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Scaling Functions And Gibbs Measures And Teichmüller Spaces Of Circle Endomorphisms

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Abstract. We study the scaling function of a C^{1+h} expanding circle endomorphism. We find necessary and sufficient conditions for a Hölder continuous function on the dual symbolic space to be realized as the scaling function of a C^{1+h} expanding circle endomorphism. We further represent the Teichmüller space of C^{1+h} expanding circle endomorphisms by the space of Hölder continuous functions on the dual symbolic space satisfying our necessary and sufficient conditions and study the completion of this Teichmüller space in the universal Teichmüller space.

1. Introduction

In the study of critical phenomena and universality in physics (see, for example, [6,10]) scaling functions have been used to describe the finer geometric structure of a dynamical system or a family of dynamical systems. The concept was introduced to the study of hyperbolic Cantor sets on the real line in [25] by Sullivan (see [14, §1.3] for a more comprehensive description). Scaling functions are also defined for Markov maps and are used in the study of the geometric structure of geometrically finite maps (see [14, Chapter 3] for a summary of this work). For a Markov map, one can construct a semiconjugacy between the map and a symbolic dynamical system of finite type. However, the scaling function (if it exists) is defined on the symbolic space dual to this shift of finite type (see §2 for the definition). The scaling function for a Markov map satisfies a summation condition (see §2). A natural question is *which functions defined on a given dual symbolic space are the scaling functions of a Markov map?*

A circle endomorphism is a Markov map with a standard Markov partition. Using this Markov partition, one can get a semiconjugacy between the one-sided full shift dynamical system on n symbols and the circle endomorphism, where n is the degree of the circle endomorphism. This allows us to define the scaling function of a circle endomorphism on the associated dual symbolic space. If this circle endomorphism is C^{1+h} expanding, then we know that its scaling function exists and is a Hölder continuous function on the dual symbolic space (see Theorem 1). In this paper, we study the above question for C^{1+h} expanding circle endomorphisms. Although our ideas work for C^{1+h} circle endomorphisms of any degree, for notational simplicity we will only formulate our results for circle endomorphisms of degree two. The most

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well-known result in the study of circle endomorphisms is that a C^{1+h}-expanding circle endomorphism has a unique absolutely continuous invariant measure. This result is closely related to Gibbs measure theory in symbolic dynamical systems (see [27] for more details). Somewhat surprisingly, without the Hölder condition on the derivative, a C^1 expanding circle endomorphism can have more than one absolutely continuous invariant measure (see [23] for more details). We will also study the geometric structure of the space of C^{1+h} circle expanding maps as a whole, that is, we will study Teichmüller theory for circle expanding endomorphisms.

The reader may refer to [5,18,23,27] for a description of the requisite Gibbs measure theory, to [14] for the scaling function theory, and to [1,2,4,7,8,11,12,13,14,19,26] for the quasiconformal mapping theory and Teichmüller theory and its application to dynamical systems.

The paper is arranged as follows. In §2, we give the definition of the scaling function of an expanding degree two orientation-preserving circle endomorphism and present some properties of scaling functions. In §3, we study two conditions satisfied by the scaling function of a C^{1+h} expanding degree two orientation-preserving circle endomorphism. One is called the summation condition and the other is the compatibility condition. This second condition allows us to lift objects from the symbolic space to the circle. The main result in this section says that a Hölder continuous function on a dual symbolic space is the scaling function of a C^{1+h} expanding orientation-preserving circle endomorphism if and only if it satisfies these two conditions. In §4, we study the geometric structure of the space of all C^{1+h} expanding degree two orientation-preserving circle endomorphisms, that is, the Teichmüller space of this space. We give a representation of the Teichmüller space by the space of scaling functions and study the completion of the Teichmüller space in the universal Teichmüller space. This part partly overlaps with the paper [13] (Theorem 6) but uses different methods.

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2. Circle expanding maps, scaling functions and Gibbs measures

Let S^1 be the unit circle equipped with normalized Lebesgue measure. Suppose f is a degree two orientation-preserving covering map from S^1 onto S^1. The map f is said to be expanding if there are constants $C > 0$ and $\lambda > 1$ such that $|f^n(I)| \geq C\lambda^n|I|$ for any sub-interval I of S^1 and any integer $n > 0$ so that $|f^n(I)| < 1$. The constant λ in this inequality is called an expanding constant of f. We will simply call a degree two expanding orientation-preserving covering map from S^1 onto S^1 *a circle map*. A circle map f has a unique fixed point, say α. The inverse image of α under f has two points, one is α and the other we denote as β. Using this two points, we get a standard Markov partition for f as follows: Let I_0 and I_1 be the closures of the two intervals in $S^1 \setminus \{\alpha, \beta\}$. Then we have

1 $S^1 = I_0 \cup I_1$,
2 $f|\mathring{I}_0$ and $f|\mathring{I}_1$ are injective,
3 $f(I_0) = f(I_1) = S^1$.

A partition of S^1 satisfying these three conditions is called a Markov partition. We call the partition $\eta_0 = \{I_0, I_1\}$ the standard partition for the circle map f.

Let g_0 and g_1 be the inverses of $f|\mathring{I}_0$ and $f|\mathring{I}_1$. Thus g_0 maps $S^1\setminus\{\alpha\}$ to $\mathring{I}_0$ and g_1 maps $S^1\setminus\{\alpha\}$ to $\mathring{I}_1$. For any string $w_n = i_0 \ldots i_n$ of 0's and 1's, let $g_{w_n} = g_{i_0}\circ\cdots\circ g_{i_n}$

and let I_{w_n} be the closure of $g_{w_n}(S^1 \setminus \{\alpha\})$. Let

$$\Sigma_n^+ = \{w_n = i_0 i_1 \dots i_{n-1}\} \qquad \text{and} \qquad \Sigma_n^- = \{w_n^* = i_n \dots i_2 i_1\}$$

be the collection of strings of 0's and 1's of length n but read differently and define

$$\sigma_{+,n}(i_0 i_1 \dots i_{n-1}) = i_1 \dots i_{n-1} : \Sigma_n^+ \to \Sigma_{n-1}^+$$

and

$$\sigma_{n,-}(i_n \dots i_2 i_1) = i_n \dots i_2 : \Sigma_n^- \to \Sigma_{n-1}^-.$$

By considering the inverse limits of

$$\left(\Sigma_n^+, \sigma_{+,n}\right)_{n=1}^{\infty} \qquad \text{and} \qquad \left(\Sigma_n^-, \sigma_{-,n}\right)_{n=2}^{\infty},$$

we have two symbolic dynamical systems

$$\Sigma^+ = (\{w = i_0 i_1 \dots\}, \sigma_+) \qquad \text{and} \qquad \Sigma^- = (\{w^* = \dots i_2 i_1\}, \sigma_-).$$

We will also make use of a third system, which is a common extension of Σ^+ and Σ^-, namely Σ defined by

$$\Sigma = (\{\dots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \dots\}, \sigma),$$

where σ is the left shift operation: $\sigma(x)_n = x_{n+1}$. There are projections π^+ and π^- from Σ to Σ^+ and Σ^- respectively defined by

$$\pi^+(\dots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \dots) = i_0 i_1 i_2 \dots$$
$$\pi^-(\dots i_{-2} i_{-1} \cdot i_0 i_1 i_2 \dots) = \dots i_{-2} i_{-1}.$$

The importance of these spaces arises as there is a natural semiconjugacy from Σ^+ to S^1:

$$p(w) = \bigcap_{n=0}^{\infty} I_{w_n}.$$

This map has the property that

$$f \circ p = p \circ \sigma_+$$

on Σ^+, where $w = i_0 i_1 i_2 \dots$ and p is a continuous onto map from Σ^+ to S^1 (refer to [14, Chapter 3]). Further p is one–one off a countable set.

The second space, Σ^- is called the dual symbolic space of f (see [14, Chapter 3]) and we define

$$s(w_n^*) = \frac{|I_{w_n^*}|}{|I_{\sigma_{-,n}(w_n^*)}|}.$$

The quantities $s(w_n^*)$ are called the pre-scalings of f. Thus we have the following definition (see [14, Chapter 3]).

Definition 1. *If for every $w^* = \cdots w_n^* \in \Sigma^-$, $\lim_{n\to\infty} s(w_n^*)$ exists, then we define a function $s(w^*) = \lim_{n\to\infty} s(w_n^*)$ on Σ^-. This function is called the scaling function of f.*

We say a circle map f is C^{1+h} if it is C^1 and its derivative is Hölder continuous and write $\mathcal{C}^{1+h}$ for the set of C^{1+h} circle maps. A function s on Σ^- is called Hölder continuous if there are constants $C > 0$ and $0 < \tau < 1$ such that

$$|s(w^*) - s(v^*)| \leq C\tau^n$$

as long as the first n digits of w^* and v^* from the right are the same. The following theorem is known (see [14, Chapter 3]) and is important in this paper. We give a full proof.

