INVARIANT DENSITIES FOR C^1 MAPS

Anthony N. Quas

Department of Pure Mathematics and Mathematical Statistics,
University of Cambridge

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Abstract. In this paper, we consider the set of C^1 expanding maps of the circle which have a unique absolutely continuous invariant probability measure whose density is unbounded, and show that this set is dense in the space of C^1 expanding maps with the C^1 topology. This is in contrast with results for C^2 or $C^{1+\epsilon}$ maps, where the invariant densities can be shown to be continuous.

For expanding maps of the circle which are C^2 or $C^{1+\epsilon}$ (that is differentiable with Hölder continuous derivative), there is always a unique absolutely continuous invariant probability measure, whose density is continuous and strictly positive. These functions will be called invariant densities. These maps with their unique absolutely continuous invariant measures form exact systems (see [4]). This paper deals with the case of C^1 expanding maps.

Throughout this paper, let $E^1(M)$ denote the space of expanding C^1 mappings of a compact manifold M to itself with the C^1 topology. In [3], Krzyżewski showed that the subset $A \subset E^1(M)$ of those mappings which have no absolutely continuous invariant probability measure with strictly positive continuous density is residual or of second category in $E^1(M)$. This means that topologically 'most' mappings fail to have absolutely continuous invariant probability measures which have continuous densities bounded above 0. Clearly there are a number of ways in which this failure can take place: One way is for there to be no absolutely continuous invariant probability measure. In the case where M is the unit circle, S^1, Góra and Schmitt showed that this can occur (see [1]). A second possibility is there may be examples which have absolutely continuous invariant densities which fail to be continuous or fail to be bounded above 0 although no examples of this type are in the literature. In particular, the question might be asked as to whether there are examples of C^1 expanding maps which have an unbounded invariant density. In this paper, we use cocycles to answer this question, showing that the set of C^1 expanding maps of the circle which have invariant densities which are not essentially bounded away from either 0 or ∞ is dense in the space of all C^1 expanding maps of the circle.

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Other possibilities of behaviour for a C^1 expanding map of the circle is that there is a non-ergodic absolutely continuous invariant measure or one which is ergodic but not weak-mixing. This is again in contrast with the C^2 or $C^{1+\epsilon}$ case (see [5] and [6] for details).

Theorem 1. *The set of C^1 expanding maps of the circle preserving a unique absolutely continuous invariant probability measure where the density has essential infimum 0 and essential supremum ∞ is dense in $E^1(S^1)$.*

To prove the theorem, first note that the set of C^2 expanding maps of the circle is dense in $E^1(S^1)$. It is therefore sufficient to show that for any C^2 expanding map, there is a map arbitrarily close to it in E^1 with unbounded invariant density. Now fix a C^2 expanding map T_0 of the circle. For simplicity, we will assume that T_0 is a degree 2 orientation-preserving map. For other degrees, the proof is similar. By the results in [4], T_0 is known to have a unique absolutely continuous invariant measure μ. The measure μ is exact with respect to T_0 and its density ρ with respect to Lebesgue measure λ is continuous. To prove the theorem, we will prove two lemmas, one showing the sufficiency of constructing a cocycle with certain properties and the other one constructing the cocycle.

Lemma 2. *Suppose F is an integrable function with the properties*
(i) $\exp(F)$ *is integrable and* $\int \exp(F)\, d\lambda = 1$
(ii) $F(T_0(x)) - F(x)$ *is equal almost everywhere to a continuous function $h(x)$, with the property that $h(x) > -\log\inf_{y\in S^1} T_0{}'(y)$ for all x.*

Then there is a C^1 expanding map of the circle with an absolutely continuous invariant measure which is measure-theoretically isomorphic and topologically conjugate to T_0 by a conjugacy θ: $S^1 \to S^1$ in such a way that the density of the absolutely continuous invariant measure for T is given by $\exp\bigl(-F(\theta(x))\bigr)\rho(\theta(x))$.

Note that in the statement of this lemma, integrability with respect to μ is equivalent to integrability with respect to Lebesgue measure λ because the density ρ is bounded above and below by positive numbers.

