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A Generic C^1 Expanding Map has a Singular S–R–B Measure

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Abstract: We show that for a generic C^1 expanding map T of the unit circle, there is a unique equilibrium state for $-\log T'$ that is an S–R–B measure for T, and whose statistical basin of attraction has Lebesgue measure 1. We also present some results related to the question of whether a generic C^1 expanding map preserves a σ-finite measure, absolutely continuous with respect to Lebesgue measure.

1. Introduction

Let $\mathcal{E}^k$ denote the set of C^k expanding maps of the unit circle S^1 onto itself, $k = 1, 2, \ldots$. Expanding maps have been widely studied in ergodic theory. In particular, various cases with $k \geq 2$ have been studied by a large number of authors including Rényi ([17], 1965), Krżyzewski ([9], 1971), Krżyzewski and Szlenk ([11], 1969). A typical result says that an expanding map with C^2 regularity has a unique absolutely continuous invariant measure with strong ergodic properties. These results have been extended to the case of $C^{1+\alpha}$ expanding maps of the circle (maps with a Hölder continuous derivative) and even to maps satisfying weaker regularity conditions. More recently Góra ([5], 1994) proved results of this type under the Dini condition.

A later result of Krżyzewski ([10], 1979) gave the first indication that the situation for C^1 expanding maps differs from that of the smoother maps. Namely, he showed that within the set of expanding C^1 self-maps of any manifold, the set of such maps for which there is an absolutely continuous invariant probability measure, with continuous density bounded away from 0, is meager. (That is, its complement is generic, i.e., contains a dense G_δ set with respect to the C^1 topology.) This theme was taken up by Góra and Schmitt ([4], 1989) who showed

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that there is an example of an expanding C^1 map of the circle that has no absolutely continuous invariant probability measure.

In further studies of C^1 expanding maps of the circle by Quas ([15,13,14], all 1996) maps with respectively more than one absolutely continuous invariant measure and a non-weak-mixing invariant measure were constructed; and it was shown that a dense set of C^1 expanding maps have a unique absolutely continuous invariant probability with unbounded density. In [2] (1998), Bruin and Hawkins constructed an example of an expanding C^1 map of the circle with no σ-finite absolutely continuous invariant measure (finite or infinite). In a more recent paper of Quas ([16], 1999) it was shown that a generic C^1 expanding map of the circle has no absolutely continuous invariant probability measure. Our main result shows that despite this result, there is (generically) a singular invariant probability from which properties of Lebesgue almost every orbit can be obtained.

Theorem 1. *For a generic $T \in \mathcal{E}^1$, there is a unique equilibrium measure μ_T for the potential $-\log T'$. This T- invariant probability measure has the following properties:*

1. *For a set of points S of Lebesgue measure 1, for all $f \in C^0(S^1)$, the averages $1/n \sum_{k=0}^{n-1} f(T^k x)$ converge to $\int f \, d\mu_T$ for all $x \in S$.*
2. *The measure μ_T is singular with respect to Lebesgue measure.*
3. *For each non-empty open set U, $\mu_T(U) > 0$.*

In other words, a generic $T \in \mathcal{E}^1$ possesses a fully supported singular Sinai-Ruelle-Bowen measure whose statistical basin of attraction has Lebesgue measure 1.

A natural question is whether the result from [16] may be extended from probability measures to σ-finite measures; i.e., is it true that generically in $\mathcal{E}^1$, there is no absolutely continuous invariant measure? At the moment, we do not know the answer, but we include the following trio of results that give some information about this situation.

Silva [19] introduced a notion of recurrence for a measure with respect to a non-singular transformation. To define this in our setting, let h be the density of $\lambda \circ T^{-1}$ with respect to Lebesgue measure ($h = d\lambda \circ T^{-1}/d\lambda$), and set $\omega_n(x) = \prod_{j=1}^n \frac{1}{h \circ T^j}$. Then $\omega_n > 0$ on S^1 and $\int \omega_n d\lambda = 1, n = 1, 2, \ldots$. Lebesgue measure is *recurrent* for T if the quantity $\sum_{n=1}^\infty \omega_n(x)$ is infinite for λ-a.e. $x \in S^1$. (We caution the reader that this notion of recurrence is much stronger than Poincaré recurrence. For example there exist C^2 expanding maps of S^1 for which Lebesgue measure is not recurrent in this sense.)

This recurrence property is relevant to the question of the existence of invariant measures as follows. If one can establish that a measure is recurrent for a non-invertible map, then existence or non-existence of absolutely continuous, σ-finite invariant measures for the map can be decided using a version of Krieger's ratio set (see Hawkins and Silva [6] for a proof of this result).

Theorem 2. *For a generic subset of $\mathcal{E}^1$, Lebesgue measure is not recurrent.*

Recall that a measure μ is *locally infinite* if $\mu(I) = \infty$ for each open interval I.

Theorem 3. *For a generic $T \in \mathcal{E}^1$, any absolutely continuous invariant measure is locally infinite.*

To describe the next result in this direction, let $h_n(x)$ be the density of $\lambda \circ T^{-n}$ with respect to λ: $h_n(x) = d\lambda \circ T^{-n}/d\lambda(x)$. Set

$$S_{\epsilon,n,a} = \left\{T \in \mathcal{E}^1\colon \lambda\{x\colon h_n(x) \in [a, 2a]\} < \epsilon\right\},$$

and consider the collection

$$S = \bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{a>0} S_{\epsilon,n,a}\,.$$

If $T \in S$, we say *the densities of $\lambda \circ T^{-n}$ have no characteristic scale.* This is because for such a T and for any $\epsilon > 0$, there exists an n such that for each $a > 0$, the set $\{x\colon h_n(x) \in [a, 2a]\}$ has Lebesgue measure less than ϵ.

It is known that there exist mappings with an infinite invariant measure so that the above densities h_n, when appropriately rescaled, converge in measure to the invariant density (see Aaronson's book [1] for examples). One can see that when T belongs to the class S defined above, this is impossible. Therefore, when T belongs to S, a natural way of producing an absolutely continuous invariant measure is lost.

Theorem 4. *The set S constructed above is a dense G_δ subset of $\mathcal{E}^1$.*

In the next section we give some notation and definitions, in Sect. 3 we state and prove some preliminary lemmas, in Sect. 4 we prove Theorems 1, 2, 3, and in Sect. 5 we prove Theorem 4.

