PRODUCT AND MARKOV MEASURES OF TYPE III

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Abstract

0 Introduction

Dye's celebrated theorem ([5]) states that any ergodic non-singular action of a countable amenable group is orbit equivalent to a measured odometer action. Hence, a complete classification of these amenable group actions up to orbit equivalence will follow from a classification up to orbit equivalence of measured odometer actions.

There is a well-known classification of ergodic non-singular group actions into classes I, II and III respectively according to whether the measure is concentrated on a single orbit, the measure is equivalent to an invariant measure (finite or $\sigma-$finite) or neither of the above holds. The case I is a relatively trivial case. The case II is fairly well understood, so the remaining interesting case (which in some sense is the most prevalent) is case III.

Krieger introduced a subdivision of case III using an invariant which he called the ratio set (see [9] and [10]). The ratio set may be informally defined as the set of limits of ratios $d\mu \circ \gamma / d\mu$ for γ in the group Γ of transformations and it may be shown to be a closed multiplicative subgroup of $\mathbb{R} \cup \{0, \infty\}$. There are then three possibilities for the ratio set. It can be $\{0, 1, \infty\}$, $\mathbb{R}$ or $\{\lambda^n : n \in \mathbb{Z}\}$ for a fixed $\lambda \in (0, 1)$. The corresponding classes of transformations are known as III_0, III_1 and III_λ. It may be shown that any two systems in III_1 are orbit equivalent (see [12]). Similarly for any fixed $\lambda \in (0,1)$, any two III_λ systems are orbit equivalent. The situation for III_0 is much less well understood and it is this category upon which we shall focus.

We will, as described above, be considering measured odometer actions. A natural class of examples is given by the actions where the measure is a product measure. An action which is orbit equivalent to one of this type is said to be product type. A necessary and sufficient condition for an odometer action to be of product type was introduced by Connes and Woods ([4]) who use a proof based on operator algebras. The condition which they introduced is that the Poincaré flow associated to the action has a property which they call approximately transitive or AT. Hawkins ([8]) showed necessity of Connes and Woods' condition with a simpler ergodic theoretic proof and Hamachi [7] was able to show sufficiency, by using purely ergodic techniques.

It nevertheless remains a difficult task to give examples of measures of type III_0 or, given a product measure to decide whether it is of type III_0. (Moore's criterion [15] allows us readily to decide when a product measure is of type I, II_1, III_∞ or III.)

Hamachi, Oka and Osikawa [10] produced examples of product measures of type III_0 and Krieger [14] gave an example of a non-AT action.

Brown and Dooley [2] introduced the notion of a $G-$measure. (Their formalism favoured the use of the groups of finite coordinate changes over the odometer; these two actions have the same orbits.) These provide an explicit description of in some senses the most general quasi-invariant measure. In [3], it was shown how to compute the ratio sets of $G-$measures in some cases, and the machinery was applied to product measures. However, there were some unresolved conjectures and some rather sketchy proofs.

The present article aims to refine the techniques of [3], resolve some conjectures therein and give full details of some results on product measures. At the same time, we are able to somewhat sharpen the examples of type III_0 product measures in [10]. From the perspective of $G-$measures, the next most complicated measures after product measures are Markov measures. A second aim is to consider a class of Markov measures on the infinite product of two point spaces, where the transition probabilities remain constant on long blocks. We are able to explicitly compute the Poincaré flow of such a measure and show that it is AT.

A detailed description of the results follows. Consider the infinite product measure $\mu = \otimes \mu_i$ or $X \prod \mathbb{Z}_2$, where $\mu_i(\{0\}) = 1 - a_i,\ \mu_i(\{1\}) = 1 + a_i \quad (-1 < a_i < 1)$. In section 2, we give an example of a measure of type III, but not of type III_0 with $a_i \nearrow 1$, disproving a conjecture in [3]. We also give a relatively easy example of a family of product measures of type III_0 on X. These improve upon the examples found in [10] where the size of the factors in the product space was unbounded. Examples are found by taking a suitable sequence $\{a_i\}$ which is constant on blocks of increasing length. In section 3, we give a detailed proof that if $a_i \to 0$ and $\sum a_i^2 = \infty$ then μ is of type III_1, providing full details of a claims made in [3]. The essential technique in these two sections is Lemma 2.1, a generalization of Theorem 3.1 of [3] and a primitive version of Theorem 1 of [9].

The final section considers Markov measures on the infinite product of two point spaces, which have the property that their transition probabilities are constant on long blocks, behaving in the same way as the probabilities in the examples of section 2. We are able to compute the Poincaré flow explicitly as an odometer with parity bit. These flows are AT, and hence the measures are orbit equivalent to product measures (although they are certainly far from being equivalent to products).

More recent work of Dooley and Hamachi [6] finds examples of non-AT Markov measures. These are realized on $\prod \mathbb{Z}_{\ell(n)}$ where $\ell(n)$ increases rapidly.

1 Definitions and Notation

We consider transformations of finite or $\sigma-$finite measure spaces. The transformations which we consider will be measurable and invertible, with measurable inverses and be non-singular: that is a set has measure 0 if and only if its image has measure 0. These transformations will be known as isomorphisms. In the case where the transformation is from a measure space X to itself, it will be called an automorphism of X. Γ will denote a countable group of automorphisms of $(X, \mathcal{B}, \mu)$. The full group $[\Gamma]$ of Γ consists of those automorphisms θ of X which have the property that for almost every $x \in X, \theta(x) = \gamma(x)$ for some $\gamma \in \Gamma$. Note that we use similar notation for the orbit of a point. Namely, $[x]$ is the orbit of the point x under the group Γ of transformations.

As an example, define X to be $\{0,1\}^{\mathbb{Z}^+}$ and Γ to be the group generated by the maps γ_n which reverses the nth coordinate (so $(\gamma_n(x))_i = \delta_{in} + x_i \bmod 1$). Then defining θ to be the standard odometer mapping obtained by regarding points of x as 2-adic integers and adding 1 (with carry) (so $\theta(\ldots 10110) = \ldots 10111; \theta(\ldots 10111) = \ldots 11000$), we see that $\theta \in [\Gamma]$.

Two group actions Γ acting on a measure space $(X_1, \mathcal{B}_1, \mu_1)$ and Γ' acting on $(X_2, \mathcal{B}_2, \mu_2)$ are orbit equivalent (sometimes also called weakly equivalent) if there exists an isomorphism Φ from X_1 to X_2 such that for almost every $x \in X_1, \Phi([x]) = [\Phi(x)]$. In the example above, the actions of Γ on X and $\{\theta^n : n \in \mathbb{Z}\}$ on X are orbit equivalent.

The ratio set R_μ, as defined in [13], is the set of r in $[0, \infty]$ such that for each $\epsilon > 0$ and set A of positive measure, there exists a subset $B \subset A$ of positive measure and a $\theta \in [\Gamma]$ such that $\theta(B) \subset A$ and $|d\mu \circ \theta / d\mu - r| < \epsilon$. In our case, where Γ is a countable group, it is equivalent to define $r \in R$ if and only if for each $\epsilon > 0$ and set A of positive measure, there exists a subset B of positive measure and $\gamma \in \Gamma$ such that $\gamma(B) \subset A$ and $|d\mu \circ \gamma / d\mu - r| < \epsilon$ on B (that is the automorphism θ may be chosen from the group itself, not the full group). We use this latter definition in what follows.

Given an action of a group Γ on a space X, we define an action of Γ on $X \times \mathbb{R}$. For $\gamma \in \Gamma$, define $\tilde{\gamma}(x,t) = (\gamma(x), t - \log(d\mu \circ \gamma / d\mu(x)))$. Then form $Y = X \times \mathbb{R}/\Gamma$, the set of $\Gamma-$orbits in $X \times \mathbb{R}$. There is a natural projection π from $X \times \mathbb{R}$ to Y. The measure on Y is taken to be projection of $\mu \times \nu$ where $d\nu(x) = \exp x \, d\lambda(x)$. The projection can be used to give Y a $\sigma-$algebra by defining a subset to be measurable if and only if its inverse image under π is a measurable subset of X. Since the action of $\mathbb{R}$ on $X \times \mathbb{R}$ given by $\theta_s(x,t) = (x, s+t)$ commutes with the action of Γ on $X \times \mathbb{R}$, it follows that the action of $\mathbb{R}$ may be pushed down to an action on Y. This is the associated flow (or Poincaré flow) of the action of Γ on X.

An important property which the associated flow may or may not possess is approximate transitivity (abbreviated to the AT property, so we often say if this property holds that the associated flow is AT). An action of a group G on a measure space X is AT if for all $\epsilon > 0$ and any sequence $f_1, f_2, \ldots, f_n$ of functions in $L^1(X)^+$, the space of positive integrable functions, there exists a function $f \in L^1(X)^+$, finitely many elements $g_{i,j}$ of G and constants $\lambda_{i,j}$ such that

$$\left\| f_i - \sum_j \left(\lambda_{i,j} f \circ g_{i,j} \frac{d\mu \circ g_{i,j}}{d\mu} \right) \right\|_1 < \epsilon$$

for each i. We will write $\mathcal{L}_g(f)$ for the function defined by

$$\mathcal{L}_g(f)(x) = f(g(x)) \frac{d\mu \circ g}{d\mu}(x).$$

Note that for positive functions $f, \mathcal{L}_g$ is an L^1 norm-preserving operator. The AT condition may be re expressed as $\|f_i - \sum_j \lambda_{i,j} \mathcal{L}_{g_{i,j}} f||_1 < \epsilon$.

2 Product measures of type III_0

Example 1. In the notation of [3], we give an example with $a_i \nearrow 1$ such that the ratio set is not a subset of $\{0, 1, \infty\}$. This disproves conjecture 6.3 of [3].

