MOST EXPANDING MAPS HAVE NO ABSOLUTELY CONTINUOUS INVARIANT MEASURE

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Abstract. We consider the topological category of various subsets of the set of expanding maps from a manifold to itself, and show in particular that a generic C^1 expanding map of the circle has no absolutely continuous invariant probability measure. This is in contrast with the situation for C^2 or $C^{1+\epsilon}$ expanding maps, for which it is known that there is always a unique absolutely continuous invariant probability measure.

Let X be a compact boundaryless Riemannian manifold with Riemannian volume λ. We will write $E^r(X)$ for the collection of all C^r expanding maps from X to itself, and will be mainly interested in $E^1(X)$. We will be interested in the existence and properties of invariant measures for maps in $E^1(X)$ which are absolutely continuous with respect to λ. We will work both with absolutely continuous invariant probability measures (which we abbreviate to a.c.i.p.) and σ-finite absolutely continuous invariant measures (which we abbreviate to a.c.i.σ.). By a.c.i.σ., we will always mean a σ-finite absolutely continuous invariant measure which is not a finite measure. An absolutely continuous invariant measure will mean either an a.c.i.p. or an a.c.i.σ.. The *density* of an absolutely continuous invariant measure is the Radon-Nikodým derivative of the measure with respect to λ.

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If T is a member of $E^2(X)$, it is known (see [7] and [5]) that T has a unique a.c.i.p.. This measure also has strong ergodic properties. Krzyżewski ([6]) showed that the set of C^1 expanding maps which have an a.c.i.p. with a continuous density bounded away from 0 is meagre in $E^1(X)$, but it is clear that there are a number of ways in which this can fail. In [10], it was shown that in the case where X is the circle, there is a dense set of C^1 expanding maps with an a.c.i.p. whose density is not bounded away from 0, nor from ∞. In [3], an example of a C^1 expanding map of the circle was produced for which there is no a.c.i.p., but there was no information about the size of the class of maps with this property. This example was later shown to have an a.c.i.σ.(see [4]). An example of a C^1 expanding map with no a.c.i.σ. or a.c.i.p.was given in [2].

In this paper, we show that the class of expanding maps of the circle which have no a.c.i.p. is a dense G_δ set (i.e. is the intersection of countably many dense open sets), and also prove a statement in the general case where X is a manifold about the class of expanding maps having more than one absolutely continuous invariant measure.

If T is an expanding map $X \to X$, let $\mathcal{L}_T$ be the corresponding Ruelle-Perron-Frobenius operator with respect to λ. This operator may be defined by the equation $\int f\mathcal{L}_T[g]\,d\lambda = \int f \circ T \cdot g\,d\lambda$ for all $g \in L^1(\lambda)$ and $f \in L^\infty(\lambda)$ and is given explicitly by the equation

$$\mathcal{L}_T[f](x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}.$$

If T preserves an absolutely continuous invariant probability measure μ, write $\mathcal{D}_T$ for its Ruelle-Perron-Frobenius operator with respect to μ. This operator satisfies $\int f\mathcal{D}_T[g]\,d\mu = \int f\circ T\cdot g\,d\mu$ for all $g \in L^1(\mu)$ and $f \in L^\infty(\mu)$ and is normalized ($\mathcal{D}_T[1] = 1$). If μ has density h, then $\mathcal{D}_T$ and $\mathcal{L}_T$ are related by the equation $\mathcal{D}_T[f](x) = \mathcal{L}_T[h\cdot f](x)/h(x)$. If ϕ: $S^1 \to S^1$ is a diffeomorphism, S is an expanding map with absolutely continuous invariant measure μ and T is the conjugate map $\phi^{-1} \circ S \circ \phi$ preserving the absolutely continuous measure $\mu \circ \phi^{-1}$, then the operators $\mathcal{D}_S$ and $\mathcal{D}_T$ are related by $\mathcal{D}_T[g] = \mathcal{D}_S[g \circ \phi] \circ \phi^{-1}$. The reader is referred to [11] for a fuller discussion of the properties of Ruelle-Perron-Frobenius operators.

A measure μ (not necessarily invariant) is called ergodic under T if $\mu(A \bigtriangleup T^{-1}A) = 0$ implies that $\mu(A) = 0$ or $\mu(A^c) = 0$.