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2. Notation & Definitions

We work on $S^1 = [0,1]/\sim$, where $\sim$ identifies 0 with 1. The Borel sigma-algebra is denoted by $\mathcal{B}$. The space of Borel measures on S^1 is denoted by $\mathfrak{M}$, with $\mathfrak{M}^1$ denoting the subspace of probabilities. If $T \in \mathcal{E}^1$, $\mathfrak{M}^1_T$ denotes the set of Borel probability measures that are invariant under T. For $\nu \in \mathfrak{M}^1_T$, the measure-theoretic entropy of T with respect to ν is denoted by $h_\nu(T)$, or h_ν if T is understood. For a continuous function $f : S^1 \to \mathbb{R}$, the *pressure* of f (with respect to T) is given by

$$P_T(f) = \sup_{\nu\in\mathfrak{M}^1_T} \left\{h_\nu(T) + \textstyle\int f\, d\nu\right\}.$$

An *equilibrium state* for f is an element $\mu \in \mathfrak{M}^1_T$ satisfying $P_T(f) = h_\mu + \int f\, d\mu$.

Recall that a Borel measure μ is called a *Sinai-Ruelle-Bowen measure* for $T \in \mathcal{E}^1$ if there exists a subset B of S^1 of positive Lebesgue measure such that for each $f \in C^0(S^1)$ and all $x \in B$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x)) \to \int f\, d\mu.$$

The set B is called the *statistical basin of attraction* of μ.

For each $T \in \mathcal{E}^1$, T' is a continuous function whose absolute value is strictly larger than 1. Since S^1 is connected, $\mathcal{E}^1$ decomposes into two disjoint open subsets, the first consisting of those T's for which $T' > 1$, the other, those T's for which $T' < -1$. Each of these sets has countably many open components, corresponding to the maps of degree k ($k = 2, 3, \ldots$, and $k = -2, -3, \ldots$, respectively). In some of our arguments, we want to prove, say, that a subset of $\mathcal{E}^1$ with a certain property is generic. We proceed by supposing that $T' > 1$ and the degree is a fixed but arbitrary integer $k > 1$, and proving that within the corresponding component, the set is generic. Since a practically identical argument (with only the obvious minor modifications) will hold for $T' < -1$ and $k \leq -2$, and the components partition $\mathcal{E}^1$, the general result will follow.

With these conventions in place we set $\Psi_T = \Psi = -\log(T') < 0$.

We define the *Perron-Frobenius operator*, or transfer operator $\mathcal{L}_T$ by

$$\mathcal{L}_T f(x) = \sum_{Ty=x} \frac{f(y)}{|T'(y)|} \quad .$$

For now we do not specify the space containing f or $\mathcal{L}_T f$. These will depend upon the context in which they are being used, and will be designated as needed in the development.

We repeatedly use the fact (proved in [9]) that for each $T \in \mathcal{E}^2$, there exists a unique, absolutely continuous $\mu \in \mathfrak{M}^1_T$, whose density is strictly positive and continuous.

3. Preliminary Lemmas

We state and prove some lemmas that lead to the main results.

Following [7], for each natural number $k \geq 2$, let $E_k : S^1 \to S^1$ denote the linear expanding map $E_k(x) = kx \bmod 1$. For $T \in \mathcal{E}^1$ of degree k, it is well-known that E_k is conjugate to T; that is, there exists a homeomorphism γ of S^1 such that $T \circ \gamma = \gamma \circ E_k$. In fact, in general there is more than one such homeomorphism (although only finitely many). For a degree k map $T \in \mathcal{E}^1$, we shall write $\mathrm{Conj}(T)$ for the set of conjugacies between E_k and T.

For our purposes, it will be necessary to study and control the dependence of the conjugacy on the map T. To do this, we shall exploit the construction in [7] of such a conjugacy. Specifically, in their construction, they start with a point p that is fixed by T and use the Markov partition of the circle given by the intervals whose endpoints are the points of $T^{-1}\{p\}$. For our modification, we need to control the choice of p.

For $z \in S^1$, set $U_z = \{T \in \mathcal{E}^1 : T(z) \neq z\}$. Note that U_z is a dense open subset of $\mathcal{E}^1$.

Lemma 1. *For each $z \in S^1$, there is a continuous map $\Pi_z : U_z \to Homeo(S^1)$ such that $\Pi_z(T) \in Conj(T)$ for each T.*

In particular, given $T \in \mathcal{E}^1$ of degree k, there is a neighborhood U of T on which there is a continuous choice of conjugacies to the map E_k.

Proof. The proof is essentially that given in the proof of Theorem 2.4.6 in [7]. For a map $T \in U_z$, we choose the fixed point p of T that is the first fixed point on the circle 'to the right' of z. That is, considering the circle to be the set $[0, 1)$, p is chosen to be the first fixed point to the right of z or if there is none, the first fixed point to the right of 0. This choice of fixed point determines a conjugacy $\Pi_z(T)$. The fixed point may be seen to depend continuously on the map, and so do its preimages. This allows one to show the required continuity of Π_z.

To show that in a neighborhood of any given map $T \in \mathcal{E}^1$, there is a continuous family of conjugacies, we argue as follows: Let z be any point not fixed by T, then U_z is the required neighborhood and $\Pi_z(S)$ is the continuous choice of conjugacy for $S \in U_z$.

Note that if $\gamma \in \mathrm{Conj}(T)$ and $f \in \mathrm{C}^0(S^1)$, then $P_{E_k}(f \circ \gamma) = P_T(f)$. Indeed, γ induces a bijection between $\mathfrak{M}^1_{E_k}$ and $\mathfrak{M}^1_T$ by $\nu \mapsto \nu \circ \gamma^{-1}$. Then $\int f \circ \gamma \, d\nu = \int f \, d\nu \circ \gamma^{-1}$, and since γ is a measure-theoretic isomorphism, $h_\nu(E_k) = h_{\nu\circ\gamma^{-1}}(T)$. The pressure equality follows.

Lemma 2. *For all $T \in \mathcal{E}^1$, $P_T(\Psi_T) = 0$.*

Proof. If $T \in \mathcal{E}^2$, this is well-known as the Ruelle-Ledrappier-Young entropy formula (see [12]). Given a degree k map $T \in \mathcal{E}^1$, by Lemma 1, we may find a neighborhood V of T and a choice of conjugacies γ_S for all $S \in V$ so that the map $S \mapsto \gamma_S$ is continuous on V. With these choices, if $\{T_i\} \subset \mathcal{E}^2$ and $T_i \to T$ in $\mathcal{E}^1$, then $\Psi_{T_i} \circ \gamma_{T_i} \to \Psi_T \circ \gamma_T$ in $C^0(S^1)$. Since pressure is continuous on $C^0(S^1)$, and $0 = P_{T_i}(\Psi_{T_i}) = P_{E_k}(\Psi_{T_i} \circ \gamma_{T_i})$ for all i, it follows by taking limits that $0 = P_{E_k}(\Psi_T \circ \gamma_T) = P_T(\Psi_T)$.

Corollary 1. *If μ is any equilibrium state for Ψ_T, then μ is non-atomic.*

Proof. Let μ be an ergodic equilibrium state; then it must be either purely atomic, or continuous. If it is purely atomic, then $h_\mu(T) = 0$ and $\int \Psi_T \, d\mu < 0$, contradicting $P(\Psi_T) = 0$. The result follows since the equilibrium states form a convex set, of which the extreme points are the ergodic states.