Proof. In this proof, we consider the circle to be labelled by points in the interval [0,1). Define θ by specifying $\theta^{-1}(x) = \int_{[0,x]} \exp(F)\, d\lambda$. Then since F is finite almost everywhere, we see that θ is a homeomorphism of the circle. Then T is defined by $\theta^{-1}\circ T_0 \circ \theta$. It follows that T preserves the push-forward of μ under θ^{-1}, that is the measure $\mu\circ\theta$.

Clearly, θ^{-1} is absolutely continuous. We now show that θ is also absolutely continuous. Define the absolutely continuous measure ν by $d\nu/d\lambda(x) = \exp(F(x))$, where λ is Lebesgue measure. Then $\nu([0,x]) = \theta^{-1}(x) = \lambda[0,\theta^{-1}(x)] = \lambda\circ\theta^{-1}([0,x])$. It follows that $\nu = \lambda\circ\theta^{-1}$. Since $0 < \exp F < \infty$ almost everywhere, it follows that $\nu(A) = 0$ if and only if $\lambda(A) = 0$. It follows that A has Lebesgue measure 0 if and only if $\theta^{-1}(A)$ has measure 0 and taking $A = \theta(B)$, it follows that B has measure 0 if and only if $\theta(B)$ has measure 0. It follows that θ is absolutely continuous. It therefore follows that T is absolutely continuous.

A quick calculation shows that the derivative of T is given almost everywhere by $T_0{}'(\theta(x))\exp\bigl(F(T_0(\theta(x)))\bigr)/\exp\bigl(F(\theta(x))\bigr) = T_0{}'(\theta(x))\exp h(\theta(x))$. Since T is absolutely continuous, it follows that the derivative of T is equal to $T_0{}'(\theta(x))\exp h(\theta(x))$ everywhere. The density of $\mu\circ\theta$ at x is given by $\rho(\theta(x))\theta'(x)$, which is equal to $\rho(\theta(x))\exp\bigl(-F(\theta(x))\bigr)$ proving the lemma. □

Lemma 3. *There exists an integrable function F whose essential supremum is ∞ and whose essential infimum is 0 on the circle such that $F(T_0(x)) - F(x)$ is equal to a continuous function $h(x)$ almost everywhere and $\exp F$ is integrable.*

Proof. We may assume (by relabelling if necessary) that 0 is a fixed point of T_0. The preimages of 0 are then 0 and c for some point $c \in (0,1)$. Write τ_L and τ_R for the inverse branches of T_0, mapping the circle onto $[0,c)$ and $[c,1)$ respectively. Let $g\colon S^1 \to [-1,1]$ be a continuous function such that

(i) The conditional expectation $\mathbb{E}_\mu[g|T^{-1}\mathcal{B}]$ is 0;

(ii) $g|_{[0,\tau_L(c))} > 0$ and $g|_{(\tau_L(c),\tau_R(c))} < 0$.

We now check that such a function exists. To satisfy (i), it is necessary that for any x, with preimages y_L and y_R, that $g(y_L)\rho(y_L) + g(y_R)\rho(y_R) = 0$. This is satisfied if we set $g(\tau_L(x)) = k(x)/\rho(\tau_L(x))$ and $g(\tau_R(x)) = -k(x)/\rho(\tau_R(x))$, where k is any continuous function on [0,1). For continuity of g at 0 and c, we require $k(0) = -k(1)$. To satisfy (ii), we require k is positive on $[0,c)$ and negative on $[c,1)$. We may ensure that $\|g\|_\infty = 1$ by scaling by a positive constant.