Theorem 1. *If X is a compact Riemannian manifold, then the set of C^1 expanding maps $T\colon X \to X$ which are ergodic and conservative with respect to Lebesgue measure is residual.*

This should be compared with a result of [8], which is stated in terms of g-measures (the set of g for which there is more than one g-measure is meagre).

Since X is assumed to be a compact space, it is known that $C(X)$ has a countable dense subset. Let $f_1, f_2, \dots$ be a countable dense set of functions in $C(X)$ with the uniform norm topology. Set

$$A_{m,k,n} = \left\{ T \in E^1(X)\colon \sup_{x\in X} \frac{\mathcal{L}_T^k[f_n](x)}{\mathcal{L}_T^k[1](x)} - \inf_{x\in X} \frac{\mathcal{L}_T^k[f_n](x)}{\mathcal{L}_T^k[1](x)} < \frac{1}{m} \right\}.$$

This is clearly an open set in $E^1(X)$.

Next, set

$$\mathcal{R} = \bigcap_m \bigcap_n \bigcup_k A_{m,k,n}.$$

Since $\bigcup_k A_{m,k,n}$ is open, $\mathcal{R}$ is a countable intersection of open sets. We now prove some lemmas about $\mathcal{R}$ and the apply them to complete the proof of Theorem 1.

Lemma 2. *The set $\mathcal{R}$ is dense in $E^1(X)$ so $\mathcal{R}$ is a residual set.*

Proof. To show this, we show that any C^2 expanding map belongs to $\mathcal{R}$. Let T be any C^2 expanding map. Then there exists a continuous function h such that $\mathcal{L}_T^k[1]$ converges uniformly to h (see [11] for a proof). We have also that for each n, $\mathcal{L}_T^k[f_n]$ converges uniformly to $c_n h$ for some constant c_n (see [5]). Since h may be shown to be uniformly bounded away from 0, it follows that for each m and n, $T \in \bigcup_k A_{m,k,n}$, and so $T \in \mathcal{R}$ as required. □

Lemma 3. *The set $\mathcal{R}$ (defined above) consists of those T in $E^1(X)$ for which $\mathcal{L}_T^n[f]/\mathcal{L}_T^n[1]$ converges uniformly to $\int f \, d\lambda$ for each continuous function f.*

Proof. Given T as described in the statement of the lemma, it is clear that for each n,

$$\lim_{k\to\infty} \sup_{x\in X} \frac{\mathcal{L}_T^k[f_n]}{\mathcal{L}_T^k[1]} - \inf_{x\in X} \frac{\mathcal{L}_T^k[f_n]}{\mathcal{L}_T^k[1]} = 0.$$

It follows that for each $m > 0$ and $n > 0$, there exists a k such that $T \in A_{m,k,n}$ and accordingly, we see that $T \in \mathcal{R}$.

Conversely, if $T \in \mathcal{R}$ and $f \in C(X)$, let $0 < \epsilon < 1$ be given. We first note that $\mathcal{L}_T^k[f]/\mathcal{L}_T^k[1]$ is uniformly convergent to $\int f\, d\lambda$ if and only if the same conclusion holds for $f+c$ for any constant c. We may therefore assume without loss of generality that $f(x) \geqslant 1$ for all x.

Let $a_k(f)$ and $b_k(f)$ be defined by $a_k(f) = \inf \frac{\mathcal{L}_T^k[f]}{\mathcal{L}_T^k[1]}$ and $b_k(f) = \sup \frac{\mathcal{L}_T^k[f]}{\mathcal{L}_T^k[1]}$. We have then

$$a_k(f)\mathcal{L}_T^k[1] \leqslant \mathcal{L}_T^k[f] \leqslant b_k(f)\mathcal{L}_T^k[1]. \tag{1}$$

