RIGIDITY OF CONTINUOUS COBOUNDARIES

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Abstract. We consider the functional equation $F\circ T - F = f$ where T is a measure-preserving transformation and f is a continuous function. We show that if there is an L^∞ function F which satisfies this equation, then F is constrained to satisfy a number of regularity conditions and in particular, if T is a one-sided Bernoulli shift, then we show that there is a continuous function F satisfying this equation. We show that this is not the case for the two-sided shift.

In this paper, we work with a continuous measure-preserving transformation T of a compact metric probability space $(X, \mathcal{B}, \mu)$ with the Borel σ-algebra. We will assume that μ is ergodic and of full support. We will require that T is a local homeomorphism in a neighbourhood about each point of x.

If the equation $F\circ T(x) - F(x) = f(x)$ is satisfied for almost every x with respect to an invariant measure μ, then we say that f is the coboundary of F with respect to μ. If F is integrable with respect to μ, then it follows that $\int f\, d\mu = 0$ since the measure μ is invariant.

A cocycle is a map from $\mathbb{Z}^+ \times X$ to $\mathbb{R}$ satisfying $\alpha(n+m, x) = \alpha(n, x) + \alpha(m, T^n x)$. It is straightforward to see that for discrete dynamical systems such as those which we consider here, the cocycle is determined by the function $f(x) = \alpha(1, x)$. Given a function f, one is often interested in the asymptotic properties of the cocycle determined by it. A coboundary gives rise to a particularly simple cocycle, namely $\alpha(n, x) = F(T^n x) - F(x)$. Two cocycles are said to be equivalent if they differ by a coboundary. If α and β are two equivalent cocyles determined by the functions f and g, then it follows from the definition that there exists a function F such that $f = g + F\circ T - F$. If such an equation holds, the functions f and g are said to be cohomologous.

We work here in the space of L^∞ functions (with the usual norm denoted by $\|\cdot\|_\infty$). In general imposing different regularity conditions on the space of cocycles will lead to completely different structures of the quotient space of equivalence classes of cocycles. However, in the context of Theorem 6, we show that a continuous function is a coboundary of an L^∞ function if and only if it is a coboundary of a

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continuous function. This implies that for continuous functions f and g, they are cohomologous as L^∞ functions if and only if they are cohomologous as continuous functions. This is in contrast with the situation for cohomology as L^1 functions. This class of result is referred to as a rigidity result as it shows that a map having certain (weak) properties necessarily also has stronger properties. The first rigidity results for cocycles were due to Livšic (see [3],[4]), who showed that showed that for maps T which have hyperbolicity properties, and for a Hölder continuous functions f, if there is an L^∞ function F satisfying $F \circ T - F = f$, then in fact there is a Hölder continuous F satisfying the same equation. For further information about cocycles, including proofs of some of the results of Livšic, an excellent reference is [2].

In what follows, we will investigate the set of L^∞ functions which have continuous coboundaries and show that these functions have some continuity properties. We will write $f^{(n)}$ for the function $f + f\circ T + \ldots + f \circ T^{n-1}$ and note that if $F\circ T - F = f$ almost everywhere then $F \circ T^n - F = f^{(n)}$ almost everywhere. Note that if the coboundary equation is satisfied by two separate functions F and G then $F - G$ satisfies $(F - G) \circ T - (F - G) = 0$ almost everywhere, so by ergodicity F and G differ almost everywhere by a constant.

In an earlier paper, [7], a class of continuous functions is given which are coboundaries of L^1 functions F. The functions F constructed there are everywhere discontinuous, in contrast with the situation for L^∞ functions. These functions are then used to construct C^1 expanding maps of the circle which have absolutely continuous invariant measures whose densities are unbounded and everywhere discontinuous.

We start with an example which will contrast with Theorem 6. Let X be the space $\{0, 1\}^{\mathbb{Z}}$, the two-sided full shift on two symbols, let T be the shift map and μ the Bernoulli$(\frac{1}{2}, \frac{1}{2})$ measure on X. Then given $x \in X$, define $N_+(x) = \inf\{n > 0\colon x_n = x_{n+1}\}$ and $N_-(x) = \sup\{n \leqslant 0\colon x_n = x_{n+1}\}$. They should be taken to be respectively ∞ and $-\infty$ if either set is empty. Then we define an L^∞ function as follows:

$$F(x) = \frac{N_+(x)x_{N_-(x)} - N_-(x)x_{N_+(x)}}{N_+(x) - N_-(x)}.$$

This function is easily seen to take values between 0 and 1. Further, a simple calculation shows that