Fix $\rho > 0$. Let $\{\sigma_i\}$ be a sequence of the form

$$\rho, \rho, \ldots, \rho, 2\rho, 2\rho, \ldots, 2\rho, 3\rho, \ldots$$

such that if n_k denotes the number of terms of type k_e, then

$$n_k \geq e^{k\rho} \quad \text{and} \quad n_{k+1} \geq n_k.$$

The $\{a_i\}$ are then determined as in [3] by

$$\frac{1+a_i}{1-a_i} = e^{\sigma_i}, \quad \text{so} \quad a_i = \frac{e^{\sigma_i}-1}{e^{\sigma_i}+1} \nearrow 1.$$

Let μ denote the resulting measure. We prove:

Proposition 2.1 $e^{-\rho} \in R_\mu$.

We need the following sufficient condition. It is a generalization of Theorem (3.2) (i) of [3] to the case of more than one $\gamma \in \Gamma^n$. In this case, we need to assume the disjointness of the sets of u's for these γ's, and that of their images under the γ's:

Lemma 2.1 *Let $r \in (0, \infty)$. Suppose that $\forall \epsilon > 0, \exists \beta > 0$ such that $\forall n, \forall \gamma_0 \in \Gamma_n$ there exist:*

$$\begin{array}{ll} \gamma_1, \gamma_2, \ldots, \gamma_k & \in \Gamma^n \\ \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k & \subset \gamma_0 X^n \end{array}$$

such that:

- *the $\{\mathcal{U}_j\}$ are disjoint,*
- *the $\{\gamma_j \mathcal{U}_j\}$ are disjoint,*
- *we have $\mu(\cup_{j=1}^k \mathcal{U}_j) \geq \beta \mu(\gamma_0 X^n)$,*
- *$\forall j$,*

$$u \in \mathcal{U}_j \iff \left|\frac{d\mu \circ \gamma_j}{d\mu}(u) - r\right| < \epsilon.$$

Then $r \in r(X, \Gamma, \mu)$.

Proof of Lemma 2.1 The proof is based on the method of proof of theorem (3.2) (i) of [3].

Let $r > 0$ and let $\epsilon \ll r$ (this will be specified). Let $\beta > 0$ be fixed as in the statement. Let A be an arbitrary set of positive $\mu-$measure. Then there exists n and $\gamma_0 \in \Gamma_n$ such that

$$\mu(A \cap \gamma_0 X^n) > (1-c)\mu(\gamma_0 X^n)$$

where c to be specified.

For this n and γ_0, there exist $\{\gamma_j\}$ and $\{\mathcal{U}_j\}$ as stated. It is given that $\cup\mathcal{U}_j$ fills up a proportion β of $\gamma_0 X^n$. Let us check that $\cup\gamma_j\mathcal{U}_j$ also has this property for some constant β':

Since $|\frac{d\mu_0\gamma_j}{d\mu}(u) - r| < \epsilon$ on $\mathcal{U}_j$, it follows that $\forall j\ \mu(\gamma_j\mathcal{U}_j) \geq (r-\epsilon)\mu(\mathcal{U}_j)$. So by disjointness,

$$\mu(\cup\gamma_j\mathcal{U}_j) \geq (r-\epsilon)\mu(\cup\mathcal{U}_j) \geq (r-\epsilon)\beta\mu(\gamma_0 X^n)$$

Thus we can take $\beta' = (r-\epsilon)\beta$.

We now claim that for at least one index $j = j_0$, we have

$$\mu(A \cap [\gamma_{j_0}(A \cup \mathcal{U}_{j_0})]) > 0.$$

We have $\mu(A \cap (\cup\mathcal{U}_j)) \geq \mu(\cup\mathcal{U}_j) - \mu([\gamma_0 X^n]\backslash A) \geq (\beta - c)\mu(\gamma_0 X^n)$.

Hence, using disjointness:

$$\begin{aligned}
\mu(\cup_j \gamma_j(A \cap \mathcal{U}_j)) &= \sum_j \mu(\gamma_j(A \cap \mathcal{U}_j)) \\
&\geq \sum_j (r-\epsilon)\,\mu(A \cap \mathcal{U}_j) \\
&= (r-\epsilon)\,\mu(\cup(A \cap \mathcal{U}_j)) \\
&\geq (r-\epsilon)(\beta - c)\,\mu(\gamma_0 X^n)
\end{aligned}$$

By definition, the set $S = \cup_j\gamma_j(A \cap \mathcal{U}_j)$ is contained in $\gamma_0 X^n$ and we have shown that

$$\mu(S) \geq (r-\epsilon)(\beta - c)\,\mu(\gamma_0 X^n)$$

Hence

$$\mu(A \cap S) \geq \mu(S) - \mu([\gamma_0 X^n]\backslash A) \geq ((r-\epsilon)(\beta - c) - c)\,\mu(\gamma_0 X^n).$$

This is positive if we ensure that $\epsilon < r$ and $c < \min(\beta/2,\ (r-\epsilon)\beta/2)$.

But the statement $\mu(A \cap S) > 0$ gives us:

$$0 < \mu(A \cap \cup_j\gamma_j(A \cap \mathcal{U}_j)) = \sum_j \mu(A \cap [\gamma_j(A \cap \mathcal{U}_j)]) \quad \text{by disjointness of } \ \gamma_j\,\mathcal{U}_j.$$

Hence $\exists j = j_0$ with

$$\mu(A \cap [\gamma_{j_0}(A \cap \mathcal{U}_{j_0})]) > 0.$$

It follows that letting

$$B = \{a \in A \cap \mathcal{U}_{j_0} : \ \gamma_{j_0} a \in A\} \subset A$$

we get $\gamma_{j_0} B = A \cap [\gamma_{j_0}(A \cap \mathcal{U}_{j_0})] \subset A$ and $\mu(\gamma_{j_0} B) > 0$, which also implies $\mu(B) > 0$.

Also, since $B \subset \mathcal{U}_j$, we have

$$\left| \frac{d\mu \circ \gamma_{j_0}}{d\mu}(u) - r \right| < \epsilon \quad \forall u \in B.$$

Since $\epsilon > 0$ was arbitrary, we have $r \in r(X, \tau, \mu)$. □

Proof of Proposition 2.1 We shall in effect take $\epsilon = 0$ and find $a\beta > 0$. Let any $n, \gamma_0 \in \Gamma_n$ be given. Then, in $\{\sigma_i\}_{i=n}^{\infty}$ there is a block of the form

$$\sigma_i = \rho k, \ldots, \rho k, \rho(k+1), \rho(k+1), \ldots$$

for some k. Fix such a block and k. Since $n_{k+1} \geq n_k$, there are available at least n_k terms of the next constant, $\rho(k+1)$, to the right of this block.

The following table defines our choices of $\{\gamma_j\}$ and $\{\mathcal{U}_|\}$ $(k = n_k)$ (below $k = 4$)

$\sigma_i =$	$\ldots,$	$\rho k,$	ρk	ρk	ρk	$\rho(k+1),$	$\rho(k+1),$	$\rho(k+1),$	$\rho(k+1)$	$\ldots$	
$(\gamma_1)_i =$	0	1	0	0	0	1	0	0	0	0	$\ldots$
$(\gamma_2)_i =$	0	0	1	0	0	0	1	0	0	0	$\ldots$
$(\gamma_3)_i =$	0	0	0	1	0	0	0	1	0	0	$\ldots$
$(\gamma_k)_i =$	0	0	0	0	1	0	0	0	1	0	$\ldots$
$\mathcal{U}_1$	X	1	X	X	X	0	X	X	X	X	$\ldots$
$\mathcal{U}_2$	X	0	1	X	X	0	0	X	X	X	$\ldots$
$\mathcal{U}_3$	X	0	0	1	X	0	0	0	X	X	$\ldots$
$\mathcal{U}_4$	X	0	0	0	1	0	0	0	0	X	$\ldots$

Here the convention is that $\mathcal{U}_j$ consists of all u having the coordinates shown, and X denotes either 0 or 1. (And we assume all this in $\gamma_0 X^n$ of course).

Clearly, applying the γ_j's to the $\mathcal{U}_j$'s gives

$\gamma_1\mathcal{U}_1$	X	0	X	X	X	1	X	X	X	X
$\gamma_2\mathcal{U}_2$	X	0	0	X	X	0	1	X	X	X
$\gamma_3\mathcal{U}_3$	X	0	0	0	X	0	0	1	X	X
$\gamma_4\mathcal{U}_4$	X	0	0	0	0	0	0	0	1	X

The disjointness of the $\{\mathcal{U}_j\}_{j=1}^{n_k}$ is guaranteed by the 1's on the diagonal, preceded by the 0's. Similarly for the $\{\gamma_j\mathcal{U}_j\}$.

Next, let us verify that $u \in \mathcal{U}_j \Longrightarrow \dfrac{d\mu \circ \gamma_j}{d\mu}(u) = e^{-\rho}$.

[We are following the convention

$$\begin{aligned} \mu_i(\{0\}) &= \frac{1+a_i}{2} = \frac{1}{1+e^{-\sigma_i}} := 1 - p_i := q_i \\ \mu_i(\{1\}) &= \frac{1-a_i}{2} = \frac{e^{-\sigma_i}}{1+e^{-\sigma_i}} := p_i] \end{aligned}$$

Thus $u \in \mathcal{U}_j \Longrightarrow \dfrac{d\mu \circ \gamma_j}{d\mu}(u) = \dfrac{\mu_{(1)}(0)\mu_{(2)}(1)}{\mu_{(1)}(1)\mu_{(2)}(0)} = \dfrac{1}{e^{-\sigma_{(1)}}}\dfrac{e^{-\sigma_{(2)}}}{1} = e^{-\sigma_{(1)}-\sigma_{(2)}} = e^{k\rho-(k+1)\rho} = e^{-\rho}$ where $\sigma_{(1)},\ \sigma_{(2)}$ denote k_ρ and $(k+1)\rho$ for convenience.