Theorem 1. *The scaling function of a map in $\mathcal{C}^{1+h}$ exists and is a Hölder continuous function on Σ^-.*

The proof of this theorem uses the naive distortion lemma.

Lemma 1 (Naive Distortion Lemma). *Suppose $f \in \mathcal{C}^{1+h}$. Then there are constants $C > 0$ and $0 < \gamma \leq 1$ such that for any integer $k > 0$, any open interval $I \neq S^1$ such that $f^k|I$ is injective, and any ξ and η in I,*

$$\left|\log\left(\frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|}\right)\right| \leq C|f^k(\xi) - f^k(\eta)|^\gamma.$$

Proof of Theorem 1. Let w^* be a point in Σ^-. Suppose $w^* = \cdots w_m^*$ and $w_m^* = i_{m-1}\cdots i_0$. Then $f^k(I_{w_m^*}) = I_{w_n^*}$ for $n = m - k \geq 0$. Thus we have

$$|s(w_m^*) - s(w_n^*)| = \left|1 - \frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|}\right| s(w_n^*) = \left|1 - \frac{|(f^k)'(\xi)|}{|(f^k)'(\eta)|}\right| \frac{|(f^n)'(\xi')|}{|(f^n)'(\eta')|} s(w_0^*),$$

where ξ and η are in $I_{\sigma_{-,m}(w_m^*)}$ and ξ' and η' are in $I_{\sigma_{-,n}(w_n^*)}$. Applying the naive distortion lemma, we have that

$$|s(w_m^*) - s(w_n^*)| \leq C_0|f^k(\xi) - f^k(\eta)|^\gamma \leq C_0|I_{w_n^*}|^\gamma \leq C_1\lambda^{-n\gamma},$$

where C_0 and C_1 are constants and $\lambda > 1$ is an expanding constant of f. This means that $\{s(w_n^*)\}_{n=0}^\infty$ is a Cauchy sequence. So

$$s(w) = \lim_{n\to\infty} s(w_n^*)$$

exists. Similarly, if $w^* = \cdots w_n^*$ and $v^* = \cdots w_n^*$ are two points in Σ^- we also have

$$|s(w^*) - s(v^*)| \leq C_0|I_{w_n^*}|^\gamma \leq C_1\lambda^{-n\gamma}.$$

This implies that s is Hölder continuous. $\square$

The pre-scalings $\{s(w_n^*)\}$ of f satisfy

$$s(w_n0) + s(w_n1) = 1.$$

Taking the limit, it follows that

$$s(w^*0) + s(w^*1) = 1 \tag{*}$$

for each $w^* \in \Sigma^-$. If a function s satisfies (*), we say that it satisfies the *summation condition.*

We mention one more property of scaling functions. Suppose f is a C^{1+h} circle map. A point a is called a periodic point of period n if $f^n(a) = a$ but $f^i(a) \neq a$ for all $0 < i < n$. The eigenvalue e_a of f at a periodic point a of period n is by definition $e_a = (f^n)'(a)$. In [14, Chapter 3], there is a relation between the scaling function of f and all eigenvalues of f as follows. Suppose $f^j(a) \in I_{i_j\ldots i_{n+j-1}}$. Let $w_n = i_0\ldots i_{n-1}$, $w_n^* = i_0\ldots i_{n-1}$. Then $w = w_n^\infty$ is a point in Σ^+ corresponding to a. Let $w^*(a) = (w_n^*)^\infty$.

Proposition 1. *For any periodic point a of f of period n,*

$$\frac{1}{e_a} = \prod_{j=0}^{n-1} s(\sigma_-^j(w^*(a))).$$

Since a major issue in what follows is whether a given function can arise as the scaling function of a circle map, it is natural that one needs to have a notion of which continuous functions on a shift space can be lifted to give continuous functions on the circle. We call a continuous function $h\colon \Sigma^+ \to \mathbb{R}$ *circle-continuous* if

$$\begin{aligned} h(11\dots) &= h(00\dots), \\ h(011\dots) &= h(100\dots) \text{ and} \\ h(w011\dots) &= h(w100\dots) \text{ for any finite string } w. \end{aligned}$$

A function h is circle continuous if and only if it can be written as $g \circ p$ where p is the natural semiconjugacy introduced earlier.

In §3, we will make extensive use of results about Gibbs measures. We outline here the necessary background.

Sets of the form $\{x \in \Sigma^+ \colon x_0 = i_0, \dots x_n = i_n\}$ will be called cylinder sets in Σ^+ and will be denoted $[i_0 \dots i_n]$. Similarly, cylinder sets in Σ^- will be denoted by $[i_n \dots i_1]$.

A σ_+-invariant measure ν^+ on Σ^+ will be called a Gibbs measure if there is a continuous function $g\colon \Sigma^+ \to (0,1)$ such that

$$\lim_{n\to\infty} \frac{\nu^+([i_0 \dots i_n])}{\nu^+([i_1 \dots i_n])} = g(i_0 i_1 \dots)$$

for all sequences $i_0 i_1 \dots \in \Sigma^+$. Such a function g necessarily satisfies $g(0x)+g(1x) = 1$, a condition analogous to the summability condition above. The function g is called the conditional probability function of ν^+. Measures satisfying this property have in the past (see [Ke]) been called g-measures. Gibbs measures often refer to measures on Σ. There is however a very close connection between the kinds of measures as mentioned below.

Similarly, a σ_--invariant measure ν^- on Σ^- will be called a dual Gibbs measure if there is a function $s\colon \Sigma^- \to (0,1)$ such that

$$\lim_{n\to\infty} \frac{\nu^-([i_n \dots i_1])}{\nu^-([i_n \dots i_2])} = s(\dots i_2 i_1).$$

The function s is called the dual conditional probability function of ν^-.

In all that follows, all conditional probability functions and dual conditional probability functions will be Hölder continuous. In this case, it may be shown (see [Bo]) that if g is a conditional probability function satisfying a summability condition ($g(0x)+g(1x) = 1$ for all x), then there is a unique measure ν^+ with this function as its conditional probability distribution function. Further, the measure ν^+ is the unique invariant measure satisfying the inequality

$$C_1 \le \frac{\nu^+[x_n \dots x_0]}{\exp(\log g(x_0 00\dots) + \log g(x_1 x_0 00\dots) + \dots + \log g(x_n \dots x_0 00\dots))} \le C_2$$

for appropriate constants C_1 and C_2. Note that the denominator is written as the exponential of a sum of logarithms rather than as a product in order to make explicit the connection with the definition in [Bo]. There is a similar characterization of ν^- in terms of the exponential of sums of terms involving the logarithm of s. This characterization of Gibbs measures is used as the definition in [Bo].

It is well-known that there is a bijection between the set of σ-invariant measures on Σ and the set of σ_+-invariant measures on Σ^+, namely $\nu \mapsto \nu \circ {\pi^+}^{-1}$. Similarly, there is a bijection between the σ-invariant measures on Σ and the set of σ_--invariant measures on Σ^-. If ν^+ and ν^- are the measures corresponding on Σ^+ and Σ^- corresponding to ν, then we have $\nu^+[i_0 \dots i_{n-1}] = \nu^-[i_0 \dots i_{n-1}]$, where the subset $[i_0 \dots i_{n-1}]$ of Σ^- consists of those $w^* = \dots x_n \dots x_1$ such that $x_n = i_0$, $x_{n-1} = i_1, \dots, x_1 = i_{n-1}$.

Suppose now that $f \in \mathcal{C}^{1+h}$ with unique absolutely continuous invariant measure μ. Let ρ denote the density of μ with respect to Lebesgue measure. Since p is one–one off a countable set, there is a unique invariant measure ν^+ on Σ^+ such that $\mu = \nu^+ \circ p^{-1}$. The measure ν^+ will be called the *symbolic measure* of f. It is a Gibbs measure with conditional probability function

$$g(x) = \frac{\rho(p(x))}{f'(p(x))\rho(f(p(x)))}.$$

Since ρ is known to be Hölder continuous, we see that g is Hölder continuous and circle-continuous. Applying the above-mentioned bijections, there are measures ν and ν^- on Σ and Σ^- corresponding to ν^+. The measure ν^- is called the *dual symbolic measure* of f.

