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Invariant measures for families of circle maps

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Abstract

In this paper, we establish conditions for a continuous family of circle maps to have a continuous family of invariant measures in the appropriate topologies, and demonstrate the necessity of some of the conditions.

1. *Background*

We reserve the symbol π for the projection $\mathbb{R} \to S^1$ given by $x \mapsto \exp(2\pi i x)$. We will denote in the usual way intervals on the circle (for example the interval $[a, b]$ is the closed interval starting at a and going anticlockwise around to b). By a circle map, we will always mean an orientation-preserving homeomorphism $T: S^1 \to S^1$. For a detailed introduction to the theory of circle maps, the reader is referred to [**2**], §3·3. The main results, however, are summarized below for convenience. The dynamical behaviour of circle maps is very well understood, and may be principally characterized by the rotation number of the map. This is a measure of the 'average rotation' that the map imparts to a point. To define the rotation number of an orientation-preserving homeomorphism $T: S^1 \to S^1$, we first need its lift $F: \mathbb{R} \to \mathbb{R}$. The lift of any continuous map ϕ of the circle (not necessarily a circle map) is a continuous map defined by the equation $\pi \circ F = \phi \circ \pi$. This is uniquely defined up to an additive integer constant. The degree of the map ϕ is given by $F(x+1) - F(x)$. This is always an integer and is independent of the point $x \in \mathbb{R}$ and the lift chosen. In the case of a circle map, the degree is always 1. The rotation number of the circle map T is then given by

$$\rho(T) = \lim_{n \to \infty} \frac{F^n(x) - x}{n},$$

where F is a lift of T. The rotation number may be shown to be independent of the point $x \in \mathbb{R}$ chosen in the definition, and is unique for a map T up to an additive integer constant (depending on the particular lift chosen to represent T). The notion of a circle map with rational rotation number is therefore well-defined since this property does not depend on the lift chosen. It can be shown that if a circle map has a periodic orbit of some given period, then all periodic orbits of the circle map have the same period. It may also be shown that a circle map has rational rotation number with denominator q say (with the fraction expressed in its lowest terms), if and only if it has periodic points of period q. Further, if this is the case, then each point converges monotonically to a periodic orbit under iteration of the map. From this, it follows that a circle map with irrational rotation number has no periodic orbits. Here, the dynamics are also well-understood: each point has the same ω-limit point

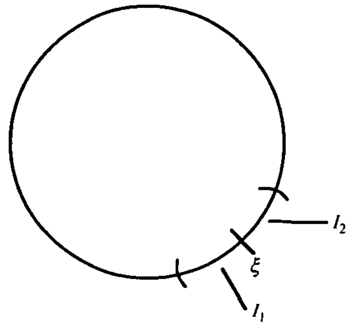

Fig. 1. Possible configuration of intervals about a fixed point.

set and this set is either a Cantor set, or the whole circle. In the former case, the map is semi-conjugate to a rotation through 2π times the rotation number, and in the latter case, the map is conjugate to a rotation through 2π times the rotation number. It may be shown by elementary means that the map taking a circle map to its rotation number is continuous with respect to the C^0-topology on the space of circle maps; see for example [**2**], §3·3, theorem 2.

This paper was motivated by the problem of finding an alternative proof of this fact, but the conditions needed for the results below to work are in fact stronger than necessary for the proof of this fact alone. The idea is roughly as follows. The rotation number is numerically equal to the rotation amount at each point integrated with respect to an invariant measure for the circle map in question, so proving that the invariant measures for a family of maps vary weak*-continuously with respect to a parameter gives an alternative proof that the rotation number varies continuously with the C^0-topology on circle maps.

In the statement of the theorem, we will need some definitions. We say that a family $(T_\alpha)_{\alpha\in J}$ of circle maps with J a compact subinterval of $\mathbb{R}$ is a continuous family of circle maps if the map $T: J\times S^1 \to S^1$; $(\alpha, \xi)\mapsto T_\alpha(\xi)$ is continuous.

Given a circle map T with rotation number p/q, let S be the lift of T^q fixing the preimages of the periodic points. Define $u(x) = S(x)-x$. Note that u satisfies the equation $u(x) = u(x+1)$, since the degree of T^q is 1. The function $v: S^1\to\mathbb{R}$; $\xi\mapsto u(\pi^{-1}(\xi))$ is then well-defined. Note that the zeros of v are precisely the periodic points of the map T. Then given a periodic point ξ, there may be a neighbourhood of ξ on which v takes the value 0 only at ξ itself. If such a neighbourhood exists, we say the periodic point is of definite type, and conversely, if no such neighbourhood exists, we say the periodic point is of indefinite type. If the periodic point is of definite type, it follows that there is an open interval I_1 clockwise from ξ with ξ as an endpoint on which the sign of v is constant, and a similar interval I_2 anticlockwise from ξ (Figure 1).

We say that ξ is of type $++$, $+-$, $-+$ or $--$ according to the sign of v on these two intervals. A hyperbolic periodic point is one of type $+-$ or $-+$ (these are stable and unstable respectively). The types $++$ and $--$ of periodic point are non-hyperbolic and have stability on one side only. We call a map with non-hyperbolic periodic points (or sometimes its parameter value) critical. Note that if a point on an orbit is of a particular type, then all the points of the orbit are of that type (this arises since the maps are orientation-preserving homeomorphisms), so that it makes sense

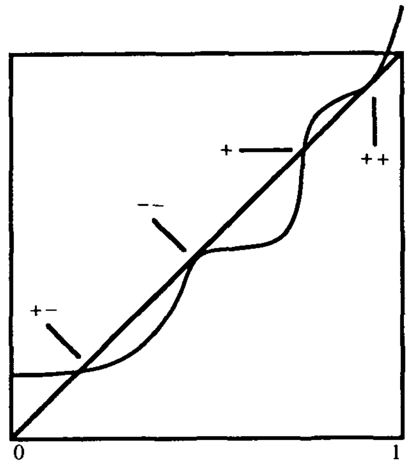

Fig. 2. Lift of an iterate of a circle map showing the types of periodic point.

to say that an orbit is of a specific type, or in particular hyperbolic, or non-hyperbolic (Figure 2).

