INFINITE PATHS IN A LORENTZ LATTICE GAS MODEL

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Abstract. We consider infinite paths in an illumination problem on the lattice $\mathbb{Z}^2$, where at each vertex, there is either a two-sided mirror (with probability $p \geqslant 0$) or no mirror (with probability $1-p$). The mirrors are independently oriented NE-SW or NW-SE with equal probability. We consider beams of light which are shone from the origin and deflected by the mirrors. The beam of light is either periodic or unbounded. The novel feature of this analysis is that we concentrate on the measure on the space of paths. In particular, under the assumption that the set of unbounded paths has positive measure, we are able to establish a useful ergodic property of the measure. We use this to prove results about the number and geometry of infinite light beams. Extensions to higher dimensions are considered.

1. Introduction and Statement of Theorems

There has been much study recently of the trajectory of a light beam subject to reflections at the surfaces of randomly positioned objects. In this paper, we consider a lattice model of this process which has attracted much attention over the last two decades, but in which relatively little progress has been made.

The model will be referred to as the 'mirror model'. In this model, there is assumed to be an arrangement of mirrors and crossings on the lattice and a light beam is shone from the origin in a direction parallel to one of the axes. The crossings have no effect on the direction of a light beam so the light carries on, leaving in the same direction as that in which it entered, while the mirrors are two-sided and placed at lattice points at $\pm 45°$ to the positive x-axis. This ensures that a light beam shone from the origin in the lattice will be a sequence of connecting line segments between adjacent lattice points. The aim is to describe the typical long-term behaviour of the beam for a random configuration of mirrors (as chosen by some probability measure) in terms of boundedness of beam etc.

In the related family of Lorentz lattice gas models (see [14]), there is a configuration of massive scatterers (atoms) which deflect a particle of small mass (electron) (or possibly infinitely many non-interacting particles). The Lorentz lattice gas models with infinitely many particles may be described in terms of Cellular Automata

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mappings. Another related model in the physics literature is the Ehrenfest wind-tree model (see [8]). There are a number of variants of the mirror model where the mirrors are replaced by different kinds of scatterer. Of interest are flipping and random scatterers (see [15], [5], [1], [2] and [3] for a description of flipping scatterers and [10] for some results about random scatterers). For some results on a continuum version of this problem, the reader is referred to [11]. In higher dimensions, it is clear that the number of possible deterministic scatterers is far higher than in 2 dimensions.

We will consider configurations on the lattice $\mathbb{Z}^2$ of crossings (which allow light to pass without changing direction) and mirrors. There will be two kinds of mirror in our model, denoted symbolically by / and \. A crossing will be denoted by +. The set of configurations is given by $\mathcal{C} = \{/, \backslash, +\}^{\mathbb{Z}^2}$. Given a configuration, we illuminate a set of vertices by shining a beam of light in an axial direction from the origin. The light beam is then deflected by the mirrors as illustrated in Figure 1.

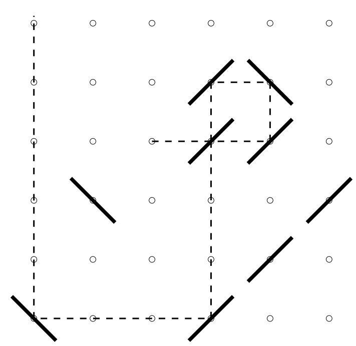

Figure 1. A typical configuration and light beam

For a light beam governed by the deterministic rules depicted in Figure 1, since the light never traverses an edge in two different directions, there is a dichotomy: either the light beam follows an unbounded path; or it returns to the origin, leaving it in the same direction as the initial one (from then the light never leaves this finite closed path). In the case where the light follows a finite closed path, it is said to be localized. This possibility is illustrated in Figure 2. In the case that the light beam is infinite, we see that the light can visit each vertex at most twice, and it follows that the light beam eventually leaves every finite set.

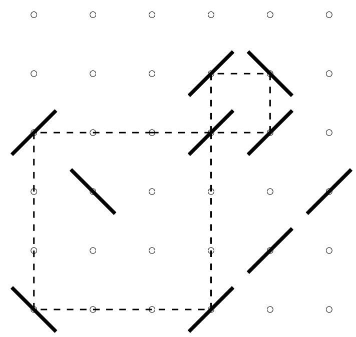

Figure 2. A typical localized light beam

We introduce a probability measure on the space of configurations by letting p be a parameter with $0 \leqslant p \leqslant 1$ and setting $\mathbb{P}_p$ to be the product measure of the measures ρ_p at each lattice site where $\rho_p(/) = \rho_p(\backslash) = \frac{1}{2}p$ and $\rho_p(+) = 1 - p$. This means that the configurations at distinct vertices are independent, and that at each vertex, there is a mirror with probability p which is then oriented with equal probability in either of the two directions. There is a natural action of the group $\mathbb{Z}^2$ on $\mathcal{C}$ by the translation maps $(\sigma_{(m,n)}\xi)_{i,j} = \xi_{(m+i,n+j)}$. It is a well-known fact that $\mathbb{P}_p$ is an ergodic invariant measure under this group action.

The central question is whether or not there exist values of $p > 0$, for which there is a positive probability that the light beam starting from the origin in a given direction is infinite. Grimmett ([9]) indicated that if $p = 1$, then the light beam from the origin is almost surely localized. The proof is by observing that there is an embedded critical bond percolation problem, where it is known that there are almost surely circuits surrounding any finite set. Bunimovich and Troubetzkoy ([1]) observed that this also holds if $p = 1$, but the proportion of NW-SE and SW-NE mirrors is different from $(\frac{1}{2}, \frac{1}{2})$ (it is essential that the proportion of each is strictly positive). Some numerical results are presented in [6], [7], [12] [18] and [19] which suggest that even for $p < 1$, all light beams are localized with probability 1.