The key observation which makes Gibbs measures of relevance to questions involving scaling functions is the following.

Proposition 2. *Suppose $f \in \mathcal{C}^{1+h}$, ν^- is the dual symbolic measure of f and $s: \Sigma^- \to (0,1)$ is the scaling function of f. Then the dual conditional probability function of ν^- is s.*

Proof. This follows from the observation that $\nu^-[i_n \dots i_1] = \nu^+[i_n \dots i_1] = \mu(I_{i_n \dots i_1})$. It follows that

$$\frac{\nu^-[i_n \dots i_1]}{\nu^-[i_n \dots i_2]} = \frac{\mu(I_{i_n \dots i_1})}{\mu(I_{i_n \dots i_2})}.$$

Since the density ρ is a continuous function, $I_{i_n \dots i_1} \subset I_{i_n \dots i_2}$ and $|I_{i_n \dots i_2}| \to 0$ as $n \to \infty$, we see that the dual conditional probability function is given by

$$s'(\dots i_n \dots i_1) = \lim_{n\to\infty} \frac{|I_{i_n \dots i_1}|}{|I_{i_n \dots i_2}|}.$$

This is equal to $s(\dots i_n \dots i_1)$. $\square$

We will make use at one stage of Ruelle-Perron-Frobenius operators, and offer the following introduction. For proofs and more details, the reader is referred to [Bo, W]. Given a continuous function ϕ on Σ^+, the *Ruelle-Perron-Frobenius operator* with *potential* ϕ is defined by

$$\mathcal{L}_\phi : C(\Sigma^+) \to C(\Sigma^+);\ \mathcal{L}_\phi[g](x) = \sum_{i\in\{0,1\}} e^{\phi(ix)} g(ix).$$

A potential $\psi \in C(\Sigma^+)$ is called *normalized* if $\mathcal{L}_\psi[1] = 1$. This is equivalent to saying that $\exp(\psi)$ satisfies the summability condition. For any Hölder continuous ϕ on Σ, there always exists a constant $P(\phi)$ (known as the pressure of ϕ) and a normalized Hölder continuous function ψ on Σ^+ such that $\psi \circ \pi^+ = \phi - P(\phi) - h + h \circ \sigma_+$ for a Hölder continuous function h. This says that any Hölder continuous function on Σ is cohomologous to a normalized Hölder continuous function up to an additive constant. A version of this also holds for Hölder continuous functions on Σ^+: Given any Hölder continuous function ϕ on Σ^+, there is a normalized Hölder continuous function ψ on Σ^+ and a Hölder h such that $\psi = \phi - P(\phi) - h + h \circ \sigma_+$. Both ψ and $P(\phi)$ are uniquely determined by ϕ and h is determined up to an additive constant. The potential ψ is called the *normalization* of ϕ. If ψ is normalized, then $P(\psi) = 0$. If ψ is normalized and Hölder continuous, then it may be shown that ${\mathcal{L}_\psi}^n[g]$ converges uniformly to a constant for all continuous functions g. This constant turns out to be $\int g \, d\nu^+$ where ν^+ is the unique Gibbs measure with conditional probability function e^ψ. If $\phi' = \phi - H + H \circ \sigma_+$, then one can check that ${\mathcal{L}_{\phi'}}^n[g] = e^H {\mathcal{L}_\phi}^n[e^{-H} g]$.

3. Characterization of the scaling functions of circle maps

If f is in $\mathcal{C}^{1+h}$, then Theorem 1 says that its scaling function s exists and is a Hölder continuous function on Σ^-. As we have seen the scaling function satisfies the summation condition,

$$s(w^*0) + s(w^*1) = 1$$

for all $w^* \in \Sigma^-$. We will show in the next theorem that there is another condition for s to be a scaling function of a map in $\mathcal{C}^{1+h}$. Suppose s is a Hölder continuous function on Σ^-. We define a sequence of functions on Σ^-:

$$C_{s,N}(w^*) = \prod_{n=0}^{N} \frac{s(w^*1\overbrace{0\ldots0}^{n})}{s(w^*0\underbrace{1\ldots1}_{n})}.$$

Provided that $s(\ldots 000) = s(\ldots 111)$, the sequence of functions $C_{s,N}(w^*)$ is necessarily uniformly geometrically convergent to a limit function $C_s(w^*)$. We say that s satisfies the *compatibility condition* if

$$s(\ldots 000) = s(\ldots 111);$$

and

$$C_s(w^*) \text{ is independent of } w^*.$$

(In other words, s satisfies the compatibility condition if $s(\ldots 000) = s(\ldots 111)$ and

$$\prod_{n=0}^{\infty} \frac{s(w^*1\overbrace{0\ldots0}^{n})}{s(w^*0\underbrace{1\ldots1}_{n})}$$

is independent of w^*).

Theorem 2. *Let s be a Hölder continuous function on Σ^-. Then s is the scaling function of a map in $\mathcal{C}^{1+h}$ if and only if s satisfies the summation and compatibility conditions.*

Before proving Theorem 2, we will need a number of lemmas. These address the relationship between a Gibbs measure and its dual Gibbs measure, showing that a Gibbs measure has a circle-continuous conditional probability function if and only if the dual Gibbs measure has a compatible dual conditional probability function. It will then follow that if $f \in \mathcal{C}^{1+h}$, the symbolic measure is a Gibbs measure with a Hölder and circle-continuous conditional probability function so the dual symbolic measure has a compatible dual continuous conditional probability function. Conversely if s satisfies the conditions, we construct a dual Gibbs measure, get the corresponding Gibbs measure, which then has to have a circle-continuous conditional probability function and finally lift this to a map of the circle with the required scaling function.

Lemma 2. *Let ν be an invariant measure on Σ and let $\nu^+ = \nu \circ {\pi^+}^{-1}$ and $\nu^- = \nu \circ {\pi^-}^{-1}$. Then ν^+ is a Gibbs measure on Σ^+ with Hölder continuous conditional probabilities if and only if ν^- is a dual Gibbs measure on Σ^- with Hölder continuous conditional probabilities. If either of these properties holds, then $\log g \circ \pi^+$ is Hölder continuously cohomologous to $\log s \circ \pi^-$ where g is the conditional probability function of ν^+ and s is the dual conditional probability function of ν^-.*

If either of the properties in the lemma holds, then we call ν a two-sided Hölder Gibbs measure. It should be noted that no theorem of this type is true if the Hölder condition is removed (see [Ka] for an example where this fails).

Proof. By symmetry, it is clearly sufficient to demonstrate that if ν^- is a dual Gibbs measure on Σ^- with Hölder continuous conditional probabilities, then ν^+ is a Gibbs measure on Σ^+ with Hölder continuous conditional probabilities.

Let s denote the dual conditional probability function of ν^-. Then s is a Hölder continuous function on Σ^-. It follows that $s \circ \pi^-$ is a Hölder continuous function on Σ. Since s is a conditional probability function, we have $P(\log s \circ \pi^-) = 0$. It follows (see [3]) that $\log s \circ \pi^-$ is Hölder continuously cohomologous to a Hölder continuous function ϕ which is normalized and depends only on the positive coordinates. Write $\phi = \log g \circ \pi^+$. We now show that ν^+ is a Gibbs measure on Σ^+ with conditional probability function g.

To see this, note that $\log s \circ \pi^- = \log g \circ \pi^+ - k \circ \sigma + k$ for a Hölder continuous function k. Since ν^- is a dual Gibbs measure with conditional probability function s, there are constants C_1 and C_2 such that

$$C_1 \leq \frac{\nu^-[x_n \dots x_0]}{\exp(\log s(\dots 00x_n) + \log s(\dots 00x_n x_{n-1}) + \dots + \log s(\dots 00x_n \dots x_0))} \leq C_2.$$

Since $\log s \circ \pi^-$ is cohomologous to $\log g \circ \pi^+$, we have

$$\begin{aligned}
&\log s(\dots 00x_n) + \log s(\dots 00x_n x_{n-1}) + \dots + \log s(\dots 00x_n \dots x_0) \\
&= \sum_{i=1}^{n+1} \log s \circ \pi^-(\sigma^i(\dots 00 \cdot x_n \dots x_0 00 \dots)) \\
&= \sum_{i=1}^{n+1} \log g \circ \pi^+(\sigma^i(\dots 00 \cdot x_n \dots x_0 00 \dots)) \\
&\quad - k \circ \sigma^{n+2}(00 \cdot x_n \dots x_0 00 \dots) + k \circ \sigma(00 \cdot x_n \dots x_0 00 \dots) \\
&= \log g(x_{n-1}x_{n-2} \dots x_0 00 \dots) + \log g(x_{n-2} \dots x_0 00 \dots) + \dots + \log g(00 \dots) \\
&\quad - k(00x_n \dots x_0 0 \cdot 00 \dots) + k(00x_n \cdot x_{n-1} \dots x_0 00 \dots).
\end{aligned}$$

Since $\nu^+[x_n \dots x_0] = \nu^-[x_n \dots x_0]$, and $\exp k$ and g are bounded away from 0 and ∞, it follows that there are constants C_3 and C_4 such that

$$C_3 \leq \frac{\nu^+[x_n \dots x_0]}{\exp(\log g(x_0 00\dots) + \log g(x_1 x_0 00\dots) + \dots + \log g(x_n \dots x_0 00\dots))} \leq C_4.$$

This shows that ν^+ is a Gibbs measure with Hölder continuous conditional probability function g as required. □

We will now give necessary and sufficient conditions under which a Hölder continuous function ϕ is continuously cohomologous to a circle-continuous function.