We are now ready to state the theorem:

THEOREM. *Suppose that* $(T_\alpha)_{\alpha\in J}$ *is a continuous family of circle maps such that* (i) *for each non-trivial interval K on which the rotation number has a constant rational value, there are at most finitely many values of* α *in K with* T_α *critical, and* (ii) *for each* $\alpha \in J$, T_α *has either a hyperbolic periodic orbit, or has a single periodic orbit which is non-hyperbolic. Then there is a weak*-continuously varying family of probability measures* μ_α *such that* μ_α *is an invariant measure for* T_α *(that is, such that* $\mu_\alpha(T_\alpha^{-1}(B)) = \mu_\alpha(B)$ *for all Borel sets B).*

Part of this theorem was previously known to Herman. In particular, Herman showed that the map taking a circle map with irrational rotation number to its unique invariant probability measure is weak*-continuous on the sets F_ρ, the collection of circle maps with rotation number equal to ρ (irrational). He in fact shows (see [**3**], proposition X·6·1) that the (semi-)conjugacy h conjugating a circle map f to the rotation by ρ depends weak*-continuously on f. But then the invariant measure is given by $\mu(A) = \lambda(h(A))$, so this implies that the invariant measure μ associated with f depends weak*-continuously on f as stated above when considered as a map from F_ρ to the probability measures. In the course of the proof below, we will in fact show that the map taking a uniquely ergodic circle map (that is, one with a unique invariant Borel probability measure) to its unique invariant Borel probability measure is continuous (that is, for uniquely ergodic maps, the conditions (i) and (ii) of the theorem above are unnecessary).

2. *Two examples showing necessity of some conditions*

Before embarking on a proof of the theorem, we first present two examples to show that some restrictions are necessary for the conclusion of the theorem to hold. In particular, we exhibit families which fail to satisfy condition (ii) for which the conclusion fails. It seems likely that condition (i) is unnecessary for the conclusion of the theorem to hold, although any significant relaxation of this condition will

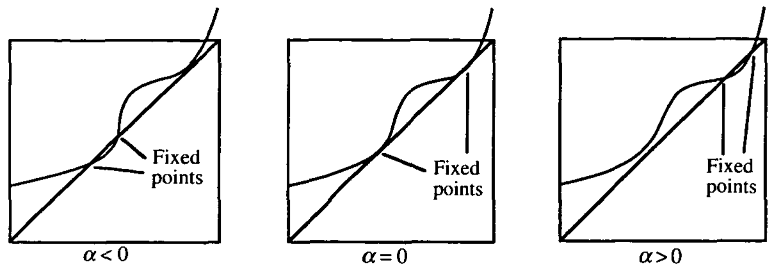

Fig. 3. A family for which the conclusion of the Theorem fails.

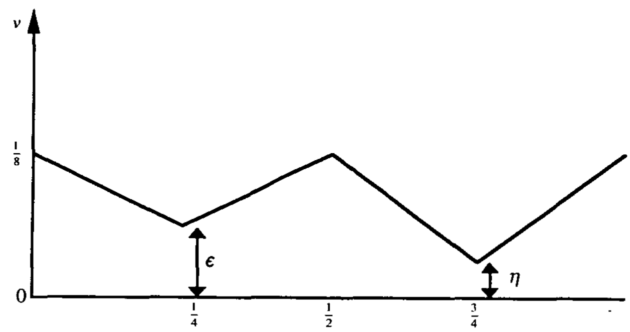

Fig. 4. 'Speed Function' for counter-example of Section 2.

necessarily make the construction of the invariant measures much harder than the one given in this proof.

The first example for which the conclusion fails is illustrated graphically in Figure 3. The process which takes place is that a pair of fixed points vanish simultaneously with the birth of a second pair of fixed points. The limits from the two sides in parameter space of the invariant measures are concentrated on the dying pair (respectively new pair) for parameter values lower than (respectively greater than) the critical value (see Lemma 1).

This example shows that even in the one-dimensional case there exist examples for which the generalized version of this theorem fails. The reliance of the proof on properties of circle maps suggests that this would fail more spectacularly in higher dimensions.

The next example, which is more complicated, shows that the conclusion of the theorem need not hold even if the condition only fails on the boundary of a component on which the rotation number is constant. To write down the example, we regard the circle as the interval $[0, 1)$ mod 1. The maps which we consider are then of the form $Tx = x + v(x)$ mod 1. The form of the functions v which we are considering is shown above in Figure 4. The function v depends on the parameters ϵ and η. It is clear that if ϵ and η are allowed to vary continuously with respect to some parameter α say, then the family of circle maps given by $T_\alpha(x) = x + v_\alpha(x)$ is in fact a continuous family of circle maps. The family v_α is given explicitly by the expression

$$v_\alpha(x) = \begin{cases} \epsilon(\alpha) + 4(\frac{1}{8} - \epsilon(\alpha))\,|x - \frac{1}{4}| & (x \in [0, \frac{1}{2}]) \\ \eta(\alpha) + 4(\frac{1}{8} - \eta(\alpha))\,|x - \frac{3}{4}| & (x \in [\frac{1}{2}, 1]). \end{cases}$$

We then consider a family with the properties that $\epsilon(\alpha) \to 0$ and $\eta(\alpha) \to 0$ as $\alpha \to \alpha_0$,