It remains to estimate the measures $\mu(\mathcal{U}_j)/\mu(\gamma_0 X^n)$ We get

$$\begin{aligned} \widetilde{\mu}(\mathcal{U}_1) &= p_{(1)}q_{(2)} && \geq p_{(1)}q_{(1)} \\ \widetilde{\mu}(\mathcal{U}_2) &= q_{(1)}p_{(1)}q_{(2)}^2 && \geq p_{(1)}q_{(1)}^3 \\ \widetilde{\mu}(\mathcal{U}_3) &= q_{(1)}^2p_{(1)}q_{(2)}^3 && \geq p_{(1)}q_{(1)}^5 \end{aligned}$$

(where $\widetilde{\mu}$) denotes $\mu/\mu(\gamma_0 X^n)$). Thus, using the notation

$$p_{(1)} = e^{-k\rho}/(1+e^{-k\rho}),\ q_{(1)} = 1 - p_{(1)};$$

$$\begin{aligned} \widetilde{\mu}\Big(\bigcup_{j=1}^{n_k}\mathcal{U}_j\Big) &\geq p_{(1)}q_{(1)} + p_{(1)}q_{(1)}^3 + p_{(1)}q_{(1)}^5 + \ldots \\ &= p_{(1)}q_{(1)}(1 + q_{(1)}^2 + q_{(1)}^4 + \ldots) \\ &= p_{(1)}q_{(1)}\left(\frac{1-q_{(1)}^{2n_k}}{1-q_{(1)}^2}\right) \\ &= \frac{q_{(1)}}{1+q_{(1)}}\left(1 - \left(\frac{1}{1+e^{-k\rho}}\right)^{2n_k}\right) \end{aligned}$$

[Note: it is clear that the infinite series gives $\dfrac{q_{(1)}}{1+q_{(1)}} \sim \dfrac{1}{2}$. Therefore we could have defined n_k more conveniently by taking it such that the sum of the first n_k terms is $\geq \frac{1}{3} =: \beta$ say.]

Now $n_k \geq e^{k\rho}$, so $\left(\frac{1}{1+e^{-k\rho}}\right)^{2n_k} \leq \left(\frac{1}{1+e^{-k\rho}}\right)^{2e^{k\rho}} \sim \frac{1}{e^2}$ (for $k \to \infty$). Thus we get say

$$\beta = \frac{1}{3}(1 - \frac{1}{e^2}) > 0.$$

Hence the Lemma applies and the proposition is proved. $\square$

Remark 1 It turns out that Example 1 would have been easier if we could have used Theorem 4.4 $(a) \Longrightarrow (b)$ of [3]. In particular, this application implies the truth of the above Lemma 1, and moreover the condition that the $\{\gamma_j \mathcal{U}_j\}$ be disjoint **appears** unnecessary. Unfortunately, [3] Theorem 4.4 $(a) \Longrightarrow (b)$ is false. We give a counter example, which was motivated by this observation. It was in fact found by first looking for a situation where the $\{\gamma_j \mathcal{U}_j\}$ are not disjoint.

We present this counterexample after presenting a method for construction μ's with ratio set contained in $\{0, 1, \infty\}$. (This method will also be needed for the counterexample.)

Remark 2 Here is a special case of Example 1.

Let $e^\rho = 2$.

Let $\sigma_i = 2^k$ for $i \in [2^k, 2^{k+1})$ (thus $n_k = 2^{k+1} - 2^k = 2^k = e^{\rho k}$).

This gives $a_i = (2^k - 1)/(2^k + 1) = 1 - (\frac{2}{2^k + 1}),\ i \in [2^k, 2^{k+1})$.

We have $\frac{1}{2} \in r(X, \Gamma, \mu)$ by the above.

It is easy to verify that any $d\mu \circ \gamma / d\mu$ takes on only the values $2^m,\ m \in \mathbb{Z}$ (or possibly $0, \infty$).

Thus in fact μ is type III_λ, with $\lambda = 1/2$.

Remark 3 By choosing rationally independent ρ_1, ρ_2 and including infinitely many block pairs $(k\rho_1, k\rho_1, (k+1)\rho_i, \ldots (k+1)\rho_i)$ of both types, we clearly get a μ of type III_1.

Example 2. We now give a family examples of product measure of type III_0 on an infinite product of two-point spaces. The examples of [10] are somewhat more elaborate and are not realized on products of two point spaces.

In the notation of [3], it is clear that everything is determined if the sequence $\{\sigma_i\}$ is specified.

Let $\{\sigma_i\}$ be of the form

$$\dots, 2^k, 2^k, \dots, 2^k, 2^{k+1}, 2^{k+1}, \dots 2^{k+1}, 2^{k+2}, \dots$$

where the n_k are chosen large enough, to ensure that:

$$\sum_{i=1}^{\infty}(1 - a_i) = \infty.$$

For example, since $1 - a_i = 2/(1 + e^{\sigma_i}) = 2/(1 + e^{2^k})$ on the kth block, the choice

$$n_k \geq \frac{1}{2}(e^{2^k} + 1), (n_k \in \mathbb{N})$$

will do. By Moore's criterion [15], this ensures μ is of type III, for eventually,

$$\min\left(\frac{2a_i}{1 - a_i}, 1\right) = 1 \qquad (\text{since} \;\; a_i \nearrow 1)$$

so $\sum \min\left(\frac{2a_i}{1-a_i}, 1\right)^2(1 - a_i) = \sum(1 - a_i) = \infty.$

Proposition 2.2 *The ratio set of μ is contained in $\{0, 1, \infty\}$.*

Proof Suppose not. Let $r \in r(X, \Gamma, \mu),\ r > 2$. Let $\epsilon = 1$.

Choose k such that $\{\sigma_i\}_{i=n}^{\infty}$ takes values in $\{2^k, 2^{k+1}, \dots\}$. Let $A = \gamma_0 X^n$ for this n and any γ_0 (say $\gamma_0 = 0$)

Clearly for any $u \subseteq A$, and γ such that $\gamma u \subseteq A$ we get

$$\frac{d\mu \circ \gamma}{d\mu}(u) = \exp \sum_{i=n}^{m} \alpha_i \sigma_i$$

for some m depending on γ and $\alpha_i \in \{-1, 0, 1\}$. Thus

$$\left|\sum_{i=n}^{m} \alpha_i \sigma_i\right| \in \{0\} \cup 2^k\mathbb{N},$$

so clearly

$$\left|\frac{d\mu \circ \gamma}{d\mu}(u) - r\right| > 1 \qquad (\text{since } r > 2.)$$

Thus there is no set B as required by the definition of R_μ. $\square$

Example 3 (Counterexample to [3] Theorem 4.4 $(a) \implies (b)$)

To define $\{\sigma_i\}$, take blocks as in example 2 and insert the value $2^k - 1$ in front of the kth block for each k. Thus $\{\sigma_i\}$ is a sequence of the form

$$2^{k-1}, 2^{k-1}, \ldots, 2^{k-1}, 2^k - 1, 2^k, 2^k, \ldots, 2^k, 2^{k+1} - 1, 2^{k+1}, 2^{k+1}, \ldots$$

The condition on n_k is the same as before, e.g:

$$n_k \geq \exp(2^k).$$

or, more conveniently, define n_k instead by considering the series

$$1 + q + q^2 + \ldots = \frac{1}{1-q}$$

where $q = \mu_i(\{0\}) = \dfrac{1}{(1+\exp(-2^k))}$ (on the block k) and let n_k be the number of terms needed to get a partial sum of say at least

$$\frac{1}{2}(\frac{1}{1-q}).$$

The idea in the form of the sequence is that it is essentially like example 2, but with a tiny nick on each big step.

Proposition 2.3 *Let μ be the resulting measure. Then*

(i) The ratio set of μ is contained in $\{0, 1, \infty\}$.

(ii) $\forall \epsilon > 0, \exists \beta > 0,$ such that $\forall n \in \mathbb{N}, \forall \gamma_0 \in \Gamma_n$ $\exists L \geq n$ such that:

$\mu(\{u \in \gamma_0 X^n : \exists v \in \gamma_0 X^n,$ *eventually equal to u, such that*

$$\ell > L \quad \textit{implies} \quad \left| \prod_{i=n}^{\ell} \frac{g_i(v)}{g_i(u)} - e \right| < \epsilon\}) \geq \beta\mu(\gamma_0 X^n).$$

Remark 1 (ii) is the case $r = e$ of Theorem (4.4)(a). But by (i), e is not in the ratio set of μ, contradicting (4.4) (b).

Remark 2 It will be evident from the proof that if (4.4) (a) is altered to read, instead of $\forall n\ \exists L$, to "$\exists L\ \forall n \ldots \ell \geq n + L \ldots$", then this would not be a counterexample. In fact, the proof given in [3] is valid with this change. Furthermore, this is the version adopted later, in Theorem (5.2) (where it should read $\sup_n(N(n) - n) < K(\epsilon)$).

