MAPPINGS OF GROUP SHIFTS

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ABSTRACT. A group shift is a proper closed shift-invariant subgroup of $G^{\mathbb{Z}^2}$ where G is a finite group. We consider a class of group shifts in which G is a finite field and show that mixing is a necessary and sufficient condition on such a group shift for all codes from it into another group shift to be affine and all codes from another group shift into it to be affine. As a corollary, it will follow for $G = \mathbb{Z}_p$ that two mixing group shifts are topologically conjugate if and only if they are equal.

1. INTRODUCTION

We consider a class of higher-dimensional shifts of finite type which are groups. It is known that in general higher-dimensional shifts of finite type are badly behaved (see for example [13] where it is shown that there is no algorithm to decide whether a given shift of finite type is empty or not). Very little is known about the class of all higher-dimensional shifts of finite type. For this reason, it is of interest to find classes of higher-dimensional shifts for which one can say something, but with enough generality to include a good range of properties. The group shifts which we consider do not suffer from the undecidability problems mentioned earlier (see [6] for an early paper in which properties of algebraic shifts were considered) and exhibit highly interesting behavior with regard to mixing properties as was pointed out by Ledrappier in [9].

In [7], Kitchens and Schmidt introduced the class of Markov subgroups. These are essentially what we are calling $\mathbb{Z}_p$-group shifts. They define a Markov subgroup to be a proper closed shift-invariant subgroup of ${\mathbb{Z}_2}^{\mathbb{Z}^2}$ (where the operation is pointwise addition modulo 2). These are generalizations of the well-known three-dot system introduced by Ledrappier [9] of all points in ${\mathbb{Z}_2}^{\mathbb{Z}^2}$ satisfying $\xi_{i,j} + \xi_{i+1,j} + \xi_{i,j+1} = 0$ for each $(i, j) \in \mathbb{Z}^2$.

In [7], the algebraic properties of $\mathbb{Z}_p$-group shifts are studied using duality results and are related to their dynamical properties. A key result is that a $\mathbb{Z}_p$-group shift is necessarily of finite type and is the intersection of finitely many

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group shifts of a simple form. We will offer alternative proofs of some of the basic facts from [7] which are more dynamical in spirit.

Codes (continuous, shift-commuting mappings) between symbolic dynamical systems have been studied for a number of reasons. First, they are used to classify systems up to relations such as topological conjugacy or almost topological conjugacy. Second, they are themselves interesting examples of dynamical systems. Lastly, they have important applications to information theory. (See [10] for further references on symbolic dynamical systems and codes). A surjective code is also known in the literature as a factor map.

In this paper, we consider codes between K-group shifts and show that if either the domain or the range of a code is mixing, the code is necessarily affine (i.e. a group homomorphism plus a constant) - see Theorems 3.2 and 3.3. This is in sharp contrast to the general case for codes between shifts of finite type, which can be far from affine. The first result of this type was in [7] (Observation 4.1), where it was shown that if θ is a factor map from a Markov subgroup to the Ledrappier three-dot system with certain ergodicity properties, then θ is necessarily a group homomorphism. Kitchens and Schmidt then made a pair of conjectures, which essentially stated that surjective measurable shift-commuting mappings of group shifts are affine. In a recent paper [8], they proved the conjecture in what was called the irreducible case, where the mapping was also assumed to be bijective. The results in this paper are essentially a complete answer to the continuous versions (i.e. where the maps are assumed to be continuous) of these conjectures, where in addition no requirements about surjectiveness are made. Our results also generalize a result of Coven, Hedlund and Rhodes ([2]) concerning the 'commuting block maps problem' where they show that the codes of ${\mathbb{Z}_2}^{\mathbb{Z}}$ commuting with a given affine map are themselves affine (see Theorem 3.8).

It follows from Theorem 3.2 that the image under a code of a mixing group shift in $\mathbb{Z}_p^{\mathbb{Z}^2}$ (p a prime) must either be a subgroup of the domain or a coset of such a subgroup. If the map is a conjugacy, then the range must equal the domain (Corollary 3.7).

We view the fact that codes are affine as a rigidity property: where a weak condition forces a much stronger condition to hold. Other examples of rigidity properties occur in [12] and [8]. It appears that rigidity is a phenomenon which occurs relatively commonly in $\mathbb{Z}^d$ actions, but less often in $\mathbb{Z}$ actions. Certainly, although our examples have some properties in common with one-dimensional full shifts (see Theorem 2.2), there is no analog of our results for one-dimensional full shifts.

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2. Background

In this section, we summarize some basic results about symbolic dynamics and group shifts. Many of the results in this section are versions or minor extensions of results contained in [7] but the proofs are mostly new and more dynamical. We begin with a finite set of symbols K. In this paper, we will assume that K is a finite field (this assumption is not generally made in symbolic dynamics). The case $K = \mathbb{Z}_p$, the integers modulo a prime p is of particular interest. We form the set $K^{\mathbb{Z}^2}$, consisting of all two dimensional arrays of symbols from K, indexed by $\mathbb{Z}^2$. If $\xi \in K^{\mathbb{Z}^2}$ and $S \subset \mathbb{Z}^2$, we let ξ_S denote the restriction of ξ to S. We call ξ_S a *configuration* on S.

Let Λ_n denote the set of points $(i,j) \in \mathbb{Z}^2$ such that $-n \leq i,j \leq n$. We choose a metric on $K^{\mathbb{Z}^2}$ in which two points ξ and η are "close" if $\xi_{\Lambda_n} = \eta_{\Lambda_n}$ for large n (there are many possibilities for such a metric and any will do). In this metric $K^{\mathbb{Z}^2}$ is a compact space which is homeomorphic to a Cantor set. For any $v \in \mathbb{Z}^2$, we define the *shift by* v to be the map $\sigma_v : K^{\mathbb{Z}^2} \to K^{\mathbb{Z}^2}$ defined by $(\sigma_v(\xi))_a = \xi_{a+v}$, for all $a \in \mathbb{Z}^2$. Clearly σ_v is a homeomorphism. This defines a $\mathbb{Z}^2$-action on $K^{\mathbb{Z}^2}$, called the *full shift.* A *subshift* is a non-empty closed subset of $K^{\mathbb{Z}^2}$ which is invariant under all shift maps. Throughout this paper, we will refer to continuous shift-commuting mappings between subshifts as codes.

Given a finite set $\Lambda \subset \mathbb{Z}^2$ and a map b from K^{Λ} to another symbol set T, b may be used to define a map F_b from $K^{\mathbb{Z}^2}$ to $T^{\mathbb{Z}^2}$ by

$$F_b(\xi)_{i,j} = b(\xi_{(i,j)+\Lambda}).$$

The map F_b is clearly a code. If X is a subshift of $K^{\mathbb{Z}^2}$, then the image of F_b, restricted to X, is a subshift of $T^{\mathbb{Z}^2}$.

Definition 1. *A map* $F\colon S^{\mathbb{Z}^2} \to T^{\mathbb{Z}^2}$ *is called a* block map *if there exists a* Λ *such that* F *is equal to* F_b *for some* $b\colon S^{\Lambda} \to T$.

By a well-known generalization of a theorem of Curtis, Hedlund and Lyndon, every code between any two subshifts is a block map ([4]).

Definition 2. *A group shift is a proper closed shift-invariant subgroup of* $G^{\mathbb{Z}^2}$ *where* G *is a finite group.*

We will restrict to the special case where G is in fact a finite field.

Definition 3. *For a finite field* K*, a* K*-group shift is a proper closed shift-invariant subgroup of* $K^{\mathbb{Z}^2}$ *(considered as an additive group) which is closed under scalar multiplication by elements of* K*. We will write* p *for the characteristic of* K*.*

We remark that the above should probably be called a K-module shift, but we chose the algebraically less daunting K-group shift. In this paper, we will almost exclusively be considering the case of K-group shifts and not more general group shifts.

We next show that $K^{\mathbb{Z}^2}$ has a natural structure as a module over the ring $\mathcal{L}$ of Laurent polynomials in two variables over K. Any Laurent polynomial $f \in \mathcal{L}$

can be written as

$$f = \sum_{(i,j)\in P_f} a_{i,j} X^i Y^j,$$

where P_f is a finite subset of $\mathbb{Z}^2$, called the *shape* of f. For $\xi \in K^{\mathbb{Z}^2}$, we define

$$(f\xi)_{i,j} = \sum_{(k,l)\in P_f} a_{k,l}\xi_{i+k,j+l}.$$

It is easy to check that with this definition of multiplication, $K^{\mathbb{Z}^2}$ is a module over $\mathcal{L}$. We verify only the property that $f(g\xi) = (fg)\xi$. Write $f = \sum_{(i,j)\in P_f} a_{i,j} X^i Y^j$ and $g = \sum_{(i,j)\in P_g} b_{i,j} X^i Y^j$. Then

$$\begin{aligned}
(f(g\xi))_{i,j} &= \sum_{(k,l)\in P_f} a_{k,l}(g\xi)_{i+k,j+l} \\
&= \sum_{(k,l)\in P_f} a_{k,l} \sum_{(m,n)\in P_g} b_{m,n}\xi_{i+k+m,j+l+n} \\
&= \left(\left(\sum_{(k,l)\in P_f}\sum_{(m,n)\in P_g} a_{k,l}b_{m,n}X^kY^lX^mY^n\right)\xi\right)_{i,j} \\
&= ((fg)\xi)_{i,j}.
\end{aligned}$$

Observe that $X\xi = \sigma_{(1,0)}\xi$ and $Y\xi = \sigma_{(0,1)}\xi$, so that multiplication by X and Y correspond to applying the horizontal and vertical shifts, respectively. Consequently, the group shifts are precisely the proper closed submodules of $K^{\mathbb{Z}^2}$. If M is a submodule of $K^{\mathbb{Z}^2}$, then the map $M \to M$ defined by $\xi \mapsto f\xi$ is a module homomorphism, and is therefore shift-commuting, since $fX = Xf$ and $fY = Yf$. We see from the definition that $\xi \to f\xi$ is a block map and it then follows that f is a code.