Before doing this, we will need to introduce the following quantities which measure the deviation of a function on Σ^+ from a circle-continuous function. If ϕ is a continuous function on Σ^+, then its *deviations from circle-continuity* are given by

$$\begin{aligned}\delta_0(\phi) &= \phi(00\dots) - \phi(11\dots)\\ \delta_1(\phi) &= \phi(100\dots) - \phi(011\dots) \text{ and}\\ \Delta_{x_n x_{n-1}\dots x_1}(\phi) &= \phi(x_n x_{n-1}\dots x_1 100\dots) - \phi(x_n x_{n-1}\dots x_1 011\dots).\end{aligned}$$

A function is circle-continuous if and only if its deviations from circle-continuity vanish.

Lemma 3. *Let ϕ be a Hölder continuous function defined on Σ^+. Then ϕ is continuously cohomologous to a circle continuous function if and only if*

(1) $\phi(00\dots) = \phi(11\dots)$*; and*
(2) *there exists a constant $C(\phi)$ such that $\sum_{n=1}^{\infty} \Delta_{x_n\dots x_1}(\phi) = C(\phi)$ for all $w^* = \dots x_n \dots x_1 \in \Sigma^-$.*

Proof. Suppose first that ϕ is continuously cohomologous to a circle-continuous function ψ. Then $\psi(x) = \phi(x) - \chi(x) + \chi(\sigma_+(x))$ for a continuous function χ. Then we have $\phi(000\dots) = \psi(000\dots) = \psi(111\dots) = \phi(111\dots)$, so (1) holds. Since ψ is circle-continuous, we have $\Delta_a(\psi) = 0$ for all finite strings a. From the cohomology equation, we now see

$$\begin{aligned}\Delta_{x_n x_{n-1}\dots x_1}(\chi) &= \Delta_{x_n x_{n-1}\dots x_1}(\phi) + \Delta_{x_{n-1}\dots x_1}(\chi) \text{ and}\\ \Delta_{x_1}(\chi) &= \Delta_{x_1}(\phi) + \delta_1(\chi).\end{aligned}$$

for any non-empty finite string $x_n x_{n-1} \dots x_1$. We also have $\delta_1(\chi) = \delta_1(\phi) + \delta_0(\chi)$. Combining the above equations, we see that

$$\Delta_{x_n x_{n-1}\dots x_1}(\chi) = \Delta_{x_n x_{n-1}\dots x_1}(\phi) + \Delta_{x_{n-1}\dots x_1}(\phi) + \dots + \Delta_{x_1}(\phi) + \delta_1(\phi) + \delta_0(\chi).$$

for each finite string $x_n \dots x_1$.

Since χ is required to be continuous, we have $\Delta_{x_n\dots x_1}(\chi)$ converges uniformly to 0 as $n \to \infty$. This implies that $\sum_{n=1}^{\infty} \Delta_{x_n\dots x_1}(\phi) = -\delta_1(\phi) - \delta_0(\chi)$ which is independent of $w^* = \dots x_2 x_1$ which is condition (2).

Conversely, suppose $\phi(000\dots) = \phi(111\dots)$ and

$$\Delta_{x_1}(\phi) + \Delta_{x_2 x_1}(\phi) + \dots \Delta_{x_n x_{n-1}\dots x_1}(\phi) + \dots = C(\phi)$$

for each $x = \dots x_n x_{n-1} \dots x_1 \in \Sigma^-$. Then set

$$\begin{aligned}\delta_0 &= -\delta_0(\phi) - C(\phi);\\ \delta_1 &= -C(\phi); \text{ and}\\ \Delta_{x_n\dots x_1} &= \Delta_{x_n\dots x_1}(\phi) + \Delta_{x_{n-1}\dots x_1}(\phi) + \dots + \Delta_{x_1}(\phi) - C(\phi).\end{aligned}$$

Then by assumption, $\Delta_{x_n\ldots x_1} \to 0$ as $n \to \infty$. Moreover, since ϕ is Hölder continuous, there are constants $K > 0$ and $\lambda > 1$ such that $\Delta_{x_n\ldots x_1}(\phi) \le K/\lambda^n$ for all strings $x_n \ldots x_1$. We then have $\Delta_{x_n\ldots x_1} \le K/(\lambda-1)\cdot 1/\lambda^n$. We now construct a continuous function χ on Σ^+ such that $\delta_0(\chi) = \delta_0$, $\delta_1(\chi) = \delta_1$ and $\Delta_{x_n\ldots x_1}(\chi) = \Delta_{x_n\ldots x_1}$ for each string $x_n \ldots x_1$. Then $\phi - \chi + \chi\circ\sigma_+$ will be a continuous function and will have vanishing deviations. To construct χ, define $h : \Sigma^+ \to \mathbb{R}$ by

$$h(x) = \frac{x_0}{2} - \sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}}.$$

and $k\colon \Sigma^+ \to \mathbb{R}$ by

$$k(x) = 1 - \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}$$

where $x = x_0x_1\ldots$. Then h is a continuous function on Σ^+ with $\|h\|_\infty = 1/2$, $\delta_1(h) = 1$, $\Delta_a(h) = 0$ for all strings a and $\delta_0(h) = 0$. Similarly, $\delta_0(k) = 1$, $\delta_1(k) = 0$ and $\Delta_a(k)$ is 0 for all finite strings a. Now for each finite string a, let

$$h_a(x) = 1_{[a]}(x)h(\sigma_+^{|a|}(x))\Delta_a$$

where $|a|$ means the length of the string a and 1_S denotes the characteristic function of the set S. Then $\Delta_a(h_a) = \Delta_a$ and all other deviations vanish. Now forming $H_n = \sum_{|a|=n} h_a$ gives a function H_n with

$$\|H_n\|_\infty = \sup_{|a|=n} \|\delta_a\| \le \frac{K}{(\lambda-1)\lambda^n}.$$

Now let $H = (\delta_0)k + (\delta_1)h + \sum_{n=1}^{\infty} H_n$. Then H is the uniform limit of continuous functions, and hence is continuous. Further, H has the property that $\Delta_a(H) = \Delta_a$ for all strings a, $\delta_1(H) = \delta_1$ and $\delta_0(H) = \delta_0$. Now set $\psi = \phi - H + H\circ\sigma_+$. Then ψ is a continuous function. Its deviations are given by

$$\begin{aligned}
\delta_0(\psi) &= \delta_0(\phi) = 0;\\
\delta_1(\psi) &= \delta_1(\phi) - \delta_1 + \delta_0 = \delta_1(\phi) - (-C(\phi)) + (-\delta_1(\phi) - C(\phi)) = 0;\\
\Delta_{x_1}(\psi) &= \Delta_{x_1}(\phi) - \Delta_{x_1} + \delta_1 = \Delta_{x_1}(\phi) - (\Delta_{x_1}(\phi) - C(\phi)) - C(\phi) = 0;\ \text{and}\\
\Delta_{x_n\ldots x_1}(\psi) &= \Delta_{x_n\ldots x_1}(\phi) - \Delta_{x_n\ldots x_1} + \Delta_{x_{n-1}\ldots x_1} = 0.
\end{aligned}$$

So ψ has vanishing deviations and is therefore circle-continuous. We now know that ϕ is cohomologous to a circle-continuous function ψ. $\square$

We now show that if ϕ is Hölder continuous and continuously cohomologous to a circle continuous function ψ, then ϕ is in fact Hölder continuously cohomologous to a circle continuous function ψ'. We note that in fact the 'transfer function' H produced above can be shown to be Hölder continuous by performing a sequence of elementary estimates. We choose however to present an alternative argument, which might have other applications. The argument essentially shows that if a Hölder continuous function ϕ is continuously cohomologous to a function possessing some symmetry (in this case, circle-continuity), then the normalized Hölder continuous function corresponding to ϕ also possesses that symmetry.