Proof of Proposition 3 (i) Suppose not. Let $r < 2$ be in the ratio set. Choose k such that $\exp(2^k) > r + 1$. Now consider the set A defined as follows. Choose $n \in \mathbb{N}$ such that $\{\sigma_i\}_{i=n}^{\infty}$ is the tail starting from block k. Let $i = i_\ell$ be the indices where the nicks occur, i.e. where

$$\sigma_{i_\ell} = 2^\ell - 1, \qquad \ell = 1, 2, \ldots$$

Fix a $\gamma_0 \in \Gamma_n$, define

$$A = \{u \in \gamma_0 X^n :\ 0 = u_{i_k} = u_{i_{k+1}} = \ldots\} = \{u \in \gamma_0 X^n :\ u_{i_\ell} = 0 \quad \forall \ell \geq k\}.$$

Let us first observe that if $u \in A$ and $\gamma u \in A$ then clearly

$$\frac{d\mu \circ \gamma}{d\mu}(u) = \exp\left(\sum_{\substack{i=n \\ i \notin \{i_\ell\}}}^{m} \alpha_i \sigma_i\right)$$

because by the definition of A, there can be no change in the coordinates $i = i_\ell$, $\ell = k, k+1, \ldots$ (i.e. all those i_ℓ which are $\geq n$).

Here $m \geq n$ and $\alpha_i \in \{-1, 0, 1\}$ depend on γ and u of course.

So, since these σ_i take values in $\{2^k, 2^{k+1}, \ldots\}$ (i.e. no $2^\ell - 1$ type values), we have, as in example 2,

$$\left|\frac{d\mu \circ \gamma}{d\mu}(u) - r\right| > 1 \quad \text{for all such } \ u.$$

Hence for this set A, and for $\epsilon = 1$, there is no set B as required in the definition of $r \in R_\mu$.

It remains to check, however, that $\mu(A) > 0$. Clearly

$$\mu(A) = \prod_{i_\ell \geq n} \mu_{i_\ell}(\{0\}) = \prod_{\ell=k}^{\infty} \mu_{i_\ell}(\{0\}).$$

But $\mu_{i_\ell}(\{0\}) = \dfrac{1}{1+\exp(-\sigma_{i_\ell})} = \dfrac{1}{1+\exp(-(2^\ell - 1))}$ since this is the definition of σ_i on the "nicks". This certainly gives a convergent product (since $\sum_{\ell=k}^{\infty} \exp(-2^\ell) < \infty$.)

Thus $\mu(A) > 0$ as required. □

Proof of 3 (ii) Fix $\epsilon \geq 0$ (it may as well be 0 in fact). Fix $n \in \mathbb{N}$ and $\gamma_0 \in \Gamma_n$. Consider a k such that the kth block, including its "nick" occurs inside the tail $\{\sigma_i\}_{i>n}$. We shall do coordinate changes within this block k only, hence we can take the L to be the end of block k.

Consider $u \in \gamma_0 X^n$ of the following types:

		i_k						$(i_k + n_k)$	
X	X	0	1	X	X	X	X		$(\mathcal{U}_1)$
X	X	0	0	1	X	X	X	⋮	$(\mathcal{U}_2)$
X	X	0	0	0	1	X	X	⋮	$(\mathcal{U}_3)$
X	X	0	0	0	0	1	X	⋮	⋮
⋮	⋮	⋮						⋮	⋮
X	X	0	0	…	…	…	…	0.1	$(\mathcal{U}_{n_k})$

For each $\mathcal{U}_j$ consider the γ_j shown below:

0	0	1	1	0	0	0	0	γ_1
0	0	1	0	1	0	0	0	γ_2
0	0	1	0	0	1	0	0	…
0	0	1	0	0	0	1	0	γ_{n_k}

Clearly $u \in \mathcal{U}_j \longrightarrow \gamma_j u =: v \in \gamma_0 X^n$ since we have agreed that all our $\mathcal{U}_j$ are in $\gamma_0 X^n$.

Let us compute $\dfrac{d\mu \circ \gamma_j}{d\mu}(u)$ for $u \in \mathcal{U}_j$. This is just $\dfrac{\mu_{i_\ell}(\{1\})}{\mu_{i_\ell}(\{0\})} \dfrac{\mu_*(\{0\})}{\mu_*(\{1\})}$ where $*$ denotes an irrelevant index between i_k and L (i.e. in the block k).

This of course, by the way, coincides with $\prod_{i=n+1}^{L} \dfrac{g_i(v)}{g_i(u)}$. It reduces to

$$\frac{e^{-\sigma_{i_\ell}}}{1} \frac{1}{e^{-\sigma_*}} = e^{-(2^k-1)} \; e^{+(2^k)} = e.$$

In other words, it is true, for all $u \in \bigcup_{j=1}^{n_k} \mathcal{U}_j$, that $\exists v$ in $\gamma_0 X^n$, eventually, equal to u, such that $\ell > L$ implies

$$\left| \prod_{i=n}^{\ell} \frac{g_i(v)}{g_i(u)} - e \right| < \epsilon \geq \beta\mu(\gamma_0 X^n) \qquad \text{holds.}$$

It remains to check that the measure satisfies

$$\widetilde{\mu}(\bigcup_{j=1}^{n_k} \mathcal{U}_j) \geq \beta \quad \text{for some absolute } \beta > 0.$$

We have

$$\begin{aligned}
\widetilde{\mu}\,(\mathcal{U}_1) &= \mu_{i_k}(\{0\})\mu_*(\{1\}) \\
\widetilde{\mu}\,(\mathcal{U}_2) &= \mu_{i_k}(\{0\})\mu_*(\{0\})\mu_*(\{1\}) \\
\widetilde{\mu}\,(\mathcal{U}_3) &= \mu_{i_k}(\{0\})\mu_*(\{0\})\mu_*(\{0\})\mu_*(\{1\}) \\
\ldots & \quad \text{etc.}
\end{aligned}$$

and they are all disjoint. Thus, we get a total of

$$q_{i_k} p_{(k)} (1 + q_{(k)} + q_{(k)}^2 + \ldots)$$

Where $q_{i_k} = \mu_{i_k}(\{0\}),\ p_{(k)} = \mu_*(\{1\}),\ q_k = 1 - p_{(k)}$.

We agreed that n_k was large enough to give us at least

$$q_{i_k} p_{(k)} \frac{1}{2} \frac{1}{1 - q_{(k)}} = q_{i_k} \frac{1}{2} \geq \frac{1}{3}$$

since $q_{i_k} = \dfrac{1}{1 + \exp(-(2^k - 1))}$. is nearly equal to 1.

Hence we can take $\beta = \frac{1}{3}$. This proves 3 (ii). □

Remark Notice how this fits in with Lemma 2.1. There it was required also that $\{\gamma_j \mathcal{U}_j\}$ be disjoint. Here we can check directly how badly they fail to be disjoint (of course, they must fail, because otherwise Lemma 2.1 would be contradicted).

The $\gamma_j \mathcal{U}_j$ are:

$$\begin{array}{ccccccc} 1 & 0 & X & X & X & & \gamma_1 \mathcal{U}_1 \\ 1 & 0 & 0 & X & X & & \gamma_2 \mathcal{U}_2 \\ 1 & 0 & 0 & 0 & X & & \vdots \\ 1 & 0 & 0 & 0 & 0 & & \vdots \end{array}$$

etc; clearly all are subsets of the first one, which has $\widetilde{\mu}(\gamma_1 \mathcal{U}_1)$ very small ($\leq \mu_{i_k}(\{1\})$).

□

3 Proof of a Proposition

The following proposition was given as Proposition 6.2 in [3]. Unfortunately, its proof used Theorem 4.4, which we have just disproved! Here is a corrected proof, not without interest in its own right.

Proposition 3.1 *If $a_i \to 0$ and $\sum a_i^2 = \infty$, then the measure μ is of type* III_1*. i.e., its ratio set is* $[0, \infty]$*.*

We need the following probabilistic lemma. First let us fix some notation as on pp. 13-14 of [3]. For given $0 \leq a_i \leq 1/2,\ i = 1, 2, \ldots$ we associate the sequence $\sigma_i = \log\{(1 + a_i)/(1 - a_i)\}\quad (a_i \leq \sigma_i \leq 4a_i)$ and vice versa (i.e. if a statement refers to $\{\sigma_i\}$ first, we assume $\{a_i\}$ defined in terms of $\{\sigma_i\}$).

We also associate with a given $\{a_i\}$ independent random variables as follows. These are $\{u_i\}$ and $\{v_i\}$, where

$$\begin{aligned} P(u_i = 0) &= P(v_i = 0) = \frac{1 + a_i}{2} \\ P(u_i = 1) &= P(v_i = 1) = \frac{1 - a_i}{2}, \end{aligned}$$

and all are taken to be independent (thus $\{u_i\}$ are independent and $\{v_i\}$ is a second independent copy of the sequence $\{u_i\}$). Here as usual $P(*)$ is the probability of the event $*$.

Now put $\Delta_i = v_i - u_i$.

Then $\Delta_i \in \{-1, 0, 1\}$ and $E(\Delta_i) = 0$. A simple calculation shows that $E(\Delta_i)^2 = \frac{1 - a_i^2}{2}$.

Also define for $1 \le n \le m$, (and for given $\{a_i\}$),

$$S_n^m = \sum_{i=n}^{m} \sigma_i \Delta_i = \sum_{i=n}^{m} \sigma_i (v_i - u_i).$$

Lemma 3.1 *Given* $-\infty < a < b < \infty$, *there exist* $p : p(a,b) > 0$ *and* $1 \ge \delta := \delta(a,b) > 0$ *such that given any* $\{a_i\}$ *and any* $m \in \mathbb{N}$ *satisfying:* $0 \le \sigma_i \le \delta$, $1 \le i \le m$, *and* $2 \le \sum_{i=1}^{m} \sigma_i^2 \le 2 + \delta^2$, *it follows that:*

$$P(a \le S_1^m \le b) \ge p.$$

Remarks:

1. We emphasize that p and δ depend only on the given interval $[a, b]$. So the conclusion holds "uniformly", whenever $\{a_i\}_{i=1}^m$ has the stated properties.