$$F(T(x)) - F(x) = f(x) \equiv \frac{x_{N_+(x)} - x_{N_-(x)}}{N_+(x) - N_-(x)}$$

for every x. It follows that the coboundary of F is continuous. We then show that there is no continuous function G such that $G \circ T - G = F \circ T - F$. To show this, if such a function G existed, we could assume (by adding a constant if necessary) that $G = F$ almost everywhere. We now demonstrate that in any neighbourhood of the point $a = \ldots 010101\ldots$, there are sets of positive measure A_+ and A_- on which F is respectively bigger than $1 - \epsilon$ and smaller than ϵ. To show this, pick $n > 0$. Let N denote the neighbourhood about a consisting of those elements of X for which $x_i = a_i$ for all $|i| \leqslant n$. Let $M > (n + 2)/\epsilon$ and set $A_+ = \{x\colon x_i = a_i,\ \forall -n \leqslant i \leqslant M, x_{M+1} = x_{M+2} = 0, x_{-n-1} = x_{-n-2} = 1\}$ and $A_- = \{x\colon x_i = a_i,\ \forall -n \leqslant i \leqslant M, x_{M+1} = x_{M+2} = 1, x_{-n-1} = x_{-n-2} = 0\}$. Then it is straightforward to check that F is greater than $1-\epsilon$ on A_+ and less than ϵ on A_-. It follows that any function G which differs from F on a set of measure 0 must also have points in N which take values respectively bigger than $1-\epsilon$ and smaller than ϵ. This demonstrates that such a function cannot be continuous at a. It is however easy to verify that F is continuous at a set of points of measure 1. This is shown to hold in general by the next theorem.

Theorem 1. *Suppose T is a continuous measure-preserving transformation of the compact metric space $(X, \mathcal{B}, \mu)$ with the Borel σ-algebra and that μ is ergodic and of full support. If F is an L^∞ function on X and $F \circ T - F = f$ almost everywhere (μ) where f is a continuous function on X, then there is a function $G\colon X \to \mathbb{R}$ with the following properties:*

(1) *$G \circ T - G = f$ everywhere;*
(2) *G is bounded; and*
(3) *G is continuous at each point of $S = \{x \in X\colon x \text{ has a dense orbit}\}$.*

Note that by (3), we mean that about each point in S, for each $\epsilon > 0$, there is a neighbourhood of that point on which G varies by less than ϵ. This is a stronger statement than the fact that $G|_S$ is continuous. We note also that since μ is fully supported and ergodic, it follows from Birkhoff's ergodic theorem that $\mu(S) = 1$ so this shows that G has some fairly strong regularity properties. Since, as noted above, G differs from F by a constant almost everywhere, this gives corresponding regularity properties for the initial function F.

We prove the theorem in a sequence of lemmas. In what follows f will denote a fixed continuous function.

Lemma 2. *Suppose $F_1 \in L^\infty$ and $F_1 \circ T - F_1 = f$ almost everywhere with respect to μ. Then there exists a bounded measurable function F_2 such that $F_2 \circ T - F_2 = f$ everywhere.*

Proof. Let $S_1 = \{x\colon |F_1(x)| > \|F_1\|_\infty \text{ or } F_1 \circ T(x) - F_1(x) \neq f(x)\}$ and let $A_1 = \left(\bigcup_{n\in\mathbb{Z}^+} T^{-n} S_1\right)^c$. Then $\mu(S_1) = 0$ and so $\mu(A_1) = 1$. If $x \in A_1$ then we have $f^{(n)}(x) = F_1(T^n x) - F_1(x)$ and since both terms on the right hand side are smaller in modulus than $\|F_1\|_\infty$, it follows that $|f^{(n)}(x)| \leqslant 2\|F_1\|_\infty$ for every $x \in A_1$ and $n \in \mathbb{N}$. Since μ is of full support and A_1 is of full measure, A_1 is a dense set. As $f^{(n)}$ is continuous it follows that $|f^{(n)}(x)| \leqslant 2\|F_1\|_\infty$ for all x and n. Now following the proof in [2], (theorem 2·9·3), define $F_2(x)$ to be $\limsup_{n\to\infty} -f^{(n)}(x)$. Then $F_2(x) = \limsup_{n\to\infty} -f^{(n)}(x) = \limsup_{n\to\infty}(-f(x) - f^{(n-1)}(Tx)) = -f(x) + F_2(Tx)$ for all x and $\|F_2\| \leqslant 2\|F_1\|_\infty$ where $\|\cdot\|$ denotes the supremum norm, which completes the proof. □

Let $M = \text{ess sup } F_2 - \text{ess inf } F_2$. We now show that we may modify F_2 so that $\sup F_2 = \text{ess sup } F_2$ and $\inf F_2 = \text{ess inf } F_2$.