We next show that conversely, any K-linear code $\theta : K^{\mathbb{Z}^2} \to K^{\mathbb{Z}^2}$ is given by multiplication by an element of $\mathcal{L}$. Observe that the map given by $\xi \to (\theta\xi)_{(0,0)}$ is a continuous linear functional. It follows from uniform continuity that

$$(\theta\xi)_{(0,0)} = \sum_{(k,l)\in P_f} a_{k,l}\xi_{k,l}$$

for some finite set P_f in $\mathbb{Z}^2$ and constants $a_{i,j} \in K$. Let g be the polynomial $\sum_{(k,l)\in P_f} a_{k,l}X^kY^l$, it is easy to see, using the shift-commuting property of θ, that $\theta\xi = g\xi$, so that θ is given by a polynomial. It follows that any two K-linear maps $K^{\mathbb{Z}^2} \to K^{\mathbb{Z}^2}$ commute. If $K = \mathbb{Z}_p$, then it is easy to see that any continuous shift-commuting group homomorphism ${\mathbb{Z}_p}^{\mathbb{Z}^2} \to {\mathbb{Z}_p}^{\mathbb{Z}^2}$ is actually K-linear, and is therefore given by a polynomial. Consequently, any two group homomorphisms commute, for $K = \mathbb{Z}_p$. This is no longer true in general if K is a finite extension of $\mathbb{Z}_p$.

The module structure gives rise immediately to a class of group shifts. For $f \in \mathcal{L}$, define $\langle f \rangle_K^\perp = \{\xi \in K^{\mathbb{Z}^2} \colon f\xi = 0\}$. When it is clear from context, we will drop the subscript and simply write $\langle f \rangle^\perp$. Since $gf\xi = 0$ for all $\xi \in \langle f \rangle^\perp$ and $g \in \mathcal{L}$, it follows that $\langle f \rangle^\perp$ is the set of points annihilated by the ideal generated by f, and that $\langle f \rangle^\perp$ is a submodule of $K^{\mathbb{Z}^2}$. Since $\langle f \rangle^\perp$ is the kernel of f, regarded as a continuous map on $K^{\mathbb{Z}^2}$, it follows that $\langle f \rangle^\perp$ is closed. In the case where $K = \mathbb{Z}_2$, a point ξ belongs to $\langle f \rangle^\perp$ if the sum of $\xi_{i,j}$ over any translation of P_f is zero. Consequently, if we translate P_f by any integer vector to obtain a new finite set $P_{f'}$, corresponding to $f' \in \mathcal{L}$, then $\langle f \rangle^\perp = \langle f' \rangle^\perp$. It turns out that all K-group shifts are defined by ideals in $\mathcal{L}$ (not necessarily principal), as we will prove shortly (Theorem 2.6).

Note that the three-dot system described earlier is just $\langle 1 + X + Y \rangle^\perp \subset {\mathbb{Z}_2}^{\mathbb{Z}^2}$.

As was noted in [7] and developed in [1], the group shifts satisfy an interesting generalization of the ordinary expansiveness property.

Definition 4. *A direction $v = (m, n)$ in $\mathbb{Z}^2$ (assumed to have the property that $\operatorname{hcf}(m, n) = 1$) is called an expansive direction for M if there is a $t > 0$ such that the configuration of ξ on $L_t(v)$ determines the whole point ξ, where $L_t(v) = \{x \in \mathbb{Z}^2 : d(x, \{zv \colon z \in \mathbb{R}\}) \le t\}$ is a strip of width $2t$ about the line parallel to v through 0.*

Definition 5. *If f is a Laurent polynomial in $\mathcal{L}$, the* convex hull *of f, $C(f)$ is defined to be the convex hull of the set of (i, j) such that f has a non-zero coefficient of X^iY^j.*

Definition 6. *If f is a Laurent polynomial in $\mathcal{L}$ with two or more non-zero coefficients, f is called* collinear *if $C(f)$ has empty interior.*

A polynomial is therefore collinear if and only if it is of the form X^iY^jh where h is a polynomial in a single monomial X^mY^n (where m and n are assumed to be coprime). In this case, we call the vector (m, n) the *direction* of f. It will turn out that it is the collinear polynomials which cause problems later on.

Let $\mathcal{L}_0$ denote the set of Laurent polynomials which have no collinear factors (i.e. $\mathcal{L}_0$ is the set of Laurent polynomials, each of whose irreducible factors has the property that the interior of its shape is non-empty). Results from [7], Proposition 2.11, and [6], Theorem 2.4 can be used to prove that $f \in \mathcal{L}_0$ if and only if $\langle f \rangle^\perp$ is mixing of order two, which in this case is equivalent to $\sigma_{m,n}$ being mixing in every direction (m, n).

If v is a vector in $\mathbb{Z}^2$ (where we assume the coordinates are coprime), a complementary vector is a vector u such that u and v generate $\mathbb{Z}^2$ i.e. such that area of the parallelogram with vertices 0, u, v and $u + v$ is 1. Such vectors always exist.

Definition 7. *If $f \in \mathcal{L}$ and v is an expansive direction for $\langle f \rangle^\perp$, then the height of f above the direction v is defined as follows: Let u be a complementary vector. The points in $C(f)$ may be expressed in terms of u and v. The* height *of f (*above *v), $\mathrm{ht}_v(f)$ is defined to the difference between the maximum coefficient of u and the minimum coefficient of u of points in $C(f)$. A strip in the v direction of height h is then given by $S_h(v) = \{xv + yu \colon x \in \mathbb{Z}, 0 \le y < h\}$.*

Note that the height is independent of the particular complementary vector chosen as any two complementary vectors differ up to orientation by a multiple of v. In [7], this was called the width, but here we want to emphasize that the height is in a direction complementary to v so are calling it the height above v.

Note also that with this definition, $\mathrm{ht}_v(fg) = \mathrm{ht}_v(f) + \mathrm{ht}_v(g)$. In particular, $\mathrm{ht}_v(f^k) = k \cdot \mathrm{ht}_v(f)$. It may also be verified that $C(fg) = C(f) + C(g)$ where the sum is the usual set of all sums of each element of $C(f)$ with each element of $C(g)$.

Theorem 2.1. *A direction is expansive for $\langle f \rangle^{\perp}$ if and only if it is not parallel to a face of $C(f)$.*

Theorem 2.2. *If v is an expansive direction for $\langle f \rangle^{\perp}$, then letting S be given by $S_{\mathrm{ht}_v(f)}(v)$, any configuration on S may be uniquely extended to give a point of $\langle f \rangle^{\perp}$. All points of $\langle f \rangle^{\perp}$ arise in this way. The map σ_v on $\langle f \rangle^{\perp}$ is then conjugate to a one-dimensional full shift on a finite set of symbols.*

Theorems 2.1 and 2.2 are straightforward generalizations of results proved in [7]. We illustrate them with an example which captures the essence of the proofs.

Example Consider the three-dot system and the direction $v = (1, 1)$. A complementary vector is $u = (1, 0)$. The Laurent polynomial is $f = 1 + X + Y$ so $C(f)$ is the convex hull of $(0, 0)$, $(1, 0)$ and $(1, 1)$. Expressed in terms of u and v, these points are $0u + 0v$, $1u + 0v$ and $-1u + 1v$. We see therefore that the difference between the maximum and minimum u coefficient is 2, so the height of the Laurent polynomial above the $(1, 1)$ direction is 2. Since $(1, 1)$ is not parallel to any face of $C(f)$, the system is expansive in this direction. By Theorem 2.2, there is a strip of height 2 in the $(1, 1)$ direction such that a point $\xi \in \langle 1 + X + Y \rangle^{\perp}$ is uniquely determined by $\xi|_S$. This is illustrated below.

$$\begin{array}{ccccccccccc}
 & & & & & & & & & . & . \\
 & & & & & & & & 0 & . & \\
 & & & & & & & 0 & 0 & & \\
 & & & & & & 1 & 1 & \dagger & & \\
 & & & & * & 1 & 1 & & & & \\
 & & & & 0 & 0 & & & & & \\
 & & & 0 & 1 & & & & & & \\
 & & 1 & 1 & & & & & & & \\
 & . & 0 & & & & & & & & \\
 . & . & & & & & & & & &
\end{array}$$

It is clear that the symbol marked with a $*$ is determined to be a 0 by the requirement that it along with the two 0s on the row below must add to 0 modulo 2. Similarly, the whole diagonal above the given strip is determined. This procedure may then be iterated to fill in the NW corner of the plane. Likewise, the symbol marked † is determined by the requirement that it should sum with the 0 and the 1 in the column immediately to the left to 0 modulo 2, so the symbol marked † has to be replaced by a 1. This process allows the entire point to be determined. Since each translate of the original shape is used to determine the value of some symbol, it follows that the resulting point is in $\langle 1+X+Y\rangle^{\perp}$. Conversely, given $\xi \in \langle 1+X+Y\rangle^{\perp}$, then the restriction $\xi|_S$ determines a unique point in $\langle 1+X+Y\rangle^{\perp}$. This point then has to be equal to ξ.

We next prove some facts about ideals in $\mathcal{L}$ and their duals. We begin by defining $\langle f_1, f_2, \ldots, f_n\rangle$ to be the ideal in $\mathcal{L}$ generated by $f_1, f_2, \ldots, f_n$. For any ideal I, $I^{\perp}$ is defined to be $\{\xi \in \mathbb{Z}_p{}^{\mathbb{Z}^2} \colon f\xi = 0 \text{ for each } f \in I\}$. We note that

$$\langle f_1, f_2, \ldots, f_n\rangle^{\perp} = \langle f_1\rangle^{\perp} \cap \langle f_2\rangle^{\perp} \cap \ldots \cap \langle f_n\rangle^{\perp}.$$

To see this, note that if $\xi \in \langle f_1, f_2, \ldots, f_n\rangle^{\perp}$, then $f_i\xi = 0$ for each i so $\xi \in \bigcap_i \langle f_i\rangle^{\perp}$. Conversely, if $\xi \in \bigcap_i \langle f_i\rangle^{\perp}$, then ξ is annihilated by anything in $\langle f_1, f_2, \ldots, f_n\rangle$ so we see $\xi \in \langle f_1, f_2, \ldots, f_n\rangle^{\perp}$.