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and investigate the limit of the invariant measures of the maps T_α as $\alpha \to \alpha_0$ and in particular show that the limit exists if and only if $\log \epsilon / \log \eta$ has a well-defined limit. In this case the limit is a measure μ concentrated at the points $\frac{1}{4}$ and $\frac{3}{4}$ with

$$\mu(\{\tfrac{1}{4}\}) = \lim_{\alpha \to \alpha_0} \frac{\log \epsilon(\alpha)}{\log \epsilon(\alpha) + \log \eta(\alpha)}, \quad \mu(\{\tfrac{3}{4}\}) = \lim_{\alpha \to \alpha_0} \frac{\log \eta(\alpha)}{\log \epsilon(\alpha) + \log \eta(\alpha)}. \tag{1}$$

It is clear that there exist examples of continuous functions $\epsilon(\alpha)$ and $\eta(\alpha)$ with the property that $\epsilon(\alpha) \to 0$ and $\eta(\alpha) \to 0$ as $\alpha \to \alpha_0$ with $\epsilon(\alpha) > 0$ and $\eta(\alpha) > 0$ for all $\alpha < \alpha_0$ such that the limit of $\log \epsilon / \log \eta$ fails to exist as $\alpha \to \alpha_0$. The particular point of interest in this result is then that if condition (ii) of the theorem fails even on the boundary of a region where the rotation number is a constant rational, then the conclusion of the theorem can fail to hold. It is therefore the case that the natural weakening of condition (ii) which only applies at the interior points of the intervals on which the rotation number is a constant rational is insufficient to prove the theorem.

It is a well-known fact of ergodic theory that each circle map T_α has some invariant Borel probability measure μ_α say (see [**4**], corollary 6·9·1). To evaluate the limiting measure μ described above, we take a small set containing $\frac{1}{4}$, say $A = [\frac{1}{4} - \delta, \frac{1}{4} + \delta)$, and one containing $\frac{3}{4}$, say $B = [\frac{3}{4} - \delta, \frac{3}{4} + \delta)$, and estimate $\mu_\alpha(A)$ and $\mu_\alpha(B)$ for $\alpha \to \alpha_0$. To do this, we note that if it takes between n and $n+1$ steps 'for a point to go all the way round the circle' (that is, if $0 \leqslant T_\alpha^{n+1}(0) < T_\alpha(0)$ and this is the first such n), and if it takes between m and $m+1$ steps for a point to go through A (that is, if $T_\alpha^{m+1}(\frac{1}{4} - \delta) \geqslant \frac{1}{4} + \delta$ and this is the smallest such m), then

$$\mu_\alpha(A) = \int \chi_A \, d\mu_\alpha = \int \frac{1}{n} \sum_{i=0}^{n-1} \chi_A \circ T_\alpha^i \, d\mu_\alpha.$$

This follows by invariance of the measure. But for each point we have

$$\frac{m-1}{n} \leqslant \frac{1}{n} \sum_{i=0}^{n-1} \chi_A \circ T_\alpha^i(x) \leqslant \frac{m+1}{n},$$

from which it follows that $|\mu_\alpha(A) - m/n| \leqslant 1/n$. There is of course a similar result for $\mu_\alpha(B)$. If we then show that the amount of steps in each cycle spent outside sets A and B is bounded above by some constant, then it is clear that we can evaluate the limit of $\mu_\alpha(\{\frac{1}{4}\})$ as $\alpha \to \alpha_0$, by estimating the values of m and n, since these tend to ∞ as $\alpha \to \alpha_0$. The only calculation which we need to perform is to solve a recurrence relation to estimate the time spent in certain sets of a very simple form. Suppose then we are considering a set C of the form $[0, a)$ and the 'speed' function is given by $v(x) = c - (c-b)\,x/a$ where $c - b < a$, and we have $T(x) = x + v(x)$; then the recurrence relation is $x_{n+1} = c + (1 - (c-b)/a)\,x_n$. Let $\rho = 1 - (c-b)/a$. Then we are solving $x_{n+1} = c + \rho x_n$. The solutions are $x_n = c/(1-\rho) + A\rho^n$. By substituting the initial conditions, we see that in fa t $x_n = c(1-\rho^n)/(1-\rho)$. The number of steps thus spent in the set C is thus the rounded-up value of

$$\log(1 - (1-\rho)\,a/c)/\log \rho = (\log b/c)/\log \rho.$$

We can now apply this to the collection of circle maps described above. The first thing to show is that the amount of steps per cycle spent outside the sets A and B is bounded above by a constant as ϵ and η tend to 0. To show this, we note the symmetry of the situation: the steps taken to get from 0 to $\frac{1}{4}-\delta$ is the same as the number of steps taken to get from $\frac{1}{4}+\delta$ to $\frac{1}{2}$. This number is given by the round-up of $(\log 8.v(\frac{1}{4}-\delta))/\log(\frac{1}{2}+4\epsilon)$. This is bounded above by $\log(4\delta)/\log(\frac{3}{4})$ provided that $\epsilon < \frac{1}{16}$, so we see that the number of steps spent outside A and B is bounded above by $4\log(4\delta)/\log(\frac{3}{4})$ provided that $\epsilon, \eta < \frac{1}{16}$. The number of steps in A is given by $-2\log\epsilon/\log(\frac{1}{2}+4\epsilon)$ plus a term which is bounded, and similarly the number of steps in B is given by $-2\log\eta/\log(\frac{1}{2}+4\eta)$ plus a bounded term. Set $m = -2\log\epsilon/\log(\frac{1}{2}+4\epsilon)$ and $p = -2\log\eta/\log(\frac{1}{2}+4\eta)$. Then given a constant $\sigma > 0$, there exists a τ such that if $\epsilon, \eta < \tau$, we get $|\mu_\alpha(A) - m/(m+p)| < \sigma$ and $|\mu_\alpha(B) - p/(m+p)| < \sigma$. By elementary analysis, we see that the assertion of equation (1) is now proved, and thus the example is complete.