Lemma 3. *The set of $T \in \mathcal{E}^1$ for which Ψ_T has a unique equilibrium state is generic.*

The lemma is a version of the Gibbs Phase Rule for the class of expanding maps of the circle. The original Gibbs Phase Rule for the case of a shift was proved by Ruelle [18] and Gallavotti and Miracle-Sole [3].

Proof. For any expansive T, there is at least one equilibrium state for each $h \in C^0(S^1)$ (see Walters [20], p. 224). Since expanding maps are expansive, every Ψ_T possesses at least one equilibrium state.

To prove uniqueness for a dense G_δ, we work with equilibrium states for the map $E_k\colon S^1 \to S^1$ given by $E_k(x) = kx \bmod 1$.

We now show that the set B of potentials for which there is a unique E_k-equilibrium state forms a G_δ set. Theorems 4.3.3 and 4.3.5 of [8] characterize those potentials with unique equilibrium states as the set of f such that for all $g \in C^0(S^1)$, $\lim_{t\to 0}(P_{E_k}(f + tg) - P_{E_k}(f))/t$ exists. For fixed f and g, define $H(t) = (P_{E_k}(f+tg)-P_{E_k}(f))/t$. Since the map $t \mapsto P_{E_k}(f+tg)$ is convex, H is an increasing function. The above limit then exists if and only if $\liminf_{t\to 0^+} H(t) - H(-t) = 0$. Hence f has a unique equilibrium state if and only if

$$\liminf_{t\to 0^+} \frac{P_{E_k}(f+tg) + P_{E_k}(f-tg) - 2P_{E_k}(f)}{t} = 0 \quad \text{for all } g \in C^0(S^1). \qquad (1)$$

To show that these f form a G_δ set, we need to show that it is sufficient to calculate the lim inf for a collection of g belonging only to a countable set. To this end, let $(g_n)_{n\in\mathbb{N}}$ be a countable collection of continuous functions that is dense in $C^0(S^1)$. We note that

$$\big| \left(P_{E_k}(f+tg) + P_{E_k}(f-tg) - 2P_{E_k}(f)\right)/t - \left(P_{E_k}(f+tg_n) + P_{E_k}(f-tg_n) - 2P_{E_k}(f)\right)/t\big| \le 2\|g-g_n\|_\infty .$$

Hence (1) holds if and only if

$$\liminf_{t\to 0^+}(P_{E_k}(f+tg_n) + P_{E_k}(f-tg_n) - 2P_{E_k}(f))/t = 0 \text{ for all } n \in \mathbb{N}.$$

The set of B of functions f satisfying this condition may be written as

$$\bigcap_{n\in\mathbb{N}}\bigcap_{p\in\mathbb{N}}\bigcap_{m\in\mathbb{N}}\bigcup_{t\in(0,\frac{1}{m})}\left\{f : |(P_{E_k}(f+tg_n) + P_{E_k}(f-tg_n) - 2P_{E_k}(f))/t| < 1/p\right\},$$

which is easily seen to be a G_δ subset of C^0.

From Lemma 1, there is a continuous choice of conjugacies for maps in U_0. For a map $T \in U_0$, we shall call this choice of conjugacy γ_T. Letting Φ be the map $U_0 \to C^0(S^1)$ defined by $\Phi(T) = \Psi_T \circ \gamma_T$, we see that Φ is continuous. It follows that $\Phi^{-1}(B)$ is a G_δ subset of U_0. We now have $T \in \Phi^{-1}(B)$ if and only if $\Psi_T \circ \gamma_T$ has a unique E_k-equilibrium state. Since there is a bijection between E_k-equilibrium states for $\Psi_T \circ \gamma_T$ and T-equilibrium states for Ψ_T, we see that $T \in \Phi^{-1}(B)$ if and only if Ψ_T has a unique T-equilibrium state.

We have established that the set $S \subset \mathcal{E}^1$ consisting of those T for which Ψ_T has a unique equilibrium state, contains a G_δ subset of U_0. Since $\mathcal{E}^2 \cap U_0$ is a dense subset of U_0 that is contained in S, it follows that S contains a dense G_δ subset of U_0. Since U_0 is a dense open subset of $\mathcal{E}^1$, we conclude that S contains a dense G_δ subset of $\mathcal{E}^1$.

Set $\Lambda = \{T \in \mathcal{E}^1 : \text{there exists a unique equilibrium state for } \Psi_T\}$.

Lemma 4. *Equip Λ with the (relative) C^1-topology, and $\mathfrak{M}^1$ with the (relative) weak*-topology. Then $M : \Lambda \to \mathfrak{M}^1$, given by $M(T) = \mu_T$, is continuous.*

Proof. Suppose $T_0 \in \Lambda$ is of degree k, $T_i \in \Lambda$ and $T_i \to T_0$ in C^1. We shall show that μ_{T_0} is the limit of the μ_{T_i}. As in Lemma 1, fix a neighborhood V of T_0 such that there is a continuous family of conjugacies γ_T for $T \in V$.

Suppose that μ is any limit point of the μ_{T_i}. We shall show that $\mu = \mu_{T_0}$, and this is sufficient, by weak*-sequential compactness, to show that the original sequence must converge to μ_{T_0}.

Replacing the original sequence with a subsequence if necessary, we suppose that $\mu_{T_i} \to \mu$. Set $\nu_i = \mu_{T_i} \circ \gamma_{T_i}$ and $\nu = \mu \circ \gamma_{T_0}$. Then ν and the ν_i are all

E_k-invariant measures on S^1. By continuity of the family of conjugacies, we see that $\nu_i \to \nu$ in the weak*-topology. For each i, since ν_i is an E_k-equilibrium state for $\Psi_{T_i} \circ \gamma_{T_i}$ which by Lemma 2 has pressure 0, we have

$$0 = h_{\nu_i} + \int \Psi_{T_i} \circ \gamma_{T_i} \, d\nu_i.$$

Since the entropy map is upper semi-continuous, $\limsup h_{\nu_i} \leq h_\nu$. Since $T_i \to T_0$ in C^1 and $\nu_i \to \nu$ we have

$$\begin{aligned} 0 &= \limsup \left\{ h_{\nu_i} + \textstyle\int \Psi_{T_i} \circ \gamma_{T_i} \, d\nu_i \right\} \\ &\leq \quad h_\nu + \textstyle\int \Psi_{T_0} \circ \gamma_{T_0} d\nu \quad \leq \quad 0\,, \end{aligned}$$

where the last inequality is true because the pressure is 0. Thus, all of the inequalities are equalities and ν is an equilibrium state for $\Psi_{T_0} \circ \gamma_{T_0}$, so that μ is an equilibrium state for Ψ_{T_0}. Since $T_0 \in \Lambda$, there is only one such state. Thus any limit point of the μ_{T_i} is μ_{T_0}, the unique equilibrium state for Ψ_{T_0}, and the lemma is proved.