Lemma 4. *If a Hölder continuous function on Σ^+ is continuously cohomologous to a circle-continuous function, then its normalization is a circle-continuous function*

Proof. Let ϕ be a Hölder continuous function which is continuously cohomologous to a circle continuous function ψ. From this we deduce that ϕ is Hölder cohomologous to a circle-continuous function as follows: We know that $\phi = C + \phi' - K + K \circ \sigma_+$ for a constant C and a Hölder continuous function K where ϕ' is a normalized potential. It follows that $\psi - C = \phi' - G + G \circ \sigma_+$ for a continuous function G. Since $\psi - C$ is circle-continuous, it follows that $\mathcal{L}_{\psi - C}$ sends the space of circle-continuous functions to itself. But $\mathcal{L}^n_{\psi-C}1(x) = \exp(G(x))\mathcal{L}^n_{\phi'}[\exp(-G)](x)$, which (by continuity of G) converges uniformly to a constant (non-zero) multiple of $\exp(G)$. However, since the circle-continuous functions form a closed subspace of $C(X)$, it follows that $\exp(G)$ is circle-continuous. It then follows that ϕ' is circle-continuous. Since ϕ was Hölder cohomologous to ϕ' (up to a constant), it follows that the normalization of ϕ is circle-continuous as required. □

Lemma 5. *If ν is a two-sided Hölder Gibbs measure such that ν^- has conditional probability function s defined on Σ^- and ν^+ has conditional probability function g defined on Σ^+, $\log g$ is Hölder cohomologous to $\Phi\colon \Sigma^+ \to \mathbb{R}$ where*

$$\Phi(x_0x_1\dots) = \sum_{n=1}^{\infty}\big(\log s(\dots 00x_0\dots x_n) - \log s(\dots 00x_1\dots x_n)\big) + \log s(\dots 00x_0)$$

Proof. Define $\delta_n\colon \Sigma \to \mathbb{R}$ by

$$\delta_n(x) = \log s(\dots 0x_{-n}\dots x_{-1}) - \log s(\dots 0x_{-(n-1)}\dots x_{-1})$$

for $n > 1$ and $\delta_1(x) = \log s(\dots 0x_{-1})$. Note that by Hölder continuity of s, we have $\|\delta_n\|$ converges geometrically to 0 and $\sum_{n=1}^{\infty}\delta_n = \log s \circ \pi^-$.

We also have $\delta_n(\sigma^n(x)) = \log s(\dots 0x_0\dots x_{n-1}) - \log s(\dots 0x_1\dots x_{n-1})$ for $n > 1$, while for $n = 1$, $\delta_1(\sigma(x)) = \log s(\dots 0x_0)$. Summing, we get

$$\sum_{n=1}^{k}\delta_n \circ \sigma^n(x) = \sum_{n=2}^{k}(\log s(\dots 0x_0\dots x_{n-1}) - \log s(\dots 0x_1\dots x_{n-1})) + \log s(\dots 0x_0).$$

Taking the limit, we have $\sum_{n=1}^{\infty}\delta_n \circ \sigma^n(x) = \Phi \circ \pi^+(x)$. That this limit exists and is continuous follows from the fact that $\|\delta_n\|_\infty$ converges to 0 exponentially. The function Φ may also be seen to be Hölder continuous.

It remains to verify that $\Phi \circ \pi^+$ is Hölder continuously cohomologous to $\log s \circ \pi^-$. To see this, note that

$$\Phi \circ \pi^+ - \log s \circ \pi^- = \sum_{n=1}^{\infty}\big(\delta_n \circ \sigma^n - \delta_n\big).$$

This may be written as $H \circ \sigma - H$ where $H = \sum_{n=1}^{\infty} h_n$ and $h_n = \sum_{i=0}^{n-1}\delta_n \circ \sigma^i$. Again, it is possible to verify directly that H is Hölder continuous, but in this case, the result follows more simply from Livšic's theorem (see [20,21]). We check that $\|h_n\|_\infty$ converges to 0 geometrically. It then follows that H is continuous. Livšic's theorem then applies as $\Phi \circ \pi^+$ and $\log s \circ \pi^-$ are two Hölder continuous functions on Σ which are cohomologous by an L^∞ function, allowing us to deduce that in fact $\Phi \circ \pi^+$ and $\log s \circ \pi^-$ are Hölder continuously cohomologous.

Since we have already seen (Lemma 2) that $\log s\circ\pi^-$ and $\log g\circ\pi^+$ are Hölder continuously cohomologous, it now follows that $\log g\circ\pi^+$ and $\Phi\circ\pi^+$ are Hölder continuously cohomologous.

A further application of Livšic's theorem allows us to deduce that $\log g$ and Φ are Hölder continuously cohomologous (as functions on Σ^+). Let $x = www\dots$ be any periodic point in Σ^+, where w is any finite string. Let y be the corresponding periodic point in Σ: $\dots www{\cdot}www\dots$. Since $\log g\circ\pi^+$ and $\Phi\circ\pi^+$ are cohomologous, it follows that

$$\sum_{i=0}^{\mathrm{per}(w)-1} \log g\circ\pi^+(\sigma^i(y)) = \sum_{i=0}^{\mathrm{per}(w)-1} \Phi\circ\pi^+(\sigma^i(y)).$$

This implies

$$\sum_{i=0}^{\mathrm{per}(w)-1} \log g(\sigma^i(x)) = \sum_{i=0}^{\mathrm{per}(w)-1} \Phi(\sigma^i(x)),$$

for any finite word w which is known by Livšic's theorem to be a necessary and sufficient condition for Φ and $\log g$ to be Hölder continuously cohomologous. $\square$

Lemma 6. *If ν is a Hölder Gibbs measure, ν^+ has conditional probability function g and ν^- has conditional probability function s, then g is circle-continuous if and only if s is compatible.*

Proof. Let Φ be as in the statement of Lemma 5. We know that Φ is Hölder continuously cohomologous to $\log g$. It follows that $\log g$ is the normalization of Φ. From Lemmas 3 and 4, we see that $\log g$ is circle-continuous if and only if Φ satisfies the conditions of Lemma 3.

From the definition of Φ, we see that $\Phi(00\dots) = \log s(\dots 00)$ and $\Phi(11\dots) = \log s(\dots 11)$. It follows s satisfies the first part of the compatibility condition if and only if Φ satisfies the first condition from Lemma 3.

To finish the proof, we show that under the assumption that s satisfies the first compatibility condition, s satisfies the second compatibility condition if and only if Φ satisfies the second condition from Lemma 3.

We now calculate the quantities involved in condition (2) of Lemma 3. Set $C_n(x) = \Delta_{x_n\dots x_1}(\Phi) + \dots \Delta_{x_1}(\Phi)$ and $C(x) = \lim_{n\to\infty} C_n(x)$. The condition is then that $C(x)$ is independent of x. We have

$$\begin{aligned}\Delta_{x_n\dots x_1}(\Phi) &= \Phi(x_n\dots x_1 100\dots) - \Phi(x_n\dots x_1 011\dots)\\ &= \sum_{k=0}^{\infty}\Big[\log s(\dots 00x_n\dots x_1 1\overbrace{0\dots 0}^{k\text{ times}}) - \log s(\dots 00x_{n-1}\dots x_1 1\overbrace{0\dots 0}^{k\text{ times}})\\ &\qquad - \log s(\dots 00x_n\dots x_1 0\overbrace{1\dots 1}^{k\text{ times}}) + \log s(\dots 00x_{n-1}\dots x_1 0\overbrace{1\dots 1}^{k\text{ times}})\Big].\end{aligned}$$

Summing, we see

$$\begin{aligned}C_n(x) = \sum_{k=0}^{\infty}\Big[&\log s(\dots 00x_n\dots x_1 1\overbrace{0\dots 0}^{k\text{ times}}) - \log s(\dots 001\overbrace{0\dots 0}^{k\text{ times}})\\ &- \log s(\dots 00x_n\dots x_1 0\overbrace{1\dots 1}^{k\text{ times}}) + \log s(\dots 000\overbrace{1\dots 1}^{k\text{ times}})\Big].\end{aligned}$$

Since $\log s$ is Hölder and $\log s(\dots 00) = \log s(\dots 11)$, the sum can be split as:

$$C_n(x) = \Big[\sum_{k=0}^{\infty} \log s(\dots 00x_n \dots x_1 1 \overbrace{0 \dots 0}^{k \text{ times}}) - \log s(\dots 00x_n \dots x_1 0 \overbrace{1 \dots 1}^{k \text{ times}})\Big]$$
$$- \Big[\sum_{k=0}^{\infty} \log s(\dots 001 \overbrace{0 \dots 0}^{k \text{ times}}) - \log s(\dots 000 \overbrace{1 \dots 1}^{k \text{ times}})\Big].$$

The second summation is clearly independent of x, so we see that $C(x)$ is independent of x if and only if

$$\sum_{k=0}^{\infty} \log s(\dots 00x_n \dots x_1 1 \overbrace{0 \dots 0}^{k \text{ times}}) - \log s(\dots 00x_n \dots x_1 0 \overbrace{1 \dots 1}^{k \text{ times}})$$

is independent of x, but this is just the second part of the compatibility condition on s.