I should like to thank Geoffrey Grimmett for introducing me to the problem and for many helpful discussions. I am grateful to Mike Keane for offering insight in re-interpreting the proofs. I should also like to thank Matthew Harris for pointing out an error in an earlier version and for some useful suggestions.

Given a configuration $\xi \in \mathcal{C}$, there is an induced equivalence relation on the bond set $\mathbb{B}^2 = \{\{x, x + e_i\}\colon x \in \mathbb{Z}^2, i \in \{1, 2\}\}$. We say two bonds $b, b' \in \mathbb{B}^2$ are ξ-*neighbours* if they touch at exactly one end and if the configuration ξ is such that b and b' form adjacent edges in a light beam. Taking the transitive closure of this relation gives an equivalence relation $\sim_\xi$, where two bonds are related if and only if they lie on the same path. For $\xi \in \mathcal{C}$, one may then ask how many infinite light beams occur in the configuration ξ. This number $n(\xi)$ may be shown to depend measurably on ξ and is clearly invariant under the action of $\mathbb{Z}^2$ on $\mathcal{C}$. It follows by ergodicity that for $\mathbb{P}_p$-almost every configuration $\xi \in \mathcal{C}$, $n(\xi)$ takes on a fixed value $N(p)$. Note that since the measures $\mathbb{P}_p$ are mutually singular, $N(p)$ may vary with p and indeed it is easy to see that $N(0) = \infty$ and it is known that $N(1) = 0$ (This follows from the observation that if the light beams starting from the origin are almost surely localized then with probability 1, the light beams starting from any given $x \in \mathbb{Z}^2$ have probability 0 of being unbounded. By countable additivity, the probability that there is an unbounded light beam is 0.)

Theorem 1. *For each p with $0 \leqslant p \leqslant 1$, the quantity $N(p)$ is equal to one of the* *values* $0, 1$ *and* ∞.

Some problems about the system may be formulated dynamically in the following framework: Let D be the set of directions $\{\pm e_1, \pm e_2\}$ in which a light beam may leave the origin. Then given a point (ξ, d) in $\mathcal{C} \times D$, write $T(\xi, d) = (\sigma_d(\xi), d')$ where d' is the d' is the resulting direction if the direction d is deflected by the mirror at the origin of the new configuration $\sigma_d(\xi)$ (if there is one). The effect of the map T may be described as: move the origin one step in the direction specified by d and adjust the direction if there is a mirror present at the new origin to get the direction in which light bounces off the mirror. This means that iterating T corresponds to moving the configuration so that the origin at each stage moves along the path of the light beam.

It may be seen that T preserves the measure $\mu_p = \mathbb{P}_p \times c$ on $\mathcal{C} \times D$ where c is the normalized counting measure on D. One can ask whether the measure μ_p is ergodic, but it is easy to show that for any value of $p > 0$, the measure has infinitely many invariant sets, each of positive measure; one corresponding to any closed path which may arise as the path of a light beam. Let $\mathcal{I}$ denote the set of points (ξ, d) in $\mathcal{C} \times D$ such that the light beam starting at the origin in direction d deflected by the mirrors in ξ is infinite. The set $\mathcal{I}$ is clearly invariant under T. If $\mu_p(\mathcal{I}) > 0$, write ν_p for the conditional measure of μ_p conditioned on $\mathcal{I}$: $\nu_p(A) = \mu_p(A \cap \mathcal{I})/\mu_p(I)$. This measure is invariant and concentrated on the configurations and directions for which the light beam through the origin is infinite. We see that $\mu_p(\mathcal{I}) > 0$ if and only if $N(p) > 0$. The following theorem addresses the ergodicity of the measure ν_p (when it exists). It is known that any invariant measure may be uniquely expressed as an integral combination of ergodic invariant measures. This is known as ergodic decomposition. The ergodic measures appearing in the decomposition are called the ergodic components of the invariant measure (see [17] for a general discussion and [16] for a more detailed exposition). Given (ξ, d) in $\mathcal{C} \times D$, write $R(\xi, d) = (\xi, \hat{d})$ for the same configuration, but with the direction d reflected in the normal to the mirror at the origin if it exists or rotated by $180°$ if it doesn't. This gives the direction to follow in order to follow the path backwards. We note that $T \circ R = R \circ T^{-1}$. The map R is a measure-preserving involution of $\mathcal{C} \times D$ and it clearly leaves ν_p invariant also.

Theorem 2. *Suppose* $0 < p < 1$ *and* $N(p) > 0$. *If* $N(p) = \infty$, *then the measure* ν_p *is ergodic. If* $N(p) = 1$ *then* ν_p *is either ergodic or has exactly two ergodic components* $\nu_p|A$ *and* $\nu_p|A^c$ *where* A *is an invariant set which satisfies* $R(A) = A^c$.

The interpretation of this latter case when $N(p) = 1$ and ν_p has exactly two ergodic components is that there is $\mathbb{P}_p$-almost surely a unique infinite path in ξ. This path has a distinguished direction: If (m, n) is a point on the path then for each direction d through (m, n) which points along the path, the point $(\sigma_{(m,n)}(\xi), d)$ belongs to A or $R(A)$. If one attaches arrows which point in the directions for which $(\sigma_{(m,n)}(\xi), d)$ belongs to A, these match up by the invariance of A, so give a distinguished direction of the unique infinite path in ξ.