Since $\mathcal{L}$ is a unique factorization domain, the highest common factor of two polynomials f and g, denoted $\mathrm{hcf}(f,g)$, exists.

Theorem 2.3. *Suppose $f_1, \ldots, f_n$ are Laurent polynomials in $\mathcal{L}$ and $\mathrm{hcf}(f_1, \ldots, f_n) = 1$. Then $\langle f_1, f_2, \ldots, f_n\rangle^{\perp}$ is a finite set.*

Note that this is a version of Lemma 2.5 in [7].

Proof. First, note that $\langle f_1, f_2, \ldots, f_n\rangle = \langle Y^k f_1, Y^k f_2, \ldots, Y^k f_n\rangle$ as Y is a unit in $\mathcal{L}$. We may therefore assume that $f_1, \ldots, f_n$ contain only positive powers of Y. Similarly, we assume that they contain only positive powers of X. We regard $f_1, \ldots, f_n$ as members of $K(X)[Y]$, the set of polynomials in Y whose coefficients are rational polynomials in X. Since $K(X)$ is a field, and $f_1, \ldots, f_n$ are coprime, there are members $a_1, \ldots, a_n$ of $K(X)[Y]$ such that $a_1 f_1 + \ldots + a_n f_n = 1$. Multiplying this equation through by the least common multiple of all the denominators of the a_i (which consist only of polynomials in X), we conclude that there exist polynomials $b_1, \ldots, b_n$ in $K[X,Y]$ such that $b_1 f_1 + \ldots + b_n f_n = g$ where g is a polynomial in X alone. We conclude that $g \in \langle f_1, f_2, \ldots, f_n\rangle$. Now, given $\xi \in \langle f_1, f_2, \ldots, f_n\rangle^{\perp}$, we have $g\xi = 0$. Since g is a polynomial in X, it follows that the entries of ξ satisfy a recurrence relation in each horizontal strip. Since K is finite, there is a number n such that all solutions to the recurrence relation are periodic with period a factor of n. In particular, $X^n\xi = \xi$. Similarly, there exists a number m such that $Y^m\xi = \xi$. Clearly, there are only finitely many ξ with this property (at most $|K|^{mn}$). □

Theorem 2.4. *If $f_1, \ldots, f_n \in \mathcal{L}$ are such that $\mathrm{hcf}(f_i, f_j) = 1$, for each $i < j$, then letting $f = f_1 f_2 \ldots f_n$, $\langle f\rangle^{\perp} = \langle f_1\rangle^{\perp} + \langle f_2\rangle^{\perp} + \ldots + \langle f_n\rangle^{\perp}$.*

The proof which we give of this consists partly of a proof taken from [7], Lemma 2.4.

Proof. It is clearly sufficient to establish the claim in the case $n = 2$. Consider the map $\theta\colon \langle f_1\rangle^\perp \to \langle f_1\rangle^\perp$ given by $\theta(\xi) = f_2\xi$. This may be seen to be finite-to-one as the kernel of θ is $\langle f_1\rangle^\perp \cap \langle f_2\rangle^\perp$, which by Theorem 2.3 is a finite set. By Theorem 2.2, θ may be considered to be a finite-to-one map from a one-dimensional full shift to itself. By a theorem of Hedlund [4] (Theorem 5.13), such a map is surjective. It follows then that θ is surjective as a map $\langle f_1\rangle^\perp \to \langle f_1\rangle^\perp$. Now, given $\xi \in \langle f_1 f_2\rangle^\perp$, we have $f_2\xi \in \langle f_1\rangle^\perp$. Let $\eta \in \langle f_1\rangle^\perp$ be such that $\theta(\eta) = f_2\eta = f_2\xi$. Now, $\xi = \eta + (\xi - \eta)$ and we note that $\xi - \eta \in \langle f_2\rangle^\perp$ because $f_2(\xi - \eta) = f_2\xi - f_2\eta = 0$. This shows that $\xi \in \langle f_1\rangle^\perp + \langle f_2\rangle^\perp$ as required. □

If A and B are two subgroups of $K^{\mathbb{Z}^2}$, then we will say that B is a finite extension of A if $A \subset B$ and the index of A in B, $B\colon A$ is finite.

Theorem 2.5. *If $f_1, \ldots, f_n$ be Laurent polynomials and $d = \operatorname{hcf}(f_1, \ldots, f_n)$ then $\langle f_1, \ldots, f_n\rangle^\perp$ is a finite extension of $\langle d\rangle^\perp$.*

This is a version of Lemma 2.6 of [7]. We remark that if $I^\perp$ is a finite extension of $\langle d\rangle^\perp$, then the expanding directions for $I^\perp$ are the same as those for $\langle d\rangle^\perp$.

Proof. We observe that since d divides each f_i, we have $\langle d\rangle^\perp$ is contained in $\langle f_1, \ldots, f_n\rangle^\perp$. We see that the image under d of $\langle f_1, \ldots, f_n\rangle^\perp$ is a subset of $\langle f_1/d, \ldots, f_n/d\rangle^\perp$. Clearly the kernel of the map d is $\langle d\rangle^\perp$, so by the fundamental theorem of group homomorphisms, $\langle f_1, \ldots, f_n\rangle^\perp/\langle d\rangle^\perp$ is isomorphic to a subgroup of $\langle f_1/d, \ldots, f_n/d\rangle^\perp$. Since $d = \operatorname{hcf}(f_1, \ldots, f_n)$, $\operatorname{hcf}(f_1/d, \ldots, f_n/d) = 1$, so by Theorem 2.3, $\langle f_1/d, \ldots, f_n/d\rangle^\perp$ is finite. This proves the theorem. □

Theorem 2.6. *If M is a K-group shift, then there exists a finite set of Laurent polynomials $f_1, \ldots, f_n \in \mathcal{L}$ such that $M = \langle f_1, \ldots, f_n\rangle^\perp$.*

This is a version of Theorem 2.1 of [7].

We note that in the above situation, the set of expansive directions for M is equal to the set of expansive directions for $\langle d\rangle^\perp$, where $d = \operatorname{hcf}(f_1, \ldots, f_n)$ and so the set of non-expansive directions is finite. It also follows from this that M is a subshift of finite type, as the condition that a point ξ lies in $\langle f_1, \ldots\rangle^\perp$ is checkable by looking at a finite region.

Proof. Let G_n be the set of all configurations on Λ_n of points in M. Since M is a proper subshift of $K^{\mathbb{Z}^2}$, there is an n_0 such that Λ_{n_0} is a proper subset of $K^{\Lambda_{n_0}}$. By the K-linearity properties of M, G_n is a vector subspace over K of K^{Λ_n}. Now let ${G_n}^\perp$ be the annihilator of G_n in K^{Λ_n}, ${G_n}^\perp = \{\theta \in \mathcal{L}(K^{\Lambda_n}, K)\colon \theta(\xi) = 0 \text{ for all } \xi \in G_n\}$. By well-known properties of finite-dimensional vector spaces, we have ${G_n}^{\perp\perp} = G_n$, where $G_n^{\perp\perp} = \{\xi \in K^{\Lambda_n} : \theta(\xi) = 0 \text{ for all } \theta \in {G_n}^\perp\}$. Each linear map in ${G_n}^\perp$ corresponds to a Laurent polynomial in $\mathcal{L}$ and let I_n be the ideal generated by these Laurent polynomials. Since ${G_n}^\perp$ may be embedded as a subspace of ${G_{n+1}}^\perp$, we see that $I_n \subset I_{n+1}$. Also, we see that $M \subset {I_n}^\perp$ for each n. Next, let $d = \operatorname{hcf}(\{q\colon q \in \bigcup I_n\})$. To see that d exists, note that there is a non-trivial $q \in I_{n_0}$ so that there are finitely many possibilities for d: the factors of q. Clearly $\mathrm{hcf}(I_n)$ is a decreasing sequence of Laurent polynomials. Now there exists an n_1 such that $\mathrm{hcf}(I_{n_1}) = d$.

It follows by Theorem 2.5 that ${I_{n_1}}^{\perp}$ is a finite extension of $\langle d\rangle^{\perp}$. But now, since the ${I_n}^{\perp}$ form a decreasing sequence, ${I_n}^{\perp}\colon \langle d\rangle^{\perp}$ is eventually constant. It then follows that ${I_n}^{\perp}$ is eventually constant: ${I_n}^{\perp} = {I_{n_2}}^{\perp}$ for all $n \geq n_2$. Let $I^{\perp} = {I_{n_2}}^{\perp}$. We already know that $M \subset I^{\perp}$. Conversely, given $\xi \in I^{\perp}$, we have $\xi \in {I_n}^{\perp}$ for each n and so $\theta(\xi|_{\Lambda_n}) = 0$ for each $\theta \in {G_n}^{\perp}$. So since ${G_n}^{\perp\perp} = G_n$, we see that $\xi|_{\Lambda_n} \in G_n$. Thus we see ξ is the limit of points in M, so since M is closed, we have $\xi \in M$.

□

We remark that the above establishes the one-to-one correspondence between K-group shifts and ideals of $K[X^{\pm 1}, Y^{\pm 1}]$ as demonstrated in the case of $\mathbb{Z}_p$-group shifts in [7] and explained further in [14].

3. Results

In this section, the main results of the paper are stated. The proofs are then deferred until Section 5.

Definition 8. *A map θ between a K-group shift and a K'-group shift will be called* affine *if it can be written $\theta(\xi)_{i,j} = L(\xi)_{i,j} + c$, where L is a continuous shift-commuting group homomorphism and $c \in K'$.*

We note that in what follows all group homomorphisms of group shifts will be continuous and shift-commuting.