It follows that Φ is continuously cohomologous to a circle-continuous function if and only if s satisfies the compatibility condition and hence g is circle-continuous if and only if s satisfies the compatibility condition. $\square$

Lemma 7. *If ν^+ is a Gibbs measure with a Hölder continuous conditional probability function g, then there is a circle map $f \in \mathcal{C}^{1+h}$ whose symbolic measure is ν^+.*

For a fuller version of this proof, the reader is referred to [23]. Here is a summary of the argument.

Proof. Define a map $f_0 : S^1 \to S^1$ by $f_0(x) = 2x \pmod 1$. Let $p\colon \Sigma^+ \to S^1$ be defined by $p(x) = \sum_{n=0}^{\infty} x_n/2^{n+1}$. This is the natural semiconjugacy between σ_+ and f_0. The map p is also a measure-theoretic isomorphism between $(\Sigma^+, \nu^+, \sigma_+)$ and $(S^1, \nu^+ \circ p^{-1}, f_0)$. To create the map f, we conjugate f_0. Let $\theta : S^1 \to S^1$ be defined by $\theta(x) = \nu^+(p^{-1}([0, x]))$. The map θ is a homeomorphism from the circle to itself as Gibbs measures are automatically non-atomic and fully supported (see [W]). Then set $f(x) = \theta(f_0(\theta^{-1}(x)))$ preserving the measure $\nu^+ \circ p^{-1} \circ \theta^{-1}$. Calculating, we see that $\nu^+ \circ p^{-1}(\theta^{-1}([0, x]) = \nu^+ \circ p^{-1}([0, \theta^{-1}(x)]) = x$. It follows that $\nu^+ \circ p^{-1} \circ \theta^{-1}$ is Lebesgue measure. So f preserves Lebesgue measure. It may be checked that $f \in \mathcal{C}^{1+h}$. Let $|\cdot|$ denote Lebesgue measure. The triples $(\Sigma, \nu^+, \sigma_+)$ and $(S^1, |\cdot|, f)$ are measure-theoretically isomorphic by the map $\theta \circ p$. The map $\theta \circ p$ may also be seen to be the natural semiconjugacy so that ν^+ is indeed the symbolic measure corresponding to the map f as required. $\square$

We now combine these Lemmas to complete the proof of Theorem 2.

Proof of Theorem 2. Suppose $f \in \mathcal{C}^{1+h}$. Then let μ be the unique absolutely continuous invariant measure for T with density ρ. Let ν^+ be the corresponding symbolic measure and p be the natural semiconjugacy from Σ^+ to S^1. Then ν^+ is a Gibbs measure with conditional probability function given by

$$g(x) = \frac{\rho(p(x))}{f'(p(x))\rho(f(p(x)))}.$$

This is clearly circle-continuous and Hölder continuous. By Lemmas 2 and 6, it follows that the scaling function s (which is also the conditional probability function of ν^-) is compatible and Hölder continuous. As pointed out in §2, scaling functions automatically satisfy the summability condition.

Conversely, suppose that s is a Hölder continuous function satisfying the summability condition and the compatibility condition. By results in [W], there is a unique Gibbs measure ν^- on Σ^- which has s as its dual conditional probability function. This may be extended to give a measure ν on Σ and by Lemma 2, ν^+ is a Gibbs measure with Hölder continuous conditional probability function, g say. Since s is compatible, it follows that g is circle-continuous. Now by Lemma 7, there is a map $f \in \mathcal{C}^{1+h}$ which has ν^+ as its symbolic measure. This map necessarily has ν^- as its dual symbolic measure and so its scaling function is just the conditional probability function of ν^- which is s. $\square$

4. Teichmüller space of circle endomorphisms

In this section, we study the geometric structure of $\mathcal{C}^{1+h}$ and represent its Teichmüller space by the space of Hölder continuous functions on Σ^- satisfying the summation and compatibility conditions. We also study the completion of the Teichmüller space.

Consider a fixed element $q(z) = z^2$ in $\mathcal{C}^{1+h}$. For every element f in $\mathcal{C}^{1+h}$, there is a unique homeomorphism h of S^1 such that

$$hf = qh.$$

Here h is called the conjugacy from f to q. The conjugacy here is not only a homeomorphism but also satisfies a very important geometric property called quasisymmetry defined as follows. Suppose $h : I \to J$ is a homeomorphism. Define

$$\rho(h, x, t) = \max\left\{\frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|}, \frac{|h(x) - h(x-t)|}{|h(x+t) - h(x)|}\right\}$$

for $x \in I$ and $t > 0$ with $x \pm t \in I$,

$$\rho(h, t) = \sup_{x \in I} \rho(h, x, t),$$

and

$$\rho(h) = \sup_{t>0} \rho(h, t).$$

The map h is said to be ρ-quasisymmetric if $\rho(h) \le \rho$ and to be symmetric if $\rho(h, t) \to 1$ as $t \to 0$. A C^1-diffeomorphism is an example of a symmetric homeomorphism but a symmetric homeomorphism may be very singular; it may map a measure zero set into a positive measure set. The proof of the following theorem can be found, for example in [14, §3.5].

Theorem 3. *Suppose f is in $\mathcal{C}^{1+h}$ and h is the conjugacy from f to q. Then h is quasisymmetric.*

Now we consider the space of pairs $[f, h]$ where f is in $\mathcal{C}^{1+h}$ and h is the conjugacy from f to q. For any two pairs $[f_1, h_1]$ and $[f_2, h_2]$, $h_2^{-1}h_1$ is the conjugacy from f_1 to f_2 and is again quasisymmetric. Two pairs $[f_1, h_1]$ and $[f_2, h_2]$ are equivalent (denoted by $[f_1, h_1] \sim [f_2, h_2]$) if $h_2^{-1}h_1$ is a C^1-diffeomorphism. It is easy to verify that $\sim$ is an equivalence relation. Let $\kappa = \{[f, h]\}$ mean an equivalence class. Then the Teichmüller space, denoted as $\mathcal{TC}^{1+h}$, of $\mathcal{C}^{1+h}$ is the collection of all equivalence classes $\kappa = \{[f, h]\}$ for f in $\mathcal{C}^{1+h}$.

The following theorem is a consequence of Shub-Sullivan's theorem in [24] and Proposition 1. A more general version for geometrically finite maps is proved in [15,16].

Theorem 4. *Suppose f_1 and f_2 are in $\mathcal{C}^{1+h}$. Let h be the conjugacy from f_1 to f_2, i.e., $hf_1 = f_2h$. Then h is a C^{1+h} diffeomorphism if and only if the scaling functions of f_1 and f_2 are the same.*

This theorem says that any two elements in an equivalence class in the Teichmüller space $\mathcal{TC}^{1+h}$ have the same scaling function. This combined with Theorem 2 give us the following important corollary. Let S^h be the space of all Hölder continuous functions on Σ^- satisfying the summation and compatibility conditions. For any $\kappa = \{[f, h]\}$ in $\mathcal{T}C^{1+h}$, let $\iota(\kappa)$ mean the scaling function of f.

Corollary 1. *The map ι from $\mathcal{TC}^{1+h}$ to S^h is bijective.*

We are going to embed $\mathcal{TC}^{1+h}$ as well as S^h into the universal Teichmüller space and study its completion in the universal Teichmüller space.