Ergodicity is helpful to us as it allows us to conclude that any invariant measurable function is constant almost everywhere. Having exactly two ergodic components is nearly as good. It means that for any invariant measurable function, there is a set of measure 1 on which it takes at most two values: one on A and the other on $R(A)$. The ergodicity allows us to draw conclusions about the geometry of the paths. Since the map T is invertible, any infinite path is in fact bi-infinite. If one follows the path in either direction, the light beam goes to infinity. The two halves of the path (the forward and backward halves) are disjoint in the bonds which they occupy, but may visit the same vertices.

Theorem 3. *If for some $0 < p < 1$, $\mu_p(X_0) > 0$, then with probability 1, all bi-infinite paths have the property that their forward and backward halves intersect each other infinitely often.*

In the case where there is a positive probability of infinite light beams, it is of interest to ask about the asymptotic behaviour of $|X_n|$, where X_n is the nth vertex on the light beam and $|(i,j)|$ is defined to be $\max(|i|,|j|)$. In what follows, we will say that the process is *superdiffusive* if the expectation of $|X_n|^2/n$ is unbounded as n tends to ∞. Cohen and co-authors have compared the asymptotic diffusive behaviour with that of Markov chains (see [6], [7], [12] and [18] for a discussion of this and some numerical results).

Theorem 4. *Suppose $0 \leqslant p < 1$ and $N(p) = \infty$, then the process (X_n) given by following an infinite light beam along the path given by almost every $(\xi, d) \in \mathcal{I}$ is superdiffusive.*

2. Proofs of Theorems

We start off with a lemma which occurs in two distinct places. This lemma is a version of the finite energy argument which is used in various places in percolation theory. When we have $0 < p < 1$, an important quantity will be $\kappa = \min(\frac{p}{2}/(1-p), (1-p)/\frac{p}{2})$.

Lemma 5. *Suppose that for some value of p with $0 < p < 1$, there are least two distinct infinite paths in $\mathbb{P}_p$-almost every configuration $\xi \in \mathcal{C}$. Then there is a positive probability that two distinct such paths meet at the origin.*

Proof. Define a function $d \colon \mathcal{C} \to \mathbb{Z}^+$ by letting $d(\xi)$ be the minimum distance between lattice points, which are visited by two distinct infinite paths. This is clearly measurable and finite-valued almost everywhere. It is also shift-invariant and by ergodicity of $\mathbb{P}_p$, it is takes on a constant value d_0 for almost every ξ. We will show that d_0 must be 0. Suppose for a contradiction that $d_0 > 0$. For any ξ for which $d(\xi)$ is equal to d_0, there exists a smallest box around the origin which contains lattice points on two distinct paths at a distance d_0 from each other. By choosing a sufficiently large box size, there is a set S of ξ of positive measure on which with positive probability there are two points in the box belonging to distinct infinite paths. We may then make a modification of a single mirror/crossing at one of the two points which ensures that the paths then get closer to each other. Since this modification was performed inside a box of fixed size, the mapping effecting the modification is bounded-to-1 and has Radon-Nikodým derivative bounded below by κ. It follows that for ξ belonging to a set of positive measure, $d(\xi)$ is strictly smaller than d_0. This contradicts the assumption that d_0 is strictly positive. It follows that for almost all ξ, there is a lattice point at which two distinct infinite paths meet. By translation invariance, for a set of ξ of positive measure, two distinct infinite paths meet at the origin.

We now use this lemma in the proof of Theorem 1. The idea of the proof is that if there is a finite number (greater than one) of infinite paths, then they must all enter a sufficiently large box around the origin. However, since the paths must also intersect at places arbitrarily far away from the origin, there is the opportunity to change a single mirror at such a point to get with reasonably high probability a path which stays a long way from the origin. This will then lead to a contradiction.

Proof of Theorem 1. We have to show that for each value of p with $0 \leqslant p \leqslant 1$, $N(p)$ is equal to 0, 1 or ∞. We know already that $N(0) = \infty$ and $N(1) = 0$. It suffices then to work with $0 < p < 1$.

We will show that $N(p) = 2$ leads to a contradiction. The same argument works for all other values of $N(p)$ which are strictly between 1 and ∞. So we will suppose that $0 < p < 1$ and $N(p) = 2$.

For $\mathbb{P}_p$-almost all configurations ξ, there are exactly two infinite paths. It follows that with positive probability $\alpha > 0$, they meet at the origin. Now let $\epsilon < \kappa\alpha/6$. For any configuration ξ with exactly two distinct infinite paths, there is a smallest number $m(\xi)$ such that the box $\Lambda_{m(\xi)}$ with side $2m(\xi)$ about the origin intersects both infinite paths. As usual, there is a number M such that with probability at least $1-\epsilon/2$, both infinite paths enter Λ_M. Since there are only finitely many paths leaving this box, which later return to the box, there is a number R such that with probability at least $1-\epsilon$, both infinite paths enter Λ_M and there is no path leaving Λ_M, which hits $\partial\Lambda_R$ before returning to Λ_M. Let S denote this event. Now let x be any lattice point outside Λ_R. Let T denote the event that both infinite paths meet at x. We have $\mathbb{P}_p(T) = \alpha$. It then follows that $\mathbb{P}_p(S \cap T) > \alpha - \epsilon > \alpha/2$. For $\xi \in S \cap T$, there are two distinct infinite paths meeting at x. Exactly one side of each path enters Λ_M. It is now possible to modify the configuration at x so as to link the two halves of the infinite paths which do not enter Λ_M. This defines a mapping on $S \cap T$. Since the resulting configuration has an infinite path which does not enter Λ_M, it follows that the mapping sends $S \cap T$ into S^c. As before, this map is three-to-one and has Radon-Nikodým derivative bounded below by κ. It follows that $\mathbb{P}(S^c) > \kappa/3 \cdot \alpha/2 > \epsilon$. This is a contradiction. □

To prove Theorem 2, we now work separately in the two cases: $N(p) = 1$ and $N(p) = \infty$.