It is emphasized that the definition of affine differs to some extent from the usual definition (where the map is required to be linear plus a constant), as the homomorphism is not required to be K-linear, but merely additive. As mentioned in the previous section however, if $K = \mathbb{Z}_p$, then any group homomorphism from a $K^{\mathbb{Z}^2}$ to $K^{\mathbb{Z}^2}$ is K-linear: a code $L : \mathbb{Z}_p^{\mathbb{Z}^2} \to \mathbb{Z}_p^{\mathbb{Z}^2}$ is a homomorphism if and only if it can be written as $L(\xi) = r\xi$ for some Laurent polynomial $r \in \mathcal{L}$. We observe also that any group homomorphism from one $\mathbb{Z}_p$-group shift to another may be extended (non-uniquely) to give a group homomorphism from ${\mathbb{Z}_p}^{\mathbb{Z}^2}$ to ${\mathbb{Z}_p}^{\mathbb{Z}^2}$. Since this is $\mathbb{Z}_p$-linear, it follows that the original homomorphism must also have been $\mathbb{Z}_p$-linear. In the following lemma, we identify the homomorphisms of $K^{\mathbb{Z}^2}$ where K is a finite field of characteristic p. As before, it follows by extending the homomorphisms to maps from $K^{\mathbb{Z}^2}$ to $K^{\mathbb{Z}^2}$ that all group homomorphisms from one K-group shift to another have the same structure of being linear combinations of Frobenius automorphisms.

Lemma 3.1. *Let K be a finite field of characteristic p. A shift-commuting map θ from $K^{\mathbb{Z}^2}$ to $K^{\mathbb{Z}^2}$ is a shift-commuting group homomorphism if and only if it is a linear combination of maps of the form $\tau^n \circ \sigma_{(i,j)}$, where τ is the Frobenius map defined by $\tau\colon K^{\mathbb{Z}^2} \to K^{\mathbb{Z}^2}$, $\tau(\xi)_{i,j} = {\xi_{i,j}}^p$.*

We are now in a position to state the main theorems.

Theorem 3.2. *Suppose $f \in \mathcal{L}_0(K)$. Then any code θ from $\langle f\rangle_K^{\perp}$ into any K'-group shift is affine.*

We note that the restrictions on the range can be relaxed without altering the proof. What is necessary is that the range is a compact zero-dimensional Abelian group with a rational expansive direction in common with $\langle f\rangle_K^{\perp}$. In the case of a K'-group shift, this is automatic as by Theorems 2.6,2.5 and 2.1, all but finitely many directions are expansive.

The assumption that the domain is the annihilator of a principal ideal is crucial here. If the domain is the annihilator of a non-principal ideal, then this is a finite extension of a principal ideal and the code can be defined independently on each component showing that the conclusion fails.

Theorem 3.3. *Suppose $f \in \mathcal{L}_0(K')$, $g \in \mathcal{L}(K)$ and let θ be a code from $\langle g\rangle_K^{\perp}$ into $\langle f\rangle_{K'}^{\perp}$. Then θ is affine.*

As in the previous theorem, the assumption that the domain is the annihilator of a principal ideal is crucial (for the same reason). One could however relax the hypotheses on the range to require instead that the range is a finite extension of the annihilator of an ideal $\langle f\rangle$ for some $f \in \mathcal{L}_0(K')$.

The assumption that $f \in \mathcal{L}_0$ is equivalent to $\langle f\rangle^{\perp}$ being mixing, by simple generalizations of [7], Proposition 2.11, and [6]. We are also able to say something about the case of Laurent polynomials with a factor which is collinear.

Theorem 3.4. *Suppose $f \in \mathcal{L}(K)$. Write $f = f_0 g$ where $f_0 \in \mathcal{L}_0(K)$ and g consists of factors $g_1 \dots g_n$ each of which is collinear such that $C(g_i)$ and $C(g_j)$ are non-parallel for $i \neq j$. Then if M is any K'-group shift and θ is a code from $\langle f\rangle^{\perp}$ into M satisfying $\theta(0) = 0$, then*

$$\theta(\xi^1 + \xi^2 + \eta^1 + \dots + \eta^n) = \theta(\xi^1) + \theta(\xi^2) + \theta(\eta^1) + \dots + \theta(\eta^n),$$

for any ξ^1 and ξ^2 in $\langle f_0\rangle^{\perp}$ and η^i in $\langle g_i\rangle^{\perp}$.

Theorem 3.5. *Suppose $f \in \mathcal{L} \setminus \mathcal{L}_0$. Then there is a code $\theta\colon \langle f\rangle^{\perp} \to \langle f\rangle^{\perp}$ which is not affine.*

Corollary 3.6. *Suppose that θ is a shift-commuting map between a K-group shift M and a K'-group shift M', where K and K' have different characteristics. If $M = \langle f\rangle_K^{\perp}$ where $f \in \mathcal{L}_0(K)$ or $M' = \langle f'\rangle_{K'}^{\perp}$ where $f' \in \mathcal{L}_0(K')$ then θ is a constant mapping.*

Corollary 3.7. *Suppose $f \in \mathcal{L}_0(\mathbb{Z}_p)$. If $\theta\colon \langle f\rangle_{\mathbb{Z}_p}^{\perp} \to M$ is a code, M a $\mathbb{Z}_p$-group shift, then the image of θ is a coset of a subgroup of $\langle f\rangle^{\perp}$. If $\langle f\rangle^{\perp}$ is topologically conjugate to $\langle g\rangle^{\perp}$, then $\langle f\rangle^{\perp} = \langle g\rangle^{\perp}$ and f and g are equal up to a multiplicative unit.*

This corollary certainly does not hold for more general fields K because if γ is any $\mathbb{Z}_p$-homomorphism of K, then applying γ pointwise to points of $\langle f\rangle^{\perp}$ gives a continuous shift commuting homomorphism whose image is $\langle\gamma(f)\rangle^\perp$. However if f has coefficients in $K\setminus\mathbb{Z}_p$, then typically $\langle f\rangle^\perp \neq \langle\gamma(f)\rangle^\perp$.

As a consequence of Theorem 3.2, we can generalize a theorem of Hedlund, Coven, Rhodes ([2], Theorem 3.5), which was proven for the case $p=2$.

Theorem 3.8. *Suppose that $f:\mathbb{Z}_p^{\mathbb{Z}}\to\mathbb{Z}_p^{\mathbb{Z}}$ is an affine code which is not a conjugacy. If $\theta:\mathbb{Z}_p^{\mathbb{Z}}\to\mathbb{Z}_p^{\mathbb{Z}}$ is any code which commutes with f, then θ is affine.*

Theorem 3.2 has implications for the automorphism group of $\langle f\rangle^\perp$. A (dynamical) *automorphism* is an invertible code from a subshift to itself. By Theorem 3.2, any automorphism of $\langle f\rangle^\perp$, $f\in\mathcal{L}_0$, is of the form L+c, where L is a group isomorphism and c is a constant point in $K^{\mathbb{Z}^2}$. Let $\mathrm{Aut}(\langle f\rangle^\perp)$ denote the group of automorphisms of $\langle f\rangle^\perp$. Let $\mathcal{I}$ denote the subgroup of $\mathrm{Aut}(\langle f\rangle^\perp)$ consisting of group isomorphisms and $\mathcal{T}$ the subgroup consisting of those pointwise rotations $x\mapsto x+c$ which are automorphisms of $\langle f\rangle^\perp$. Clearly, $L+c$ maps $\langle f\rangle^\perp$ to itself if and only if $f(c)=0$, and if there exists a non-zero c with this property, then it holds for every constant point. Therefore, if $f(1)\neq 0$ then $\mathcal{T}$ contains only the identity and $\mathrm{Aut}(\langle f\rangle^\perp)=\mathcal{I}$, and if $f(1)=0$ then $\mathcal{T}$ is isomorphic to K. It can be shown that $\mathrm{Aut}(\langle f\rangle^\perp)$ is a semi-direct product of $\mathcal{I}$ and $\mathcal{T}$ ([3], Sect. 6.5).

If $K=\mathbb{Z}_p$, then by the remarks in Section 2, any element of $\mathcal{I}$ is given by a polynomial, and therefore $\mathcal{I}$ is Abelian. It is easy to see that $L\in\mathcal{I}$ commutes with every element of $\mathcal{T}$ if and only if $L(1)=1$ or $\mathcal{T}$ consists just of the identity. For $K=\mathbb{Z}_2$, one of these conditions holds for every $L\in\mathcal{I}$.

Corollary 3.9. *The automorphism group of $\langle f\rangle^\perp$, $f\in\mathcal{L}_0$ is Abelian if either (i) $K=\mathbb{Z}_p$ and $f(1)\neq 0$ (in which case* $\mathrm{Aut}(\langle f\rangle^\perp)=\mathcal{I}$*), or (ii) $K=\mathbb{Z}_2$.*

By contrast, the automorphism group of a strongly irreducible shift of finite type contains any finite group as a subgroup (see [15], Theorem 2.3, and [11]). T. Ward has shown that the only automorphisms of the three-dot system are shift maps, and so $\mathrm{Aut}(\langle f\rangle^\perp)\cong\mathbb{Z}^2$ ([15], Theorem 3.2). Before we prove the main theorem, we need to state and prove some auxiliary results.

Proof of Lemma 3.1. Clearly if the map θ is a linear combination of powers of maps of the form $\tau^n\circ\sigma_{(i,j)}$, then θ is a group homomorphism. Conversely, since θ is a block map, we see that $\theta(\xi)_{0,0}$ is determined by ξ_{Λ_n} for some n. Regarding K as a vector space over $\mathbb{Z}_p$, let a basis be given by $a_1,\ldots a_d$ where $d=[K:\mathbb{Z}_p]$. Letting $e^{(i,j)}$ be the point in $K^{\mathbb{Z}^2}$ with a 1 at position (i,j) and 0s elsewhere, we see that $\theta(\xi)_{0,0}=\sum_{(i,j)\in\Lambda_n}\theta(\xi_{i,j}e^{(i,j)})$. Since θ is a group homomorphism, it follows that the map $\theta^{(i,j)}K\to K$ given by $\theta^{(i,j)}(a)=\theta(a\cdot e^{(i,j)})_{0,0}$ is a group homomorphism. The result will then follow if we establish that any group endomorphism of K is a linear combination of powers of the Frobenius map $\tau(a)=a^p$. Regarding τ as a $\mathbb{Z}_p$-linear map of the vector space K (over $\mathbb{Z}_p$), we see that $\tau^d=1$. Taking K-linear combinations of the τ^n for $0\leq n<d$, one can show the powers are independent so that there are $|K|^d$ such combinations, all giving rise to distinct $\mathbb{Z}_p$-linear homomorphisms from K to K. However, a

homomorphism from K to K is determined by the images of $a_1, \dots, a_d$. It follows that there are $|K|^d$ such homomorphisms and the proof is complete. □

4. Mixing in $\langle f \rangle^{\perp}$

We need to show some mixing results in spaces of the form $\langle f^k \rangle^{\perp}$, for an irreducible Laurent polynomial $f \in \mathcal{L}_0$ and $k > 0$. We will assume throughout this section that $(1,0)$ is an expansive direction for f. This condition will always hold in the cases where the theorems are to be applied by a simplification made at the start of the proof of the main theorem.