Then define

$$F_a^b(x) = \sum_{k=a}^{b-1} \frac{g({T_0}^k(x))}{k+1}$$
$$F_n(x) = F_n^\infty(x);\ F^N(x) = F_0^N(x);\ F(x) = F_0^\infty(x)$$

First, note that $g \circ {T_0}^m$ is orthogonal to $g \circ {T_0}^n$ for all $n \neq m$ with respect to the invariant measure μ. To see this, note that if $m > n$, then $\int (g \circ {T_0}^n)\cdot(g \circ {T_0}^m)\, d\mu = \int g \cdot (g \circ {T_0}^{m-n})\, d\mu$. Since $g \circ {T_0}^{m-n}$ is measurable with respect to ${T_0}^{-1}\mathcal{B}$, this is equal to $\int \mathbb{E}_\mu[g|{T_0}^{-1}\mathcal{B}] \cdot (g \circ {T_0}^{m-n})\, d\mu$ which is 0 as required.

It follows that F^N converges in the L^2 norm to F. The function F is therefore integrable. Similarly, F_n exists as an L^2 function with ${\|F_n\|_2}^2$ uniformly bounded above by $\pi^2/6$ and that F_n converges in the L^2 norm to 0. In particular, $\|F_n\|_1$ is a uniformly bounded sequence. By a similar argument to the above, $\mathbb{E}_\mu[g \circ {T_0}^n|{T_0}^{-m}\mathcal{B}] = 0$ whenever $m > n$. In particular, letting $\mathcal{B}_n$ be the σ-algebra ${T_0}^{-n}\mathcal{B}$, we have $\mathbb{E}_\mu[F_n|\mathcal{B}_{n+1}] = F_{n+1}$. This and the integrability of F_n prove that the sequence F_n is a backwards martingale and so converges pointwise almost everywhere and in L^1 to a function F_∞ (see [2] for details). This function is necessarily measurable with respect to $\bigcap \mathcal{B}_n$, but since (T_0,μ) is exact, this σ-algebra is trivial. It follows that F_∞ is constant almost everywhere. Since $\int F_n\, d\mu = 0$ for all n, it follows that F_∞ is 0 almost everywhere. In particular, since $F(x) = F^n(x) + F_n(x)$, it follows that for almost all x, $F(x)$ is the L^1 and almost everywhere pointwise limit of the functions F^N.

We then show that $\exp F$ is integrable. Write $e_k(x)$ for $\exp\bigl(\frac{1}{k+1} g \circ {T_0}^k(x)\bigr)$ and note that for $x \in [-1,1], \exp x \leqslant 1 + x + x^2$. Now set $E_n^N = \exp F_n^N$ and $E^N = \exp F^N$. We then have that $\int E_n^N\, dx = \int e_n E_{n+1}^N\, dx$ and taking conditional expectations with respect to $\mathcal{B}_{n+1}$, we have

$$\begin{aligned}\int E_n^N\, d\mu &= \int \mathbb{E}_\mu\left[\exp\left(\left(\tfrac{1}{n+1} g\circ {T_0}^n(x)\right)\middle| {T_0}^{-(n+1)}\mathcal{B}\right] E_{n+1}^N\, d\mu \\ &\leqslant \int \mathbb{E}_\mu\left[1+\tfrac{1}{n+1} g\circ {T_0}^n(x) + \left(\tfrac{1}{n+1} g\circ {T_0}^n(x)\right)^2 \middle| {T_0}^{-(n+1)}\mathcal{B}\right] E_{n+1}^N\, d\mu \\ &\leqslant \int \left(1+\left(\frac{1}{n+1}\right)^2\right) E_{n+1}^N\, d\mu.\end{aligned}$$

This implies that $\int E^N\, dx$ is bounded above by $\prod_{k=1}^N (1+\frac{1}{k^2})$. This is easily seen to be bounded above as $N\to\infty$ so since E^N converges pointwise almost everywhere to $E=\exp F$, it follows by Fatou's lemma that E is integrable as required.

Since F is the almost everywhere pointwise limit of the F^N, it follows that for almost every x,

$$F(T_0(x)) - F(x) = \sum_{k=1}^\infty \frac{g({T_0}^k(x))}{k(k+1)} - g(x).$$

Let the right hand side of this equation be denoted by $h(x)$. Then this is the uniform limit of continuous functions and is therefore continuous. We have $F(T_0(x)) - F(x) = h(x)$ almost everywhere as required. Let A be the set of x on which $F({T_0}^{n+1}(x)) - F({T_0}^n(x)) = h({T_0}^n(x))$ for all n. This set has measure 1.