2. Obviously, if instead of "$1 \le i \le m$" we consider "$n \le i \le m$", we get the same result (applying the lemma): If $0 \le \sigma_i \le \delta$, $n \le i \le m$, and $2 \le \sum_{i=1}^{m} \sigma_i^2 \le 2 + \delta^2$, Then

   $$P(a \le S_n^m \le b) \ge p,$$

   since the index i plays no role in the statement. (Here $\delta = \delta(a, b)$, $p = p(a, b)$).

3. The $\{\Delta_i\}$ are independent, but not identically distributed, the distribution of Δ_i is given by a_i. However, the distributions are all "comparable" since $a_i \le 1/2$ (in fact $a_i \le \sigma_i \le \delta$ imposes an even stronger uniformity on them if δ is small.)

4. If δ is small, then m must be large (at least $1/\delta^2$). Thus S_1^m is an essentially normalized sum of a large number of independent random variables.

Proof of Lemma It follows from an exercise in [5], § 7.1 page 205, problem 5.

This exercise asserts the following: For every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that: whenever $k \in \mathbb{N}$ and $X_1, \ldots, X_k$ are independent random variables with

$$\begin{aligned} E(X_i) &= 0 \quad \forall i, \\ \sum_{i=1}^{k} E(X_i^2) &= 1, \\ \sum_{i=1}^{k} E(|X_i|^3) &\leq \delta, \text{ and } S := \sum_{i=1}^{k} X_i, \end{aligned}$$

Then $\sup_{x\in\mathbb{R}}|P(S \leq x) - \Phi(x)| \leq \epsilon$ where $\Phi(x) = \int_{-\infty}^{x} e^{-t^2/2}dt/\sqrt{2\pi}$ is the standard normal distribution function.

This is simply a quantitative version of Liapunov's Central Limit Theorem and can be verified by following the steps in the proof of it given in [5] (Theorem 7.1.2).

(A more direct proof can be given for our random variables Δ_i).

Given the above result, the lemma follows by normalizing S_1^m and substituting it for the S in this result: (indeed S_1^m is almost normalized, but not quite, because of the δ).

Given $-\infty < a < b < \infty$, choose $a < a' < b' < b$ (say $a' = \frac{3}{4}a + \frac{1}{4}b,\ b' = \frac{1}{4}a + \frac{3}{4}b$ to be definite).

Let $p' = \Phi(b') - \Phi(a') = \displaystyle\int_{a'}^{b'} e^{-t^2/2}dt/\sqrt{2\pi}$ and put $\epsilon' = p'/100,\ \delta' = \delta(\epsilon')$.

To obtain the lemma, choose $\delta_0 < 1$ and small enough to ensure

$$a \leq a'\sqrt{1 \pm \delta_0^2} < b'\sqrt{1 \pm \delta_0^2} \leq b$$

and put

$$\delta(a,b) = \frac{1}{1000}\min(\delta_0, \delta').$$

We shall compute the resulting $p = p(a,b)$: Assume $\sigma_i \leq \delta(a,b) =: \delta$ and $2 \leq \displaystyle\sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta^2$ as in the statement of the lemma.

To normalize S_1^m, we need the variance

$$E(S_1^m)^2) = \sum_{i=1}^{m} E(\sigma_i^2\Delta_i^2) = \sum_{i=1}^{m} \sigma_i^2(\frac{1-a_i^2}{2})$$

Since $\sigma_i \le \delta(a,b) =: \delta$, and $2 \le \sum_{i=1}^{m} \sigma_i^2 \le 2+\delta^2,\ a_i \le \sigma_i^2 \le \delta(a,b)^2$.

So that

$$\frac{1}{2}\sum_{i=1}^{m}\sigma_i^2 \ge \sum_{i=1}^{m}\sigma_i^2(\frac{1-a_i^2}{2}) \ge \frac{1}{2}\sum_{i=1}^{m}\sigma_i^2 - S^2\sum_{i=1}^{m}\sigma_i^2/2 \ge 1-\delta^2.$$

i.e.

$$1-\delta^2 \le E((S_1^m)^2) \le 1+\frac{1}{2}\delta^2.$$

Putting $X_i = \sigma_i\Delta_i/\sqrt{E((S_1^m)^2)}$

$S = \sum_{i=1}^{m} X_i$, we have that

$$\begin{aligned} |X_i|^3 &= \|X_i\|_\infty \cdot |X_i|^2 \\ &\le \max_i(\sigma_i)\cdot\frac{1}{\sqrt{1-\delta^2}}|X_i|^2 \\ &\le \frac{\delta}{\sqrt{1-\delta^2}}X_i^2 \end{aligned}$$

Thus

$$\sum_{i=1}^{m} E(|X_i|^3) \le \frac{\delta}{\sqrt{1-\delta^2}} \le 2\delta \le \delta' = \delta(\epsilon').$$

Hence Chung's exercise applies with $\epsilon = \epsilon'$. Thus

$$\begin{aligned} P(a' < S < b') &= P(S < b') - P(S \le a') \\ &\ge (\Phi(b') - \Phi(a')) - 2\epsilon' \\ &= p' - 2p'/100 = (0.98)p' \end{aligned}$$

But now, we show that the inequalities

$$a' < S < b'$$

imply

$$a \le S_1^m \le b:$$

If $S < b'$, then

$$S_1^m = S\sqrt{E(S_1^m)^2)} < b'\sqrt{1+\frac{1}{2}\delta^2} \le b'\sqrt{1\pm\delta_0} \le b.$$

If $a' < S$, then $a'\sqrt{1-\delta^2} < S_1^m$, so $a \le S_1^m$.

We conclude that

$$P(a \le S_1^m \le b) \ge P(a' < S < b') \ge (0.98)p'$$

and thus we can take

$$p(a,b) = (0.98)p' = (0.98)\int_{\frac{3}{4}a+\frac{1}{4}b}^{\frac{1}{4}a+\frac{3}{4}b} e^{-t^2/2}\, dt/\sqrt{2\pi}!$$

and the lemma is proved. □

Proof of Proposition 6.2 It suffices to show the following, which is a version of (2.3) (i) of [3]:

For any $0 < r < \infty, \rho > 0$, put $\rho = \log r$. Then for every $A \subset X$ with $\mu(A) > 0$, there exists $\gamma \in \Gamma$ such that

$$\mu(\{x \in A : \gamma x \in A \text{ and } |\log \frac{d\mu \circ \gamma}{d\mu}(x) - \rho| \le \epsilon\}) > 0.$$

Recall that if $u = (u_i)_{i=1}^{\infty}$, $\gamma u = ((\gamma u)_i)_{i=1}^{\infty} =: (v_i)_{i=1}^{\infty}$ then $\frac{d\mu \circ \gamma}{d\mu}(u) = \prod_{i=1}^{\infty} \frac{g_i((\gamma u)_i)}{g_i(u_0)} = \prod (\frac{1+a_i}{1-a_i})^{u_i - (\gamma u)_i}$ so $\log \frac{d\mu \circ \gamma}{d\mu}(u) = -\sum_{i=1}^{\infty} \sigma_i(\gamma(u)_i - u_i)$. Thus, consider the interval, (for given ρ, ϵ),

$$[-\rho - \epsilon, -\rho + \epsilon] =: [a, b].$$

Let the set $A \subset X$ be given, $\mu(A) > 0$. Let

$$\kappa = \frac{1}{200} p(a, b).$$

(recall that $p(a, b)$ is defined in Lemma 3.1). Let $n_1 \in \mathbb{N}$ be large enough to ensure that

$$n \ge n_1 \Longrightarrow \sigma_n \le \delta(a, b).$$

(recall : $\delta(a, b)$ is also defined in the Lemma 3.1).

Since $\mu(A) > 0$, we can find $n \geq n_1$ and $\gamma_0 \in \Gamma_n$ such that the cylinder $\gamma_0 X^n$ satisfies

$$\mu(A \cap \gamma_0 X^n) \geq (1-\kappa)\mu(\gamma_0 X^n)$$

Put $\widetilde{\mu} = \dfrac{\mu}{\mu(\gamma_0 X^n)}$ restricted to $\gamma_0 X^n$, $A_0 = A \cap \gamma_0 X^n$, $X_0 = \gamma_0 X^n$.

Consider $X_0 \times X_0$ with measure $\widetilde{\mu} \times \widetilde{\mu}$. Observe:

$$A_0 \times A_0 \subset X_0 \times X_0 \quad \text{and}$$

$$(\widetilde{\mu} \times \widetilde{\mu})(A_0 \times A_0) \geq (1-\kappa)^2 \geq 1 - 2\kappa.$$

Let $(u, v) \in X_0 \times X_0$, and identify

$$\begin{aligned} u &= (u_n, u_{n+1}, \ldots) \\ v &= (v_n, v_{n+1}, \ldots) \end{aligned}$$

Then $\{u_i\}_{i=n}^{\infty}, \{v_i\}_{i=n}^{\infty}$ are independent random variables on the probability space $(X_0 \times X_0,\ \widetilde{\mu} \times \widetilde{\mu})$ satisfying all the conditions of the Lemma (in the same notation). Since

$$\sigma_i \leq \delta(a, b) \qquad i \geq n,$$

we can also choose $m > n$ large enough (not too large) so that

$$2 \leq \sum_{i=1}^{m} \sigma_i^2 \leq 2 + \delta(a, b)^2.$$

Since $\sum_{i=n}^{\infty} \sigma_i^2 = \infty$, this is achieved by letting m be the smallest integer such that

$$2 \leq \sum_{i=n}^{m} \sigma_i^2.$$

Thus, applying the lemma to

$$S_n^m = \sum_{i=n}^{m} \sigma_i (v_i - u_i)$$

we have

$$(\widetilde{\mu} \times \widetilde{\mu})(a \leq S_n^m \leq b) \geq p(a, b).$$

Let G_n^m denote the "good set"

$$G_n^m = \{(u,v) \in X_0 \times X_0 : a \leq S_n^m(u,v) \leq b\}$$

so that

$$\widetilde{\mu} \times \widetilde{\mu}(G_n^m) \geq p(a,b).$$

Since S_n^m depends only on the coordinates $n, \dots, m$, then G_n^m is a disjoint union of cylinder sets

$$G_n^m = \bigcup_\alpha (U_\alpha \times V_\alpha)$$

where each U_α, V_α is of the form

$$\begin{aligned} U_\alpha &= (u_n^\alpha, u_{n+1}^\alpha, \dots, u_m^\alpha) \times \{0,1\} \times \{0,1\} \times \dots \\ V_\alpha &= (v_n^\alpha, v_{n+1}^\alpha, v_m^\alpha) \times \{0,1\} \times \{0,1\} \times \dots \end{aligned}$$

(where the u_i^α, v_i^α $i = n, \dots, m$ are the "good" choices of zeros and ones).