Proof of Theorem 2. First assume that $N(p) = 1$. Suppose that A is an invariant subset of $\mathcal{I}$ with $0 < \nu_p(A) < 1$. Then the set of ξ such that there is some d with $(\xi, d) \in A$ has positive $\mathbb{P}_p$ measure. By ergodicity of $\mathbb{P}_p$ under the shift, for almost every $\xi \in \mathcal{C}$, there is some z and d such that $(\sigma_z(\xi), d) \in A$. Similarly there is some z' and d' such that $(\sigma_{z'}, d') \in A^c$. However since almost every ξ has a unique infinite path, $\sigma_z(\xi)$ and $\sigma_{z'}(\xi)$ must have the origin lying at (possibly) different points on the same infinite path. If the directions d and d' are pointing along the path in the same sense, then by invariance of A, we would have $(\sigma_{z'}(\xi), d') \in A$ which is a contradiction. It follows that $R(\sigma_z(\xi), d) \in A^c$. This establishes that $R(A) = A^c$. It follows that $\nu_p(A) = 1/2$ and ν_p has exactly two ergodic components.

We deal now with the case $N(p) = \infty$. Suppose A is an invariant subset of $\mathcal{C} \times D$ with $0 < \nu(A) < 1$. Let $\mathcal{I}^*$ denote the subset of $\mathcal{I}$ for which there are two distinct infinite paths crossing at 0. We first show that for almost all $\xi \in \mathcal{I}^*$, for all d, $(\xi, d) \in A$ or for all d, $(\xi, d) \in A^c$. Suppose that this does not hold and on a set of ξ of positive measure, there are d_1 and d_2 such that (ξ, d_1) and (ξ, d_2) belong to A and A^c. For such a ξ, we can change a mirror at 0 to get a point in $\mathcal{I}$ whose forward orbit belongs to A and whose backward orbit belongs to A^c. For such a point, the forward and backward orbits are generic for two different measures. The set of points with this property is of measure 0 with respect to any invariant measure and certainly with respect to ν_p. This contradicts the fact that the Radon-Nikodým derivative of the map effecting the modification is bounded away from 0.

We conclude that for almost every ξ, if two paths meet at a point, then they either both belong to A or they both belong to A^c. We then use an argument identical to that used in Lemma 5 to get a contradiction: given ξ, consider the minimum distance between a path of type A and a path of type A^c. This is a finite invariant function which is bounded below by 1 by the above observation. However, it is clear by the argument of Lemma 5, that the constant value of this function must be 0. This contradicts the assumption that $0 < \nu_p(A) < 1$ and we conclude that ν_p is ergodic. □

We now seek to apply the above theorem to arrive at the geometric conclusion that if $N(p) > 0$ for some $0 < p < 1$, then for $\mathbb{P}_p$-almost every ξ, any infinite path in ξ has the property that its forward and backward halves intersect infinitely often. The idea of the proof is to show first that a given half of any path winds infinitely often around the origin. In order for the other half to intersect the first half only finitely many times, its winding number must keep pace with the winding number of the first path. Since the paths don't intersect infinitely often, they are essentially independent, so it is sufficient to show that two independently chosen paths do not have the property that their winding numbers 'keep pace' with each other.

We start with a lemma about winding number. To state the lemma, we need a notion of the nth lattice point on a path. Let $z_n(\xi, d) \in \mathbb{Z}^2$ be defined by the equations:

$$\begin{aligned} z_0(\xi, d) &= 0 \\ z_{n+1}(\xi, d) &= z_n(\xi, d) + d_n, \end{aligned}$$

where d_n is the second coordinate of $T^n(\xi, d)$. We then let θ_n measure the winding number of the path $z_0, z_1, \ldots, z_n$ about the point $(-\frac{1}{2}, -\frac{1}{2})$. Formally, we define

$$\begin{aligned} \theta_0(\xi, d) &= \tfrac{1}{8} \\ z_n(\xi, d) - (-\tfrac{1}{2}, -\tfrac{1}{2}) &= |z_n(\xi, d) - (-\tfrac{1}{2}, -\tfrac{1}{2})|(\cos(2\pi\theta_n(\xi, d)), \sin(2\pi\theta_n(\xi, d))) \\ -\tfrac{1}{4} \leqslant \theta_{n+1}(\xi, d) &- \theta_n(\xi, d) \leqslant \tfrac{1}{4}. \end{aligned}$$

Note that there are advantages and disadvantages to considering winding about $(-\frac{1}{2}, -\frac{1}{2})$ rather than the origin. The disadvantages are that this entails a small loss of symmetry, but we make a corresponding gain when it comes to defining a winding function later.

Lemma 6. *Suppose* $0 < p < 1$ *and* $N(p) > 0$. *For* ν_p*-almost every* (ξ, d) *in* $\mathcal{I}$, $\limsup_{n\to\infty} |\theta_n(\xi, d)| = \infty$.

