma(x))|, \frac{1}{|f(\sigma(x))|}\} \\ &= \begin{cases} \min\{|f(x)e^{2\pi\alpha i}|, \frac{1}{|f(x)e^{2\pi\alpha i}|}\} & x \in [0, 1-\eta) \\ \min\{|\frac{1}{f(x)}|, \frac{1}{|\frac{1}{f(x)}|}\} & x \in [1-\eta, 1) \end{cases} \\ &= \begin{cases} \min\{|f(x)|, \frac{1}{|f(x)|}\} & x \in [0, 1-\eta) \\ \min\{|f(x)|, \frac{1}{|f(x)|}\} & x \in [1-\eta, 1) \end{cases} \\ &= k(x). \end{aligned}$$

Therefore k must be constant almost everywhere with constant r, as σ is ergodic. Then the graph of f is everywhere contained in the union of the circles with radius r and $\frac{1}{r}$, upon removal of a σ-invariant set of measure zero from X. This may either be two distinct circles, or the unit circle. We shall consider these two cases separately; each case reduces the above skew product into one on a space which is easier to understand.

3.1 Two Circles

Suppose the graph of f is contained in two circles in $\bar{\mathbb{C}}$. Let $\mathbb{Z}_2$ denote the additive group on two elements, 0 and 1. Let $E = X \times \mathbb{Z}_2$, and consider the measure space $(E, \mathcal{B} \times \mathcal{P}(\mathbb{Z}_2), \lambda \times c)$, where c is the normalized counting measure, so that $\lambda \times c$ is the normalized Haar measure on the group. Define the map

$$\iota : \bar{\mathbb{C}} \setminus \{z \;:\; |z| = 1\} \to \mathbb{Z}_2, \quad \begin{cases} \iota(z) = 0 & |z| < 1, \\ \iota(z) = 1 & |z| > 1,\; z = \infty. \end{cases}$$

Then we consider dynamics on E given by pushing Y forward to E using ι, so that we get:

$$\hat{T}(x, a) = \begin{cases} (x + \eta, a) & x \in [0, 1 - \eta), \\ (x + \eta, a + 1) & x \in [1 - \eta, 1). \end{cases}$$

$\hat{T}$ is only taking into account on which side of the unit circle points lie. It is measure-preserving, as an invertible piecewise combination of two rotations preserving Haar measure. Define $\hat{f} = \iota \circ f : X \to \mathbb{Z}_2$, and observe that the graph of $\hat{f}$ is $\hat{T}$-invariant. Moreover, f takes value either 0 or 1, for all x, and so the graph of f has measure $\frac{1}{2}$, as a subset of the space E. We shall show that $\hat{T}$ is an ergodic map on E, thus arriving at a contradiction.

We induce on the set $F = [1 - \eta, 1) \times \mathbb{Z}_2$. Observe that the rotation angle η is equal to the interval length, so that the return time is easy to calculate. Note that

$$\bigcup_{n=0}^{\infty} \hat{T}^{-n}(F) = E,$$

so that $\hat{T}$ is ergodic if and only if the induced map $\hat{T}_F$ is ergodic with respect to the induced measure. Let us compute $\hat{T}_F$.

Let k be a positive integer such that $k\eta < 1 < (k + 1)\eta$, and let $q = 2 - (k + 1)\eta$. Then we see that the first return time to F, $n_F(x, a)$, is given by:

$$n_F(x, a) = \begin{cases} k & x \in [q, 1), \\ k + 1 & x \in [1 - \eta, q). \end{cases}$$

Hence, we obtain the first return map, $\hat{T}_F$:

$$\begin{aligned} \hat{T}_F(x, a) = \hat{T}^{n_F(x,a)}(x, a) &= \begin{cases} \hat{T}^k(x, a) & x \in [q, 1) \\ \hat{T}^{k+1}(x, a) & x \in [1 - \eta, q) \end{cases} \\ &= \begin{cases} (x + k\eta - 1, a + 1) & x \in [q, 1) \\ (x + (k + 1)\eta - 1, a + 1) & x \in [1 - \eta, q) \end{cases} \end{aligned}$$

We now perform a coordinate change, $x \mapsto \frac{1-x}{\eta}$. This flips the interval $[1 - \eta, 1)$ around, and expands it to the unit interval. Its inverse is $x \mapsto 1 - \eta x$. Consider the new dynamics on E, denoted R. The $\mathbb{Z}_2$ part of R is simple; observe this coordinate change doesn't affect what happens in the $\mathbb{Z}_2$ component at all, so we see that $a \mapsto a + 1$. Now, for the X part of R, the image of q under the coordinate change is

$$\frac{1 - (2 - (k + 1)\eta)}{\eta} = 1 + k - \frac{1}{\eta} = 1 - \left\{ \frac{1}{\eta} \right\}.$$

Denote $\beta = \left\{ \frac{1}{\eta} \right\}$. We thus obtain $x \mapsto x + \beta$. Then we have, for $y \in X, a \in \mathbb{Z}_2$:

$$R(y, a) = (y + \beta, a + 1).$$

Hence R is a rotation in the y coordinate by β, and in the a coordinate by 1.

