A C^1 EXPANDING MAP OF THE CIRCLE WHICH IS NOT WEAK-MIXING

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Abstract. In this paper, we construct an example of a C^1 expanding map of the circle which preserves Lebesgue measure such that the system is ergodic, but not weak-mixing. This contrasts with the case of $C^{1+\epsilon}$ maps, where any such map preserving Lebesgue measure has a Bernoulli natural extension and hence is weak-mixing.

1. Introduction

In this paper, we apply techniques of [6] to prove the following theorem.

Theorem 1. *There is a C^1 expanding map of the circle preserving Lebesgue measure, such that Lebesgue measure is ergodic for the map, but not weak-mixing.*

This is in contrast with results for the $C^{1+\epsilon}$ case, where it is known that if such a map preserves Lebesgue measure, then the natural extension of the transformation is Bernoulli (see [7]). Previously, Bose (in [2]) has established the existence of a piecewise monotone and continuous expansive map preserving Lebesgue measure which is weak-mixing but not ergodic. (He also found piecewise monotone and continuous maps which are weak- but not strong-mixing; and strong-mixing but not exact). These proofs were based on the construction of generalized baker's transformations (see [1] for details).

We will make extensive use of g-measures in what follows. For a fuller description of g-measures, the reader is referred to [4], [5] and [6]. Here, we will construct a g-function on the symbol space $\Sigma_{10} \equiv \{0,\dots,9\}^{\mathbb{Z}^+} = \{x_0x_1x_2\dots : x_i \in \{0,\dots,9\}\}$ with shift map σ (that is a continuous function g satisfying $0 < g(x) < 1$ for all x and $\sum_{y\in\sigma^{-1}(x)} g(y) = 1$ for all x). Given such a g, we consider sequences of random variables $(X_n) : \Omega \to \{0,\dots,9\}$ satisfying

$$\mathbb{P}(X_n = i | X_{n-1} = a_1,\ X_{n-2} = a_2,\ \dots) = g(i, a_1, a_2, \dots), \tag{1}$$

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for all n. There are then natural maps $\rho_n : \Omega \to \Sigma_{10}$ defined by $\rho_n(\omega) = X_{n-i}(\omega)$. These maps induce natural push-forward maps of probability distributions on Ω to probability measures on Σ_{10} defined by $\rho_n^*(\mathbb{P})(A) = \mathbb{P}(\rho_n^{-1}(A))$. A g-measure is a push-forward under ρ_0^* of any stationary distribution. Another way of characterizing g-measures on symbol spaces is that a g-measure is a measure ν satisfying

$$\lim_{n\to\infty} \frac{\nu([ix]^{n+1})}{\nu([x]^n)} = g(ix), \tag{2}$$

for all $x \in \Sigma_{10}$, where $[x]^n$ denotes the cylinder of those points of Σ_{10} which agree with x for the first n terms, and ix denotes the sequence in Σ_{10} which consists of the symbol i followed by the sequence x.

We will need to consider g-functions which have the property of *compatibility* introduced in [5], that is $g(000\dots) = g(999\dots)$ and $g(ai999\dots) = g(aj000\dots)$, for any $0 \leqslant i < 9$, $j = i+1$, and any finite word a. We will need the following result from [6].

Theorem 2. *Let g be a compatible g-function on Σ_r. Then if ν is a g-measure, there is a C^1 expanding map $T : S^1 \to S^1$ preserving Lebesgue measure λ, such that (σ, Σ_r, ν) is measure-theoretically isomorphic to (T, S^1, λ).*

It will then be sufficient to construct an example of a compatible g-function having a g-measure which is ergodic but not weak-mixing.

We start with some preliminary definitions. As in [6], we introduce a partial order on Σ_{10}. First define $3 \preceq i \preceq 6$ for any $0 \leqslant i \leqslant 9$. Then $x \preceq y$ if $x_i \preceq y_i$ for all $i \in \mathbb{Z}^+$. A function $f : \Sigma_{10} \to \mathbb{R}$ is called *monotonic* if $f(x) \leqslant f(y)$ whenever $x \preceq y$. We will say that a function $f : \Sigma_{10} \to \mathbb{R}$ is *precompatible* if $f(090909\dots) = f(909090\dots)$ and $f(ai090909\dots) = f(aj909090\dots)$, where a is any finite word, i is any symbol with $0 \leqslant i < 9$ and $j = i+1$. We write this second condition as $f(b, 090909\dots) = f(b+1, 909090\dots)$ for any finite word b not ending in a 9.