Proof. Set A to be the set of (ξ, d) in $\mathcal{I}$ for which $\limsup_{n\to\infty} |\theta_n(\xi, d)| = \infty$. We first note that A is a T-invariant subset of $\mathcal{I}$ as if a configuration is such that the upper limit of the absolute value of the winding number about $(-\frac{1}{2}, -\frac{1}{2})$ is infinite, then the same conclusion must hold for the winding number about any lattice point in the plane (using the fact that an infinite path may only travel along a given bond once). In particular if the path defined by (ξ, d) is such that the path winds infinitely often about $(-\frac{1}{2}, -\frac{1}{2})$, then the same conclusion must hold for $T(\xi, d)$. This establishes the T-invariance of A. It follows that $B = \mathcal{I} \setminus A$ is also invariant and has ν_p measure 0 or 1 in the ergodic case, or 0, $\frac{1}{2}$ or 1 in the non-ergodic case. We will suppose for a contradiction that B has positive measure. For (ξ, d) in B, $\limsup_{n\to\infty} \theta_n(\xi, d)$ is a finite-valued function. We see that although this function is not invariant, its fractional part, which we will call $\phi(\xi, d)$, is T-invariant: $\theta_n(\xi, d) - \theta_n(T(\xi, d))$ converges to one of 0, 1 and -1 according to the configuration of the infinite path on the immediate neighbourhood of the origin. Since ϕ is a T-invariant function, it takes on a single value almost everywhere in the ergodic case and at most two distinct values in the non-ergodic case. We will use the symmetry of the system to show that this yields a contradiction.

The original system has a degree four rotational symmetry in the origin: a configuration in $\mathcal{I}$ (and mirrors) can be rotated by $\pi/2$ about the origin giving a new configuration in $\mathcal{I}$. Clearly the effect of this rotation on the fractional part of ϕ is to add $\frac{1}{4}$ (mod 1). It follows then that ϕ must take on at least four distinct values, each on a set of positive measure. This is a contradiction. $\square$

The following lemma will give a useful way of telling whether a pair of paths cross and will be an important part of the deduction of Theorem 3 from Theorem 2.

Let R denote the ray of points through $(-\frac{1}{2}, -\frac{1}{2})$ parallel to the positive horizontal axis, $R = \{(x, -\frac{1}{2}) \colon x > -\frac{1}{2}\}$. We show how for $(\xi, d) \in \mathcal{I}$ to define a function $f_{(\xi,d)} \colon R \to \mathbb{R}$ which encodes data about the winding of $z_n(\xi, d)$ about 0. This function will be called the *winding function*. For a point $(\xi, d) \in \mathcal{I}$, we consider the times at which the path $z_n(\xi, d)$ crosses R. If the path crosses R at a point a then it does so at most once as the infinite path travels along each bond at most once. We consider the corresponding winding number of the path at the time when it crosses at a and define $f_{(\xi,d)}(a)$ to be this winding number. The function $f_{(\xi,d)}$ is then defined on the remainder of the ray R by interpolation. Let $\gamma^+(\xi, d)$ denote the (one-sided) curve followed by the light beam described by (ξ, d) (i.e. the union of the line segments joining the points $z_n(\xi, d)$ and $z_{n+1}(\xi, d)$ as n runs over $\mathbb{Z}^+$). Similarly, $\gamma^-(\xi, d)$ is the curve given by following the light beam in the opposite direction: $\gamma^-(\xi, d) = \gamma^+(\xi, \hat{d})$. We will denote by $\|\cdot\|$, the supremum norm.

Lemma 7. *Suppose $(\xi, d) \in \mathcal{I}$ is such that the one-sided paths $\gamma^+(\xi, d)$ and $\gamma^-(\xi, d)$ both cross R infinitely often, but meet each other only finitely many times. Then $\|f_{(\xi,d)} - f_{(\xi,\hat{d})}\| < \infty$.*

Proof. Let $f_{(\xi,d)} : R \to \mathbb{R}$ and $f_{(\xi,\hat{d})}$ be defined as above. Suppose the paths $\gamma^+(\xi, d)$ and $\gamma^-(\xi, d)$ only intersect within the box Λ_l about the origin. Since each of the paths only enter Λ_l finitely many times, there is an L such that neither path re-enters Λ_l after crossing the boundary of Λ_L.

We now consider the part of R lying outside Λ_L. We will work with $\gamma^+(\xi, d)$ and demonstrate that if r_1 and r_2 are a pair of crossings of $\gamma^+(\xi, d)$ over R which are adjacent in the sense that there is no point of R between r_1 and r_2, then the difference in winding number can be at most 1, as otherwise there is a loop encircling the origin. To show this, suppose that the winding numbers of a pair of adjacent crossings differ by $n > 1$.

This situation is illustrated by Figure 3. In this case, the section of the path between the two crossings together with the segment of the ray between the crossings forms a closed curve which winds around the origin n times. Such a curve must cross itself at least $n-1$ times. It is clear that there must be a curve forming a subset of this curve which consists entirely of segments which formed part of the original path (i.e. not including the segment of the ray between the two crossings) which is a closed curve winding around the origin. We now have that part of $\gamma^+(\xi,d)$ forms a closed curve around the origin, which stays outside Λ_l. Since $\gamma^-(\xi,d)$ goes out to infinity, it follows that it must cross $\gamma^+(\xi,d)$ outside Λ_l, which is a contradiction.