By the remarks following Definition 7, $C(f^k) = kC(f)$, so that the sides of $C(f^k)$ are parallel to those of $C(f)$. Therefore $(1,0)$ is an expansive direction for $\langle f^k \rangle^{\perp}$. Setting $h = \mathrm{ht}_{(1,0)}(f)$, we have $\mathrm{ht}(f^k) = kh$. (We will omit the subscript for direction in $\mathrm{ht}(f^k)$ in this section, as it is assumed to be $(1,0)$.) We will use the notation $S(f^k)$ to represent the strip $S_{\mathrm{ht}(f^k)}$ associated to $\langle f^k \rangle^{\perp}$ by Theorem 2.2. Let $S(f^k)^+$ denote the points $(i,j) \in S(f^k)$ for which $i \geq 0$.

Since $(1,0)$ is an expansive direction for $\langle f \rangle^{\perp}$, P_f has a unique highest point t_f and a unique lowest point b_f. By multiplying f by some X^iY^j, which translates P_f by (i,j), we can assume that b_f lies on the x-axis, t_f lies on the line $y = h$ (just above $S(f)$), and that the leftmost point (or points) of P_f lies on the y axis. Let L_t be the line in $\mathbb{Z}^2$ containing the side of $C(f)$ incident with t_f that is the leftmost of the two and let e_f be the other extreme point of this side. Then we have that f^k also has a unique highest point t_{f^k} given by kt_f, kL_t is the line containing the side of $C(f^k)$ adjacent to t_{f^k}, and ke_f is the other extreme point of this side.

The *(upper) shadow* of $S(f^k)^+$ is the set of points in $S(f^k)^+$, together with all points $(i,j) \in \mathbb{Z}^2$, $j \geq kh$ which are on or to the right of L_t.

Let L_t' be the line in $\mathbb{Z}^2$ parallel to L_t which passes immediately to the left of L_t.

We consider the collection $\mathcal{C}_R(f^k)$ of points in $\langle f^k \rangle^{\perp}$ for which $\xi_{i,j} = 0$ for all $(i,j) \in S(f^k)^+$.

Lemma 4.1. *Let $k > d \geq 0$. If $\xi \in \mathcal{C}_R(f^k)$, then $\xi_{i,j} = 0$ for any (i,j) in the shadow of $S(f^k)^+$. For any $q \in L_t'$, there exists a point $\xi \in \mathcal{C}_R(f^k)$ such that $(f^d\xi)_{q-dt_f} \neq 0$.*

Note that in the case $k = 1$, for which $\langle f^0 \rangle^{\perp} = \{0\}$, the second statement is simply that for each q on L_t', there is a point $\xi \in \mathcal{C}_R(f)$ such that $\xi_q \neq 0$.

Proof. By translating $C(f^k)$ to the right, along the line $y = kh$, and using the fact that $\xi_{i,j} = 0$ for all $(i,j) \in S(f^k)^+$, we see that $\xi_{i,kh} = 0$ for all points in the shadow belonging to the horizontal line $y = kh$. This in turn forces $\xi_{i,kh+1} = 0$ for points on $y = kh+1$ in the shadow. Continuing this way, we see that $x_{i,j} = 0$ for all (i,j) in the shadow.

Let v denote the vector from t_f to the closest integral point of the line L_t below t_f. Then we can write $e_f = t_f + mv$ for some $m \geq 1$. Let L be a line in

$\mathbb{Z}^2$ parallel to L_t. We then show that given an $\langle f^k \rangle^\perp$-configuration ξ to the right of L (exclusive), ξ may be extended in $|K|^{km}$ ways to give a $\langle f^k \rangle^\perp$-configuration to the right of L (inclusive). In order to be an extension to the larger region, the point must satisfy for each $u \in L$ a recurrence relation of the form

$$a_0 \xi_u + a_1 \xi_{u+v} + \ldots + a_{km} \xi_{u+kmv} = \phi_u(\xi),$$

where $\phi_u(\xi)$ is a linear combination of the terms of ξ to the right of L and a_0 and a_{km} are non-zero. In particular, given a point q on L, $\xi_q, \xi_{q+v}, \ldots, \xi_{q+(km-1)v}$ may be assigned arbitrarily. The recurrence relation may then be used to determine the values of ξ on the remainder of L.

We now consider extending an $\langle f^k \rangle^\perp$-configuration consisting entirely of zeros to the right of the line L'_t. A similar analysis to that above shows that an extension belongs to $\langle f^d \rangle^\perp$ if it satisfies a recurrence relation with $dm+1$ terms. In particular the configuration may be extended in $|K|^{dm}$ ways to give an $\langle f^d \rangle^\perp$-configuration.

We deduce that by setting $\xi_q = 1, \xi_{q+v} = \xi_{q+2v} = \ldots = \xi_{q+(km-1)v} = 0$, we can get an extension of ξ to the region to the right of L'_t inclusive which satisfies $(f^d \xi)_{q-dt_f} \neq 0$. This configuration may then inductively be extended as described above to give a point $\xi \in \mathcal{C}_R(f^k)$ satisfying the conditions of the lemma. □

The following definition is given in [7] (Definition 1.5).

Definition 9. *Two sets $E, F \subseteq \mathbb{Z}^2$ are* independent *in $\langle f \rangle^\perp$ if for any $\langle f \rangle^\perp$-configuration η_F on F, there is a point $\xi \in \langle f \rangle^\perp$ such that $\xi_F = \eta_F$ and ξ_E is the configuration of all zeros.*

By additivity, if two sets are independent, then any $\langle f \rangle^\perp$-configurations on E and F can occur in a single point. If $\langle f \rangle^\perp$ is mixing, which means that f has no collinear factors, then any two finite sets sufficiently far apart in $\mathbb{Z}^2$ are independent. However, if E or F are infinite, then it is more difficult to determine whether they are independent.

Definition 10. *A direction $w = (m, n)$ will be called an* internal direction *for f if for sufficiently small positive s, $t_f + sw$ lies in the interior of $C(f)$.*

Given an internal direction w, the line through the origin parallel to w will be called L_w. By a *box*, we mean a rectangular subset of $\mathbb{Z}^2$ with sides parallel to the coordinate axes.

Theorem 4.2. *Let f be an irreducible Laurent polynomial in $\mathcal{L}_0$ which is expansive in the $(1,0)$ direction and let $k > 0$. For any $a > 0$, and internal direction w, there is an $l > 0$ such that for any box Λ of size $a \times k \cdot \mathrm{ht}(f)$, positioned above $S(f^k)$ at least l to the left of the line through 0 parallel to w, Λ is independent in $\langle f^k \rangle^\perp$ of $S(f^k)^+$.*

The content of the theorem is illustrated in Figure 1. Note that since the height of Λ is equal to $\mathrm{ht}(f^k)$, the height of $S(f^k)$, any configuration on Λ is an

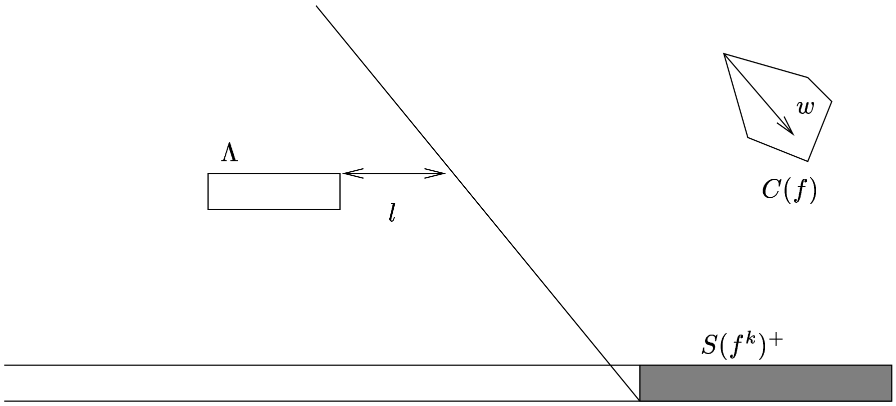

FIGURE 1. Λ is independent of $S(f^k)^+$

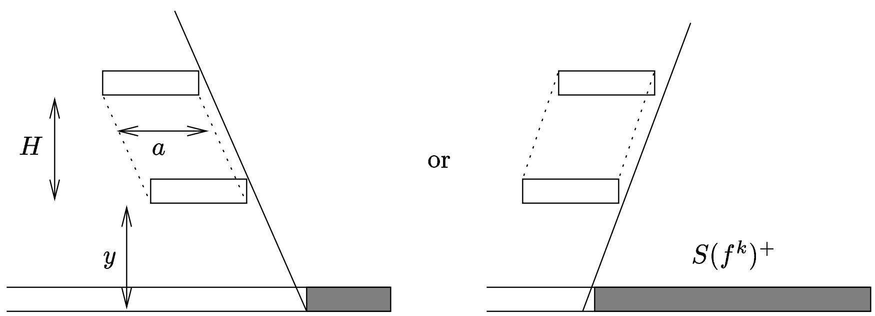

FIGURE 2. H boxes of size $a \times kh$

$\langle f \rangle^{\perp}$-configuration. We denote the set of all configurations on the $a \times kh$ box $([0, a-1] \times [0, kh-1]) \cap \mathbb{Z}^2$ by $K^{a\times kh}$. The reader may find it helpful when reading the proof to consider the case $k = 1$, which is somewhat simpler than the general case.