Applying $\mathcal{L}_T$, we see that $a_k(f)\mathcal{L}_T^{k+1}[1] \leqslant \mathcal{L}_T^{k+1}[f] \leqslant b_k(f)\mathcal{L}_T^{k+1}[1]$ and it follows that $a_k(f) \leqslant a_{k+1}(f) \leqslant b_{k+1}(f) \leqslant b_k(f)$. It follows that $a_k(f)$ is an increasing sequence and $b_k(f)$ is a decreasing sequence. Integrating (1), we see that $a_k(f) \leqslant \int f\, d\lambda \leqslant b_k(f)$. It follows that $a_k(f)$ and $b_k(f)$ are convergent sequences to $a(f)$ and $b(f)$ say, where $a(f) \leqslant \int f\, d\lambda \leqslant b(f)$. It is then sufficient to verify that $b_k(f) - a_k(f)$ tends to 0 as k tends to ∞. To see this, note that by assumption, we have $b_k(f_n) - a_k(f_n) \to 0$ as k tends to ∞, so since $a_k(f_n) \leqslant \int f_n\, d\lambda \leqslant b_k(f_n)$, we see that $a_k(f_n)$ and $b_k(f_n)$ converge to $\int f_n\, d\lambda$ as $k \to \infty$. Now as the f_n are a dense sequence, there exists an n such that $\|f - f_n\|$ is bounded above by $\epsilon/2$. Since $\min f_n \geqslant \min f - \epsilon/2 \geqslant 1 - \frac{1}{2}$, on dividing through by f_n, we see that $\|f/f_n - 1\| \leqslant \epsilon$. Then we have $(1-\epsilon)f_n \leqslant f \leqslant (1+\epsilon)f_n$. It follows that $(1-\epsilon)a_k(f_n) \leqslant a_k(f) \leqslant b_k(f) \leqslant (1+\epsilon)b_k(f_n)$, so that $b(f) - a(f) \leqslant 2\epsilon \int f_n\, d\lambda$. Since f_n may be chosen arbitrarily uniformly close to f and ϵ was arbitrary, it follows that $a(f) = b(f) = \int f\, d\lambda$ as required. □

We now apply these lemmas to complete the proof of the theorem.

Proof of Theorem 1. We show that if $T \in \mathcal{R}$ then the volume measure λ is ergodic under T. Suppose that this is not the case. Then there exists a set A such that $A = T^{-1}A$ up to a set of measure 0 and $0 < \lambda(A) < 1$. Since λ is non-singular with respect to T, we may assume that $A = T^{-1}A$ by modifying A on a set of measure 0. Applying T, we now see that $A = T(A)$ and similarly, we get $A^c = T(A^c)$. Now let $\rho_1 = \chi_A/\lambda(A)$ and $\rho_2 = \chi_{A^c}/\lambda(A^c)$. Then $\|\rho_i\|_1 = 1$ for $i = 1, 2$. Further, since ρ_1 is supported on A, the images $\mathcal{L}_T^n[\rho_1]$ are of mass 1 and are supported on $T^n(A) = A$. A similar statement is valid for ρ_2 and we deduce that $\|\mathcal{L}_T^n[\rho_1] - \mathcal{L}_T^n[\rho_2]\|_1 = 2$ for all n. However since $C(X)$ is dense in $L^1(X)$, there exist two positive continuous functions f_1 and f_2 each of mass 1 such that $\|\rho_i - f_i\|_1 < \frac{1}{2}$. By Lemma 3, we see that $\mathcal{L}_T^n[f_i]/\mathcal{L}_T^n[1]$ converges uniformly to 1 so $(\mathcal{L}_T^n[f_1] - \mathcal{L}_T^n[f_2])/\mathcal{L}_T^n[1]$ converges uniformly to 0. Multiplying through by $\mathcal{L}_T^n[1]$ and integrating shows that $\|\mathcal{L}_T^n[f_1] - \mathcal{L}_T^n[f_2]\|_1$ converges uniformly to 0. But now we have

$$\begin{aligned} 2 &= \|\mathcal{L}_T^n[\rho_1] - \mathcal{L}_T^n[\rho_2]\|_1 \\ &\leqslant \|\mathcal{L}_T^n[\rho_1] - \mathcal{L}_T^n[f_1]\|_1 + \|\mathcal{L}_T^n[f_1] - \mathcal{L}_T^n[f_2]\|_1 + \|\mathcal{L}_T^n[f_2] - \mathcal{L}_T^n[\rho_2]\|_1 \\ &\leqslant \tfrac{1}{2} + \|\mathcal{L}_T^n[f_1] - \mathcal{L}_T^n[f_2]\|_1 + \tfrac{1}{2} \\ &< 2 \text{ for large } n. \end{aligned}$$

The second inequality follows from the fact that $\|\mathcal{L}_T\|_1 = 1$. This contradiction establishes that λ is ergodic as claimed.