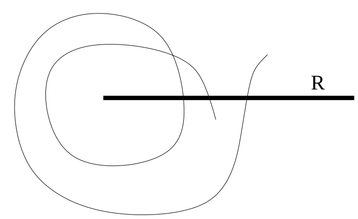

Figure 3. A path with non-consecutive winding numbers

It follows that we may assume that $\gamma^+(\xi,d)$ and $\gamma^-(\xi,d)$ have the property that between adjacent crossings of the ray R, the winding number differs by at most 1. Then consider the set $U=\mathbb{R}^2\setminus(\gamma^+(\xi,d)\cup R)$. We call the open components of U regions. Two regions are said to be separated by the ray R if there are points of the form $a+(0,\frac{1}{4})$ and $a-(0,\frac{1}{4})$ for $a=(n+\frac{1}{2},-\frac{1}{2})\in R$ which lie in the two regions.

We now introduce a system for numbering some of the regions. The region lying above and to the left of the first intersection of $\gamma^+(\xi,d)$ with R outside Λ_L is labelled 0. Thereafter, regions are numbered inductively by the rule that if two regions are separated by R, then their numbers differ by 1 in such a way that the region which touches from the upper half plane has label greater by 1. This numbering is illustrated in Figure 4.

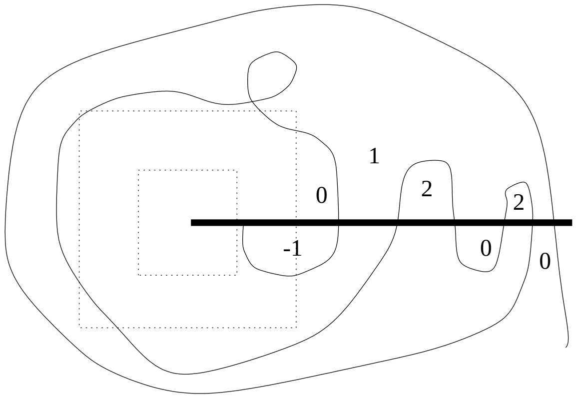

Figure 4. Numbering of the regions of $\mathbb{R}^2 \setminus (\gamma^+(\xi, d) \cup R)$

To ensure that this numbering scheme is consistent, we note that if a region is numbered in two different ways according to this scheme, then it follows that there exists a closed path which does not intersect $\gamma^+(\xi, d)$ and which has non-zero winding number about the origin. This is clearly a contradiction since $\gamma^+(\xi, d)$ goes from 0 to ∞.

Since we assume that $\gamma^-(\xi, d)$ does not cross $\gamma^+(\xi, d)$ outside the box of side l, it follows that the curve $\gamma^-(\xi, d)$ only moves from one region to another by crossing R, where the crossings are made at integer points of R. That is to say, the only crossings between regions are between adjacently numbered regions. It follows that (since $\gamma^-(\xi, d)$ and $\gamma^+(\xi, d)$ intersect R infinitely often, that infinitely many regions are numbered under the above scheme. However, since $\gamma^-(\xi, d)$ starts at the origin and does not cross $\gamma^+(\xi, d)$ outside the box of side l, then at all points on $\gamma^-(\xi, d)$, the winding number about the origin differs from the number of the region that contains it by at most a constant (which depends on the number of crossings inside Λ_l). It follows that $\|f_{(\xi,d)} - f_{(\xi,\hat{d})}\| < \infty$ as required. □

To finish the proof of Theorem 3, we will use the following lemma, which shows that if the paths cross each other only finitely many times, then the forward and backward parts of the path have some independence properties. To do this, we introduce a measure which describes only the sites on $\gamma^+(\xi, d)$. Set $M = \{/, \backslash, +\}$ and $\mathcal{P} = (M \times D)^{\mathbb{Z}^+}$. We call $\mathcal{P}$ the (one-sided) path space as it gives the sequence of mirrors and directions along a path. Let $\pi^+ \colon \mathcal{C} \times D \to \mathcal{P}$ be defined by $\pi^+(\xi, d)_n = Q(T^n(\xi, d))$, for $n \geqslant 0$ where $Q(\xi, d) = (\xi_0, d)$. Thus $\pi^+(\xi, d)$ describes the path followed by the forward light beam, by giving the mirror and direction at each stage. Note that $\pi^+ \circ T = \sigma \circ \pi^+$, where σ is the left shift map on $\mathcal{P}$, so π^+ is a semi-conjugacy (factor mapping) of T onto the map σ on the path space. We also define $\pi^- \colon \mathcal{C} \times D \to \mathcal{P}$ to be $\pi^-(\xi, d)_n = Q(T^n(\xi, \hat{d}))$. This describes the path followed by the backward light beam. Define $\bar{\mu}_p^+ = \mu_p \circ {\pi^+}^{-1}$, $\bar{\nu}_p^+ = \nu_p \circ {\pi^+}^{-1}$. Letting $\pi^{(2)}$ be the map sending (ξ, d) to $(\pi^+(\xi, d), \pi^-(\xi, d))$, we get measures $\bar{\mu}_p^{(2)}$ and $\bar{\nu}_p^{(2)}$ on $\mathcal{P}^2$ as above.