We will need to consider the involutions on Σ_{10} given by

$$F(x)_n = \begin{cases} 9 - x_n & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even} \end{cases}$$
$$R(x)_n = 9 - x_n.$$

Write $\bar{x}$ for $R(x)$, $\hat{x}$ for $F(x)$ and $\tilde{x}$ for $R \circ F(x)$. We say that a function f is *symmetric* if $f(\bar{x}) = f(x)$ for all x.

2. Construction of the Example

To construct the example, we will use the following lemma.

Lemma 3. *There exists a precompatible, compatible, symmetric, monotonic g-function h with the property that if one considers random variables (X_n) evolving as*

$$\mathbb{P}(X_n = i | X_{n-1} = a_1,\ X_{n-2} = a_2,\ \dots) = h(i, a_1, a_2, \dots),$$

conditioned upon $X_i = 6$, for all $i < 0$, then there exists a $\beta > \frac{1}{2}$ such that $\mathbb{P}(X_n = 6) \geqslant \beta$ for all n.

We will write $\mathbb{P}_6$ for the probability distribution on (X_n) defined in this way. The construction shown here differs from the construction in [6] only in the initial stages. The reader should note that that paper in turn is based on [3].

Proof. Define $\delta(x) = \chi_6(x) - \chi_3(x)$, where $\chi_i(x)$ is 1 if $x_0 = i$ and 0 otherwise. Then let $\Delta_m(x) = \sum_{i=0}^{m-1} \delta(\sigma^i(x))$. To construct h, we will need to define a collection of functions $W^i_{m,n} : \Sigma_{10} \to (0,1)$ indexed by $0 \leqslant i \leqslant 9$ and $m > n > 0$. These will be based on a family of functions $V_{m,n}$ whose existence is asserted by the following lemma.

Lemma 4. *There exists a family $V_{m,n}$ (where $m > n > 0$) of compatible, precompatible, monotonic Hölder continuous functions satisfying*

$$0 \leqslant V_{m,n}(x) \leqslant 1$$

$$V_{m,n}(x) = \begin{cases} 1 & \text{if } \Delta_m(x) > n \\ 0 & \text{if } \Delta_m(x) < n \end{cases}$$

The construction of the $V_{m,n}$ is rather involved and is (in the author's opinion) a distraction from the main flow of the paper. It has therefore been relegated to an appendix to the paper. Once the $V_{m,n}$ have been defined, the $W_{m,n}$ are defined as follows:

$$W^6_{m,n}(x) = \tfrac{1}{10} + \tfrac{1}{2} V_{m,n}(x)$$

$$W^3_{m,n}(x) = W^6_{m,n}(\bar{x})$$

$$W^i_{m,n}(x) = \tfrac{1}{10} - \tfrac{1}{16}(V_{m,n}(x) + V_{m,n}(\bar{x})) \text{ for } i \neq 3, 6.$$

Note that for each x, $\sum_{i=0}^{9} W^i_{m,n}(x) = 1$ and since we require $n > 0$, we have that for each x, only one of $V_{m,n}(x)$ and $V_{m,n}(\bar{x})$ is positive. This implies that $W^i_{m,n}(x)$ is bounded below by $\frac{3}{80}$ for $i \neq 3, 6$. The function h will then be given by $h(ix) = \sum_{j=1}^{\infty} \frac{1}{2}(\frac{2}{3})^j W^i_{m_j,n_j}$, where m_j and n_j are appropriately chosen increasing sequences with $n_j < m_j < n_{j+1}$. The proof that m_j and n_j can be chosen so as to make h have the stated properties is identical to the proof in [6]. □

3. Proof of Theorem 1

In this section, we use the results of §2 to prove Theorem 1, subject to the construction of $V_{m,n}$ in the appendix.