Lemma 3. *Let F_2 be as above. If $T^n x = T^m y$, then $|F_2(x) - F_2(y)| \leqslant M$.*

Proof. Write z for the point $T^n x = T^m y$. Then by assumption T^n is a local homeomorphism on a neighbourhood N_1 of x mapping to a neighbourhood N_1' of z. Similarly there is a neighbourhood N_2 of y mapping to a neighbourhood N_2' of z. Write h_1 and h_2 for the restrictions of T^n and T^m to N_1 and N_2.

Suppose $F_2(y) - F_2(x) > M$. Then since $F_2(x) = F_2(z) - f^{(n)}(x)$ and $F_2(y) = F_2(z) - f^{(m)}(y)$, it follows that $f^{(n)}(x) - f^{(m)}(y) > M$. This may also be written $f^{(n)}({h_1}^{-1}(z)) - f^{(m)}({h_2}^{-1}(z)) > M$. Since the left hand side is continuous, it follows that there exists a neighbourhood N of z contained in $N_1' \cap N_2'$ on which this inequality holds.

Now since h_1 and h_2 are non-singular (as T is measure-preserving), it follows that for almost every $z \in N$, $F_2({h_1}^{-1}(z)) \leqslant \text{ess sup } F_2$ and $F_2({h_2}^{-1}(z)) \geqslant \text{ess inf } F_2$. In particular, we have for almost every $z \in N$,

$$\begin{aligned} f^{(n)}({h_1}^{-1}(z)) - f^{(m)}({h_2}^{-1}(z)) &= F_2({h_2}^{-1}(z)) - F_2({h_1}^{-1}(z)) \\ &\leqslant \text{ess sup } F_2 - \text{ess inf } F_2 = M. \end{aligned}$$

This is a contradiction since N has positive measure. $\square$

Lemma 4. *Suppose the conditions of the theorem are satisfied. Then there exists a bounded measurable function G such that*

(1) *$G \circ T - G = f$ everywhere;*
(2) *$\sup G =$ ess sup G and*
(3) *$\inf G =$ ess inf G.*

Proof. Let F_2 be as above. We know that F_2 is bounded. Define a function H by

$$H(x) = \inf\Big\{F_2(y)\colon y \in \bigcup_{m,n\in\mathbb{Z}^+} T^{-m}(T^n\{x\})\Big\}.$$

It is straightforward to see that H is a bounded invariant function. To check measurability, note that since T is locally a homeomorphism about each point and X is a compact space, there are only countably many inverse branches of T so that H is the infimum of a countable collection of measurable functions. It follows that H is constant almost everywhere and in fact $H(x) = \text{ess inf } F_2$ almost everywhere.

Now let $G = F_2 - H$. Then $G \geqslant 0$ everywhere and $\text{ess inf } G = \text{ess inf } F_2 - \text{ess inf } F_2 = 0$ so $\inf G = \text{ess inf } G = 0$. Similarly, we see that $\text{ess sup } G = \text{ess sup } F_2 - \text{ess inf } F_2 = M$. Suppose $G(x) > M$ for some x. We have $G(x) = F_2(x) - \inf_{\{y\colon T^m y = T^n x,\ \text{for some } m,n\}} F_2(y)$. Then it follows that there exists a y such that $T^m y = T^n x$ for some m, n and $F_2(x) - F_2(y) > M$. This however contradicts the assertion of Lemma 3 and shows that $\sup F \leqslant \text{ess sup } F$ as required. $\square$

We now use these lemmas to complete the proof of the theorem.

Proof of theorem 1. We use G as defined in Lemma 4. Let $\epsilon > 0$ be given. Let $B_+ = \{x\colon G(x) > M - \epsilon/4\}$ and $B_- = \{x\colon G(x) < \epsilon/4\}$. Then both of these sets have positive measure. It follows by ergodicity that there exists a point $x \in B_-$ such that $T^n x \in B_+$ for some $n > 0$. Then we see that $f^{(n)}(x) = G(T^n x) - G(x) > M - \epsilon/2$. Now since $f^{(n)}$ is continuous, there exists a neighbourhood N of x such that $x' \in N$ implies $f^{(n)}(x') > M - \epsilon/2$. Since $G(T^n x) \leqslant M$, it follows that $0 \leqslant G(x') = G(T^n x') - f^{(n)}(x') < \epsilon/2$, whenever $x' \in N$.

Now we recall that S was defined to be the set of all points of X which have dense orbits. Given $y \in S$, we now exhibit a neighbourhood of y on which G remains within ϵ of $G(y)$. Since y has a dense orbit, there exists an $m > 0$ such that $T^m y \in N$. By continuity of $f^{(m)}$ and T^m, there exists a neighbourhood U of y such that $y' \in U$ implies $|f^{(m)}(y) - f^{(m)}(y')| < \epsilon/2$ and $T^m y' \in N$. Now fix $y' \in U$. It then follows that $0 \leqslant G(T^m y') < \epsilon/2$ and $0 \leqslant G(T^m y) < \epsilon/2$ so $|G(T^m y) - G(T^m y')| < \epsilon/2$. But now $|G(y) - G(y')| = |(G(T^m y) - f^{(m)}(y)) -$

$(G(T^m y') - f^{(m)}(y'))| < \epsilon$. Since y was an arbitrary point of S and $\epsilon > 0$ was arbitrary, it follows that G is continuous at each point of S as required. □

We are then able to apply the above to recover an old result of Gottschalk and Hedlund.