Our problem is to show that there is an $\alpha = \alpha_0$ such that the $\gamma = \gamma_{\alpha_0}$ (determined by sending $u_i^{\alpha_0} \mapsto v_i^{\alpha_0}$, $i = n, \dots, m$) satisfies property (6).

Put $P = \widetilde{\mu} \times \widetilde{\mu}$ for convenience. We have

$$\begin{aligned} P(G_n^m \cap (A_0 \times A_0)) &= P(G_n^m) - P(G_n^m \backslash (A_0 \times A_0)) \\ &\geq P(G_n^m) - P((X_0 \times X_0) \backslash (A_0 \times A_0)) \\ &\geq P(G_n^m) - 2\kappa = P(G_n^m) - \frac{1}{100} p(a,b) \\ &\geq P(G_n^m) - \frac{1}{100} P(G_n^m) \\ &= (0.99) P(G_n^m). \end{aligned}$$

i.e. $A_0 \times A_0$ covers 99% or more of the good set, G_n^m.

Consequently, $A_0 \times A_0$ covers 99% or more of at least one of the $U_\alpha \times V_\alpha$ (whose disjoint union is G_n^m).

Proof Put $E = A_0 \times A_0$. Suppose on the contrary that $\forall \alpha, P(E \cap (U_\alpha \times V_\alpha)) < (0.99)P(U_\alpha \times V_\alpha)$. Summing over α,

$$P(E \cap G_n^m) = \sum_\alpha P(E \cap (U_\alpha \times V_\alpha)) < \sum_\alpha (0.99)P(U_\alpha \times V_\alpha) = (0.99)P(G_n^m),$$

a contradiction.

Thus, there exists α_0 such that

$$P((A_0 \times A_0) \cup (U_{\alpha_0} \times V_{\alpha_0})) \geq (0.99)P(U_{\alpha_0} \times V_{\alpha_0})$$

where $U_{\alpha_0} \times V_{\alpha_0} \subset G_n^m$.

Put $U_0 = U_{\alpha_0},\ V_0 = V_{\alpha_0}$. We have

$$\frac{P((A_0 \cap U_0) \times (A_0 \cap V_0))}{P(U_0 \times V_0)} \geq (0.99) = \frac{\mu(A_0 \cap U_0)\mu(A_0 \cap V_0)}{\mu(U_0)\mu(V_0)} \quad (\because P = \widetilde{\mu} \times \widetilde{\mu})$$

$$\Longrightarrow \frac{\mu(A_0 \cap U_0)}{\mu(U_0)} \geq (0.99) \quad \text{and} \quad \frac{\mu(A_0 \cap V_0)}{\mu(V_0)} \geq (0.99)$$

Defined the proposed $\gamma \in \Gamma$ by the conditions

$$\begin{array}{llll} (\gamma x)_i = X_i & \text{if} \quad i < n & \text{or} & i > m \\ (\gamma x)_i = v_i^{\alpha_0} & \text{if} \quad x_i = U_i^{\alpha_0} & \text{and} & n \leq i \leq m \end{array}$$

Clearly $\gamma(U_0) = V_0$ and

$$\frac{\mu(E)}{\mu(U_0)} = \frac{\mu(\gamma(E))}{\mu(V_0)} \quad \text{whenever } E \subset U_0.$$

Thus

$$\frac{\mu(\gamma(A_0 \cap U_0))}{\mu(V_0)} = \frac{\mu(A_0 \cap U_0)}{\mu(U_0)} \geq (0.99).$$

Combining this knowledge with the above fact that $\dfrac{\mu(A_0 \cap U_0)}{\mu(V_0)} \geq (0.99)$, we get

$$\frac{\mu(\gamma(A_0 \cap U_0) \cap (a_0 \cap V_0)}{\mu(V_0)} \geq (0.99) - (0.01) = 0.98 > 0$$

Put $B = \gamma^{-1}(\gamma(A_0 \cap V_0) \cap (A_0 \cap V_0))$. Then

$$\mu(B) > 0,$$

$$B \subset A_0 \cap U_0 \subset A_0 \subset A,$$

$$\gamma(B) \subset A_0 \cap V_0 \subset A_0 \subset A,$$

and $B \times \gamma(B) \subset U_0 \times V_0 \subset G_n^m$, i.e., if $u \in B$, then

$$-\sum_{i=n}^{m}((\gamma u)_i - u_i)\sigma_i \in [\log r - \epsilon, \log r + \epsilon] \tag{$*$}$$

which is the required property. □

4 Markov measures of type III_0

We now present a class of examples which are measured odometers, but with a measures which are not a product measures, but rather a Markov measures.

We use the Daniell-Kolmogorov consistency theorem to define a measure on $X = \{0,1\}^{\mathbb{Z}^+}$ by specifying the measure of each cylinder. To be specific, let $[x_0 x_1 \dots x_n]$ denote the set of points in X whose first $n+1$ coordinates are $x_0, x_1, \dots, x_n$. A set of this kind will be called an $n-$cylinder and will also be denoted $[x]^n$. Then define

$$\mu([x_0 x_1 \dots x_n]) = \frac{1}{2} P^{(1)}_{x_0 x_1} P^{(2)}_{x_1 x_2} \dots P^{(n)}_{x_{n-1} x_n},$$

$$\text{where } P^{(k)}_{ij} = \begin{cases} 1 - q_k & \text{if } i = j; \\ q_k & \text{if } i \neq j. \end{cases}$$

This gives a measure on X. We then define the transformation group on the space: Γ is the group of all finite coordinate rotations of X generated by the γ_n as introduced above.

We will now demonstrate briefly an orbit equivalent system which in some ways resembles the more familiar systems. Define a second measure ν on X which is just a product measure:

$$\begin{aligned} \nu([x_0 x_1 \dots x_n]) &= \frac{1}{2} q^{(1)}_{x_1} \dots q^{(n)}_{x_n}, \\ \text{where} \quad q^{(k)}_i &= \begin{cases} 1 - q_k & \text{if } i = 0; \\ q_k & \text{if } i = 1. \end{cases} \end{aligned}$$

When then define a second transformation group on the space: Γ' is the group of all finite coordinate changes on X which change an even number of coordinates (this is generated by the $\gamma_m \circ \gamma_n$). Then we can show that the Γ action on (X, μ) is orbit equivalent to the Γ' action on (X, ν). The equivalence is given by a map which is in fact a measure-preserving homeomorphism. Namely, define $\Phi : X \longrightarrow X$ by $\Phi(x)_n = x_0 + x_1 + \ldots x_n$ mod 1. If $x_0, \ldots, x_n$ are given, define $y_0, \ldots, y_n$ by $y_k = x_0 + x_1 + \ldots + x_k$ mod 1. Then a quick check shows that $P^{(k)}_{y_{k-1} y_k} = q^{(k)}_{x_k}$ so it follows that $\nu([x_0 \ldots x_n]) = \mu([y_0 \ldots y_n])$. This shows that Φ is measure-preserving as required. We can write down an explicit inverse Ψ as follows: $\Psi(y)_n = y_{n-1} + y_n$ mod 1, thereby showing that Φ is a homeomorphism (and in particular is invertible). To check that Φ is an orbit equivalence, from (X, ν, Γ') to (X, μ, Γ), pick an $x \in X$ and note that $\Phi(\gamma_m \circ \gamma_n(x))$ differs from $\Phi(x)$ in all coordinates between the mth and the $n-1$st. It is thus clear that the image of a Γ orbit under Φ is exactly a Γ orbit as required.

Note that the system (X, ν, Γ) is known to have an associated flow with the AT property by Hawkins' result. (In fact, more is true: the flow on $X \times \mathbb{R}$ prior to forming the quotient also has the AT property.) But the system which we are considering, (X, μ, Γ), is not orbit equivalent to the above, but rather to (X, ν, Γ') which appears at first sight to be very similar to (X, ν, Γ) (Γ' is a subgroup of Γ of index 2), but Hawkins' proof does not seem to work in this situation where there is more dependence.

The system as defined so far has a number of parameters q_n. We now show how to choose them in such a way that the system is a III_0 system by analogy with the construction in §2. We will construct an increasing sequence of integers n_i and a rapidly decreasing sequence of real numbers p_i and define

$$q_n = \begin{cases} \frac{1}{2} & \text{if } n = 0 \\ p_i & \text{if } n_{i-1} < n \leq n_i. \end{cases}$$

Defining $n_0 = 0$ and $m_i = n_i - n_{i-1}$, the sequences are chosen so that

(1) $$\sum_{i=1}^{\infty} m_i p_i = \infty;$$

(2) $$R_j > 1 \quad \text{and} \quad \lim_{j \to \infty} R_j = \infty \quad \text{where}$$

$$R_j = \prod_{i<j} \left(\frac{p_i}{1-p_i}\right)^{m_i} \cdot \frac{1-p_i}{p_j} = \infty.$$

We remark that it is possible to simultaneously satisfy these conditions by an inductive construction. Supposing $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ are chosen. Then p_{k+1} may be chosen to ensure that $p_{k+1} > k+1$ and subsequently, n_{k+1} may be chosen so that $m_{k+1}p_{k+1} > 1$.