To show that a residual set of C^1 expanding maps are conservative with respect to Lebesgue measure, we use proposition 1.3.1 from [1], which states that $\mathcal{C}$, the conservative part of a transformation T is equal (up to a set of measure 0) to $\{x\colon \sum_{n=0}^{\infty} \mathcal{L}_T^n[f](x) = \infty\}$, where f is any strictly positive function which is integrable (with respect to λ). In this case, we take the function f to be 1. We then see that to show that T is conservative, it is sufficient to show that

$$T \in \bigcap_{N=1}^{\infty} \bigcup_{n=1}^{\infty} \{T\colon \sum_{i=0}^{n} \mathcal{L}_T^n[1] > N\}.$$

This set is clearly a G_δ set and it remains to show that it is dense. To see this, note that if T is a C^2 expanding map, then $\mathcal{L}_T^n[1]$ is uniformly convergent to a strictly positive function h. It follows that T belongs to the set. □

Corollary 4. *The set of C^1 expanding maps preserving more than one absolutely continuous invariant measure (a.c.i.p. or a.c.i.σ.) is meagre.*

Proof. We have shown that the set of ergodic conservative maps is residual. These maps have at most one absolutely continuous invariant measure (see [1], theorem 1.5.6). The corollary follows. □

We now specialize to the case where X is the unit circle. We will identify the unit circle, S^1, with $[0,1)$. In the case that $X = S^1$, we are able to show that the set of C^1 expanding maps which have absolutely continuous invariant probability measures is a meagre set. Set

$$\mathcal{S} = \left\{T \in E^1(S^1)\colon \liminf_{n\to\infty} \lambda(\{x\colon \mathcal{L}_T^n[1](x) > \tfrac{1}{2}\}) = 0\right\}.$$

Lemma 5. *The set $\mathcal{S}$ contains a dense G_δ set.*

Proof. Let $f(x)$ be the continuous function which is 0 if $x \leqslant \frac{1}{3}$; 1 if $x \geqslant \frac{1}{2}$ and linear in between. Then define a family of continuous maps from $E^1(S^1)$ to $\mathbb{R}$ by

$$\Lambda_n(T) = \int f(\mathcal{L}_T^n[1](x))\ d\lambda(x).$$

Then it is clear that

$$\liminf \Lambda_n(T) = 0 \Leftrightarrow T \in \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n \geqslant m} \{T \colon \Lambda_n(T) < \tfrac{1}{k}\}.$$

This set is a G_δ set and $\liminf \Lambda_n(T) = 0$ implies that $T \in \mathcal{S}$. It remains to prove that the set of T such that $\liminf \Lambda_n(T) = 0$ is dense in $E^1(S^1)$.

Since $E^2(S^1)$ is dense in $E^1(S^1)$, it is sufficient to show that any E^1-neighbourhood of any map in $E^2(S^1)$ contains a map T such that $\liminf \Lambda_n(T) = 0$. To show this, we use Baire's theorem and show that in any E^1-neighbourhood N of any map S_0 in $E^2(S^1)$, for any $\epsilon > 0$ and $m > 0$, there is an $n \geqslant m$ and a $T \in N$ such that $\Lambda_n(T) < \epsilon$. This then shows that $\bigcup_{n \geqslant m}\{T \colon \Lambda_n(T) < \frac{1}{k}\}$ is dense in $E^1(S^1)$.

Now pick S_0 in $E^2(S^1)$. Since S_0 is a C^2 expanding map, it preserves an absolutely continuous probability measure μ_0. Let the density of μ_0 with respect to λ be ρ. It is known (see [5]) that ρ is a C^1 function. Setting $\phi^{-1}(x) = \mu_0([0,x])$, and letting $T_0(x) = \phi^{-1} \circ S_0 \circ \phi$ makes T_0 a C^2 expanding map preserving λ which is conjugate to S_0. (To see that T_0 preserves λ, note that $\lambda = \mu_0 \circ \phi$ and to see that T_0 is expanding, note that any C^1 map preserving λ is automatically expanding.) We then find a small C^1 perturbation of T_0 and apply the conjugacy to get a map close to S_0. The map S_0, density ρ and map ϕ are fixed for the remainder of the proof.