It remains to show that F is essentially unbounded. First, note that ${\tau_L}^3([0,1)) \subset [0,\tau_L^2(c)]$ which is a compact subinterval of $[0,\tau_L(c))$. Similarly, $(\tau_L\tau_R)^2([0,1))$ and $(\tau_R\tau_L)^2([0,1))$ are subsets of the compact subinterval $[\tau_L\tau_R\tau_L(c),\tau_R\tau_L\tau_R(c)]$ of $(\tau_L(c),\tau_R(c))$. It follows that there exists a $C>0$ such that for any $x\in[0,1)$, we have $g({\tau_L}^3(x)) > C$, $g((\tau_L\tau_R)^2(x) < -C$ and $g((\tau_R\tau_L)^2(x)) < -C$. Now pick $M>0$. Then there exists an m such that $C\log(2m-4)-5>M$. Set $f^{(m)}(x) = \sum_{n=0}^\infty g(T^n x)/(n+2m+1)$.

We then have $\|f^{(m)}\|_1 \leqslant \|f^{(m)}\|_2 = (\sum_{n=2m+1}^\infty 1/n^2)^{1/2}$. This in turn may be bounded above by $1/\sqrt{(2m)}$. It follows that the set $S_m = \{x\colon |f^{(m)}(x)| \leqslant 1\}$ has positive measure.

Since μ is absolutely continuous with continuous positive density, and the maps τ_L and τ_R have derivatives bounded above 0, it follows that $A=(\tau_L)^{2m}(S_m)$ and $B=(\tau_L\tau_R)^m(S_m)$ have positive measure.

Next, note that if $x\in A$, we have $F(x) = \sum_{k=0}^{2m-1} g(T^k x)/(k+1) + f^{(m)}(T^{2m}x)$. For $k<2m-3$, $T^k x = \tau_L^{2m-k}x$, so we have $g(T^k x) > C$. It follows that $F(x) \geqslant C(1+1/2+\ldots+1/(2m-3)) - 4 > M$.

Similarly for $y\in B$, we have $F(x) \leqslant -C(1+1/2+\ldots+1/(2m-4))+5 < -M$. It follows that F is essentially unbounded above and below as required.

This completes the proof of the lemma. □

The proof of the theorem now follows:

Proof of Theorem. By the above, there exists an F which is essentially unbounded above and below with the properties in the statement of Lemma 3. Next, note that since $h(x)\equiv F(T_0(x)) - F(x)$ is continuous, it is bounded. Define $A_n(x) = F(x)/n$ and $B_n(x) = \exp A_n(x)$. Then $B_n(x)$ converges pointwise to 1 as $n\to\infty$. The sequence is dominated by $\max(\exp F(x),1)$ so we have $\|B_n - 1\|_1 \to 0$ as $n\to\infty$, where 1 denotes the constant function with value 1. Now let $C_n(x) = A_n(x) - \log \int B_n \, d\mu$.

It follows, using C_n in place of F in Lemma 2 to get a conjugation θ_n, that ${\theta_n}^{-1}$ converges uniformly to the identity as $n \to \infty$. Then θ_n converges uniformly to the identity and T_n defined by $T_n = {\theta_n}^{-1} \circ T_0 \circ \theta_n$ converges uniformly to T_0 as $n \to \infty$. Now $T_n'(x)$ is given by ${T_0}'(\theta_n(x)) \exp\big(\frac{1}{n} h(\theta_n(x))\big)$ which may be seen to converge uniformly in x to ${T_0}'(x)$ as $n \to \infty$. Then we have shown that T_n converges to T_0 in the C^1 topology. Since the invariant density of T_n is given by $\exp\big(-\frac{1}{n} F(\theta(x))\big)\rho(\theta(x))$, the conclusion of the theorem follows.
□

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Statistical Laboratory, Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge, CB2 1SB, ENGLAND

E-mail address: `A.Quas@statslab.cam.ac.uk`