Proof of Theorem 1.
Let h and $\mathbb{P}_6$ be as defined in the previous section. Take $\mu_n = \rho_n^*(\mathbb{P}_6)$ and form Cesàro sums $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} \mu_i$. Then we see (as in [6]) that if ν_{n_i} is a weak*-convergent subsequence, converging to a measure ν, then ν is an h-measure. We see also that $\nu([6])$, the measure of those members of Σ_{10} starting with a 6 is at least β. We may assume ν is ergodic, for otherwise, by ergodic decompostion, there is another h-measure with this property. If ν is not ergodic with respect to σ^2, then one can check that there exist sets A and B of measure $\frac{1}{2}$ such that $\sigma^{-1}(A) = B$ and $\sigma^{-1}(B) = A$. It then follows quickly that ν is ergodic but not weak-mixing and by Theorem 2 and the compatibility of h, Theorem 1 follows. It remains to consider the case where ν is ergodic with respect to σ^2. We note that the involution

F defined above is not shift-commuting, but that F does commute with σ^2. Define a new measure μ by $\mu(A) = \frac{1}{2}\nu(\hat{A}) + \frac{1}{2}\nu(\tilde{A})$. This is shift-invariant. Now we have

$$\begin{aligned}\frac{\mu([ix]^{n+1})}{\mu([x]^n)} &= \frac{\frac{1}{2}\nu([\widehat{ix}]^{n+1}) + \frac{1}{2}\nu([\widetilde{ix}]^{n+1})}{\frac{1}{2}\nu([\hat{x}]^n) + \frac{1}{2}\nu([\tilde{x}]^n)} \\ &= \frac{\nu([i\tilde{x}]^{n+1}) + \nu([\bar{i}\hat{x}]^{n+1})}{\nu([\tilde{x}]^n) + \nu([\hat{x}]^n)}.\end{aligned}$$

Then using the symmetry of h, we see $h(i\tilde{x}) = h(\bar{i}\hat{x})$, so we get

$$\lim_{n\to\infty} \frac{\mu([ix]^{n+1})}{\mu([x]^n)} = h(i\tilde{x}) = h \circ F(ix).$$

It follows that μ is a g-measure, where $g = h \circ F$. Note that by the precompatibility of h, g is compatible. It remains to show that μ is ergodic but not weak-mixing. Suppose for a contradiction that $\sigma^{-1}(A) = A$ and $0 < \mu(A) < 1$. Then $\mu(A) = \frac{1}{2}\nu(\hat{A}) + \frac{1}{2}\nu(\tilde{A})$, but $\sigma^{-1}(\tilde{A}) = \hat{A}$ and $\sigma^{-1}(\hat{A}) = \tilde{A}$. It follows that $\nu(\hat{A}) = \nu(\tilde{A})$, so $0 < \nu(\hat{A}) < 1$. But this is a contradiction as $\sigma^{-2}(\hat{A}) = \hat{A}$ and ν is assumed to be ergodic with respect to σ^2, proving that μ is ergodic.

Next, note that μ is not ergodic with respect to σ^2 as $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$, where μ_1 and μ_2 are σ^2-invariant measures defined by $\mu_1(A) = \nu(\hat{A})$ and $\mu_2(A) = \nu(\tilde{A})$. These are not equal as $\mu_1([6]) > \frac{1}{2} > \mu_2([6])$. It follows that μ is not weak-mixing, thus completing the proof of Theorem 1 subject to the proof of Lemma 4 in the appendix. □

Appendix. Construction of $V_{m,n}$.

Proof of Lemma 4. In this appendix, we give the construction of the function $V_{m,n}$, which was introduced in §2. First we define a contraction map $\mathcal{L}$ on the subspace X of $(C[0,1])^4$ with the metric induced by the uniform norm:

$$\begin{aligned}X = \{(f_1, f_2, f_3, f_4) : &f_i : [0,1] \to [0,1];\ f_1(0) = f_3(0) = 0,\ f_1(1) = f_3(1) = 1, \\ &f_2(0) = f_4(0) = 1,\ f_2(1) = f_4(1) = 0\}\end{aligned}$$