Proof of Theorem 4.2. Let $kh = \mathrm{ht}(f^k)$ and let y be an arbitrary positive integer. Consider a tower (parallel to w) of H overlapping boxes of size $a \times kh$ in successive rows positioned adjacent to L_w at heights $y, y+1, \ldots, y+H-1$ as shown in Figure 2. We denote these boxes by $\Lambda_y, \ldots \Lambda_{y+H-1}$. Assume that the lower left corner of Λ_j has coordinates $(x(j), j)$. Let V_j denote the possible configurations on the jth box which occur for points in $\mathcal{C}_R(f^k)$; that is $V_j = \{\xi_{\Lambda_j} : \xi \in \mathcal{C}_R(f^k)\}$, for $y \leq j \leq y+H-1$. Each set V_j is clearly closed under addition and under scalar multiplication by elements of K so each V_j forms a vector space over K. Since the boxes are of the same size, there is a natural notion of adding configurations on two of them (i.e. we treat all of them as subsets of a single copy of $K^{a\times kh}$).

We will show that for large enough H (independent of y), the vector space sum $V_y + V_{y+1} + \ldots + V_{y+H-1}$ consists of the set of all configurations in $K^{a\times kh}$. The result will then follow fairly easily.

To show this, we suppose for a contradiction that $V_y+V_{y+1}+\ldots+V_{y+H-1}$ is not all of $K^{a\times kh}$. Then as in Theorem 2.6, there exists a non-trivial K-linear map on $K^{a\times kh}$ which vanishes on $V_y + V_{y+1} + \ldots + V_{y+H-1}$. This linear map corresponds to a Laurent polynomial $g \in \mathcal{L}$, with vanishing coefficients for monomials X^iY^j with (i,j) outside $[0,a-1]\times[0,kh-1]$. We will show that the existence of g implies that for some $d < k$, there is a right infinite strip of height slightly less than H on which $f^d\xi$ vanishes for all $\xi \in \mathcal{C}_R(f^k)$. (When $k = 1$, ξ itself vanishes). For large enough H, this will contradict Lemma 4.1.

Recall from Section 2 that we can regard g as a map $\langle f^k\rangle^\perp \to \langle f^k\rangle^\perp$, where $g(\xi)_{i,j}$ is determined by the values of $\xi_{i+m,j+n}$ for $0 \le m < a$ and $0 \le n < kh$. Then given $\xi \in \mathcal{C}_R(f^k)$, we have $(g\xi)_{(x(j),j)} = 0$ for $y \le j < y+H$, corresponding to the lower left corners of the boxes Λ_j . Note however that it follows from the definition of $\mathcal{C}_R(f^k)$ that $\sigma_{m,0}(\mathcal{C}_R(f^k)) \subseteq \mathcal{C}_R(f^k)$ for $m \ge 0$. Therefore $\xi_{\Lambda_j+m} \in V_j$ for $m \ge 0$, and so $(g\sigma_{m,0}\xi)_{x(j),j} = 0$ for $y \le j < y + H$. However $(g\sigma_{m,0}\xi)_{x(j),j} = (g\xi)_{x(j)+m,j}$ so we see that $(g\xi)_{i,j} = 0$ on the set R_1 defined by

$$R_1 = \{(i,j)\colon y \le j < y+H,\, i \ge x(j)\}$$

(see Figure 3). Also, $(f^k\xi)_{i,j} = 0$ (for all i,j), since ξ is by assumption in $\langle f^k\rangle^\perp$.

Since f is irreducible, we have $\mathrm{hcf}(g, f^k) = f^d$ for some $d \le k$. Since $\mathrm{ht}(g) \le kh-1$, f^k cannot divide g, and so $d < k$. Write $g = f^dg'$ where $\mathrm{hcf}(g', f) = 1$, and hence $\mathrm{hcf}(g', f^{k-d}) = 1$. From the proof of Theorem 2.3, we see that there exist a and b in $\mathcal{L}$ with the property that $af^{k-d}+bg'$ is a polynomial p in positive powers of X alone. Since f and g are assumed to have only positive powers of X and Y, we see also that a and b may be chosen to have only positive powers of X and Y. Now $af^k + bg = pf^d$. Since $(g\xi)_{i,j} = 0$ for $(i,j) \in R_1$ and $\xi \in \mathcal{C}_R(f^k)$, it follows that $(bg\xi)_{i,j} = 0$ provided $(i,j) + P_b \subset R_1$. In particular, we deduce that there is a $\delta > 0$ (depending on the polynomial b and the slope of w) so that $(bg\xi)_{i,j} = 0$ for $(i,j) \in R_2$ where

$$R_2 = \{(i,j)\colon y \le j < y+H-\delta \text{ and } i \ge x(j)+\delta\}$$

(see Figure 3). Further as we need a bound independent of y, we note that there are only finitely many choices of the polynomial g and hence only finitely many different b and so only finitely many different δ. We will take δ to be the maximum of all of these. Note that the δ is independent of H so we may assume that H is chosen to be larger than δ so that R_2 is non-empty.

With this definition, we see that $(pf^d\xi)_{i,j} = 0$ for $\xi \in \mathcal{C}_R(f^k)$ and $(i,j) \in R_2$ as $pf^d = bg + af^k$.

Write $p = \sum_{k=m}^{M} c_kX^k$, where $m \ge 0$ is the smallest non-zero coefficient. Again in order to get a bound independent of y, we let m_0 the maximum such number taken over all the finite set of all possible polynomials g. Then since $(pf^d\xi)_{i,j} = 0$ for $(i,j) \in R_2$, we have $\sum_{k=m}^{M} c_k(f^d\xi)_{i+k,j} = 0$, so that the jth row of $f^d\xi$ satisfies

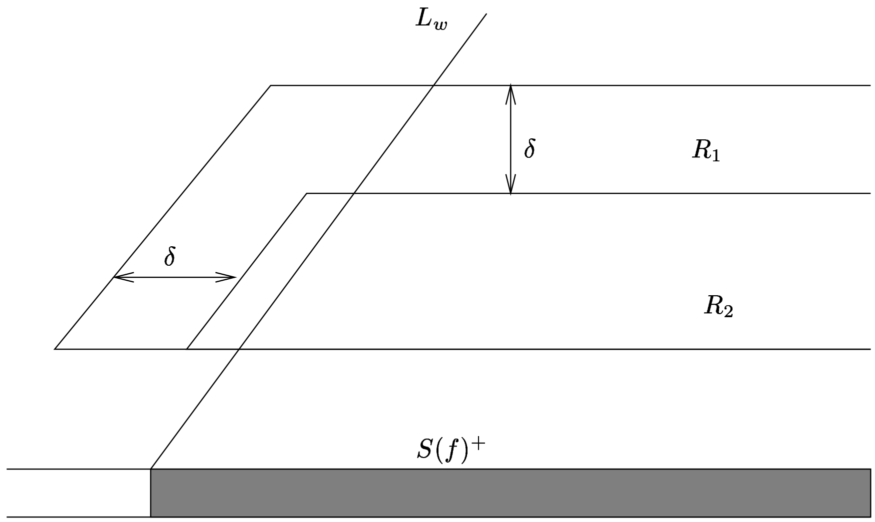

FIGURE 3. The region R_2 where $(af^k + bg)\xi$ vanishes

a recurrence relation from coordinates $(i + m, j)$ to $(i + M, j)$. It follows that $(f^d\xi)_{i+m,j}$ is determined by the values of $(f^d\xi)_{i+m+1,j}, \ldots, (f^d\xi)_{i+M,j}$, However as the line $y = j$ intersects the shadow of $S(f)^+$, it follows from Theorem 4.1 that $(f^d\xi)_{i,j} = 0$ for all sufficiently large i so using the recurrence relation inductively from right to left, we see that $(f^d\xi)_{i,j} = 0$ for $(i,j) \in R_2$ satisfying $i \geq x(j) + \delta + m_0$. This fact imposes an upper bound on the number H, as follows. Let y_0 be the height of the top row of R_2, $y_0 = y + H - \delta - 1$ and define the subset T of R_2 by $T = \{(i, y_0) \colon i \geq x(y_0) + \delta + m_0\}$.

This is the strip of height 1 at the top of R_2, to the right of $x(y_0) + \delta + m_0$ (see Figure 4). Let H be sufficiently large to ensure that $(x(y_0) + \delta + m_0, y_0) + kt_f$ lies to the left of the line L'_t described in Lemma 4.1. The larger the value of y, the smaller H must be to ensure this because L_w lies to the left of L_t. This means that we can choose an H which is sufficient for all positive y. Now, let q be the point on L'_t at height $y_0 + dh$. Then $q - dt_f$ lies in T. By Lemma 4.1, there exists a point $\xi \in \mathcal{C}_R(f^k)$ such that $(f^d\xi)_{q-dt_f} \neq 0$. This contradicts the fact that $(f^d\xi)_{i,j} = 0$ for $(i,j) \in T$. It follows that for sufficiently large H (chosen independently of y), the vector space sum $V_y + V_{y+1} + \ldots + V_{y+H-1}$ consists of the set of all configurations in $K^{a\times kh}$.