3. *Proof of the Theorem*

A useful lemma is the following:

LEMMA 1. *The invariant Borel probability measures for a circle map T with rational rotation number p/q are precisely those measures which can be expressed in the form*

$$\mu(A) = \frac{1}{q}\sum_{i=0}^{q-1} \nu(T^{-i}A),$$

where ν is a probability measure concentrated on the fixed points of T^q.

Proof of Lemma 1. Certainly any Borel probability measure of the form described is invariant for the circle map in question. Conversely, in the preliminary discussion, it was noted that if the rotation number of a circle map is rational, then each point of the circle converges monotonically under iteration of the map to a periodic orbit. From this, it follows that the only non-wandering points of the map are the periodic points. There is then a standard theorem telling us the non-wandering set has full measure (that is, measure 1) with respect to any invariant Borel probability measure (see [**4**], theorem 6·15). The remainder of the proof follows easily from the invariance of the measure.

LEMMA 2. *Suppose that $(T_\alpha)_{\alpha\in J}$ is a continuous family of circle maps such that T_{α_0} has a hyperbolic periodic orbit of period q through a point $\xi \in S^1$. Then for each neighbourhood M of ξ, there exists a neighbourhood N of α_0 such that if $\beta \in N$, then T_β has a periodic point of period q in M.*

Proof. Suppose that we are given a neighbourhood M of ξ. There must then exist a closed subinterval I of M with the property that $T^q_{\alpha_0}(I) \subset \text{Int}(I)$ or $T^q_{\alpha_0}(\text{Int}(I)) \supset I$ according to whether ξ is stable or unstable. But then for any T which is sufficiently close to T_{α_0}, the appropriate containment property persists ($T^q(I) \subset \text{Int}(I)$ or $T^q(\text{Int}(I)) \supset I$ respectively). But then it follows by the fixed point theorem of Brouwer that T has a periodic point of period q in $\text{Int}(I)$.

We now proceed to the proof of the main theorem.

Proof of Theorem. Set $C = \text{Cl}(\{\alpha \in J : \rho(T_\alpha) \notin \mathbb{Q}\})$. We will show that for those values of α in C, the map T_α is uniquely ergodic (that is, there is exactly one invariant Borel probability measure for T_α). The proof divides into two cases: $\rho(T_\alpha)$ irrational and $\rho(T_\alpha)$ rational.

The case where $\rho(T_\alpha)$ is irrational is a standard ergodic theorem and is shown in [**4**], theorem 6·18.

If T_α has a hyperbolic periodic point, of period q, say, then by Lemma 2, there is a neighbourhood of parameter values about α such that for maps in the neighbourhood there is a periodic point, and hence the rotation number is rational on a whole neighbourhood of parameter values about α. In particular, $\alpha \notin C$.

It therefore follows that if $\alpha \in C$, but $\rho(T_\alpha)$ is rational, then T_α has no hyperbolic periodic points, and therefore, since we know by the rotation number characterization that T_α does indeed have periodic points, we see that T_α has only non-hyperbolic periodic points. It then follows by hypothesis (ii) of the Theorem that T_α has a unique periodic orbit. By Lemma 1, we see that there is a unique invariant probability measure. We are now in a position to describe the invariant measure for T_α for $\alpha \in C$. Let Ω be the Borel σ-algebra of S^1. Given $f \in C(S^1)$, and a Borel probability measure μ, define

$$F_\alpha^{(n)}(x) = \frac{1}{n}\left(\sum_{i=0}^{n-1} f(T_\alpha^i(x))\right), \quad \mu_\alpha^{(n)}(B) = \frac{1}{n}\left(\sum_{i=0}^{n-1} \mu(T_\alpha^{-i}(B))\right) \quad \text{for } B \in \Omega. \tag{2}$$

Elementary integration shows that $\int f\,d\mu_\alpha^{(n)} = \int F_\alpha^{(n)}\,d\mu$.

Then for a uniquely ergodic map T_α, the sequence of functions $(F_\alpha^{(n)})_{n\in\mathbb{N}}$ is known to converge uniformly to a constant (see [**4**], theorem 6·19), so it follows that $(\int f\,d\mu_\alpha^{(n)})_{n\in\mathbb{N}}$ is a convergent sequence (to this same constant). It follows (by the Riesz Representation Theorem – see [**4**], theorem 6·3), that the sequence of measures $\mu_\alpha^{(n)}$ is weak*-convergent. Let μ_α be the weak*-limit of this sequence. This is clearly T_α-invariant, and independent of the original probability measure μ, so we have for all $\alpha \in C$ and for all Borel probability measures μ,

$$\mu_\alpha = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=0}^{n-1} \mu \circ T_\alpha^{-i}\right). \tag{3}$$

We proceed by defining μ_α for $\alpha \in J\backslash C$. First, we consider the structure of the set $J\backslash C$. Now $J\backslash C$ is an open subset of J, and as such consists of a countable disjoint union of open subintervals of J, say $J_1, J_2, J_3 \ldots$. Now fix such an open interval J_i. Unless J_i is one of the end intervals, J_i is open in $\mathbb{R}$, so we write $J_i = (\alpha_i, \beta_i)$. In this case, T_{α_i} and T_{β_i} are uniquely ergodic, so the invariant measures are determined at the end-points of the interval. If J_i is one of the end intervals, then we just have that it is closed at one end or the other. Now set $K_i = [\alpha_i, \beta_i]$. The idea behind the construction is as follows. Plotting the periodic points of T_α against α gives a graph similar to Figure 5. The invariant measure, being concentrated on the periodic points, must be chosen to be a superposition of δ-measures, moving along the periodic point curves. Since these curves can terminate, it may be necessary to transfer to a new periodic curve. This must also be done continuously, so in the construction, one curve is being phased in on some transitional region of parameter space, while another curve is being phased out.

The arguments used below will be familiar to many, but for completeness, a detailed construction of a continuous choice of invariant measure on the interval K_i follows.