We will identify I with Σ_{10} so σ^2 will denote the map $x \mapsto 100x \bmod 1$. The map $\mathcal{L}$ is defined by $\mathcal{L}(f_1, f_2, f_3, f_4) = (g_1, g_2, g_3, g_4)$, where

$$g_1(x) = \begin{cases} 0 & 0 \leqslant x < .04 \\ \frac{1}{2}f_1(\sigma^2(x)) & .04 \leqslant x < .05 \\ \frac{1}{2} + \frac{1}{2}f_1(\sigma^2(x)) & .05 \leqslant x < .06 \\ 1 & .06 \leqslant x < .09 \\ \frac{1}{2} + \frac{1}{2}f_4(\sigma^2(x)) & .09 \leqslant x < .10 \\ \frac{1}{2} - \frac{1}{2}f_4(1-\sigma^2(x)) & .10 \leqslant x < .11 \\ 0 & .11 \leqslant x < .15 \\ \frac{1}{2}f_1(\sigma^2(x)) & .15 \leqslant x < .16 \\ \frac{1}{2} & .16 \leqslant x < .17 \\ \frac{1}{2}f_2(\sigma^2(x)) & .17 \leqslant x < .18 \\ 0 & .18 \leqslant x < .40 \\ \frac{1}{2} - \frac{1}{2}f_3(1-\sigma^2(x)) & .40 \leqslant x < .41 \\ \frac{1}{2}f_2(\sigma^2(x)) & .41 \leqslant x < .42 \\ 0 & .42 \leqslant x < .45 \\ \frac{1}{2}f_1(\sigma^2(x)) & .45 \leqslant x < .46 \\ \frac{1}{2} & .46 \leqslant x < .47 \\ \frac{1}{2}f_2(\sigma^2(x)) & .47 \leqslant x < .48 \\ 0 & .48 \leqslant x < .49 \\ \frac{1}{2}f_3(\sigma^2(x)) & .49 \leqslant x < .50 \\ 1 - g_1(1-x) & .50 \leqslant x \leqslant 1 \end{cases}$$

$$g_2(x) = \begin{cases} 1 - \frac{1}{2}f_4(1-\sigma^2(x)) & 0 \leqslant x \leqslant .01 \\ \frac{1}{2}f_2(\sigma^2(x)) & .01 \leqslant x \leqslant .02 \\ 0 & .02 \leqslant x \leqslant .04 \\ \frac{1}{2}f_1(\sigma^2(x)) & .04 \leqslant x < .05 \\ \frac{1}{2} + \frac{1}{2}f_1(\sigma^2(x)) & .05 \leqslant x < .06 \\ 1 & .06 \leqslant x < .07 \\ \frac{1}{2} + \frac{1}{2}f_2(\sigma^2(x)) & .07 \leqslant x < .08 \\ \frac{1}{2}f_2(\sigma^2(x)) & .08 \leqslant x < .09 \\ 0 & .09 \leqslant x < .15 \\ \frac{1}{2}f_1(\sigma^2(x)) & .15 \leqslant x < .16 \\ \frac{1}{2} & .16 \leqslant x < .19 \\ \frac{1}{2}f_4(\sigma^2(x)) & .19 \leqslant x < .20 \\ g_1(x) & .20 \leqslant x \leqslant .80 \\ 1 - g_2(1-x) & .80 \leqslant x \leqslant 1 \end{cases}$$

$$g_3(x) = \begin{cases} g_1(x) & 0 \leqslant x \leqslant .07 \\ g_2(x) & .07 \leqslant x \leqslant .15 \\ 0 & .15 \leqslant x \leqslant .2 \\ g_1(x) & .2 \leqslant x \leqslant 1 \end{cases}$$

$$g_4(x) = \begin{cases} g_2(x) & 0 \leqslant x \leqslant .07 \\ g_1(x) & .07 \leqslant x \leqslant .15 \\ g_2(x) & .15 \leqslant x \leqslant 1. \end{cases}$$

It is then straightforward to check that $\mathcal{L}$ is indeed a contraction map from X to X, and it follows that there is a unique fixed point, $e = (e_1, e_2, e_3, e_4)$. Using the fact that these form a fixed point of $\mathcal{L}$, it is straightforward to check that if x and y agree for $2n$ digits, then the difference between $e_i(x)$ and $e_i(y)$ is at most 2^{-n}. It follows that the functions e_i are Hölder continuous when considered as functions $\Sigma_{10} \to [0,1]$. Since the functions are continuous as maps $[0,1] \to [0,1]$, it follows that considered as functions $\Sigma_{10} \to [0,1]$, they are compatible.