Corollary 5. *(Gottschalk, Hedlund, [1]) Suppose X is a compact metric space and $T\colon X \to X$ is minimal (that is each orbit is dense) and continuous. If f is a continuous function such that there is a point x_0 and a number $M > 0$ such that $|f^{(n)}(x_0)| \leqslant M$ for each n then there is a continuous G such that $G \circ T - G = f$.*

Proof. First note that if $|f^{(n)}(x)| > 2M$ for some x and n, then by minimality there is a point $T^m x_0$ on the orbit of x_0 such that $|f^{(n)}(T^m x_0)| > 2M$ and this contradicts the assumptions made, so we see that $|f^{(n)}(x)| \leqslant 2M$ for all x and M. Let F be as constructed in Lemma 2. Then let μ be any ergodic invariant measure for T. Since the support of μ is a compact invariant set, it follows that μ is of full support and since each point has a dense orbit, it follows that the set S in the statement of Theorem 1 is all of X. Now Theorem 1 gives the required result. □

We now show that this theorem can be applied in some other situations to show that the function G is continuous on the whole space X. To do this, we recall a definition: A map T is called locally eventually onto if for any non-empty open set U, there exists an $n > 0$ such that $T^n U = X$. This definition is due to Parry. We will consider maps T with the properties given in the introduction which in addition are locally eventually onto. Examples of maps with this property are one-sided full shift spaces, one-sided irreducible aperiodic subshifts of finite type and expanding maps of the circle.

Theorem 6. *Suppose the map T is locally eventually onto and a local homeomorphism with an ergodic invariant measure of full support and that F is an L^∞ function satisfying $F \circ T - F = f$ almost everywhere where f is a continuous function. Then there exists a continuous function G such that $G \circ T - G = f$.*

Proof. Let G be as constructed in the course of Theorem 1. Let x be any point which has a dense orbit in X and let $\epsilon > 0$ be given. By Theorem 1, G is continuous at x so there exists a $\delta > 0$ such that $d(x, x') < \delta$ implies that $|G(x) - G(x')| < \epsilon/4$. Take n such that $T^n\big(B_\delta(x)\big) = X$, where $B_\delta(x)$ denotes $\{x' \colon d(x, x') < \delta\}$. Then for each y, there exists a point $p(y)$ such that $T^n(p(y)) = y$ and $d(p(y), x) < \delta$. It follows that $|G(x) - G(p(y))| < \epsilon/4$. Now fix $y \in X$. Since T is locally a homeomorphism about each point, there exists a neighbourhood $N_1 \subset B_\delta(x)$ about $p(y)$ on which T^n acts homeomorphically. Further, since $f^{(n)}$ is continuous, there exists a neighbourhood N of $p(y)$ which is contained in N_1 such that $z \in N$ implies that $|f^{(n)}(z) - f^{(n)}(p(y))| < \epsilon/2$. Then define $U = T^n N$. This is a neighbourhood of y. Write S for $\big(T^n|_N\big)^{-1}$. Then S is a local inverse of T^n and given $y' \in U$, we see that $G(y') = G(S(y')) + f^{(n)}(S(y'))$. It follows that if $y' \in U$,

$$
\begin{aligned}
&|G(y') - G(y)| \\
&= \Big|\big(G(S(y')) - G(x)\big) - \big(G(S(y)) - G(x)\big) + \big(f^{(n)}(S(y')) - f^{(n)}(S(y))\big)\Big| \\
&\leqslant \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon.
\end{aligned}
$$

This completes the proof. □

We observe that it follows from the above that the function f constructed in the example at the start of the paper is not continuously cohomologous to a one-sided function. If this were the case, then f would be of the form $g + h \circ T - h$ for some continuous function h and we would have $(F - h) \circ T - (F - h)$ would be equal to g. By Theorem 6, it would then follow that there is a continuous function G such that $G \circ T - G = g$ and so in particular, $G + h$ would be a continuous function such that $(G + h) \circ T - (G + h) = f$ which contradicts the result shown for this example. It is however the case (as can be shown using the techniques of [7]) that there exists a continuous function $\tilde{f}$ which is one-sided such that $\tilde{f}$ is the limit of a sequence of functions, each of which is continuously cohomologous to f.

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