Lemma 5. *$\tilde{\Lambda} = \{T \in \Lambda : \mu_T$ is fully supported$\}$ is a generic subset of Λ (and hence of $\mathcal{E}^1$).*

Proof. From Corollary 1, for each $T \in \Lambda$, μ_T must be non-atomic. By Lemma 4, for a non-empty open interval $I \subset S^1$, the map $T \mapsto \mu_T(I)$ is continuous on Λ. Choose any collection $\{I_i\}$ of non-empty open intervals that forms a countable basis for the topology of S^1. Then $\tilde{\Lambda} = \bigcap_i \{T \in \Lambda : \mu_T(I_i) > 0\}$, a G_δ that contains $\mathcal{E}^2$ (and is therefore dense).

4. Proofs of Theorems 1, 2, and 3

Proof (Proof of Theorem 1). Lemma 3 establishes that for T belonging to the residual set Λ, there is a unique equilibrium state μ_T for the potential $-\log T'$.

To prove Statement 1, we use a result of Keller. Any fixed $T \in \mathcal{E}^1$, together with the Markov partition for T, forms what Keller [8] calls a *continuous $e^{-\psi}$-conformal fibred system.* He shows ([8], Theorem 6.1.8)[1] that in such a system, for λ-almost every x, the weak*-limit points of the averages $\frac{1}{k}(\delta_x + \ldots + \delta_{T^{k-1}x})$ are contained in the set of measures satisfying $h_\mu + \int(-\log T')\, d\mu \geq 0$. Since $P_T(-\log T') = 0$, these measures are precisely the equilibrium states. Hence for $T \in \Lambda$, for λ-almost every x, the sequence $\frac{1}{k}(\delta_x + \ldots + \delta_{T^{k-1}x})$ has at most one weak-$*$ limit point, namely μ_T. By weak-$*$ sequential compactness, the entire sequence must converge to μ_T.

To see that μ_T must be singular (with respect to λ), we first note that each $T \in \mathcal{E}^1$ is a non-singular transformation (with respect to λ). Thus, if $\mu_T = \mu_{si} + \mu_{ac}$ is the decomposition of μ_T into singular and absolutely continuous components, the map $\mu_T \mapsto \mu_T \circ T^{-1}$ preserves μ_{si} and μ_{ac}, so that μ_{ac} is a finite, absolutely continuous T- invariant measure. But we have seen that a

[1] In fact the quoted theorem, as stated in the book, contains a mistake, although an irrelevant one for the present setting. The interested reader may go to http://www.mi.uni-erlangen/~keller/publications/equibook.html, where the needed correction to the proof of the theorem is given.

generic $T \in \mathcal{E}^1$ possesses no such invariant measure ([16]); that is, $\mu_{ac} = 0$. This proves Statement 2.

Lemma 5 implies that generically, μ_T is fully supported, showing Statement 3.

This completes the proof of Theorem 1.

Before proving Theorem 2, we state and prove a lemma. There is a reference to a similar lemma in [2] although we have been unable to find the proof in the papers cited there. Recall that if $T \in \mathcal{E}^2$, μ_T is an absolutely continuous probability measure with strictly positive Radon-Nikodym derivative $\rho = d\mu_T/d\lambda$.

Lemma 6. *Suppose* $T \in \mathcal{E}^2$. *Then* $\int \log \mathcal{L}_T \mathbb{1}\, d\mu_T \geq 0$, *with equality if and only if* ρ *is* $T^{-1}\mathcal{B}$*-measurable.*

Proof. Fix $T \in \mathcal{E}^2$. In this case, the equilibrium state μ_T is absolutely continuous. We write ρ for the density of μ_T with respect to Lebesgue measure.

Let $\mathcal{P}$ denote the Perron-Frobenius operator for T with respect to $\mu_T \in \mathfrak{M}^1_T$. Then $\mathcal{P}(f) = \dfrac{\mathcal{L}_T(\rho\cdot f)}{\rho}$. In particular $\mathcal{L}_T(\mathbb{1}) = \rho\mathcal{P}(\frac{1}{\rho})$. Thus,

$$\begin{aligned}\int \log \mathcal{L}_T(\mathbb{1})\, d\mu_T &= \int \log \rho\, d\mu_T + \int \log \mathcal{P}(1/\rho)\, d\mu_T \\ &= -\int \log(1/\rho)\, d\mu_T + \int \log \mathcal{P}(1/\rho)\, d\mu_T \\ &= -\int \mathcal{P}(\log(1/\rho))\, d\mu_T + \int \log \mathcal{P}(1/\rho)\, d\mu_T\ ,\end{aligned}$$

where the last equality follows because $\mathcal{P}$ preserves μ_T-integrals. It is well-known that $\mathcal{P}(\cdot) \circ T = \mathbb{E}_{\mu_T}(\cdot|T^{-1}\mathcal{B})$. Since T preserves μ_T, we may continue the above calculations as follows:

$$\begin{aligned}&-\int \mathcal{P}(\log(1/\rho))\, d\mu_T + \int \log \mathcal{P}(1/\rho)\, d\mu_T \\ &= -\int \mathcal{P}(\log(1/\rho)) \circ T\, d\mu_T + \int \log \mathcal{P}(1/\rho) \circ T\, d\mu_T \\ &= -\int \mathbb{E}_{\mu_T}\left(\log(1/\rho)|T^{-1}\mathcal{B}\right)\, d\mu_T + \int \log\left(\mathbb{E}_{\mu_T}(1/\rho|T^{-1}\mathcal{B})\right)\, d\mu_T \geq 0,\end{aligned}$$

where the last inequality follows from Jensen's inequality, from which it also follows that equality holds in the last step if and only if $\log(\frac{1}{\rho})$ is $T^{-1}\mathcal{B}$-measurable, which holds if and only if ρ is $T^{-1}\mathcal{B}$-measurable. This concludes the proof of Lemma 6.

Proof (Proof of Theorem 2). Since $\log \omega_n(x) = -\sum_{j=1}^n \log \mathcal{L}_T \mathbb{1} \circ T^j(x)$, by Theorem 1 we have that $\frac{1}{n} \log \omega_n(x) \to -\int \log \mathcal{L}_T \mathbb{1}\, d\mu_T$ for λ-a.e. $x \in S^1$ and $T \in \Lambda$. If $\int \log \mathcal{L}_T \mathbb{1}\, d\mu_T > 0$, then for large n, $\omega_n(x) = O(a^n)$ for λ-a.e. x, where a is any number such that $-\int \log \mathcal{L}_T \mathbb{1}\, d\mu_T < \log a < 0$. That is, the sequence $\omega_n(x)$ is asymptotically comparable to a geometric sequence, and hence summable (for λ-a.e. x), so that Lebesgue measure is not recurrent for T.