We construct an open cover for K_i. Suppose the rotation number of the maps with parameters in K_i is p/q. This may be assumed by the continuity of ρ. Then let S_α be

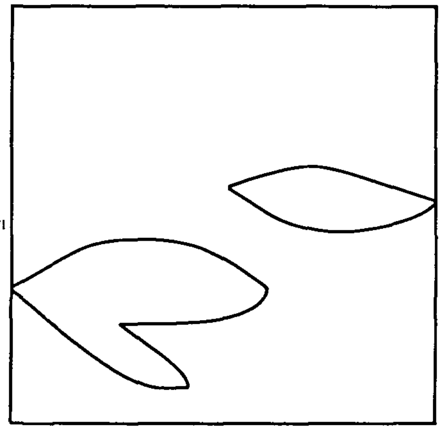

Fig. 5. Typical diagram of periodic points against parameter.

a lift of T_α^q fixing the preimages (under π) of the periodic points, and set $u_\alpha(x) = S_\alpha(x) - x$. Then we see easily that the maps $(\alpha, x) \mapsto S_\alpha(x)$ and $(\alpha, x) \mapsto u_\alpha(x)$ are continuous maps $K_i \times \mathbb{R} \to \mathbb{R}$. Given $\alpha \in K_i$, we seek a connected open neighbourhood N_α containing α, and a continuous map $\phi_\alpha : \mathrm{Cl}\,(N_\alpha) \to \mathbb{R}$ taking each parameter value to a fixed point of S for that parameter value that is such that $S_\beta(\phi_\alpha(\beta)) = \phi_\alpha(\beta)$.

If the periodic points of T_α are all non-hyperbolic, choose ξ to be any periodic point of T_α. If however, T_α has a hyperbolic orbit, choose ξ to be a hyperbolic periodic point of T_α. Then there is a neighbourhood of ξ in which there are no other periodic points of T_α. Let x be a preimage (under π) of ξ. There is then a neighbourhood of x which contains no fixed points of S_α. Choose τ small such that $[x-\tau, x+\tau]$ is in this neighbourhood. Then set $\epsilon = \min(|u_\alpha(x-\tau)|, |u_\alpha(x+\tau)|)$. Then by continuity of u, there exists a $\delta_1 > 0$ such that $|\beta - \alpha| < \delta_1$ implies that $u_\beta \neq 0$ at $x-\tau$ and $x+\tau$. We know also that there are finitely many values of β in the above neighbourhood with T_β critical, so it follows that there is a $\delta > 0$ such that $0 < |\beta - \alpha| < \delta$ implies that $u_\beta \neq 0$ at $x-\tau$ and $x+\tau$, and that T_β has no non-hyperbolic periodic points (note that there may be a non-hyperbolic orbit at α itself, but if so, it either lies outside $\pi([x-\tau, x+\tau])$, or there are no hyperbolic periodic points for T_α).

We then define $N_\alpha = \{\beta : |\alpha - \beta| < \delta\} \cap K_i$ and we define ϕ_α on this reduced interval by the equation

$$\phi_\alpha(\beta) = \sup\{y \in [x-\tau, x+\tau] : u_\beta(y) = 0\}.$$

We claim that ϕ_α is continuous. If ϕ_α is not continuous, there exists a sequence $(\beta_i)_{i \in \mathbb{N}}$ of points in N_α tending to some $\beta \in N_\alpha$ such that $\phi_\alpha(\beta_j)$ fails to converge to $\phi_\alpha(\beta)$. By passing to a subsequence, we may assume that the $\phi_\alpha(\beta_j)$ converge to some other value. But if $\phi_\alpha(\beta)$ is smaller than this limit, then we get a contradiction by noting that $u_\beta(\lim \phi_\alpha(\beta_j)) = 0$, so that $\phi_\alpha(\beta)$ was not in fact the supremum of those fixed points in the range of interest. Conversely, if $\phi_\alpha(\beta)$ is greater than the limit, then T_β certainly has at least two periodic orbits in $[x-\tau, x+\tau]$, and so by construction we see that T_β has no non-hyperbolic points. It follows that the periodic orbit through $\pi(\phi_\alpha(\beta))$ is hyperbolic, and, therefore, that there exist periodic points arbitrarily close to this one for all $T_{\beta'}$ with β' sufficiently close to β, by Lemma 2. This then gives the required contradiction, since it shows that we should have had a larger value for $\phi_\alpha(\beta_j)$ for large j. We have therefore established as required that for each $\alpha \in K_i$, there

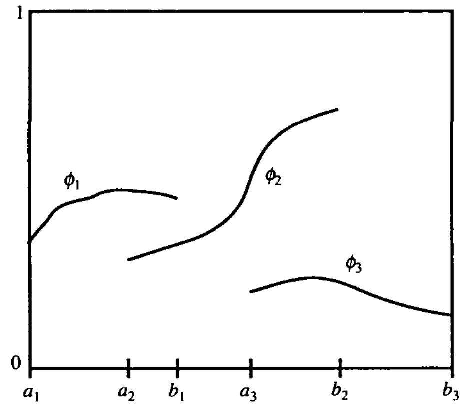

Fig. 6. Possible arrangement of chosen periodic points.

exists a neighbourhood N_α in K_i, containing the point α, and a continuous function ϕ_α defined on N_α, such that $\phi_\alpha(\beta)$ is a fixed point of S_β, where defined. By reducing each N_α, we may further assume that ϕ_α is continuous on the closure of N_α, that the only neighbourhood containing α_i is N_{α_i} and the only neighbourhood containing β_i is N_{β_i}. We then have found an open cover of K_i, and so may apply compactness of K_i to pick out a finite subcover. Note that by construction N_{α_i} and N_{β_i} must be in the subcover. We may assume that this subcover is minimal by inclusion (that is, that there is no smaller subcover, each of whose sets is a member of our chosen subcover). We label the sets in the open cover in the order of the left-most point from left to right as $N_1, N_2, \ldots, N_k$, and write $N_j = (a_j, b_j)$ for $1 < j < k$; $N_1 = [a_1, b_1)$; $N_k = (a_k, b_k]$, where we have taken $b_k = \beta_i$ and $a_1 = \alpha_i$. Let ϕ_j be the ϕ function associated to the interval N_j. We then have

$$a_1 < a_2 < b_1 \leqslant a_3 < b_2 \leqslant a_4 < \ldots \leqslant a_k < b_{k-1} < b_k$$

by the minimality of the cover. To see this, note that clearly the sequence of a's is increasing by construction. The sequence of b's must also be increasing, since otherwise one of the intervals is completely contained in another. We need that $N_j \cup N_{j+2} \not\supset N_{j+1}$ giving the condition that $b_j \leqslant a_{j+2}$, and the condition $a_{j+1} < b_j$ arises from the requirement that the collection be a cover.