Fix $\epsilon < 1$ and let $\mathcal{P}$ be the natural partition of S^1 into the branches of T_0. We now introduce a collection of quantities which depend on the choice of ϵ. These will be denoted by for instance $s(\epsilon)$ when they are first introduced, and then simply by s for tidiness.

Since T_0 is expanding, there exists $s(\epsilon) > 0$ such that the intervals of $\mathcal{P}_s = \bigvee_{i=0}^{s-1} T_0^{-i}\mathcal{P}$ are of length at most ϵ. Next, let $\mathcal{L}$ be the Ruelle-Perron-Frobenius operator for the map T_0 with respect to λ. Since T_0 is conjugate to S_0, the relationship between the Ruelle-Perron-Frobenius operators shows that $\mathcal{L}^n f$ converges uniformly to $\int f\ d\lambda$ for any continuous function f. Since the functions χ_A for $A \in \mathcal{P}_s$ may be approximated above and below by continuous functions with arbitrarily small L^1 difference, it follows that $\mathcal{L}^n\chi_A$ converges uniformly to $\lambda(A)$ for $A \in \mathcal{P}_s$. Since $|\mathcal{P}_s|$ is finite, there exists a $k(\epsilon) > 0$ such that for each $A \in \mathcal{P}_s$, $|\mathcal{L}^k\chi_A - \lambda(A)|$ is uniformly bounded by $\epsilon\lambda(A)$. It is known that $U_{T_0}^k\mathcal{L}^k(f) = \mathbb{E}_\lambda[f|T_0^{-k}\mathcal{B}]$ where $U_T(f) = f \circ T$ (see [9]) so it follows that $|\mathbb{E}_\lambda[\chi_A|T_0^{-k}\mathcal{B}] - \lambda(A)|$ is bounded above by $\epsilon\lambda(A)$. Now pick a $\delta(\epsilon) > 0$ such that $(1+\delta)^k < 1+\epsilon$ (it then follows that $(1-\delta)^k > 1-\epsilon$) and let $f(x)$ be a continuous function with $\|f\|_\infty = 1$ and $\mathbb{E}_\lambda[f|T_0^{-1}\mathcal{B}] = 0$. Set $G(x) = 1+\delta f(x)$ and write $G^{(n)}(x) = G(x)G(T_0(x))\ldots G(T_0^{n-1}(x))$. We will specify n later, but note that $G^{(n)}$ has the property that $\int G^{(n)}\, d\lambda = 1$ because

$$\begin{aligned}
\int G^{(n)}\, d\lambda &= \int G(x)G(T_0(x))\ldots G(T_0^{n-1}(x))\, d\lambda(x)\\
&= \int \mathbb{E}_\lambda[G|T_0^{-1}\mathcal{B}](x)G(T_0(x))\ldots G(T_0^{n-1}(x))\, d\lambda(x)\\
&= \int G(T_0(x))\ldots G(T_0^{n-1}(x))\, d\lambda(x)\\
&= \int \mathbb{E}_\lambda[G\circ T_0|T_0^{-2}\mathcal{B}](x)G(T_0^2(x))\ldots G(T_0^{n-1}(x))\, d\lambda(x)\\
&\ldots\\
&= 1.
\end{aligned}$$

We have also that if $A \in \mathcal{P}_s$, then

$$\begin{aligned}
\int_A G^{(n)}\, d\lambda &\leqslant \int (1+\delta)^k\chi_A G\circ T_0^k \ldots G\circ T_0^{n-1}\, d\lambda\\
&< (1+\epsilon)\int \mathbb{E}_\lambda[\chi_A|T_0^{-k}\mathcal{B}]G\circ T_0^k\ldots G\circ T_0^{n-1}\, d\lambda\\
&\leqslant (1+\epsilon)^2\lambda(A)\int G\circ T_0^k\ldots G\circ T_0^{n-1}\, d\lambda\\
&= (1+\epsilon)^2\lambda(A).
\end{aligned}$$