Let $\mathcal{QS}$ be the set of all quasisymmetric homeomorphisms of the circle modulo the space of all Möbius transformations of the circle ($\mathcal{QS}$ may be identified with the set of all quasisymmetric homeomorphisms of the circle fixing three points). Let $\mathcal{S} \subset \mathcal{QS}$ be the subset of all symmetric homeomorphisms of the circle. For any $h \in \mathcal{QS}$, let $\mathcal{E}_h$ be the set of all quasiconformal extensions H of h into the unit disk. Let K_H be the quasiconformal dilatation of $H \in \mathcal{E}_h$ (see [19]). Using quasiconformal dilatation, one defines a distance in $\mathcal{QS}$ by

$$d(h_1, h_2) = \frac{1}{2} \inf\{\log K_{H_2^{-1}H_1} | H_1 \in \mathcal{E}_{h_1}, H_2 \in \mathcal{E}_{h_2}\}.$$

It is known that the space $(\mathcal{QS}, d)$ is a complete space under the metric d. This space is called the universal Teichmüller space (see [19]). The topology coming from the metric d on $\mathcal{QS}$ induces a topology on $\mathcal{QS}/\mathcal{S}$. Defining

$$\overline{d}(\mathcal{S}h_1, \mathcal{S}h_2) = \inf_{k_1, k_2 \in \mathcal{S}} d(k_1h_1, k_2h_2).$$

gives a metric on $\mathcal{QS}/\mathcal{S}$ (see [12]). The topology on $(\mathcal{QS}/\mathcal{S}, \overline{d})$ is the finest topology which makes the projection $\Pi : \mathcal{QS} \to \mathcal{QS}/\mathcal{S}$ continuous. An equivalent topology can be defined as follows. For any $h \in \mathcal{QS}$, let H_{loc} be a quasiconformal extension of h to a small neighborhood U of S^1 in the complex plane. Let

$$\mu = \mu_{H_{loc}} = \frac{(H_{loc})_{\overline{z}}}{(H_{loc})_z}.$$

Let

$$k_{H_{loc}} = \|\mu(z)\|_\infty$$

and

$$C_{H_{loc}} = \frac{1 + k_{H_{loc}}}{1 - k_{H_{loc}}}.$$

Then

$$C_h = \inf C_{H_{loc}}$$

where the infimum is taken over all quasiconformal extensions H_{loc} of h to a neighborhood of the circle. The number C_h is called the boundary dilatation of h. It is known that h is symmetric if and only if $C_h = 1$. Define

$$\tilde{d}(h_1, h_2) = \frac{1}{2} \log C_{h_2^{-1}h_1}.$$

The topology of $(\mathcal{QS}/\mathcal{S}, \tilde{d})$ is equivalent to the topology of $(\mathcal{QS}/\mathcal{S}, \overline{d})$. For any $\kappa = \{[f,h]\}$ in $\mathcal{TC}^{1+h}$, define $\Phi(\kappa) = h\mathcal{S}$, where $h\mathcal{S}$ means the equivalence class of h in $\mathcal{QS}/\mathcal{S}$. Then Φ embeds $\mathcal{TC}^{1+h}$ into $\mathcal{QS}/\mathcal{S}$ and induces a topology on the Teichmüller space $\mathcal{TC}^{1+h}$. By Corollary 1, we also get an embedding of S^h into $\mathcal{QS}/\mathcal{S}$. Now we are going to discuss the completion of $\mathcal{TC}^{1+h}$ in $\mathcal{QS}/\mathcal{S}$.

Let $\mathbb{R}$ be the real line. Then $E(x) = e^{2\pi ix} : \mathbb{R} \to S^1$ is a covering map. For any circle map, let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f, i.e., F is an orientation-preserving homeomorphism such that $E \circ F = f \circ E$. Then $F(x+1) = F(x) + 2$. If we impose the additional condition $F(0) = 0$, then F is uniquely defined. A circle map f, is said to be uniformly symmetric (refer to [26]) if

$$\rho_f(t) = \sup_n \left\{\rho\big((F^{\circ n})^{-1}, t\big)\right\} \to 0 \qquad \text{as} \qquad t \to 0.$$

It is equivalent to say that all inverse branches of $f^n|I$ for all $n > 0$ and all open intervals $I \neq S^1$ are uniformly symmetric. An example of a uniformly symmetric circle map is a C^{1+h} circle map. This fact follows from the naive distortion lemma directly. Let $\mathcal{SC}$ be the space of all uniformly symmetric circle maps. Then $\mathcal{C}^{1+h} \subset \mathcal{SC}$. The following lemma can be found in [19]. A normalized homeomorphism h of the real line is a homeomorphism such that $h(0) = 0$, $h(1) = 1$, and $h(\infty) = \infty$.

Lemma 8. *Let $[a,b]$ be a closed interval on the real line, and $\epsilon > 0$. Then there is a $\delta > 0$ such that for any normalized quasisymmetric function h,*

$$|h(x) - x| < \epsilon, \quad x \in [a,b],$$

whenever h is $(1+\delta)$-quasisymmetric.

By using this lemma and referring to the proof of Theorem 1, we get the following theorem.

Theorem 5. *Suppose f is a uniformly symmetric circle map. Then its scaling function exists. Moreover, this scaling function is continuous.*

By applying Lemma 8 again, we see that the scaling function is left invariant if the map is conjugated by a symmetric homeomorphism.

Lemma 9. *Suppose f_1 and f_2 are two uniformly symmetric circle maps. Let h be the conjugacy between f_1 and f_2. If h is symmetric, then the scaling functions of f_1 and f_2 are the same.*

Similarly we define the Teichmüller space of $\mathcal{SC}$ as follows. Consider a fixed element $q(z) = z^2$ in $\mathcal{SC}$. For every element f in $\mathcal{SC}$, there is a homeomorphism h of S^1 such that

$$hf = qh.$$

The conjugacy h here is again quasisymmetric. The proof of this fact is similar to the proof of Theorem 3 by applying bounded geometry (refer to [14, §3.5]). Now we consider the space of pairs $[f,h]$ where f is in $\mathcal{SC}$ and h is the conjugacy from f to q. Two pairs $[f_1,h_1]$ and $[f_2,h_2]$ are equivalent (denote as $[f_1,h_1] \sim [f_2,h_2]$ again) if $h_2^{-1}h_1$ is symmetric. It is easy to verify that $\sim$ is an equivalence relation. Let $\kappa = \{[f,h]\}$ denote an equivalence class. Then the Teichmüller space, denote as $\mathcal{TSC}$, of $\mathcal{SC}$ is the collection of all equivalence classes $\kappa = \{[f,h]\}$ for f in $\mathcal{TSC}$. For any $\kappa = \{[f,h]\}$ in $\mathcal{SC}$, $\Phi(\kappa) = h\mathcal{S}$, where $h\mathcal{S}$ means the equivalence class of h in $\mathcal{QS}/\mathcal{S}$, defines an embedding of $\mathcal{TSC}$ into $\mathcal{QS}/\mathcal{S}$ and induces a topology on the Teichmüller space $\mathcal{TSC}$. Following Lemma 9 and Theorem 4, if both of f_1 and f_2 are in $\mathcal{C}^{1+h}$ and $[f_1,h_1]$ and $[f_2,h_2]$ are equivalent in $\mathcal{SC}$, then they are also equivalent in $\mathcal{C}^{1+h}$. The following lemma also appeared in [13], but we offer a different proof.

Lemma 10. *The space $\mathcal{TSC}$ is a complete space.*

Proof. Take any Cauchy sequence $\{\kappa_n\}_{n=1}^{\infty} = \{[f_n, h_n]\}_{n=1}^{\infty}$ in $\mathcal{TSC}$. Then

$$\overline{d}(\mathcal{S}h_n, \mathcal{S}h_m) \to 0 \qquad \text{as} \qquad m, n \to \infty.$$

We may assume by working modulo $\mathcal{S}$ that $h_n^{-1}h_m$ tends to the identity map as m and n go to infinity. Therefore, $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the universal Teichmüller space and h_n tends to a quasisymmetric map h as n goes to infinity. Let $l_n = h^{-1}h_n$ and $f = h^{-1}qh$. Define maps f_n by $f = l_n f_n l_n^{-1}$ for all $n > 0$. Let $\rho(l_n)$ be the quasisymmetric constant of l_n. Then $\rho(l_n) \to 1$ as $n \to \infty$. Let g_{n,w_k} be the inverse of $f_n^k : \mathring{I}_{n,w_k} \to S^1 \setminus \{\alpha_n\}$, where α_n is the fixed point of f_n. Let g_{w_k} be the inverse of $f^k : \mathring{I}_{w_k} \to S^1 \setminus \{\alpha\}$ where α is the fixed point of f. Let $\rho(g_{n,w_k}, t)$ be the symmetric distortion of g_{n,w_k}. Then there is a function $\epsilon_n(t) \to 1$ as $t \to 0$ such that $\rho(g_{n,w_k}, t) \leq \epsilon_n(t)$ for all w_k. Let $\rho(g_{w_k}, t)$ be the symmetric distortion of g_{w_k}. Since $g_{w_k} = l_n f_{n,w_k} l_n^{-1}$,

$$\rho(g_{w_k}, t) \leq (\rho(l_n))^2 \epsilon_n(t)$$

for all w_k and all $n > 0$. Therefore,

$$\rho_f(t) \leq \inf_{n>0} \{(\rho(l_n))^2 \epsilon_n(t)\}.$$

So $\rho_f(t) \to 1$ as $t \to 0$, this means that f is uniformly symmetric and $[f, h] \in \mathcal{TSC}$. So $\mathcal{TSC}$ is complete. □