First we observe that $\{T : \int \log \mathcal{L}_T \mathbb{1}\, d\mu_T > 0\}$ is open in Λ. To see this, if $T \in \Lambda$ satisfies $\int \log \mathcal{L}_T \mathbb{1}\, d\mu_T > 0$ and $S \in \Lambda$ is C^1-close to T, then $\mathcal{L}_S \mathbb{1}$ is C^0-close to $\mathcal{L}_T \mathbb{1}$. By Lemma 4, μ_S is weak*-close to μ_T, proving the observation.

Thus by Lemma 6, it is sufficient to show that for maps T belonging to a dense subset of $\mathcal{E}^2$ (and hence a dense subset of $\mathcal{E}^1$), the invariant density ρ_T is not $T^{-1}\mathcal{B}$-measurable.

Choose $T \in \mathcal{E}^2$ for which ρ_T is $T^{-1}\mathcal{B}$-measurable. We shall show that there is an $S \in \mathcal{E}^2$ arbitrarily close to T (in the C^1 topology) for which ρ_S is not $S^{-1}\mathcal{B}$-measurable.

Since ρ_T is $T^{-1}\mathcal{B}$-measurable, $Tx = Ty$ implies that $\rho(x) = \rho(y)$. Given a Markov partition for T, we call the atoms of the partition the *branches* of T. We shall construct a C^2-homeomorphism $\pi : S^1 \to S^1$ in such a way that

1. π is arbitrarily (C^1-) close to the identity, and
2. The map $\tilde{T} = \pi \circ T \circ \pi^{-1}$ has the property that $\rho_{\tilde{T}} = \tilde{\rho}$ is *not* $\tilde{T}^{-1}\mathcal{B}$-measurable.

Establishing Items 1 and 2 will finish the proof.

Suppose for the moment that π is any C^2-homeomorphism of the circle, and $\tilde{T}(\tilde{x}) = \pi \circ T \circ \pi^{-1}(\tilde{x})$. Then $\tilde{\rho}(\tilde{x}) = \frac{\rho(\pi^{-1}\tilde{x})}{\pi'(\pi^{-1}\tilde{x})}$, so that $\tilde{\rho}$ will be $\tilde{T}^{-1}\mathcal{B}$-measurable precisely when $\tilde{T}(\tilde{x}) = \tilde{T}(\tilde{y})$ implies that $\tilde{\rho}(\tilde{x}) = \tilde{\rho}(\tilde{y})$. Suppose $\tilde{x} \neq \tilde{y}$ and $\tilde{T}(\tilde{x}) = \tilde{T}(\tilde{y})$. Then, since ρ is $T^{-1}\mathcal{B}$-measurable, $\rho(\pi^{-1}\tilde{y}) = \rho(\pi^{-1}\tilde{x})$. Hence $\tilde{\rho}(\tilde{x})$ will differ from $\tilde{\rho}(\tilde{y})$ precisely when $\pi'(\pi^{-1}\tilde{x}) \neq \pi'(\pi^{-1}\tilde{y})$.

Hence, if π is chosen so that π' is not $T^{-1}\mathcal{B}$-measurable, these terms will be different. Now we specify that π is a C^2-homeomorphism of S^1 with the property that $\pi' \equiv 1$ on one branch of T, and different from 1, yet arbitrarily close to 1, on the other branches. This completes the proof of Theorem 2.

Proof (Proof of Theorem 3). Suppose that T satisfies the conditions of Theorem 1. We show that in this case, any absolutely continuous invariant measure for T is locally infinite. Suppose ν is an absolutely continuous invariant measure for T. Then $\nu(S^1) = \infty$. Suppose, for the purpose of obtaining a contradiction, that I is any open interval with $\nu(I) < \infty$. Let f be any non-negative continuous function supported on I that is positive on some subinterval of I. Clearly $f \in L^1(\nu)$. By Birkhoff's ergodic theorem for an infinite invariant measure, for ν-almost every x, $\frac{1}{n}(f(x) + \ldots + f(T^{n-1}x)) \to 0$. This holds in particular on a set of positive Lebesgue measure. On the other hand, since μ_T is a Sinai-Ruelle-Bowen measure, we have for λ-almost every x, $\frac{1}{n}(f(x) + \ldots + f(T^{n-1}x)) \to \int f \, d\mu_T$. Since f is strictly positive on a subinterval of I and μ_T is fully supported, this quantity is strictly positive. This contradiction completes the proof of the theorem.

5. No Characteristic Scale

In this section we prove that if

$$S_{\epsilon,n,a} = \left\{T \in \mathcal{E}^1 \colon \lambda\{x \colon \mathcal{L}^n \mathbb{1}(x) \in [a, 2a]\} < \epsilon\right\},$$

and

$$S = \bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{a>0} S_{\epsilon,n,a}$$

then S is a dense G_δ subset of $\mathcal{E}^1$.

Proof (Proof of Theorem 4). We can replace the uncountable intersections in the definition of S by countable intersections over the rationals without changing the set. Define

$$F_n(T) = \lambda \times \lambda \left\{ (x, y) \colon \frac{1}{2} \le \frac{\mathcal{L}_T^n \mathbb{1}(x)}{\mathcal{L}_T^n \mathbb{1}(y)} \le 2 \right\}.$$

Clearly, $F_n(T) < \epsilon^2$ implies that for all positive a, the measure of the set of points with $\mathcal{L}_T^n \mathbb{1}(x) \in [a, 2a]$ is less than ϵ. Letting $R_{\epsilon,n} = \{T \colon F_n(T) < \epsilon^2\}$, it is clear that $R_{\epsilon,n} \subset \bigcap_{a>0} S_{\epsilon,n,a}$. Conversely, for fixed x, let $a_1 = \mathcal{L}_T^n \mathbb{1}(x)/2$ and $a_2 = 2a_1$. If $T \in \bigcap_{a>0} S_{\epsilon^2/2,n,a}$, then for each x, by considering $\cup_{i=1}^2 \{y \colon \mathcal{L}_T^n \mathbb{1}(y) \in [a_i, 2a_i]\}$ we have

$$\lambda\{y \colon \mathcal{L}_T^n \mathbb{1}(y) \in [\mathcal{L}_T^n \mathbb{1}(x)/2, 2\mathcal{L}_T^n \mathbb{1}(x)]\} < \epsilon^2 .$$

By Fubini's theorem, we see that $F_n(T) \le \epsilon^2$ so that $T \in R_{\epsilon,n}$. It follows that

$$S = \bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{a>0} S_{\epsilon,n,a} = \bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} R_{\epsilon,n}.$$