Similarly, $\int_A G^{(n)}\, d\lambda \geqslant (1-\epsilon)^2\lambda(A)$. Now, define a homeomorphism $\theta\colon S^1 \to S^1$ by $\theta(x) = \int_{[0,x]} G^{(n)}\, d\lambda$. Since the interval $[0,x]$ consists of a union of elements of $\mathcal{P}_s$ together with a remainder interval which is contained in a single element of $\mathcal{P}_s$, we see from the above that $|\theta(x) - x| < 6\epsilon$ for each $x \in S^1$.

Since we have $\int G\, d\lambda = 1$, and G is non-constant, we have, by Jensen's inequality, that $\int \log G\, d\lambda < 0$. Let $\alpha = -\int \log G\, d\lambda$. Note that $\log G^{(n)}(x) = \log G(x) +$

$\ldots + \log G(T_0^{n-1}(x))$. By ergodicity of λ under T_0, we have that for λ-almost all x, $1/n \log G^{(n)}(x) = 1/n(\log G(x) + \ldots + \log G(T_0^{n-1}(x)))$ converges to $-\alpha$. In particular, there exists an N such that for $n \geqslant N$, there exists a set of measure at least $1-\epsilon \inf \rho/8$ on which $\log G^{(n)}(x) < -n\alpha/2$. Now choose $n \geqslant N$ such that $\exp(-n\alpha/2) < \epsilon/2$. Now for x in a set of measure $1-\epsilon \inf \rho/8$, $G^{(n)}(x) < \epsilon/2$. It follows that $\int \min(G^{(n)}(x), 4/\inf \rho)\, d\lambda \leqslant \epsilon$. But we have

$$G^{(n)} \leqslant \min(G^{(n)}, 4/\inf \rho) + G^{(n)} \chi_{\{x\colon G^{(n)}(x) \geqslant 4/\inf \rho\}}.$$

Integrating, we get

$$\int G^{(n)} \chi_{\{x\colon G^{(n)}(x) \geqslant 4/\inf \rho\}}\, d\lambda \geqslant \int (G^{(n)} - \min(G^{(n)}, 4/\inf \rho))\, d\lambda \geqslant 1 - \epsilon. \tag{2}$$

We now use θ to define a map $T(\epsilon)$ conjugate to T_0 by setting $T = \theta \circ T_0 \circ \theta^{-1}$. This technique was used similarly in [10]. Since θ can be made arbitrarily close to the identity and the map sending a homeomorphism to its inverse is continuous, it follows that as ϵ is reduced to 0, T approaches T_0 in the uniform norm. To establish C^1 convergence, it is also necessary to establish that T' approaches T_0' as ϵ is reduced to 0. To see this, note that

$$\begin{aligned} T'(x) &= \frac{\theta'(T_0(\theta^{-1}x))}{\theta'(\theta^{-1}(x))} T_0'(\theta^{-1}(x)) \\ &= \frac{G^{(n)}(T_0(\theta^{-1}(x)))}{G^{(n)}(\theta^{-1}(x))} T_0'(\theta^{-1}(x)) \\ &= \frac{G(T_0^n(\theta^{-1}(x)))}{G(\theta^{-1}(x))} T_0'(\theta^{-1}(x)) \\ &= \frac{1 + \delta f(\theta^{-1}(T_0^n(x)))}{1 + \delta f(\theta^{-1}(x))} T_0'(\theta^{-1}(x)). \end{aligned}$$

As ϵ is reduced to 0, $T_0'(\theta^{-1}(x))$ converges to $T_0'(x)$ and δ tends to 0, so the first term converges to 1. It follows that as ϵ is reduced to 0, the map $T(\epsilon)$ converges to T_0 in the C^1 norm. The map T preserves the absolutely continuous invariant measure $\nu_T = \lambda \circ \theta$. This has density $\rho_T(x) = 1/G^{(n)}(\theta^{-1}(x))$, so in particular, $\rho_T(x) < \inf \rho/4$ when $G^{(n)}(\theta^{-1}(x)) > 4/\inf \rho$. That is $\{x\colon \rho_T(x) < \inf \rho/4\}$ is $\theta(A)$ where $A = \{x\colon G^{(n)}(\theta^{-1}(x)) > 4/\inf \rho\}$. It is not hard to see that $\lambda(\theta(A)) = \int_A G^{(n)}\, d\lambda$, but by (2), we see $\lambda(\theta(A)) \geqslant 1 - \epsilon$. It follows that ρ_T is less than $\inf \rho/4$ on a set of Lebesgue measure at least $1 - \epsilon$.