Lemma 11. *For any $[f, h] \in \mathcal{TSC}$ and any $\epsilon > 0$, there is an analytic circle map f_ϵ in $\mathcal{C}^{1+h}$ such that $[f_\epsilon, h_\epsilon]$ is in the ϵ-neighborhood of $[f, h]$ in $\mathcal{TSC}$.*

Proof. For any $[f, h] \in \mathcal{TSC}$, we construct an analytic circle map approximating f in the quasiconformal metric. The technique in the construction was developed in complex dynamics (see [9]).

Consider a quasiconformal extension H of h to the complex plane. Then $T = HqH^{-1}$ is a quasiregular map of the complex plane. Let

$$\mu_{T^n}(z) = \frac{(T^n)_{\overline{z}}}{(T^n)_z}$$

be the Beltrami coefficient of T^n. Since f is uniformly symmetric, we can pick an extension T (equivalently, pick an extension H of the conjugacy h) such that there is a function $\gamma(t) \to 0$ as $t \to 0$ and such that $|\mu_{T^n}(z)| \leq \gamma(|z|^{2n} - 1)$ for all $n > 0$ and a.e. z. (This can be taken as an equivalent definition of a uniformly symmetric circle map (refer to [12]).) Let

$$\mu_H(z) = \frac{H_{\overline{z}}}{H_z}$$

be the Beltrami coefficient of H. By calculus,

$$\mu_{T^n}(z) = \frac{\mu_H(q^n(z)) - \mu_H(z)}{1 + \mu_H(q^n(z))\overline{\mu_H(z)}}.$$

This implies that

$$|\mu_H(q^n(z)) - \mu_H(z)| \leq C\gamma(|z|^{2^n} - 1)$$

for all $n > 0$ and a.e. z where $C > 0$ is a constant. For any $\epsilon > 0$, we have a $\delta > 0$ such that $\gamma(t) < \epsilon/C$ for all $0 \leq t < \delta$. Let

$$A_0 = \{z \in \mathbb{C} \mid 1 - \delta < |z| < (1-\delta)^{1/2}\} \cup \{z \in \mathbb{C} \mid (1+\delta)^{1/2} < |z| < 1 + \delta\}$$

and set $A_n = q^{-n}(A_0)$. Define $\mu(z) = \mu_H(z)$ for $z \in \mathbb{C} \setminus (\cup_{n=1}^{\infty} A_n)$ and $\mu = \mu_H(q^n(z))$ for $z \in A_n$ and $n > 0$. Then μ is a Beltrami coefficient defined on the complex plane. Let R be a quasiconformal homeomorphism solving the Beltrami equation $R_{\overline{z}} = \mu(z) R_z$ and let $T_\epsilon = RqR^{-1}$. By calculus,

$$\mu_{T_\epsilon}(z) = \frac{\mu(q(z)) - \mu(z)}{1 + \mu(q(z))\overline{\mu(z)}}.$$

So $(T_\epsilon)_{\overline{z}} = 0$ for $(1-\delta)^{1/2} < |z| < (1+\delta)^{1/2}$, that is, $f_\epsilon = T_\epsilon | S^1$ is analytic. Because $|\mu(z) - \mu_T(z)| < \epsilon$ for all $z \in \mathbb{C}$, T_ϵ is ϵ-approximate to F in the quasiconformal metric.

Since $R|S^1$ is quasisymmetric, f_ϵ has bounded geometry meaning that all pre-scalings $s_\epsilon(w_n)$ of f_ϵ are greater than a constant. A real analytic degree two circle endomorphism having bounded geometry is expanding (refer to [14, Chapter 3]). This completes the proof. □

Theorem 6. *The completion of $\mathcal{TC}^{1+h}$ is $\mathcal{TSC}$*

Proof. It follows from Lemmas 10 and 11.

The study of $\mathcal{TSC}$ and its representation by the space of scaling functions is still an important problem.

REFERENCES

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, D. Van Nostrand-Reinhold Company, Inc., Princeton, New Jersey, 1966.
[2] V. Baladi, Y. Jiang, and O. E. Lanford III *Transfer operators acting on Zygmund functions*, Trans. Amer. Math. Soc., Vol. **348** (1996), 1599–1615.
[3] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer-Verlag, Berlin, 1975.
[4] E. Cawley, *The Teichmüller space of an Anosov diffeomorphism of T^2*, Invent. Math. Vol **112** (1993), 351–376.
[5] Z. Coelho and A. N. Quas, *Criteria for $\overline{d}$-continuity*, Trans. Amer. Math. Soc., to appear.
[6] P. Coullet and C. Tresser, *Itération d'endomorphismes et groupe de renormalisation*, C. R. Acad. Sci. Paris Ser., A-B, Vol **287** (1978), A577–A580.
[7] G. Cui, *Linear model of circle expanding maps*, Preprint.
[8] G. Cui, *Circle expanding maps and symmetric structures*, Preprint.
[9] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. Éc. Norm. Sup., Paris, Vol **18** (1985), 287–343.
[10] M. Feigenbaum, *The universal metric properties of non-linear transformations*, J. Stat. Phys., Vol **21** (1979), 669–706.
[11] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, Wiley, New York, 1987.
[12] F. Gardiner and D. Sullivan, *Symmetric and quasisymmetric structures on a closed curve*, Amer. J. Math., Vol **114**, 4 (1992), 683–736.
[13] F. Gardiner and D. Sullivan, *Lacunary series as quadratic differentials in conformal dynamics*, Contemporary Mathematics (AMS Series), Vol **169** (1994), 307-330.
[14] Y. Jiang, *Renormalization And Geometry in One-Dimensional And Complex Dynamics*, World Scientific, Singapore-New Jersey-London-Hong Kong, Vol **10**, 1996.
[15] Y. Jiang, *Smooth classification of geometrically finite one-dimensional maps*, Trans. Amer. Math. Soc., Vol **348**, 6 (1996), 2391–2412.
[16] Y. Jiang, *On rigidity of one-dimensional maps*, Lipa's Legacy (Contemp. Math.), 1997, Amer. Math. Soc., Providence, RI, 319–341.
[17] S. Kalikow, *Random Markov processes and uniform martingales*, Israel J. Math., Vol **71** (1990), 33–54.
[18] M. Keane, *Strongly mixing g-measures*, Invent. Math., Vol **16** (1972), 309-324
[19] O. Lehto, *Univalent Functions and Teichmüller Spaces*, Springer-Verlag, New York, Berlin, 1987.
[20] A. Livšic, *Some homology properties of Y-systems*, Math. Notes of the USSR Acad Sci., Vol **10** (1971), 758-763.
[21] A. Livšic, *Cohomology of dynamical systems*, Math. USSR Izvestija, Vol **6** (1972), 1278-1301.
[22] A. Pinto and D. Sullivan, *The circle and solenoid*, Preprint.
[23] A. Quas, *Non-ergodicity for C^1-expanding maps and g-measures*, Ergodic Theory Dynamical Systems, Vol *16* (1996), 531-543.
[24] M. Shub and D. Sullivan, *Expanding endomorphisms of the circle revisited*, Ergod. Th & Dynam. Sys., Vol **5** (1985), 285-289.
[25] D. Sullivan, *Differentiable structure on fractal like sets determined by intrinsic scaling functions on dual Cantor sets*, The Proceedings of Symposia in Pure Mathematics, Vol **48** (1988).
[26] D. Sullivan, *Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers*, Topological Methods in Modern Mathematics, A Symposium in Honor of John Milnor's Sixtieth Birthday, 543-564.
[27] P. Walters, *Ruelle's operator theory and g-measures*, Trans. Amer. Math. Soc., Vol **214** (1975), 375-387.

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