The first condition, (1) is to ensure that the system is not of type I. The condition is that the expected number of transitions (i.e. places at which $x_i \neq x_{i-1}$) is infinite). By the Kolmogorov 0-1 law, this ensures that the probability of having a sequence which is eventually all 1s or eventually all 0s is 0 and this is sufficient to guarantee that the system is not of type I.

The second condition, (2) is to ensure that the only ratios occurring in the ratio set are 0, 1 and ∞. Further, it can be checked as in §2, that all these occur so that the system really is of type III$_0$. Given $\gamma \in \Gamma$ and $x \in X$, we see that the ratio $d\mu \circ \gamma / d\mu$ is constant on any n_k−cylinder about x where γ only affects coordinates before the n_kth and so $d\mu \circ \gamma / d\mu(x) = \mu([\gamma(x)]^{n_k})/\mu([x]^{n_k})$.

Given $x \in X$, define its block type as follows: the block type is a sequence of numbers $(a_0, a_1, a_2, \ldots)$ where $a_0 = x_0$ and a_i denotes the number of transitions in block i (that is the number of n with $n_{i-1} < n \leq n_i$ such that $a_n \neq a_{n-1}$, where n_0 is taken to be 0). The number a_0 is to be interpreted as the number of transitions in the 0th block (i.e. we have a notional initial state of 0 and then a_0 denotes the number of transitions from 0 in the 0th block.) Next, we note that the measure of a cylinder set of the form $[x]^{n_k}$ is determined by its block type. If the block type is $(a_0, a_1, a_2 \ldots)$ then the measure of $[x]^{n_k}$ is $\prod_{i=1}^{k} p_i^{a_i}(1-p_i)^{m_i - a_i}$. If γ is as above and $\gamma(x)$ has block type $(b_0, b_1, \ldots, b_k, a_{k+1}, a_{k+2}, \ldots)$ then we see that

$$\frac{d\mu \circ \gamma}{d\mu}(x) = \mu([\gamma(x)]^{n_k})/\mu([x]^{n_k}) = \left(\frac{p_1}{1-p_1}\right)^{b_1 - a_1} \cdots \left(\frac{p_k}{1-p_k}\right)^{b_k - a_k}.$$

We then see that either all of the terms $b_i - a_i$ are 0 (in which case the ratio is 1) or there is a largest i for which $b_i - a_i$ is non-zero. It is then straightforward to check that the ratio is either larger than R_k or smaller than $1/R_i$ according to whether $b_i - a_i$ is negative or positive. Since in the definition of a ratio set, the ratios are required to be found in any set A of positive measure, letting A be an n_k-cylinder, if B is a subset of A and $\gamma(B) \subset A$, then the ratio $d\mu \circ \gamma d\mu$ on B is either 1 or larger than R_k or smaller than $1/R_k$. Conversely, inside any set of positive measure, there are ratios which are arbitrarily close to 0 and ∞. This proves that the system is of class III_0).

We use a construction of Hamachi and Osikawa ([8]) to give an explicit description of the associated flow of the system (X, ν, Γ) and show that it still has the AT property. To construct the flow, it is first necessary to get an explicit description of a quotient space which arises in their construction. We now give this description.

Since the functions $d\mu \circ \gamma / d\mu$ are continuous, they are defined on the whole space and not just on sets of measure 1. This means that the following definition makes sense. Given x and y in X, write $x \sim y$ if $y = \gamma(x)$ for some $\gamma \in \Gamma$ and $d\mu \circ \gamma / d\mu(x) = 1$. Then let Y denote the collection of equivalence classes $X/\sim$ and let Π denote that natural projection from X to Y. The $\sigma-$algebra on Y is then given by $\mathcal{F} = \{A \subset Y : \Pi^{-1}A \in \mathcal{B}\}$. Let $\mathcal{G}$ denote the collection of measurable subsets of X which are unions of $\sim$ $-$equivalence classes. Then we see that $\mathcal{F} = \Pi(\mathcal{G})$.

We are then able to identify certain elements of $\mathcal{G}$. We first note that if x has block type $(a_0, a_1, a_2, \ldots)$ and y has block type $(b_0, b_1, b_2, \ldots)$ then

$$\frac{\mu([y]^{n_k})}{\mu([x]^{n_k})} = (\frac{p_1}{1-p_1})^{b_1 - a_1} \ldots (\frac{p_k}{1-p_k})^{b_k - a_k}.$$

In particular, by the remarks made in the section on the III_0 property, we see that $x \sim y$ if and only if x and y lie in the same orbit and have $a_i = b_i$ for all $i \geq 1$. We now show that if $x \sim y$ then $a_0 = b_0$ as well. To see this, suppose $x \sim y$. Then as noted above, $a_i = b_i$ for each $i \geq 1$. This means that $x_{n_i} = x_{n_{i-1}} + a_i \pmod 2$ and $y_{n_i} = y_{n_{i-1}} + a_i \pmod 2$ for each $i \geq 1$. In particular it follows that $y_{n_i} - x_{n_i} \bmod 2$ is independent of i for $i \geq 0$. Since x and y live in the same orbit, we require that x and y differ only in finitely many places, so in particular, $x_{n_i} = y_{n_i}$ for all $i \geq 0$. This implies that $a_0 = b_0$. Denote the block type of x by $B(x)$. We have shown that $x \sim y \Longrightarrow B(x) = B(y)$. This allows us to identify certain elements of $\mathcal{G}$ as follows. Set $Z = \{(a_0, a_1, \ldots) : 0 \le a_0 \le 1;\ 0 \le a_i \le m_i\}$. Given $b \in Z$, write $C_k(b)$ for $\{x : B(x)_i = b_i,\ \forall i \le k\}$. Then $C_k(b)$ is a finite union of cylinder sets of length n_k so is certainly a measurable set. If $x \in C_k(b)$ and $x \sim y$, then since y has the same block type as x, we see $y \in C_k(b)$ so $C_k(b)$ is a union of $\sim$-equivalence classes as required. It will be useful to note that by the above arguments, if x and y are members of $C_k(b)$ then $x_{n_i} = y_{n_i}$ for each $i \le k$.

We have therefore identified a collection of cylinder type sets which belong to $\mathcal{G}$. It is then possible to show that these sets generate $\mathcal{G}$. We demonstrate this by showing that the algebra consisting of finite unions of sets of the form $C_k(b)$ may be used to approximate any element of $\mathcal{G}$.

Let A be any element of $\mathcal{G}$. We will show that A may be arbitrarily closely approximated by taking a union of sets of the form $C_k(b)$. Let $\epsilon > 0$ be given. Then pick $\delta < \min(1, \epsilon/2)$. Then let $\mathcal{A}$ denote the algebra of all finite unions of cylinder sets in X. Then since $\mathcal{A}$ generates the $\sigma-$algebra $\mathcal{B}$, any element of $\mathcal{B}$ may be arbitrarily closely approximated by an element of $\mathcal{A}$. In particular, there exists a finite union of cylinders S such that $\mu(A \triangle S) < \delta^2$. Since S consists of a finite union of cylinders, one of these cylinders has a maximum length and in particular, there exists a k such that all of the cylinders forming S have length less than n_k. We may then assume that S is formed of cylinders of length exactly n_k, say $C_1, \ldots C_r$. A cylinder will be called good if it satisfies $\mu(C \backslash A)/\mu(C) < \delta$ and bad otherwise. We let $\mathcal{G}$ be the union of the good cylinders forming S and B be the union of the bad cylinders forming S. Since the cylinders forming B are disjoint and for each, we have $\mu(C) \le \mu(C \backslash A)/\delta$, it follows that $\mu(B) \le \mu(B \backslash A)/\delta \le \mu(S \backslash A)/\delta < \delta$. Now, we have that $G = S \backslash B$ consists of a finite disjoint union of good n_k-cylinders. Further, we have $\mu(G \triangle A) \le \mu(G \triangle S) + \mu(S \triangle A) < \delta + \delta^2 < \epsilon$.

Now if C is an $n - k-$cylinder forming part of G, then C is one of the n_k-cylinders forming $C_k(b)$ for some $b \in Z$. We now show that any other cylinder making up $C_k(b)$

is also good. To show this, let D be another n_k-cylinder which is a subset of $C_k(b)$. Then there exists a $\gamma \in \Gamma$ which only affects coordinates up to the n_k-1st such that $\gamma(C) = D$. Further, restricted to C, γ is a measure-preserving map. Since we assumed that S consisted of a union of $\sim -$equivalence classes, it follows that $\mu(S \cap D) = \mu(S \cap C)$ from which it follows that D is good as required.

Finally, let $\widetilde{G}$ be the union of those $C_k(b)$ which intersect G. Then from the above, it follows that $\mu(\widetilde{G} \cap S)/\mu(\widetilde{G}) > 1 - \delta$. In particular, $\mu(\widetilde{G} \backslash S) < \epsilon$. But we have also that $\mu(S \backslash \widetilde{G}) < \mu(S \backslash G) < \epsilon$ so we see that $\mu(S \triangle \widetilde{G}) < 2\epsilon$, proving the claim that any element of $\mathcal{G}$ may be arbitrarily well approximated by unions of sets of the form $C_k(b)$. From this, it follows that these sets generated the $\sigma-$algebra $\mathcal{G}$.