We shall show that $F_n \colon \mathcal{E}^1 \to \mathbb{R}$ is an upper semi-continuous map so that S is a G_δ set. To prove this, suppose that $F_n(T) < \alpha$. We have

$$\lambda \times \lambda \left(\left\{ (x, y) \colon \tfrac{\mathcal{L}^n \mathbb{1}(x)}{\mathcal{L}^n \mathbb{1}(y)} \in [\tfrac{1}{2}, 2] \right\} \right) =$$
$$\lim_{k\to\infty} \lambda \times \lambda \left(\left\{ (x, y) \colon \tfrac{\mathcal{L}^n \mathbb{1}(x)}{\mathcal{L}^n \mathbb{1}(y)} \in [\tfrac{1}{2} - \tfrac{1}{k}, 2 + \tfrac{1}{k}] \right\} \right).$$

One can therefore find a k such that $\lambda \times \lambda(\{(x, y) \colon \mathcal{L}_T^n \mathbb{1}(x)/\mathcal{L}_T^n \mathbb{1}(y) \in [1/2 - 1/k, 2 + 1/k]\}) < \alpha$. Since the map $\Phi \colon \mathcal{E}^1 \to C^0(S^1 \times S^1)$ given by $\Phi(T)(x, y) = \mathcal{L}_T^n \mathbb{1}(x)/\mathcal{L}_T^n \mathbb{1}(y)$ is continuous (with the C^1 and C^0-topologies on the respective spaces), there exists a neighborhood U of T such that if $\tilde{T} \in U$, then $\|\Phi(T) - \Phi(\tilde{T})\| < 1/k$. It follows that if $\tilde{T} \in U$, then $F_n(\tilde{T}) < \alpha$, proving the upper semi-continuity of F_n.

It then remains to demonstrate the density of S. To do this, we shall establish that for any $\epsilon > 0$, any $T_0 \in \mathcal{E}^2$ and any neighborhood U of T_0 (in the C^1 topology), there is a $T \in U$ and an $n \in \mathbb{N}$ such that for each a, $\lambda\{x \colon \mathcal{L}^n \mathbb{1}(x) \in [a, 2a]\} < \epsilon$. This will be accomplished by conjugating T_0 using a homeomorphism constructed via a cocycle.

We shall therefore assume $\epsilon > 0$, $T_0 \in \mathcal{E}^2$ and $\delta > 0$ are given. Let $\eta > 0$ be such that $(1 + \eta)/(1 - \eta) < 1 + \delta$. Then we also have $(1 - \eta)/(1 + \eta) > 1 - \delta$.

Since T_0 belongs to $\mathcal{E}^2$, T_0 preserves an absolutely continuous invariant probability measure, μ, with a strictly positive continuous density, ρ. Let m be such that $\frac{1}{m} \le \rho(x) \le m$ for all x. Let $\bar{T}_0 \colon X \to X$ be a natural extension of $T_0 \colon S^1 \to S^1$ preserving the measure $\bar{\mu}$. From [21], $\bar{\mu}$ is Bernoulli, so we may find a non-trivial independent partition $\mathcal{P} = \{A_0, A_1\}$ of X. Write p for $\bar{\mu}(A_0)$ and q for $\bar{\mu}(A_1)$. We then define a function $\bar{G}_0$ on X as follows:

$$\bar{G}_0(x) = \begin{cases} 1 + \eta q \text{ if } x \in A_0 \\ 1 - \eta p \text{ if } x \in A_1. \end{cases}$$

Let $n > 0$ be an integer. We then form the multiplicative cocycle $\bar{G}_0^{(n)}$ defined by

$$\bar{G}_0^{(n)}(x) = \bar{G}_0(x)\bar{G}_0(\bar{T}_0 x)\dots G_0(\bar{T}_0^{n-1}x).$$

The function $\bar{G}_0^{(n)}$takes on the value $v_k = (1+\eta q)^k(1-\eta p)^{n-k}$ on a set of measure $\begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}$.

Let $K \in \mathbb{N}$ be the least integer so that

$$\left(\frac{1+\eta q}{1-\eta p}\right)^K > 2m^2.$$

Since $v_{k+1}/v_k = \left(\frac{1+\eta q}{1-\eta p}\right)$, for each a there are at most K values taken by $\bar{G}_0^{(n)}$ in $[a, 2m^2 a]$.

We then have the estimate

$$\bar{\mu}\{x\colon \bar{G}_0^{(n)}(x) \in [a, 2m^2 a]\} \le K \max_{\{k: v_k \in [a, 2m^2 a]\}} \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}.$$

Since for the values of k in the range over which the maximum is taken have the property that $v_k \ge a$, we see

$$a\bar{\mu}\{x\colon \bar{G}_0^{(n)}(x) \in [a, 2m^2 a]\} \le K \max_{\{k: v_k \in [a, 2m^2 a]\}} \begin{pmatrix} n \\ k \end{pmatrix} v_k p^k q^{n-k}$$
$$= K \max_{0 \le k \le n} \begin{pmatrix} n \\ k \end{pmatrix} (p + \eta pq)^k (q - \eta pq)^{n-k} < \tfrac{CK}{\sqrt{n}},$$

where C is a constant that depends only on the values of p and q.

Now fix an n so that $a\bar{\mu}(\{x\colon \bar{G}_0^{(n)}(x) \in [a, 2m^2 a]\}) < \epsilon/4$ for all a. It will turn out that an inequality of this type will be what is needed for the conjugate map to have the desired property. At this point, the function $\bar{G}_0^{(n)}$is defined not on the circle, but on the natural extension space. We shall apply a conditional expectation and approximation argument to $\bar{G}_0^{(n)}$to obtain a function on the circle as needed.

Let $\mathcal{Q}$ be a Markov partition for T_0 consisting of intervals. There exists a k such that $\bigvee_{s=0}^{k-1} T_0^{-s}\mathcal{Q}$ consists of intervals of length less than δ. Denote these intervals by I_j and write $\bar{I}_j$ for $\pi^{-1}I_j$, where π denotes the natural projection from the natural extension $(X, \bar{T}_0, \bar{\mu})$ to (S^1, T_0, μ).

Write $\bar{\rho} = \rho \circ \pi$ and define the natural extension of λ, $\bar{\lambda}$ by $\bar{\lambda}(A) = \int_A (1/\bar{\rho})\, d\bar{\mu}$.