We now deduce the statement of the theorem as follows: Given a configuration $\eta \in K^{a\times kh}$, η may be expressed as $\eta^y + \ldots + \eta^{y+H-1}$, where $\eta^j \in V_j$. Now, let ξ^j be a configuration in $\mathcal{C}_R(f^k)$ having configuration η^j on Λ_j. Then the point $\sigma_{x(j)-x(y),j-y}\xi^j$ has the configuration η^j on Λ_y. Since $\xi^j \in \mathcal{C}_R(f^k)$, it follows from the expansiveness of the horizontal direction for $\langle f\rangle^\perp$ that ξ^j is equal to 0 on a cone containing the positive x axis. In particular, there is an l_0 depending

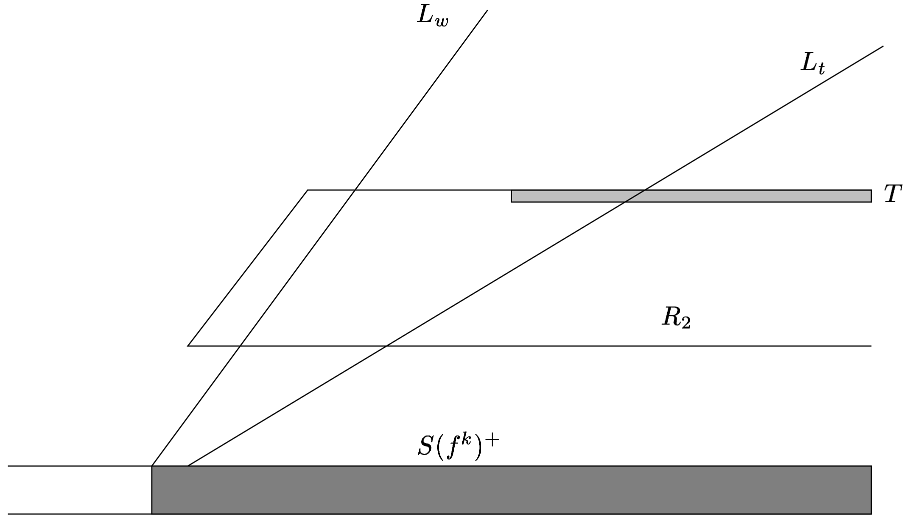

FIGURE 4. The subset T of R_2 which extends to the left of L_t

on $x(j)-x(y)$ and $j-y$ such that $\sigma_{x(j)-x(y),j-y}\xi^j$ is equal to 0 on the part of the strip with x coordinates to the right of l_0. Taking the maximum of these l_0 over the finite set of vectors $(x(j)-x(y), j-y)$, for $y \le j < y+H$, gives a number l such that for each j, $\sigma_{x(j)-x(y),j-y}\xi^j$ is equal to 0 on the part of the strip with x coordinates to the right of l. (Note that l too can be chosen independently of y as there are only a finite number of finite sets $(x(j)-x(y), j-y)_{y\le j<y+H}$ as y runs over the positive integers.) Now summing the $\sigma_{x(j)-x(y),j-y}\xi^j$ gives a configuration ξ which agrees with η on Λ_y and which is equal to 0 at all points of the strip with x coordinates to the right of l. In particular, forming $\sigma^l\xi$ gives a point of $\mathcal{C}_R(f^k)$ with the desired configuration on a box at height y, l to the left of the line L_w. □

5. Proof of the main results

In this section, we give a proof of the main results, Theorems 3.2 and 3.3. We first state and prove a key lemma.

Lemma 5.1. *Suppose M and M' are respectively K- and K'-group shifts which are expansive in the $(1,0)$ direction. Let θ be a code from M to M' satisfying $\theta(0)=0$. In this case a point in M is determined by its configuration on a horizontal strip S of some height h. There exists a length W such that for any i_0, if $\xi \in M$ is identically zero on $\{(i,j) \in S\colon i > i_0\}$ and $\eta \in M$ is identically zero on $\{(i,j) \in S\colon i < i_0+W\}$, then $\theta(\xi+\eta) = \theta(\xi)+\theta(\eta)$.*

Note that the condition that M' is a K'-group shift is more than what is needed here. In fact, the proofs all work under the condition that the range is a compact

Abelian zero-dimensional group with $(1,0)$ as an expansive direction. This also applies to the results which follow. This would give rise to the strengthening of Theorem 3.2 mentioned earlier.

Proof. Since M' is expansive in the $(1,0)$ direction, there is a strip S' of height h' such that a point in M' is uniquely determined by its configuration on S'. (Note that if M' is not given by $\langle f'\rangle^{\perp}$ for some polynomial f', it may not be the case that each configuration on S' gives rise to a point of M'). Let V_i be the part of S' with x coordinate i: $V_i = \{(i,j)\colon 0 \le j < h'\}$. Since θ is given by a block map, there is a box B such that the configuration of ξ on B determines $\theta(\xi)_{V_0}$. Since S is an expansive strip for M, the M-configuration of ξ on B is determined by the M-configuration of ξ on a finite subset Λ of S (which we assume to be a box). Let W be the width of this box and m be the leftmost x-coordinate of Λ (so the rightmost x-coordinate of Λ is $m+W-1$).

Now for any i_0, let ξ and η be as in the statement of the lemma. Then for any k, the configuration of $\theta(\xi+\eta)$ on V_k is determined by the configuration of $\xi+\eta$ on $\Lambda+(k,0)$. If $m+k \le i_0$, then $\Lambda+(k,0)$ is contained in $\{(i,j) \in S\colon i < i_0+W\}$ so that $(\xi+\eta)_{\Lambda+(k,0)} = \xi_{\Lambda+(k,0)}$. In this case, $\theta(\xi+\eta)_{V_k} = \theta(\xi)_{V_k}$ and $\theta(\eta)_{V_k} = 0$.

If $m+k > i_0$ then $\Lambda+(k,0)$ is contained in $\{(i,j) \in S\colon i > i_0\}$ so $(\xi+\eta)_{\Lambda+(k,0)} = \eta_{\Lambda+(k,0)}$. In this case, $\theta(\xi+\eta)_{V_k} = \theta(\eta)_{V_k}$ and $\theta(\xi)_{V_k} = 0$.

Putting these together, we see that for each k, $\theta(\xi+\eta)_{V_k} = \theta(\xi)_{V_k} + \theta(\eta)_{V_k}$. This implies that $\theta(\xi+\eta)_{S'} = \theta(\xi)_{S'} + \theta(\eta)_{S'}$. As M' is a group, $\theta(\xi)+\theta(\eta)$ belongs to M' and so we see that $\theta(\xi+\eta)$ and $\theta(\xi)+\theta(\eta)$ are two points of M' which agree on S'. But since points in M' are determined by their configurations on S', we see that $\theta(\xi+\eta) = \theta(\xi)+\theta(\eta)$ as required. □

Next, we will state and prove a special case of the main theorem. The general case will be deduced from this below.

Theorem 5.2. *Let f be an irreducible polynomial in $\mathcal{L}_0(K)$ and M be a K'-group shift such that $(1,0)$ is an expansive direction for $\langle f\rangle_K^{\perp}$ and M. Let $k \ge 1$. If θ is a code from $\langle f^k\rangle_K^{\perp}$ to M satisfying $\theta(0) = 0$, then θ is a group homomorphism.*

Proof. Let the box Λ be a subset of $S(f^k)$ as in the proof of Lemma 5.1. Pick two distinct internal directions w_L and w_R with w_R lying to the left of w_L in $C(f^k)$. Let L_L be the line parallel to w_L touching the right side of Λ and L_R be the line parallel to w_R touching the left side of Λ. Then by Theorem 4.2, there is a length l_L so that if a translate of Λ is at least l_L to the right of L_L then it is independent of the subset S_L of $S(f^k)$ consisting of Λ and everything to the left of Λ. Similarly, there is a length l_R so that if a translate of Λ is at least l_R to the left of L_R then it is independent of the subset S_R of $S(f^k)$ consisting of Λ and everything to the right. By the geometry of the lines L_L and L_R (see Figure 5), for all sufficiently large y, there is an x so that the translate $\Lambda' = \Lambda + (x,y)$ of Λ is simultaneously at least l_R to the left of L_R and at least l_L to the right of L_L. It follows that this translate is independent of both of the semi-infinite strips discussed above.

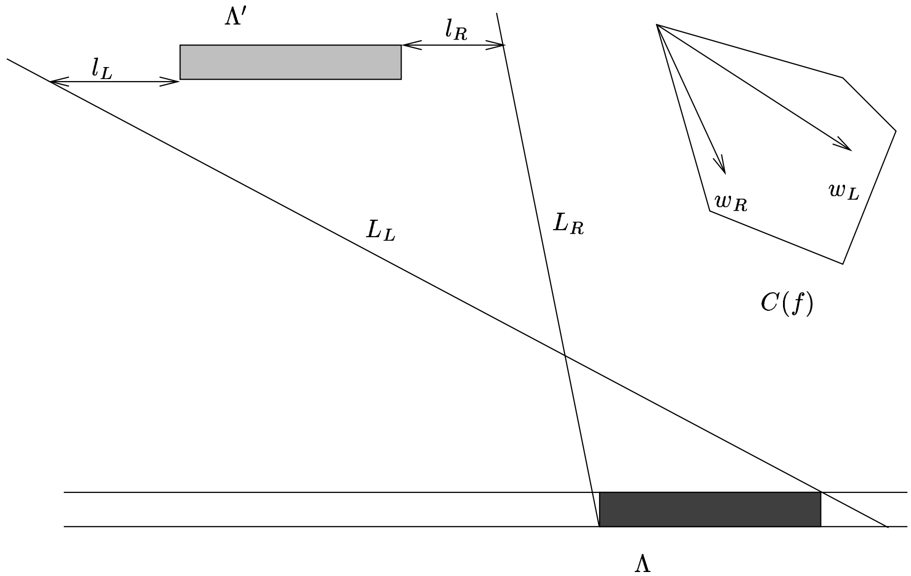

FIGURE 5. Λ' is independent of S_R and S_L.

By the discussion in Lemma 5.1, we know that the configuration of a point ξ in $\langle f^k\rangle^\perp$ on Λ' determines $\theta(\xi)_{x,y}$. Now, let $\bar{\theta}$ be the block map sending the configuration of ξ on Λ' to $\theta(\xi)_{x,y}$. We show that $\bar{\theta}$ is linear as follows: Let c_1 and c_2 be two configurations on Λ'. By independence, there are points ξ and η of $\langle f^k\rangle^\perp$ which restricted to Λ' are c_1 and c_2 respectively, but which are 0 on S_L and S_R respectively.