Now, letting $S(\epsilon) = \phi \circ T \circ \phi^{-1}$, S is a C^1 expanding map preserving the absolutely continuous invariant measure $\nu_S = \nu_T \circ \phi^{-1}$. The density of this measure is given by

$\rho_S(x) = \rho_T(\phi^{-1}(x))/\phi'(\phi^{-1}(x))$. This is less than 1/4 on the image under ϕ of a set whose ν_T-measure is at least $1-\epsilon$, so we see that ρ_S is less than 1/4 on a set of ν_S-measure at least $1-\epsilon/\inf\rho$. Clearly as ϵ is reduced to 0, $S(\epsilon)$ converges in the C^1 topology to S_0 (as the map $T \mapsto \phi\circ T\circ\phi^{-1}$ is C^1 continuous) and the density ρ_S is less than 1/4 on a set of measure arbitrarily close to 1.

Now S is a C^1 expanding map and may be chosen to be arbitrarily close in the C^1 topology to S_0. It is also conjugate to S_0 and it follows from this that the Perron-Frobenius operator $\mathcal{D}_S$ for S with respect to ν_S is conjugate to the Perron-Frobenius operator for S_0 with respect to μ_0. This operator inherits the property that $\mathcal{D}_S^n[f]$ converges uniformly to $\int f\ d\nu_S$ as $n\to\infty$. Now the Perron-Frobenius operator for S with respect to λ is given by $\mathcal{L}_S = M_{\rho_S}\mathcal{D}_S M_{\rho_S}^{-1}$ where M_f is the multiplication operator sending a function g to fg. It follows that $\mathcal{L}_S^n[f]$ converges uniformly to $\rho_S\int f\ d\lambda$ as $n\to\infty$. In particular, $\Lambda_n(S) < \epsilon/\inf\rho$ for sufficiently large n, so for small ϵ, we have

(i) S is in the original neighbourhood

(ii) $\liminf \Lambda_n(S) < 1/k$.

This completes the proof of the lemma. □

We are now able to use this to prove our main theorem.

Theorem 6. *The set of T in $E^1(S^1)$ which have no absolutely continuous invariant probability measure contains a dense G_δ set.*

Proof. We will show that the C^1 expanding maps belonging to $\mathcal{R}\cap\mathcal{S}$ have no absolutely continuous invariant probability measures. Pick $T\in\mathcal{R}\cap\mathcal{S}$ and suppose that T has an absolutely continuous invariant probability measure μ. The density of μ with respect to λ is then an L^1 function ρ which satisfies $\mathcal{L}_T[\rho]=\rho$. Given ϵ, there is then a continuous function f such that $\|f-\rho\|_1 < \epsilon$ and $\int f\ d\lambda = 1$. Now, by Lemma 3, there exists an n_0 such that for $n\geqslant n_0$, $(1-\epsilon)\mathcal{L}_T^n[1] \leqslant \mathcal{L}_T^n[f] \leqslant (1+\epsilon)\mathcal{L}_T^n[1]$. It follows that $\|\mathcal{L}_T^n[1]-\mathcal{L}_T^n[f]\|_1 < \epsilon$. Since $\mathcal{L}_T$ is a contraction in the L^1 norm, we see that $\|\mathcal{L}_T^n[1]-\rho\|_1 \leqslant \|\mathcal{L}_T^n[1]-\mathcal{L}_T^n[f]\|_1+\|f-\rho\|_1$. It follows that $\|\mathcal{L}_T^n[1]-\rho\|_1\leqslant 2\epsilon$ for n sufficiently large. Since ϵ was arbitrary, it follows that $\mathcal{L}_T^n[1]$ converges in the L^1 norm to ρ. We now show that this contradicts the fact that $T\in\mathcal{S}$.