Lemma 8. *Under the assumption that $N(p) > 0$ and that for ν_p-almost every $(\xi, d) \in \mathcal{I}$, the number of crossings of $\gamma^+(\xi, d)$ and $\gamma^-(\xi, d)$ is finite, the measure $\bar{\nu}^{(2)}$ is absolutely continuous with respect to $\bar{\nu}_p^+ \times \bar{\nu}_p^+$.*

Proof. To prove the lemma, we first give a description of $\bar{\mu}_p^+$ and $\bar{\nu}_p^+$. Let C be a cylinder set in $(M \times D)^{\mathbb{Z}}$. Such a set is described by a sequence $(m_n, d_n)_{0 \leqslant n \leqslant s}$ of mirrors and directions. The set C then consists of those sequences of mirrors and directions which agree with (m_n, d_n) when n is between 0 and s. It may be seen that

$$\bar{\mu}_p^+(C) = \prod_{n=0}^{s}{}^{*} \rho_p(m_n), \tag{1}$$

where $\rho_p(m)$ is $\frac{p}{2}$ if m is / or \; and $1-p$ otherwise and the product is taken once over all the sites which are visited. This means that if the cylinder corresponds to a light path which crosses itself one or more times, then at those sites, the corresponding weight is only taken into account on the first visit to the site. The interpretation of this is that each time we arrive at a new (previously unvisited site), the state of the mirror at that site is dynamically chosen with the correct probabilities.

Letting $I = \pi^+(\mathcal{I})$, we have $\bar{\nu}_p^+(C) = \bar{\mu}_p^+(C \cap I)/\bar{\mu}_p^+(I)$. Now, let A_n be the set of (ξ, d) for which the forward and backward paths meet exactly n times and set $B_n = \pi(A_n)$. Then since $B_n \subset I$, we have $\bar{\nu}_p^+(B_n) = \bar{\mu}_p^+(B_n)/\bar{\mu}_p^+(I)$.

Notice that on B_n, $\bar{\mu}_p^{(2)}$ is absolutely continuous with respect to $\bar{\mu}_p^+ \times \bar{\mu}_p^+$ as the number of omitted factors in (1) is bounded above by n. It follows that on B_n, $\bar{\nu}_p^{(2)}$ is absolutely continuous with respect to $\bar{\nu}_p^+ \times \bar{\nu}_p^+$. Since $\bigcup_{n\geqslant 0} B_n = I$, the conclusion of the lemma follows. □

We are now able to complete the demonstration of the fact that there are infinitely many crossings of $\gamma^-(\xi, d)$ and $\gamma^+(\xi, d)$. We observe that the function $f_{(\xi,d)}$ depends only on the one-sided path $\gamma^+(\xi, d)$. This is exactly the information given by a point of $\mathcal{P}$, so we abuse notation and write f_γ for γ in $\mathcal{P}$ to mean the winding function introduced above for the path corresponding to the point γ. The idea of the proof is the following: we show that for $\bar{\nu}_p^+ \times \bar{\nu}_p^+$-almost every pair of infinite paths, their winding functions f_γ and f_δ satisfy $\|f_\gamma - f_\delta\| = \infty$. It will then follow that any two such paths intersect infinitely often. The above lemma will then yield the desired conclusion.

Proof of Theorem 3. We will need to consider the involution of $\mathcal{C} \times D$ given by reflecting the configuration, mirrors and initial direction in the x-axis. Clearly, this has the effect of reflecting the whole path in the x-axis. It makes sense to consider this involution on $\mathcal{P}$, the path space. We call it S. Clearly S preserves the measure $\bar{\nu}_p^+$. We note that if $\gamma \in \mathcal{P}$, then $S(\gamma)$ winds around the plane in the opposite direction. In particular, there is a relationship between f_γ and $f_{S(\gamma)}$, namely, $f_\gamma \approx -f_{S(\gamma)}$. This is not an equality because the ray R is through the line $y = -\frac{1}{2}$ rather that $y = 0$. It is sufficient for our purposes to note that $\|f_\gamma - (-f_{S(\gamma)})\| < \infty$.

We now use this involution to complete the proof of the theorem. We deal first with the case where ν_p is ergodic. We suppose for a contradiction that for ν_p-almost every point (ξ, d) in $\mathcal{I}$, the number of crossings of $\gamma^+(\xi, d)$ and $\gamma^-(\xi, d)$ is finite (the set of points with this property is invariant so has measure 0 or 1). Since ν_p is ergodic, it follows that $\bar{\nu}_p^+$ is ergodic. For a given path γ, we consider $A_\gamma = \{\delta\colon \|f_\gamma - f_\delta\| = \infty\}$. This is an invariant set under σ and so has measure 0 or 1. Since for $\bar{\nu}_p^+$-almost every δ, $\|f_\delta\| = \infty$, we have $\|f_\delta - f_{S(\delta)}\| = \infty$ and so at least one of δ and $S(\delta)$ belongs to A_γ. It follows that $\bar{\nu}_p^+(A_\gamma) = 1$ and $\bar{\nu}_p^+ \times \bar{\nu}_p^+$-almost every pair of paths (γ,δ) satisfy $\|f_\gamma - f_\delta\| = \infty$. However, under the hypothesis that for almost every $(\xi,d)\in\mathcal{I}$, $\gamma^+(\xi,d)$ and $\gamma^-(\xi,d)$ intersect finitely many times, this shows using the above lemma that $\|f_{(\xi,d)} - f_{(\xi,\hat{d})}\| = \infty$ for almost every $(\xi,d)\in\mathcal{I}$ and hence the two halves intersect infinitely often. This is a contradiction.