We then calculate

$$\int_{\bar{I}_j} \bar{G}_0^{(n)} \circ \bar{T}_0^i \, d\bar{\lambda} = \int \frac{\chi_{\bar{I}_j}}{\bar{\rho}} \cdot \bar{G}_0^{(n)} \circ \bar{T}_0^i \, d\bar{\mu}.$$

Since $\bar{T}_0$ is mixing, we see that

$$\lim_{i\to\infty} \int_{\bar{I}_j} \bar{G}_0^{(n)} \circ \bar{T}_0^i \, d\bar{\lambda} = \int \tfrac{\chi_{\bar{I}_j}}{\bar{\rho}} \, d\bar{\mu} \int \bar{G}_0^{(n)} \, d\bar{\mu}$$
$$= \bar{\lambda}(\bar{I}_j) \left(\int \bar{G}_0 \, d\bar{\mu}\right)^n$$
$$= \lambda(I_j),$$

where we used the fact that $\mathcal{P}$ is an independent partition to get the second equality.

We recall that n is chosen so that

$$\bar\mu(\{x\colon \bar G_0^{(n)}(x)\in[a,2m^2a]\})<\epsilon/(4a), \tag{2}$$

for each $a>0$. We now choose an i_0 such that for $i\ge i_0$,

$$\left|\int_{\bar I_j}\bar G_0^{(n)}\circ \bar T_0^i\,d\bar\lambda-\lambda(I_j)\right|<\tfrac{\delta}{3}\lambda(I_j), \tag{3}$$

for each j.

We now show that similar inequalities persist for functions $\bar G^{(n)}$ if $\bar G$ is chosen to be an appropriate perturbation of $\bar G_0$.

It is useful to note that because the values taken by $\bar G_0^{(n)}$ are in the range $[(1-\eta p)^n,(1+\eta q)^n]$, the inequality (2) holds trivially for a outside this range.

We define N to be a subset of $L^1(\bar\mu)$ as follows and equip it with the L^1 subspace topology:

$$N=\{\bar G\colon 1-\eta p\le \bar G\le 1+\eta q;\ \|\bar G-\bar G_0\|_1<\zeta\}.$$

Since composition with $\bar T$ is an isometry on $L^1(\bar\mu)$, and because N consists of bounded functions, the map from N to L^1 given by $\bar G\mapsto \bar G^{(n)}$ is continuous. Clearly, for $\bar G\in N$, the values taken by $\bar G^{(n)}$ are in the range $[(1-\eta p)^n,(1+\eta q)^n]$. By choosing ζ appropriately small, we can ensure that $|\bar G^{(n)}-\bar G_0^{(n)}|<(1-\eta p)^n/2$ on a set of measure at least $1-\epsilon/(8(1+\eta q)^n)$. For a given a in the range $[(1-\eta p)^n/2m^2,(1+\eta q)^n]$, let $a_1=a/2$ and $a_2=2a$. Then

$$\begin{gathered}\{x\colon \bar G^{(n)}(x)\in[a,2m^2a]\}\subset\\ \{x\colon \bar G_0^{(n)}(x)\in[a_1,2m^2a_1]\}\cup\{x\colon \bar G_0^{(n)}(x)\in[a,2m^2a]\}\\ \cup\{x\colon \bar G_0^{(n)}(x)\in[a_2,2m^2a_2]\}\cup\{x\colon |\bar G_0^{(n)}(x)-\bar G^{(n)}(x)|>(1-\eta p)^n/2\}.\end{gathered}$$

We shall denote the four sets on the right-hand side by A_1, A_2, A_3 and A_4 respectively. By our previous estimates on $\bar G_0^{(n)}$we have $\bar\mu(A_1)<\epsilon/(2a)$, $\bar\mu(A_2)<\epsilon/(4a)$ and $\bar\mu(A_3)<\epsilon/(4a_2)<\epsilon/(8a)$. We chose ζ above to ensure that $\bar\mu(A_4)<\epsilon/(8(1+\eta q)^n)<\epsilon/(8a)$, so that

$$\bar\mu(\{x\colon \bar G^{(n)}(x)\in[a,2m^2a]\})<\epsilon/a$$

for each a in the range $[(1-\eta p)^n/(2m^2),(1+\eta q)^n]$. As before, the inequality holds trivially for a outside this range, so we have established that for sufficiently small ζ, a similar inequality to (2) persists for all a and functions $\bar G^{(n)}$, if $\bar G$ is chosen from N.

Since

$$\begin{aligned}\int_{\bar I_j}|\bar G_0^{(n)}\circ\bar T_0^i-\bar G^{(n)}\circ\bar T_0^i|\,d\bar\lambda\quad&\le\textstyle\int|\bar G_0^{(n)}\circ\bar T_0^i-\bar G^{(n)}\circ\bar T_0^i|\,d\bar\lambda\\ &\le m\textstyle\int|\bar G_0^{(n)}\circ\bar T_0^i-\bar G^{(n)}\circ\bar T_0^i|\,d\bar\mu,\end{aligned}$$

we see that provided ζ is sufficiently small, (3) holds for $\bar{G} \in N$.

We have therefore shown that there exists a $\zeta > 0$ such that for $\bar{G} \in N$,

$$\bar{\mu}(\{x : \bar{G}^{(n)}(x) \in [a, 2m^2 a]\}) < \epsilon/a \text{ for each } a, \quad \text{and} \tag{4}$$

$$\left| \int_{\bar{I}_j} \bar{G}^{(n)}(x) \circ \bar{T}_0^i \, d\bar{\lambda} - \lambda(I_j) \right| < \tfrac{\delta}{3} \lambda(I_j) \text{ for each } j, \text{ and } i \geq i_0. \tag{5}$$

We note that since $\bar{T}_0 : X \to X$ is a natural extension of $T_0 : S^1 \to S^1$, the σ-algebras $\bar{T}_0^k \pi^{-1} \mathcal{B}_{S^1}$ increase to $\mathcal{B}_X$. It follows that $\mathbb{E}_{\bar{\mu}}(\bar{G}_0 \,|\, \bar{T}_0^i \pi^{-1} \mathcal{B}_{S^1})$ converges to $\bar{G}_0$ in L^1. By the monotonicity of conditional expectation, these functions also satisfy the inequality $1 - \eta p \leq \mathbb{E}_{\bar{\mu}}(\bar{G}_0 \,|\, \bar{T}_0^i \pi^{-1} \mathcal{B}_{S^1}) \leq 1 + \eta q$. It follows that for sufficiently large $i \geq i_0$, (i_0 as above), (4) and (5) are satisfied with $\mathbb{E}_{\bar{\mu}}(\bar{G}_0 \,|\, \bar{T}_0^i \pi^{-1} \mathcal{B}_{S^1})$ in place of $\bar{G}$. Fix some such i and write $\bar{G}_1$ for $\mathbb{E}_{\bar{\mu}}(\bar{G}_0 \,|\, \bar{T}_0^i \pi^{-1} \mathcal{B}_{S^1})$.