In Figure 6, an example of such a configuration is shown. We are then in a position to construct the invariant measures for the T_α with $\alpha \in K_i$. Define

$$\nu_\alpha = \begin{cases} \dfrac{b_j - \alpha}{b_j - a_{j+1}} \delta_{\pi(\phi_j(\alpha))} + \dfrac{\alpha - a_{j+1}}{b_j - a_{j+1}} \delta_{\pi(\phi_{j+1}(\alpha))} & (\alpha \in [a_{j+1}, b_j]) \\ \delta_{\pi(\phi_{j+1}(\alpha))} & (\alpha \in [b_j, a_{j+2}]) \end{cases}$$

where δ_ξ is the δ-measure with unit mass concentrated at ξ. Given a continuous function $f \in C(S^1)$,

$$\int f\, d\nu_\alpha = \begin{cases} \dfrac{b_j - \alpha}{b_j - a_{j+1}} f(\pi(\phi_j(\alpha))) + \dfrac{\alpha - a_{j+1}}{b_j - a_{j+1}} f(\pi(\phi_{j+1}(\alpha))) & (\alpha \in [a_{j+1}, b_j]) \\ f(\pi(\phi_{j+1}(\alpha))) & (\alpha \in [b_j, a_{j+2}]). \end{cases}$$

Continuity is clear everywhere except at the a_j and b_j, and this can be checked by comparing the expressions and using the fact that the ϕ functions were chosen to be continuous on the closures of the interval. We can then see that v_α is a continuous family of probability measures concentrated on periodic points of T_α, so that forming

$$\mu_\alpha = \frac{1}{q}\sum_{i=0}^{q-1} v_\alpha \circ T_\alpha^{-i}$$

gives a continuous family of measures with the measure concentrated on periodic orbits, and with the same value for each point of the orbit, so by Lemma 1, a continuous family of invariant measures on K_i. Notice that by the construction, since we forced $N_1 = N_{\alpha_i}$ and $N_k = N_{\beta_i}$, the limit of the measures as they approach the end-points is just the required measure in the case that the map at the end-point is uniquely ergodic.

Repeating this process inductively, we will be able to define a family of invariant Borel probability measures, one measure for each parameter in the set $J\backslash C$, and so since we have already determined the uniqueness of invariant probability measures for parameter values in C, we have defined the whole family of invariant measures with parameters in J. The family thus constructed has already been shown to be continuous on all intervals contained in $J\backslash C$, and therefore, since $J\backslash C$ is an open set, it follows that the map $M:\alpha \mapsto \mu_\alpha$ defined on J is continuous at points of $J\backslash C$. It remains therefore to show that this map is also continuous at points of C.

Now fix $\alpha\in C$. To show continuity of the family at α, we need to show that we can bound $|\int f\,d\mu_\alpha - \int f\,d\mu_\beta|$ to be as small as we like by restricting β to lie near α for a fixed $f\in C(S^1)$. We proceed by estimating $|\int f\,d\mu_\alpha - \int f\,d\mu_\beta|$. Note that by the T_β-invariance of μ_β, we have $\int f\,d\mu_\beta = \int F_\beta^{(n)}\,d\mu_\beta$ for all $n\in\mathbb{N}$. Also, by (3), we see that $\int f\,d\mu_\alpha = \lim\int F_\alpha^{(n)}\,d\mu_\beta$. We therefore have

$$\int f\,d\mu_\alpha - \int f\,d\mu_\beta = \lim\int (F_\alpha^{(n)} - F_\beta^{(n)})\,d\mu_\beta,$$

so that a bound on $|F_\alpha^{(n)}(x) - F_\beta^{(n)}(x)|$ for each $x\in S^1$ and for each n in some increasing sequence of natural numbers gives rise to a bound on $|\int f\,d\mu_\alpha - \int f\,d\mu_\beta|$.

LEMMA 3. *Suppose that we are given a parameter value α, an increasing sequence (q_n) of natural numbers and an algebra $\mathscr{F}$ of continuous functions on S^1 with the following properties*: (i) *T_α is uniquely ergodic*; (ii) *$q_n = r_n q_{n-1} + s_n q_{n-2}$ with r_n and s_n non-negative integers*; (iii) *$\mathscr{F}$ contains the constant functions and separates points of S^1* (*that is, given two distinct points x, y of S^1, there exists $f\in\mathscr{F}$ such that $f(x) \neq f(y)$*); (iv) *for each $f\in\mathscr{F}$*,

$$\sum_{n=1}^{\infty} E_n(f) < \infty, \tag{4}$$

where $E_n(f)$ is defined by the equations

$$E_n(f) = \frac{1}{q_n}(r_n q_{n-1} B_{q_{n-1}}(f) + s_n q_{n-2} B_{q_{n-2}}(f)), \quad B_n(f) = \sup_{x,y\in S^1} |F_\alpha^{(n)}(x) - F_\alpha^{(n)}(y)|. \tag{5}$$

Then the map $M:\beta\mapsto\mu_\beta$ is weak-continuous at the parameter value α.*

Proof. Suppose that the conditions of the lemma hold. Then take $g\in C(S^1)$, the space of continuous functions on the circle, and $\epsilon > 0$. We will show that there exists

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a $\delta > 0$ such that if $|\beta - \alpha| < \delta$, then $|\int g\,d\mu_\alpha - \int g\,d\mu_\beta| < \epsilon$. This will demonstrate the weak*-continuity of the family of measures as required.