In the case where ν_p has two ergodic components, we show that with for almost every pair of one-sided paths, one from each component, the equality $\|f_\gamma - f_\delta\| = \infty$ is satisfied. This will be sufficient to prove the theorem as above. Let the two components of ν_p be $\nu_{p,1}$ and $\nu_{p,2}$. These project to ergodic measures $\bar\nu_1^+$ and $\bar\nu_2^+$ on $\mathcal{P}$ as before. If these measures are equal, then the above proof works. Otherwise, $\mathcal{P}$ splits into two disjoint parts $\mathcal{P}_1$ and $\mathcal{P}_2$ on which the two measures are supported. As before, for any path γ, $\bar\nu_p^+(A_\gamma)\geqslant\frac{1}{2}$, so A_γ must be one of $\mathcal{P}_1$, $\mathcal{P}_2$ or $\mathcal{P}$. Let $B = \{\gamma\in\mathcal{P}_1\colon \mathcal{P}_2\subset A_\gamma\}$. As before, this is an invariant set. If B has measure 0, then for $\bar\nu_1^+$-almost every γ in $\mathcal{P}_1$ and for $\bar\nu_2^+$-almost every δ in $\mathcal{P}_2$, we have $\|f_\gamma - f_\delta\| < \infty$. Now by the triangle inequality, for $\bar\nu_1^+$-almost every $\gamma'\in\mathcal{P}_1$, $\|f_\gamma - f_{\gamma'}\| < \infty$ and so we have A_γ is of measure 0 for almost all γ. This is a contradiction, so we conclude that B has positive measure so $B = \mathcal{P}_1$. We therefore conclude that for almost all $\gamma\in\mathcal{P}_1$ and $\delta\in\mathcal{P}_2$, $\|f_\gamma - f_\delta\| = \infty$. However, the measure $\bar\nu_p^{(2)}$ may be seen to be supported on $\mathcal{P}_1\times\mathcal{P}_2\cup\mathcal{P}_2\times\mathcal{P}_1$ (as almost every infinite path has one direction belonging to one component and the other direction belonging to the other component). This means that if we make as before the assumption that the two halves of an infinite path meet only finitely many times, then for $\bar\nu_p^{(2)}$-every pair (γ,δ), we have $\|f_\gamma - f_\delta\| = \infty$, so we conclude that the paths γ and δ intersect infinitely often. This contradicts our original assumption and we conclude that for ν_p-almost every (ξ,d) in $\mathcal{I}$, $\gamma^+(\xi,d)$ and $\gamma^-(\xi,d)$ intersect infinitely often. □

We now turn to the case where $N(p)=\infty$ and show that in this case, there is superdiffusion.

Proof of Theorem 4.

The first part of this is an observation due to M. Harris, that in the case where $N(p)>0$, a well-known theorem of Burton and Keane ([4]) applies, showing that all the infinite paths individually have well-defined densities. This means that the function r sending a a point of $\mathcal{I}$ to the density of the corresponding light beam (i.e. the density of the points in the lattice visited by the light beam) is well-defined. Clearly r is invariant under the action of the shift. If the measure ν_p is ergodic, then it follows immediately that the density is an invariant function and hence equal for ν_p-almost all infinite paths. If ν_p has exactly two ergodic components, then the same conclusion holds as the density is unaffected by reversing the sense of the path. We write $\rho(p)$ for the ν_p-almost sure constant value of $r(\xi,d)$. From this, it follows that for any value of p, for which $N(p)>0$, almost all configurations in $\mathcal{C}$ have $N(p)$ infinite light beams, all of density $\rho(p)$ in the lattice. Clearly if $N(p)=\infty$, then $\rho(p)$ must be 0. This allows us to deduce the conclusion of the theorem as follows:

Fix $\epsilon>0$. Since the density of any infinite path is 0 in $\mathbb{P}_p$-almost every configuration, there exists a number M such that the ν_p measure of the set of (ξ,d) in $\mathcal{I}$ for which the infinite path has density less than ϵ in Λ_M is at least $\frac{1}{2}$. For n large, consider the largest box Λ_r containing less than $n/(4\epsilon)$ vertices (i.e. $r = \lfloor\sqrt{n/(16\epsilon)} - 1\rfloor$). We will assume n is sufficiently large that $r > M$. Let A be the set of (ξ,d) in $\mathcal{I}$ for which the density in Λ_r is less than ϵ. By the above, we have

$\nu_p(A) > \frac{1}{2}$. For $(\xi, d) \in A$, consider $z_0(\xi, d), \ldots, z_{n-1}(\xi, d)$. At most $2\epsilon \cdot n/4\epsilon = n/2$ of these points lie in Λ_r (the factor of 2 arises as a given vertex may be visited twice by a single infinite path). It follows that for $(\xi, d) \in A$,

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{|z_k(\xi,d)|^2}{n} \geqslant \frac{r^2}{2n} > \frac{1}{32\epsilon}.$$

We then have

$$\int \frac{1}{n}\sum_{k=0}^{n-1}\frac{|z_k(\xi,d)|^2}{k}\, d\nu_p \geqslant \int \frac{1}{n}\sum_{k=0}^{n-1}\frac{|z_k(\xi,d)|^2}{n}\, d\nu_p \geqslant \frac{1}{64\epsilon}.$$

Since ϵ was arbitrary, it follows that conditional on (ξ, d) belonging to an infinite path, the Cesáro averages of the expectations of $|z_k(\xi, d)|^2/k$ converge to ∞. However since $\mu_p(\mathcal{I}) > 0$, it follows that the Cesáro averages of the expectations on the whole of $\mathcal{C} \times D$ converge to ∞. In particular, the expectations of $|z_k(\xi, d)|^2/k$ are unbounded. □

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