We now show that the quotient space Y may be identified with Z. There is a natural map from Y to Z and the above shows that any measurable subset of Y agrees with the inverse image of a Borel measurable subset of Z up to a set of measure 0. This is sufficient to guarantee the identification of Y and Z. We are also able to calculate the quotient measure on Z. This is defined by

$$\nu([b]^k) = \frac{1}{2} \prod_{i=1}^{k} \binom{m_i}{b_i} (1 - p_i)^{m_i - b_i} p_i^{b_i}.$$

This is because the inverse under the projection of the cylinder set $[b]^k$ is the union of $\prod_{i=1}^{k} \binom{m_i}{b_i}$ cylinder sets in X of measure $\frac{1}{2} \prod_{i=1}^{k} (1 - p_i)^{m_i - b_i} p_i^{b_i}$.

Having identified the quotient space, define a function ϕ on X by $\phi(x) = \min\{\log d\mu \circ \gamma / d\mu(x) : \log d\mu \circ \gamma / d\mu(x) > 0\}$. Since the R_i were taken to be greater than 1, we see that this is a strictly positive quantity (and in fact bounded below by $\min R_i$). Further, it is clear that if $x \sim y$, then $\phi(x) = \phi(y)$. This shows that ϕ may be regarded as a function on the quotient Z.

The final ingredient in the construction of Hamachi and Osikawa is the construction of an automorphism U of X such that $d\mu \circ U / d\mu = \exp \phi(x)$. Again, it is clear that if $x \sim y$ then $U(x) \sim U(y)$ so once again, U may be regarded as a map of Z. Clearly from the construction of the measure on X, $U(x)$ should be a point in the orbit of x which has the transitions modified in such a way that

$$\frac{\mu([U(x)]^{n_k}}{\mu([x]^{n_k})} = \left(\frac{1-p_1}{p_1}\right)^{a_1-b_1}\left(\frac{1-p_2}{p_2}\right)^{a_2-b_2}\cdots\left(\frac{1-p_k}{p_k}\right)^{a_k-b_k}$$

is minimal but greater than 1, where n_k is defined such that $U(x)$ only disagrees with x before the n_kth terms and $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ are the block types of $U(x)$ and x. One can identify the effect of U on the cycle types. Namely, U increases b_1 by 1 to a_1 unless b_1 is already maximal, in which case a_1 is set to 0 and b_2 is increased (unless b_2 should happen to be maximal etc.). This is nothing other than an odometer action where b_i can range between 0 and m_i. This determines U apart from its effect on b_0. This is determined by the requirement that x and $U(x)$ should lie in the same $\Gamma-$orbit which determines that the total number of transitions up to n_k for x and $U(x)$ should have the same parity (even or odd). The digit a_0 is then a 'parity bit' which must be chosen to ensure that $a_0 + a_1 + \ldots + a_k$ differs from $b_0 + b_1 + \ldots + b_k$ by an even number.

We call the automorphism U of Z an odometer with parity. The ergodicity of such odometers with parity is not immediately apparent, but they turn out always to be ergodic. This will in any case follow from results about the associated flow.

Finally, the construction of Hamachi and Osikawa gives an explicit description of the associated flow. Namely, it is isomorphic to the suspension flow of $U : Z \longrightarrow Z$ with ceiling function $\phi(z)$. To describe this, let Z_ϕ denote the space $\{(z,t) : z \in Z,\ 0 \le t < \phi(z)\}$. The flow on this space is given by the maps T_s where $s > 0$,

$$T_s((z,t)) = \begin{cases} (z, s+t) & \text{if } s+t < \phi(z) \\ (U(z), s+t-\phi(z)) & \text{if } \phi(z) \le s+t < \phi(z)+\phi(U(z)) \\ \cdots & \end{cases}$$

Since the map U is invertible, the flow is also defined for negative time. An alternative description of the flow is the following. An equivalence relation is defined on the space $Z \times \mathbb{R}$, namely $\approx$ is the equivalence relation generated by (i.e. the transitive closure of) $(z,t) \approx (U(z), t-\phi(z))$. Then letting $[(z,t)]$ denote the $\approx$-equivalence class of (z,t), T_s

acts on the quotient space $Z \times \mathbb{R}/\approx$ by $T_s[(s,t)] = [(z, t+s)]$. In particular, we see that if z and z' lie differ in finitely many places, then we may pick x and x' whose block types are respectively z and z' which lie on the same orbit. There is then a $\gamma_0 \in \Gamma$ such that $\gamma_0(x) = x'$. The ratio $d\mu \circ \gamma / d\mu(x)$ is determined by the block types of x and x' (namely z and z') alone so is independent of the particular values of x and x'. Forming $\tau = \log d\mu \circ \gamma / d\mu(x)$, we show that $(z,t) \approx (z', t+\tau)$. Note that in doing this, we may assume that τ is positive. Since the ratios log $d\mu \circ \gamma / d\mu(x)$ take values in a discrete set, there can only be finitely many ratios between 0 and τ (j say). Write $\mathcal{R}(x)$ for $\{\log d\mu \circ \gamma / d\mu(x) : \gamma \in \Gamma\}$. Then by the chain rule, we see that $\mathcal{R}(\gamma(x)) = \mathcal{R}(x) - \log d\mu \circ \gamma / d\mu(x)$. It now follows that $d\mu \circ U^j / d\mu \circ \gamma_0(x) = 1$ so $U^j(x) \sim x'$. Now we see $\tau = \phi(z) + \phi(U(z)) + \ldots + \phi(U^{j-1}(z))$. In particular, we have that $(z,t) \approx (z', t-\tau)$. This is extremely important as it shows that $T_\tau(z,t) = (z',t)$. Letting K_j denote $\log((1-p_j)/p_j)$, we see that in the case where $z_j < m_j$, $T_{K_j}(z,t) = (\widehat{z}, t)$, where $\widehat{z}_i = z_i + \delta_{ij}$.

It remains to demonstrate that this flow has the AT property. We will let χ_S denote the characteristic function of a set S and use the notation $[z]^k$ or $[b_0 \ldots b_k]$ for cylinder sets in Z. Fix $k > 0$. Then we will show, taking sufficiently small δ and letting $f = \chi_{[0\ldots0]^k \times [0,\delta]}$, that we can closely approximate arbitrarily closely any given finite collection of functions of the form $\chi_{[b_0\ldots b_k]^k \times [c,d]}$. Since any positive integrable function may be arbitrarily closely approximated by a finite linear combination of functions of this form, it will follow from this that the flow has the AT property.

First, we observe that letting $\tau = \sum_{i=1}^{k} a_i K_i$ where $0 \le a_i \le m_i$, we have

$$\mathcal{L}_{-\tau} f(z,t) = \chi_{[00\ldots0]^k \times [0,\delta]}(T_{-\tau}(z,t)) \, \frac{d\mu \circ T_{-\tau}}{d\mu}(z,t).$$

But we see that $\chi_{[00\ldots0]^k \times [0,\delta]}(T_{-\tau}(z,t))$ is equal to $\chi_{T_\tau([00\ldots0]^k \times [0,\delta])}(z,t)$ and it is straightforward to check that $T_\tau([00\ldots0]^k \times [0,\delta]) = [c(a)a_1a_2\ldots a_k]$ where $c(a)$ is such that $c(a) + a_1 + \ldots + a_k = 0 \pmod 2$. This follows from the fact that if x has block type in $[00\ldots0]^k$ and $\gamma \in \Gamma$ affect only the first n_k coordinates, leaving $\gamma(x) \in [c(a)a_1a_2\ldots a_k]$

then $\log d\mu \circ \gamma / d\mu(x) = \tau$. This means that $\mathcal{L}_{-\tau} f$ is just $\chi_{[c(a)a_1a_2\ldots a_k]\times[0,\delta]}$. From this, it clearly follows that we can approximate arbitrarily closely (by taking small δ) any function of the form $\chi_{[b]^k\times[c,d]}$ provided that $\sum_{i=0}^k b_i = 0 \pmod 2$. It remains to show that we can approximate cylinders where $\sum_{i=0}^k b_i = 1 \pmod 2$.

To this end, pick a very large M and consider $\mathcal{L}_{-K_M} f$. Then as before, we have $f(T_{-K_M}(z,t)) = \chi_{T_{K_M}([0\ldots o]^k\times[0,\delta])}(z,t)$. Then we observe that if $(y,s) \in [0\ldots0]^k \times [0,\delta]$ and $y_M \neq m_M$ then $T_{K_M}(y,s) = (\widehat{y}, s)$ where $\widehat{y}_0 = 1$ and $y_i = y_i + +\delta_{iM}$ for $i \geq 1$. Setting $B = \{(z,t) \in [0\ldots0]^k \times [0,\delta] : z_M \neq m_M\}$, and $B' = ([0\ldots0]^k \times [0,\delta]) \backslash B$, we have that $f = \chi_B + \chi_{B'}$. Since $\|\chi_{B'}\|_1 = p_M^{m_M} \|f\|_1$ and $\mathcal{L}_{-K_M}$ is linear and norm-preserving, we have

$$\|\mathcal{L}_{K_M} f - \alpha\chi_{[100\ldots0]^k\times[0,\delta]}\|_1 = 2p_M^{m_M}\|f\|_1.$$

where α is chosen to ensure that $\|\alpha\chi_{[100\ldots0]^k\times[0,\delta]}\|_1 = \|f\|_1$. This shows that for large M, we can get an arbitrarily close approximation to $\chi_{[100\ldots0]^k\times[0,\delta]}$ and then by a similar argument to that for the even parity cylinders, we see that we can approximate any function of the desired type. This completes the proof.

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