By the Stone–Weierstrass theorem, there exists $f \in \mathscr{F}$ such that $|g(x) - f(x)| < \frac{1}{4}\epsilon$ for each $x \in S^1$. We then define $F_\alpha^{(n)}$ as the usual ergodic sum of f composed with powers of T_α as in (2). Let q_n and $B_n(f)$ be as in the statement of the lemma, and make the following definitions:

$$A_n(f,\beta) = \sup_{x \in S^1} |F_\alpha^{(n)}(x) - F_\beta^{(n)}(x)|, \quad D_n(f,\beta) = \max_{1 \leqslant i \leqslant n} A_{q_i}(f,\beta).$$

By (2), we see that $F_\alpha^{(q_n)}(x) - F_\beta^{(q_n)}(x)$ is equal to

$$\frac{q_{n-1}}{q_n} \sum_{i=0}^{r_n - 1} (F_\alpha^{(q_{n-1})}(T_\alpha^{iq_{n-1}} x) - F_\beta^{(q_{n-1})}(T_\beta^{iq_{n-1}} x))$$

$$+ \frac{q_{n-2}}{q_n} \sum_{i=0}^{s_n - 1} (F_\alpha^{(q_{n-2})}(T_\alpha^{r_n q_{n-1} + iq_{n-2}} x) - F_\beta^{(q_{n-2})}(T_\beta^{r_n q_{n-1} + iq_{n-2}} x)).$$

Each of the terms in the sums is of the form $F_\alpha^{(k)}(x) - F_\beta^{(k)}(y)$. By the triangle inequality, we have

$$|F_\alpha^{(k)}(x) - F_\beta^{(k)}(y)| \leqslant |F_\alpha^{(k)}(x) - F_\alpha^{(k)}(y)| + |F_\alpha^{(k)}(y) - F_\beta^{(k)}(y)|.$$

This in turn is bounded above by $A_k(f,\beta) + B_k(f)$. It follows that

$$|F_\alpha^{(q_n)}(x) - F_\beta^{(q_n)}(x)| \leqslant \frac{r_n q_{n-1}}{q_n}(A_{q_{n-1}}(f,\beta) + B_{q_{n-1}}(f)) + \frac{s_n q_{n-2}}{q_n}(A_{q_{n-2}}(f,\beta) + B_{q_{n-2}}(f)).$$

This implies that $A_{q_n}(f,\beta) \leqslant D_{n-1}(f,\beta) + E_n(f)$, but it is also true that $D_{n-1}(f,\beta) \leqslant D_{n-1}(f,\beta) + E_n(f)$, so since we have $D_n(f,\beta) = \max(D_{n-1}(f,\beta), A_{q_n}(f,\beta))$, we get $D_n(f,\beta) \leqslant D_{n-1}(f,\beta) + E_n(f)$. It therefore follows that

$$D_N(f,\beta) \leqslant D_{n-1}(f,\beta) + \sum_{i=n}^{N} E_i(f).$$

By hypothesis, the sum on the right is a convergent sum of positive terms as $N \to \infty$, so there is an n such that

$$\sum_{i=n}^{\infty} E_i(f) < \tfrac{1}{4}\epsilon.$$

Now $D_{n-1}(f,\beta)$ is a continuous function of β taking the value 0 when $\beta = \alpha$, so that there exists a neighbourhood U of α on which $D_{n-1}(f,\beta)$ takes values less than $\frac{1}{4}\epsilon$. But if $\beta \in U$, then $D_m(f,\beta) < \frac{1}{2}\epsilon$ for each $m > n$. This implies that $|\int f\,d\mu_\alpha - \int f\,d\mu_\beta| < \frac{1}{2}\epsilon$, so using the facts that $|f(x) - g(x)| < \frac{1}{4}\epsilon$ for $x \in S^1$ and $\mu_\alpha(S^1) = \mu_\beta(S^1) = 1$ gives the result $|\int g\,d\mu_\alpha - \int g\,d\mu_\beta| < \epsilon$, proving the lemma.

Applying this lemma to points of C will finally complete the proof of the theorem. Suppose $\alpha \in C$ and $\rho(T_\alpha)$ is irrational. Then the number $\rho = \rho(T_\alpha)$ has a unique continued fraction expansion. (See [1], §6·2 for fuller details.) The convergents to ρ are rationals p_n/q_n consisting of the first n terms of the partial fraction expansion. These have the property that

$$\left|\frac{p_n}{q_n} - \rho\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

This, however is the condition for the Denjoy–Koksma inequality (see [**2**], §3·4, lemma 1) to operate. The precise statement of this is that if T is a circle map with irrational rotation number, the rational p/q is such that $|p/q-\rho(T)| < 1/q^2$ and $f\in C(S^1)$ is a function of bounded variation, then

$$\left|\frac{1}{q}\sum_{i=0}^{q-1} f(T^i x) - \int f\,d\mu\right| < \frac{\operatorname{Var}(f)}{q}$$

where μ is the invariant Borel probability measure for T and $\operatorname{Var}(f)$ is the variation of the function. It follows that $|F_\alpha^{(q_n)}(x) - F_\alpha^{(q_n)}(y)| < 2\operatorname{Var}(f)/q_n$ for each $n\in\mathbb{N}$ and $x, y\in S^1$. That is, there exists a K such that $B_{q_n} < K/q_n$ for each n. Notice that by standard continued fraction theory, $q_n = a_n q_{n-1} + q_{n-2}$ with $a_n > 0$, so the sequence q_n satisfies the conditions of Lemma 3. Take $\mathscr{F}$ to be any algebra of functions of bounded variation satisfying the conditions of Lemma 3, say the trigonometric polynomials. Then calculating $E_n(f)$ as defined in (5), we get

$$E_n(f) \leqslant \frac{1}{q_n}\left(r_n q_{n-1}\frac{K}{q_{n-1}} + s_n q_{n-2}\frac{K}{q_{n-2}}\right) = K\frac{r_n+s_n}{r_n q_{n-1} + s_n q_{n-2}} \leqslant \frac{K}{q_{n-2}}.$$

From this, it follows that the sum in (4) is convergent, since the denominators q_n grow at least as fast as g^n where g is the golden ratio, and hence that the map M is continuous at those values of the parameter α which are in C and have $\rho(T_\alpha)$ irrational.