Now $\bar{G}_1 \circ \bar{T}_0^i = \mathbb{E}_{\bar{\mu}}(\bar{G}_0 \circ \bar{T}_0^i | \pi^{-1} \mathcal{B}_{S_1})$ so we see that $\bar{G}_1 \circ \bar{T}_0^i$ may be written as $g_1 \circ \pi$ for some $\mathcal{B}$-measurable function g_1 on the circle. Since $C^0(S^1)$ is dense in $L^1(S^1, \mathcal{B}, \mu)$, it follows that there exists a continuous function g_2 such that $\|g_1 - g_2\|_1$ is arbitrarily small. Since $\|\bar{G}_1 - g_2 \circ \pi \circ \bar{T}_0^{-i}\|_1 = \|g_1 - g_2\|_1$, we see that g_2 may be chosen so that $g_2 \circ \pi \circ \bar{T}_0^{-i}$ lies in N.

Equations (4) and (5) now yield

$$\left| \int_{I_j} g_2^{(n)} \, d\lambda - \lambda(I_j) \right| < \tfrac{\delta}{3} \lambda(I_j) \text{ for each } j; \quad \text{and}$$

$$\mu(\{x : g_2^{(n)}(x) \in [a, 2m^2 a]\}) < \epsilon/a \text{ for each } a > 0.$$

From the first equation, we see that $1 - \frac{\delta}{3} < \int g_2^{(n)} \, d\lambda < 1 + \frac{\delta}{3}$, so finally we rescale g_2 (i.e. multiply by a constant, that will, by our above estimates, be very close to 1) to obtain a function g that satisfies $\int g^{(n)} \, d\lambda = 1$. We then have the inequalities

$$\left| \int_{I_j} g^{(n)} \, d\lambda - \lambda(I_j) \right| < \delta \lambda(I_j) \text{ for each } j; \quad \text{and} \tag{6}$$

$$\mu(\{x : g^{(n)}(x) \in [a, 2m^2 a]\}) < 2\epsilon/a \text{ for each } a > 0. \tag{7}$$

Set $\theta(x) = \int_0^x g^{(n)}(t) \, dt$ and let $T(x) = \theta \circ T_0 \circ \theta^{-1}(x)$. Then from the above, and since each interval I_j has length less than δ, it may be verified that $|\theta(x) - x| < 2\delta$, and $\sup_{x \in S^1} |T(x) - T_0(x)| < (C + 4)\delta$, where $C = \max_{x \in S_1} |T_0'(x)|$. Hence this quantity can be made arbitrarily small by choosing δ sufficiently small. Also, differentiating, we see

$$\begin{aligned} T'(x) &= T_0'(\theta^{-1} x) \tfrac{\theta'(T_0(\theta^{-1} x))}{\theta'(\theta^{-1} x)} \\ &= T_0'(\theta^{-1} x) \tfrac{g^{(n)}(T_0(\theta^{-1} x))}{g^{(n)}(\theta^{-1} x)} \\ &= T_0'(\theta^{-1} x) \tfrac{g(T_0^n(\theta^{-1} x))}{g(\theta^{-1} x)}. \end{aligned}$$

Since g is uniformly close to 1 and T_0' is uniformly continuous, we see that $\sup_{x \in S^1} |T'(x) - T_0'(x)|$ can also be made arbitrarily small by controlling δ and η. This shows that T can be chosen arbitrarily close to T_0 in the C^1 norm.

It remains to verify that T has the property that there exists an n such that for each a, $\lambda\{x: \mathcal{L}^n \mathbb{1}(x) \in [a, 2a]\} < \epsilon$. Since T is conjugate to T_0, there is also a conjugacy relation between their Perron-Frobenius operators given by $\mathcal{L}_T = \mathcal{L}_\theta \circ \mathcal{L}_{T_0} \circ {\mathcal{L}_\theta}^{-1}$, where $\mathcal{L}_\theta f(x) = f(\theta^{-1}(x))/\theta'(\theta^{-1}x)$.

Since T_0 is a C^2 expanding map, we have that $\mathcal{L}_{T_0}^n \mathbb{1}$ converges uniformly to ρ. It follows that $\mathcal{L}_T^n \mathbb{1}$ converges uniformly to $\mathcal{L}_\theta \rho(x) = \rho(\theta^{-1}x)/\theta'(\theta^{-1}x)$. We then estimate

$$\begin{aligned}\lambda(\{x: \frac{\rho(\theta^{-1}x)}{g^{(n)}(\theta^{-1}x)} \in [a, 2a]\}) &\le \lambda(\{x: \tfrac{1}{g^{(n)}(\theta^{-1}x)} \in [\tfrac{a}{m}, 2ma]\}) \\ &= \lambda(\{x: g^{(n)}(\theta^{-1}x) \in [\tfrac{1}{2ma}, \tfrac{m}{a}]\}).\end{aligned}$$

But we see that $\{x: g^{(n)}(\theta^{-1}x) \in [\frac{1}{2ma}, \frac{m}{a}]\} = \theta(\{y: g^{(n)}(y) \in [\frac{1}{2ma}, \frac{m}{a}]\})$. Using this, we get

$$\begin{aligned}\lambda(\{x: \frac{\rho(\theta^{-1}x)}{g^{(n)}(\theta^{-1}x)} \in [a, 2a]\}) &\le \lambda \circ \theta(\{y: g^{(n)}(y) \in [\tfrac{1}{2ma}, \tfrac{m}{a}]\}) \\ &= \textstyle\int_{\{y: g^{(n)}(y) \in [\frac{1}{2ma}, \frac{m}{a}]\}} g^{(n)}(y)\, d\lambda \\ &< \tfrac{m}{a} \lambda(\{y: g^{(n)}(y) \in [\tfrac{1}{2ma}, \tfrac{m}{a}]\}) \\ &< 4m^3 \epsilon,\end{aligned}$$

where we have used (7) with a replaced by $\frac{1}{2ma}$. This completes the proof.

6. Conclusion

Few methods are known for detecting the presence of invariant measures for general non-invertible mappings. With regard to the specific question of whether a generic C^1 expanding map has a σ-finite absolutely continuous invariant measure, the known methods to try would include inducing, rescaling densities of finite measures and taking a limit, or establishing the recurrence property defined earlier. Our results may be seen as establishing that the known methods will fail. Here is why.

Since we show (Theorem 3) that any infinite invariant measure is locally infinite, any set of finite measure to perform an inducing construction would necessarily be irregular. It seems unlikely that such a set could be constructed in the uniform way required to prove existence of a σ-finite measure for a generic set of maps in $\mathcal{E}^1$.

To exploit rescaling, one typically attempts to rescale the sequence of densities of $\lambda \circ T^{-n}$ in order to obtain a limit. Theorem 4 suggests that such an approach would probably not work.

If one can establish that a measure is recurrent (in the sense defined in the introduction) for a non-invertible map, then existence or non-existence of absolutely continuous, σ-finite invariant measures for the map can be decided using a version of Krieger's ratio set (see Hawkins and Silva [6] for a proof of this result). Theorem 2 indicates that this approach will fail in the context of generic C^1 expanding maps.

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