By [6, Theorem 1.10], we see that R is ergodic if and only if the only character which remains invariant under some power of R is the trivial character. All characters of $X \times \mathbb{Z}_2$ are of the form $(x, b) \mapsto e^{2\pi nxi}(-1)^{ab}$, where $n \in \mathbb{Z}$, and $a \in \mathbb{Z}_2$; see [2, Chap. II]. Then we have, if some character γ satisfies $\gamma \circ R^m = \gamma$:

$$\gamma \circ R^m(x, b) = e^{2\pi n(x+m\beta)i}(-1)^{a(b+m)} = e^{2\pi nm\eta i}(-1)^{am} e^{2\pi nxi}(-1)^{ab} = e^{2\pi nxi}(-1)^{ab}.$$

This implies that

$$1 = e^{2\pi nm\eta i}(-1)^{am}.$$

Because η is irrational, this forces $n = 0$, and then we must have $a = 0$ also. Hence γ is the trivial character, and so R is ergodic on E. We have our desired contradiction.

3.2 One Circle

Suppose now that the graph of f is contained inside the unit circle. Then we may consider the restriction of the dynamics T (which we continue to denote as T) to the unit circle as its own system, where the unit circle is endowed with the usual Borel σ-algebra and Lebesgue measure. Denote $W = [0,1)^2$, which has the usual two-dimensional σ-algebra and Lebesgue measure. Upon an appropriate scaling of the second coordinate, we obtain:

$$T : W \to W,\ T(x,y) = \begin{cases} (x+\eta, y+\alpha) & x \in [0, 1-\eta), \\ (x+\eta, 1-y) & x \in [1-\eta, 1). \end{cases}$$

T is indeed measure-preserving, as it is an invertible piecewise combination of rigid transformations over Lebesgue measure. We consider the function f to take values now in $[0,1)$, and we remark that the graph of f is still invariant under T. We shall show that T is ergodic, but that there exists a non-constant T-invariant function, which is a contradiction.

Remark 2. If one merely wished to determine if a real cocycle of 2-by-2 matrices, satisfying all of the same hypotheses as above, was measurably equivariantly block upper-triangularizable over the *real* numbers, one would note that $\mathrm{Gr}_1(\mathbb{R}^2)$ can be put into correspondence with the interval $[0,\pi)$ by considering subspace angle, and thus to $[0,1)$. The dynamics arising from this construction are exactly the dynamics derived in the one circle case, for this map. Hence the following proof is exactly what we could do to show that A is not upper-triangularizable equivariantly over the real numbers. It must be noted that this occurred in this specific case due to the particular structure of the map T on $\bar{\mathbb{C}}$; the graph of f remained away from 0 and ∞.

Consider the distance on the 1-torus $|\cdot|_W$ (e.g. $|.1 - .8|_W = .3$, not .7). Define $g(x,y) = |y - f(x)|_W$. Then we observe that:

$$\begin{aligned} g(T(x,y)) &= \begin{cases} |y+\alpha - f(x+\eta)|_W & x \in [0,1-\eta) \\ |1-y-f(x+\eta)|_W & x \in [1-\eta,1) \end{cases} \\ &= \begin{cases} |y+\alpha - (f(x)-\alpha)|_W & x \in [0,1-\eta) \\ |1-y-(1-f(x))|_W & x \in [1-\eta,1) \end{cases} \\ &= \begin{cases} |y - f(x)|_W & x \in [0,1-\eta) \\ |f(x)-y|_W & x \in [1-\eta,1) \end{cases} \\ &= g(x,y). \end{aligned}$$

So g is T-invariant, but clearly g is not constant. We now show that T is ergodic. Similarly to the first case, we induce upon the set $H = [1-\eta, 1) \times [0,1)$, and make the identical coordinate change. First, we have:

$$\begin{aligned} T_H(x,y) = T^{n_H(x,y)}(x,y) &= \begin{cases} T^k(x,y) & x \in [2-(k+1)\eta, 1) \\ T^{k+1}(x,y) & x \in [1-\eta, 2-(k+1)\eta) \end{cases} \\ &= \begin{cases} (x+k\eta-1, (k-1)\alpha - y) & x \in [q,1) \\ (x+(k+1)\eta - 1, k\alpha - y) & x \in [1-\eta, q) \end{cases} \end{aligned}$$