The final part of the proof is then to show that the map M is continuous at the remaining parameter values (that is, those with $\alpha\in C$ and $\rho(T_\alpha)$ rational). Suppose then that $\rho(T_\alpha) = p/q$, say, but that $\alpha\in C$. In this case, we know that the map T_α is critical with a single periodic orbit which is non-hyperbolic. It follows therefore, that everything tends under iteration to this periodic orbit.

In this case, we take $q_n = 2^n q$ (with $r_n = 2$ and $s_n = 0$). This satisfies the conditions for a sequence in Lemma 3, so now we need to check that there exists an algebra of $\mathscr{F}$ of functions such that for each $f\in\mathscr{F}$,

$$\sum_{n=1}^{\infty} E_n(f) < \infty \quad \text{which is equivalent to} \quad \sum_{n=1}^{\infty} B_{2^n q}(f) < \infty.$$

To do this, pick $a_0\in S^1$ which is not periodic and set $a_i = T_\alpha^i(a_0)$. Then the sequence a_{i+nq} tends monotonically to a periodic point as $n\to\infty$ for each value of i. We may assume without loss of generality that the sequence $(a_{i+nq})_{n\in\mathbb{N}}$ moves anticlockwise. We define

$$\xi_j = \lim_{n\to\infty} a_{nq+j}, \quad \tau_j = \lim_{n\to\infty} a_{-nq+j}.$$

Then we may express S^1 as the disjoint union

$$S^1 = \bigcup_{i\in\mathbb{Z}} [a_i, a_{i+q}) \cup \{\xi_i : 1\leqslant i\leqslant q\}.$$

We set

$$\mathscr{F} = \left\{ f\in C(S^1) : \exists K \text{ such that } \begin{bmatrix} |f(\zeta)-f(\xi_j)| < K/2^n & \zeta\in[a_{j+(n-1)q}, a_{j+nq}) & \forall n\geqslant 0 \\ |f(\zeta)-f(\tau_j)| < K/2^n & \zeta\in[a_{j-nq}, a_{j-(n-1)q}) & \forall n\geqslant 0 \end{bmatrix} \text{ for } 1\leqslant j\leqslant q \right\}.$$

This may easily be shown to satisfy the conditions of Lemma 3. We then take $f \in \mathscr{F}$ with constant K and estimate $B_{2^n q}(f)$. Note that

$$B_{2^n q}(f) \leqslant 2 \sup_{\zeta \in S^1} |F_\alpha^{(2^n q)}(\zeta) - F_\alpha^{(2^n q)}(\xi_1)|$$

and $$|F_\alpha^{(2^n q)}(\zeta) - F_\alpha^{(2^n q)}(\xi_1)| \leqslant \frac{1}{2^n q} \sum_{k=0}^{2^n - 1} \left| \sum_{i=0}^{q-1} f(T_\alpha^{qk+i}(\zeta)) - \sum_{i=0}^{q-1} f(\xi_i) \right|.$$

Now assume $\zeta \in [a_{j+(m-1)q}, a_{j+mq})$. Then $T_\alpha^{qk+i}(\zeta) \in [a_{j+i+(m+k-1)q}, a_{j+i+(m+k)q})$. In particular,

$$T_\alpha^{qk+i}(\zeta) \in [a_{(m+k-1)q+((j+i) \bmod q)}, a_{(m+k+1)q+((j+i) \bmod q)}),$$

where $(j+i) \bmod q$ is the representative of the conjugacy class of $j+i$ modulo q in the range 1 to q. If $m+k \geqslant 0$, then

$$|f(T_\alpha^{qk+i}(\zeta)) - f(\xi_{i+j})| < K/2^{m+k} = K/2^{|m+k|}.$$

If $m+k < 0$, then $|f(T_\alpha^{qk+i}(\zeta)) - f(\tau_{i+j})| < K/2^{-(m+k)} = K/2^{|m+k|}$.

We see therefore that $$\left| \sum_{i=0}^{q-1} f(T_\alpha^{qk+i}(\zeta)) - \sum_{i=0}^{q-1} f(\xi_i) \right| \leqslant \frac{Kq}{2^{|m+k|}}.$$

Hence $$|F_\alpha^{(2^n q)}(\zeta) - F_\alpha^{(2^n q)}(\xi_1)| \leqslant \frac{1}{2^n q} \sum_{k=0}^{2^n - 1} \frac{Kq}{2^{|m+k|}} \leqslant \frac{2K}{2^n} \sum_{k=0}^{2^n - 1} \frac{1}{2^k} \leqslant \frac{K}{2^{n-2}}$$

and we conclude that the series $\sum B_{2^n q}(f)$ is convergent. Applying Lemma 3 completes the proof of the theorem.

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REFERENCES

[1] A Baker. *A Concise Introduction to the Theory of Numbers* (Cambridge University Press, 1984).
[2] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai. *Ergodic Theory* (Springer-Verlag, 1982).
[3] M. R. Herman. Sur la conjugaison differentiable des diffeomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.* **49** (1979), 5–233.
[4] P. Walters. *An Introduction to Ergodic Theory* (Springer-Verlag, 1981).