Since ρ is a density, the set $A=\{x\colon \rho(x) > \frac{3}{4}\}$ has positive measure, α say. As T was assumed to be in $\mathcal{S}$, we have also that there exists a sequence n_i such that $\lambda(\{x\colon \mathcal{L}_T^{n_i}[1](x) > \frac{1}{2}\}) < \alpha/2$. It then follows that $\|\mathcal{L}_T^{n_i}[1] - \rho\|_1 > \alpha/8$. This contradiction shows that T has no absolutely continuous invariant probability measure as claimed when $T \in \mathcal{R} \cap \mathcal{S}$, and $\mathcal{R} \cap \mathcal{S}$ clearly contains a dense G_δ set by Theorem 1 and Lemma 5. □

We note that in fact for $T \in \mathcal{R}$, $T \in \mathcal{S}$ if and only if T has no absolutely continuous invariant probability measure. To show this, we note that if $T \notin \mathcal{S}$, then there exists an $\alpha > 0$ such that for all n, $\lambda(\{x\colon \mathcal{L}_T^n[1](x) > \frac{1}{2}\}) > \alpha$. We have that $\mathcal{L}_T^n[1]$ is the density of $\lambda \circ T^{-n}$ with respect to λ. Setting $f_n = \frac{1}{2}\chi_{\{x\colon \mathcal{L}_T^n[1](x) > \frac{1}{2}\}}$, we have $f_n \leqslant \mathcal{L}_T^n[1]$ and $\frac{\alpha}{2} \leqslant \int f_n \, d\lambda \leqslant \frac{1}{2}$. Let ν_n be the sequence of measures defined by $\nu_n(A) = \int_A f_n(A)$ for each $A \in \mathcal{B}$. As $f_n \leqslant \frac{1}{2}$ and $f_n \leqslant \mathcal{L}_T^n[1]$, we see that $\nu_n(A) \leqslant \frac{1}{2}\lambda(A)$ and $\nu_n(A) \leqslant \lambda \circ T^{-n}(A)$ for each $n \geqslant 0$ and $A \in \mathcal{B}$.

By weak*-compactness, there is a sequence n_i such that ν_{n_i} is weak*-convergent to some measure ν (not a probability measure) and also $\lambda \circ T^{-n_i}$ is convergent to a probability measure μ say. These limiting measures satisfy $\nu(A) \leqslant \frac{1}{2}\lambda(A)$ and $\nu(A) \leqslant \mu(A)$. It follows that ν is absolutely continuous. By construction, we have $\nu(S^1) \geqslant \frac{\alpha}{2}$. Using the Lebesgue decomposition theorem, we may write μ as a sum $\mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$, where μ_{ac} is absolutely continuous with respect to Lebesgue measure and μ_{s} is singular with respect to Lebesgue measure. It follows that there exists a set $B \in \mathcal{B}$ such that $\lambda(B) = 1$ and $\mu_{\mathrm{s}}(B) = 0$. We now have $\mu_{\mathrm{ac}}(B) = \mu(B) \geqslant \nu(B) = \nu(S^1) - \nu(B^c) = \nu(S^1) \geqslant \frac{\alpha}{2}$ and it follows that μ_{ac} is non-trivial.

We then verify that μ_{ac} is invariant as follows: Since $T \in \mathcal{R}$, we have $\mathcal{L}_T^n[\mathcal{L}_T[1]]/\mathcal{L}_T^n[1]$ converges uniformly to 1. It follows that for each continuous function g,

$$\int g \circ T \; d\lambda \circ T^{-n} - \int g \; d\lambda \circ T^{-n} = \int g(\mathcal{L}_T^{n+1}[1] - \mathcal{L}_T^n[1]) \; d\lambda,$$

which converges to 0 and so μ is an invariant measure for T. Since T is non-singular with respect to λ, it follows that the absolutely continuous component μ_{ac} of μ is an invariant measure and so we see that T has a finite absolutely continuous invariant measure as claimed.

It would be interesting to know whether a generic C^1 expanding map of a manifold has an absolutely continuous invariant measure and even in the case where the manifold is the unit circle, it is unclear whether or not a generic C^1 expanding map has a σ-finite absolutely continuous invariant measure.

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