We then change coordinates the same way as before, letting $x \mapsto \frac{1-x}{\eta}$ and $\beta = \left\{\frac{1}{\eta}\right\}$, so that we obtain a new map on W:

$$\tilde{T}_H = \begin{cases} (x+\beta, (k-1)\alpha - y) & y \in [0, 1-\beta) \\ (x+\beta, k\alpha - y) & y \in [1-\beta, 1) \end{cases}.$$

Observe that if we computed the second-return map to H or of course, its coordinate-change analogue, we would eliminate the flip in the second coordinate. Moreover, it is a well-known fact that if a map T is measure-preserving and T^2 is ergodic, then so too is T. Here, to make the situation simpler, we shall assume $\beta < \frac{1}{2}$, so that $0 < 1-2\beta < 1-\beta < 1$ and $\beta < 1-\beta$. There are plenty of η for which this is the case (for example, let $\eta = \frac{1}{\sqrt{2}}$); the case $\beta > \frac{1}{2}$ leads to the same conclusion, but is notationally more complicated. If

$x \in [0, 1-\beta)$, then $x+\beta \in [\beta, 1) = [\beta, 1-\beta) \cup [1-\beta, 1)$, and if $x \in [1-\beta, 1)$, then $x \in [0, \beta)$. Thus we obtain:

$$
\begin{aligned}
(\tilde{T}_H)^2(x,y) &= \begin{cases} \tilde{T}_H(x+\beta, (k-1)\alpha - y) & x \in [0, 1-\beta) \\ \tilde{T}_H(x+\beta, k\alpha - y) & x \in [1-\beta, 1) \end{cases} \\
&= \begin{cases} (x+2\beta, (k-1)\alpha - ((k-1)\alpha - y)) & x \in [0, 1-2\beta) \\ (x+2\beta, k\alpha - ((k-1)\alpha - y)) & x \in [1-2\beta, 1-\beta) \\ (x+2\beta, (k-1)\alpha - (k\alpha - y)) & x \in [1-\beta, 1) \end{cases} \\
&= \begin{cases} (x+2\beta, y) & x \in [0, 1-2\beta) \\ (x+2\beta, y+\alpha) & x \in [1-2\beta, 1-\beta) \\ (x+2\beta, y-\alpha) & x \in [1-\beta, 1) \end{cases}
\end{aligned}
$$

We induce again, this time on the set $J = [1-2\beta, 1) \times [0, 1)$. This works exactly in the same way as the previous induction, because the rotation is the same length as the interval length. We also make the appropriate coordinate change, to simplify the situation. Let S be the resulting map on W, and let $\zeta = \left\{ \frac{1}{2\beta} \right\}$. Note that ζ is irrational. Observe easily that we obtain:

$$
S(x,y) = \begin{cases} (x+\zeta, y+\alpha) & x \in [0, \frac{1}{2}), \\ (x+\zeta, y-\alpha) & x \in [\frac{1}{2}, 1). \end{cases}
$$

Summarizing the ergodicity implications tells us that if S is ergodic, then T is ergodic. In order to conclude the proof, we shall prove the following result, and then utilize a result from the literature.

Proposition 3.2. *Suppose $\sigma : [0,1) \to [0,1)$ is measure-preserving and ergodic with respect to Lebesgue measure, and let $f : [0,1) \to \mathbb{R}$ be a measurable function, with range $f([0,1)) \subset \alpha\mathbb{Z}$, where α is irrational. Let $[0,1)^2$ have the usual Lebesgue product measure and Borel sets, and let $T_f : [0,1)^2 \to [0,1)^2$ be the skew product extension of σ and f to $[0,1)^2$, so that:*

$$
T_f(x,y) = (\sigma(x), y + f(x)).
$$

Let $\tilde{T}_f : [0,1) \times \alpha\mathbb{Z} \to [0,1) \times \alpha\mathbb{Z}$ be the skew product extension of σ and f to $[0,1) \times \alpha\mathbb{Z}$ with the product measure $\lambda \times c$ (Lebesgue and counting, with the discrete σ-algebra for the counting measure), so that:

$$
\tilde{T}_f(x, n\alpha) = (\sigma(x), n\alpha + f(x)).
$$

Then if $\tilde{T}_f$ is ergodic, then so is T_f.