Next, suppose that $x \prec y$ and x and y differ in either the zeroth or first place. Then it is easy to see that $e_i(x) \leqslant e_i(y)$ for each i just by examining the condition that e is a fixed point of $\mathcal{L}$. Then one checks that $x \prec y$ implies $e_i(x) \leqslant e_i(y)$ for each i by induction on the first place in which they differ. It follows that the functions e_i are monotonic.

We also need to check the precompatibility of the functions e_i. First note the following table of values of the functions e_i. For later use, we include also two additional functions e_5 and e_6 defined by $e_5(x) = 1 - e_3(1-x)$ and $e_6(x) = 1 - e_4(1-x)$.

(3)

	0	.0909...	.9090...	0
e_1	0	1	0	1
e_2	1	0	1	0
e_3	0	0	0	1
e_4	1	1	1	0
e_5	0	1	1	1
e_6	1	0	0	0

It is then a routine matter to check that $e_i(a0909\dots) = e_i(a+1, 9090\dots)$ for each $i \leqslant 4$, where a is any word of length 1 or 2 whose last digit is not a 9. Then by induction on the length of the word, as before, we see that the e_i are precompatible for each $i \leqslant 4$.

We have therefore checked that the e_i $(1 \leqslant i \leqslant 4)$ are monotonic, compatible, precompatible, Hölder continuous and take values as shown in (3). One can check that e_5 and e_6 also have these properties. Further the functions e_i are all equal on the range $0.2 \leqslant x \leqslant 0.8$. This implies that forming f_{ij} defined by

$$f_{ij}(x) = \begin{cases} e_i(x) & x \leqslant .5 \\ e_j(x) & x \geqslant .5 \end{cases}$$

for $3 \leqslant i, j \leqslant 6$ gives 16 functions, each of which is monotonic, compatible, precompatible and Hölder continuous. Looking at (3), we see that these functions take all combinations of values of 0 and 1 on the set $\{0, .0909\dots, .9090\dots, 1\}$. We label the functions according to their values on each of these four points as $d_{i_1 i_2 i_3 i_4}$ so for example d_{0110} takes values 0,1,1 and 0 at $0, .0909\dots, .9090\dots$ and 1 respectively, so $d_{0110} = f_{54}$.

To define $V_{m,n}$, we also need to define two further maps defined on words of $S_m = \{0,\dots,9\}^m$. We have already made implicit use of the equivalence relation $\sim$ generated by $a0909\dots \sim a+1, 9090\dots$, for any word a not ending with a 9 when discussing precompatibility. Given a word $a \in S_m$, define $\phi(a)$ by the requirement that $a0909\dots \sim \phi(a)9090\dots$ and $\psi(a)$ by the requirement that $a9090\dots \sim \psi(a)0909\dots$. We are now in a position to specify $V_{m,n}$. This is defined cylinder by cylinder. If $a \in S_m$, write $[a]$ for those elements of Σ_{10} whose first m digits are given by a. Define $\kappa : S_m \to \{0,1\}$ by $\kappa(b) = 1$ if $\Delta_m(b) > n$ and $\kappa(b) = 0$ otherwise. By $a+1$, we mean the word obtained by adding 1 (with carry if necessary). The word $a-1$ is defined similarly, so for example, $99999+1 = 00000$ and $88900-1 = 88899$. Then given $a \in S_m$, define $N(a) = \kappa(a-1), \kappa(\phi(a)), \kappa(\psi(a)), \kappa(a+1)$. Note that $|\Delta_m(a) - \Delta_m(a+1)|$, $|\Delta_m(a) - \Delta_m(a-1)|$, $|\Delta_m(a) - \Delta_m(\phi(a))|$ and $|\Delta_m(a) - \Delta_m(\psi(a))|$ are all bounded above by 1. Given this, we set

$$V_{m,n}\big|_{[a]}(x) = \begin{cases} 1 & \Delta_m(a) > n \\ 0 & \Delta_m(a) < n \\ d_{N(a)}(\sigma^m(x)) & \Delta_m(a) = n. \end{cases}$$

The function $V_{m,n}$ defined in this way is then seen to be Hölder continuous, monotonic, compatible and precompatible. Further, it satisfies $0 \leqslant V_{m,n} \leqslant 1$, $V_{m,n}(x) = 1$ when $\Delta_m(x) > n$ and $V_{m,n}(x) = 0$ when $\Delta_m(x) < n$ as required. This completes the construction and hence the proof of Lemma 4 and Theorem 1. □

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