Proof. Let $h : [0,1)^2 \to \mathbb{R}$ be a bounded measurable function invariant under T_f, so $h \circ T_f = h$. We shall show that h must be a.e. constant; this implies that T_f is ergodic, by a simple modification of [6, Theorem 1.6]. For $y \in [0,1)$, define the measurable function

$$
\pi_y : [0,1) \times \alpha\mathbb{Z} \to [0,1)^2, \ \pi_y(x, n\alpha) = (x, y + n\alpha).
$$

Then we see that $T_f \circ \pi_y = \pi_y \circ \tilde{T}_f$. In addition, define $\tilde{h}_y = h \circ \pi_y$, so that $\tilde{h}_y$ is measurable. Since π_y intertwines the dynamics on the two spaces, we get the following:

$$
\tilde{h}_y \circ \tilde{T}_y = h \circ \pi_y \circ \tilde{T}_f = h \circ T_f \circ \pi_y = h \circ \pi_y = \tilde{h}_y.
$$

Thus $\tilde{h}_y$ is invariant under $\tilde{T}_f$, and so is constant a.e. with respect to the product measure $\lambda \times c$, since $\tilde{T}_f$ is ergodic.

We wish to use the fact that $\tilde{h}_y$ is a.e. constant for each $y \in [0,1)$ to show that h is constant a.e. To do this, we make an intermediate step. Define

$$
I : [0,1) \to \mathbb{R}, \ I(y) = \int_0^1 h(x,y) \, dx = \int_0^1 \tilde{h}_y(x, 0) \, dx.
$$

Because h is bounded, I is not infinite, hence well-defined, and by Fubini's theorem, I is measurable. Moreover, we have the following, since $\tilde{h}_y$ is a.e. constant on $[0,1) \times \alpha\mathbb{Z}$:

$$\begin{aligned} I(y+\alpha) &= \int_0^1 h(x, y+\alpha)\ dx = \int_0^1 \tilde{h}_y(x,\alpha)\ dx = \int_0^1 \tilde{h}_y(x,0)\ dx \\ &= \int_0^1 h(x,y)\ dx = I(y). \end{aligned}$$

$y \mapsto y+\alpha$ is an ergodic map on $[0,1)$, thus we see that I is a.e. constant on $[0,1)$; write $I(y) = C$ for a.e. $y \in [0,1)$. Note that for all y, $\tilde{h}_y$ is a.e. constant on $[0,1) \times \alpha\mathbb{Z}$, so we see that for a.e. $x \in [0,1)$, $\tilde{h}_y(x,0) = I(y)$. Denote $Y_G = \{y \in [0,1)\ :\ I(y) = C\}$; this set has full measure in $[0,1)$. If $y \in Y_G$, then for a.e. x, $h(x,y) = \tilde{h}_y(x,0) = C$. Applying Fubini's Theorem yields the final statement: $h = C$ almost everywhere. Hence T is ergodic. □

To conclude the proof, we utilize the following proposition:

Proposition 3.3 (Schmidt, [5])**.** *Consider the space* $[0,1) \times \alpha\mathbb{Z}$ *as defined above. Define the map* $\tilde{S}$ *on* $[0,1) \times \alpha\mathbb{Z}$ *by*

$$\tilde{S}(x, n\alpha) = \begin{cases} (x+\zeta, (n+1)\alpha) & x \in [0, \frac{1}{2}), \\ (x+\zeta, (n-1)\alpha) & x \in [\frac{1}{2}, 1). \end{cases}$$

Then $\tilde{S}$ *is an ergodic measure-preserving transformation.*

Proof. The result follows from Theorem 3.9, Corollary 5.4, and Theorem 12.8 of [5], upon the relabelling of $\mathbb{Z}$ to $\alpha\mathbb{Z}$. □

Observe that $\tilde{S}$ and S are related in exactly the manner outlined in Proposition 3.2, and Proposition 3.3 states that $\tilde{S}$ is ergodic. Therefore, we see that the map T is ergodic, which means we've arrived at a contradiction.

We see that f cannot exist as assumed. This means that we've achieved the negative answer to our original question: in this situation, it is impossible for there to be an equivariant family of 1-D complex subspaces for our matrix cocycle A, and so there is no block upper-triangularization of the matrix cocycle A.

References

[1] L. Barreira and Y. Pesin. *Introduction to smooth ergodic theory*, volume 148 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013.

[2] T. Bröcker and T. tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[3] G. Froyland, S. Lloyd, and A. Quas. A semi-invertible Oseledets theorem with applications to transfer operator cocycles. *Discrete Contin. Dyn. Syst.*, 33(9):3835–3860, 2013.

[4] V. I. Oseledec. A multiplicative ergodic theorem. Characteristic Ljapunov exponents of dynamical systems. *Trudy Moskov. Mat. Obšč.*, 19:179–210, 1968. (Russian).

[5] K. Schmidt. *Cocycles on ergodic transformation groups*. Macmillan Company of India, Ltd., Delhi, 1977. Macmillan Lectures in Mathematics, Vol. 1.

[6] P. Walters. *An Introduction to Ergodic Theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.