By Lemma 5.1, we see that $\theta(\xi+\eta) = \theta(\xi)+\theta(\eta)$, but this implies that $\bar{\theta}(c_1+c_2) = \bar{\theta}(c_1)+\bar{\theta}(c_2)$ so the block map defining θ is a group homomorphism. It then clearly follows that θ is a homomorphism as required. □

To prove the main theorem, we need a way of combining the above results on spaces of the form $\langle f^k\rangle^\perp$ for an irreducible $f \in \mathcal{L}$. Given a polynomial f, we can essentially uniquely (up to units) decompose f as a product $f_1^{k_1}\ldots f_n^{k_n} g_{n+1}\ldots g_{n+m}$ where the f_α are irreducible, pairwise coprime and belong to $\mathcal{L}_0$, and the g_β are collinear with distinct directions d_β. Note that in general, the g_β are not powers of a single irreducible polynomial, but rather the product of all the collinear polynomials in the irreducible decomposition of f in direction d_β. Since the terms in the decomposition are pairwise coprime, we observe by Theorem 2.4, that $\langle f\rangle^\perp = \langle f_1^{k_1}\rangle^\perp + \ldots + \langle f_n^{k_n}\rangle^\perp + \langle g_{n+1}\rangle^\perp + \ldots + \langle g_{n+m}\rangle^\perp$. We know that any code is linear when restricted to any of the non-collinear components. We then need to know how to combine the components. The essential idea is simple: given configurations $c^1,\ldots,c^{n+m}$ on a box above the strip, use independence to find points in the respective spaces extending the given configurations which have large stretches of zeros in $S(f)$. Then use Lemma 5.1 to arrive at the desired conclusion.

Theorem 5.3. *Suppose $f \in \mathcal{L}$ and $(1,0)$ is an expansive direction for $\langle f\rangle^{\perp}$. Write $f = f_1^{k_1} \dots f_n^{k_n} g_{n+1} \dots g_{n+m}$ as above. Suppose that θ is a code from $\langle f\rangle_K^{\perp}$ into a K'-group shift M satisfying $\theta(0) = 0$ for which $(1,0)$ is an expansive direction. Then if $\xi^\alpha \in \langle f_\alpha^{k_\alpha}\rangle^{\perp}$ for $1 \le \alpha \le n$ and $\xi^\beta \in \langle g_\beta\rangle^{\perp}$ for $n < \beta \le n+m$ then*

$$\theta(\xi^1 + \dots + \xi^{n+m}) = \theta(\xi^1) + \dots + \theta(\xi^{n+m}).$$

Proof. In this proof, we will use the index α when working with non-collinear polynomials and β when working with collinear polynomials. Let $g_\alpha = f_\alpha^{k_\alpha}$ for $1 \le \alpha \le n$. Let Λ be a box in $S(f)$ as in Lemma 5.1. For each factor g_α, let B_α be a box in $S(g_\alpha)$ such that the $\langle g_\alpha\rangle^{\perp}$-configuration of a point ξ on B_α determines the $\langle g_\alpha\rangle^{\perp}$-configuration on Λ.

As in the proof of Theorem 5.2, we choose directions w_R^α and w_L^α for each factor $\langle g_\alpha\rangle^{\perp}$ and we make the additional assumption that the ranges of directions are disjoint for the different α and do not include any of the directions d_β corresponding to the collinear directions.

Let y be any height greater than the maximum of the y_α (the vertical translations of the boxes found in the proof of Theorem 5.2 applied to the $\langle g_\alpha\rangle^{\perp}$). We will fix y later. Let the translate $B_\alpha + (x_\alpha, y_\alpha)$ of B_α have the property that the translate is independent of B_α and the part of the strip lying to the right; and is also independent of B_α and the part of the strip lying to the left. Letting $\Lambda_\alpha = B_\alpha - (x_\alpha, 0)$, we see by translating that $B_\alpha + (0,y)$ is independent of Λ_α and the part of the strip lying to the right; and is also independent of Λ_α and the part of the strip lying to the left.

Write $l_\alpha = -x_\alpha + (W-1)$ and $r_\alpha = -x_\alpha$, where W is as in Lemma 5.1. Then given any $\langle g_\alpha\rangle^{\perp}$-configuration c^α on $\Lambda + (0,y)$, there are points ξ^α and η^α extending c^α which are respectively zero on the strip $S(f)$ to the right of r_α and to the left of l_α.

For each factor g_β, $(n < \beta \le n+m)$ given any $\langle g_\beta\rangle^{\perp}$-configuration c^β on Λ, let τ be an extension of c^β to a point in $\langle g_\beta\rangle^{\perp}$. Then since g_β is collinear, modifying τ to be 0 off the diagonals parallel to d_β which intersect Λ gives a new point of $\langle g_\beta\rangle^{\perp}$, which for notational convenience we denote both by ξ^β and η^β. Clearly, there are l_β and r_β such that ξ^β is zero on $S(f)$ to the right of r_β and η^β is zero on $S(f)$ to the left of l_β just as for the non-collinear factors. The only difference is that here $r_\beta > l_\beta$ but $r_\beta - l_\beta$ is a positive number depending only on $\mathrm{ht}(f)$ and the slope of d_β.

Since the directions from the box Λ to the distinguished parts of $S(f)$ for each component (g_α or g_β) belong to disjoint intervals, as y becomes large, we see that the parts of the interval become arbitrarily far apart. In particular, for sufficiently large y, there is a transitive ordering on the subscripts $1, \dots, n+m$ where $a \prec b$ means $\max(r_a, l_a) \le \min(r_b, l_b) - W$.

This enables us to finish the proof as follows: We establish that if $\gamma_1 \prec \ldots \prec \gamma_k$ and c^{γ_s} is a $\langle g_{\gamma_s}\rangle^{\perp}$-configuration on $\Lambda + (0, y)$ for $s = 1 \ldots k$ then

$$\bar{\theta}(c^{\gamma_1} + c^{\gamma_2} + \ldots + c^{\gamma_k}) = \bar{\theta}(c^{\gamma_1}) + \bar{\theta}(c^{\gamma_2} + \ldots + c^{\gamma_k}), \tag{1}$$

where $\bar{\theta}$ is the block map defining θ. This will clearly imply by induction that $\bar{\theta}$ satisfies the linearity property in the statement of the theorem and hence that θ does too.

To establish (1), we let ξ^{γ_1} be a configuration extending c^{γ_1} which is 0 on the part of $S(f)$ to the right of r_{γ_1} and for $s = 2, \ldots, k$, we let η^{γ_s} be the configuration extending c^{γ_s} which is 0 to the left of l_{γ_s}. Since $r_{\gamma_1} \leq l_{\gamma_s} - W$ for each s, we see that ξ^{γ_1} and $\eta^{\gamma_2} + \ldots + \eta^{\gamma_k}$ satisfy the conditions of Lemma 5.1 and (1) is proved.

This completes the proof of the theorem. □

To simplify the proof of the main theorem, we will use a change of basis, by means of which we can assume that $(1, 0)$ is an expansive direction for $\langle f\rangle^{\perp}$ and all group subshifts contained in it. Let $\{v, u\}$ be an integral basis for $\mathbb{Z}^2$, and define a lattice isomorphism of $\mathbb{Z}^2$ by $L(x, y) = xu + yv$. Now define $\pi : K^{\mathbb{Z}^2} \to K^{\mathbb{Z}^2}$ by $(\pi\xi)_{i,j} = \xi_{L(i,j)}$. Clearly, π is a linear homeomorphism. While it is not shift-commuting, it is easy to check that $\pi\sigma_v = \sigma_{L^{-1}(v)}\pi$. Given this, we see that if θ is a code, then the continuous map $\theta' = \pi\theta\pi^{-1}$ is shift-commuting. The linear map L induces an isomorphism on the polynomial ring $\mathcal{L}$, by defining $L(X^iY^j) = X^{i'}Y^{i'}$ if $L(i, j) = (i', j')$, and extending linearly. Since L preserves collinearity of points, it follows that L maps $\mathcal{L}_0$ to itself. A simple computation on polynomials of the form X^iY^j shows that $L^{-1}f\pi = \pi f$ for any $f \in \mathcal{L}$. It follows that $\pi(\langle f\rangle^{\perp}) = \langle L^{-1}f\rangle^{\perp}$, and more generally π takes group shifts to group shifts. By the preceding discussion, if θ maps $\langle f\rangle^{\perp}$ into M, then θ' maps $\langle L^{-1}f\rangle^{\perp}$ to the group shift $\pi(M)$. Clearly, θ is a homomorphism if and only if θ' is, and v is an expansive direction for $\langle f\rangle^{\perp}$ if and only if $L^{-1}(v) = (1, 0)$ is expansive for $\langle L^{-1}f\rangle^{\perp}$.

Proof of Theorem 3.2. Suppose θ is a code from $\langle f\rangle^{\perp}$ to M. Then clearly defining $\phi(\xi) = \theta(\xi) - \theta(0)$ gives a second shift-commuting map from $\langle f\rangle^{\perp}$ to M such that $\phi(0) = 0$. It is clear that θ is affine if and only if ϕ is a homomorphism. We will therefore assume that $\theta(0) = 0$ and show in the proof that θ is a homomorphism.

Write $f = f_1^{k_1} f_2^{k_2} \ldots f_r^{k_r}$ where the f_α are irreducible and belong to $\mathcal{L}_0$. We observe that all of the $\langle {f_\alpha}^{k_\alpha}\rangle^{\perp}$ and M are expansive in all but finitely many directions. We will therefore take $v = (m, n)$ to be a direction in which all $\langle {f_\alpha}^{k_\alpha}\rangle^{\perp}$ and M are expansive, where the m and n are chosen to be coprime. Set $L(x, y) = xu + yv$, where u is a complementary vector. Let $\pi : \langle f\rangle^{\perp} \to \langle L^{-1}(f)\rangle^{\perp}$ be the map defined in the remarks prior to the proof. Write $L^{-1}(f) = f'$, $L^{-1}(f_\alpha) = f'_\alpha$ and $\pi(M) = M'$. By the remarks above, $f' \in \mathcal{L}_0$, and the induced map $\theta' = \pi\theta\pi^{-1}$ which maps $\langle f'\rangle^{\perp}$ into M' is a code.

Now $(1,0) = l^{-1}(v)$ is an expansive direction for each $\langle f'_\alpha \rangle^\perp$ and M'. We now see from Theorems 5.3 and 5.2 that θ' is a homomorphism and hence θ is also a homomorphism as required.

$\square$

Proof of Theorem 3.4. By the observation above, without loss of generality, one may assume that $(1,0)$ is an expansive direction for $\langle f \rangle^\perp$ and M. The result follows from Theorems 5.3 and 3.2. $\square$

Proof of Theorem 3.3. Suppose $g \in \mathcal{L}$ and $f \in \mathcal{L}_0$ and let θ be a code from $\langle g \rangle^\perp$ into $\langle f \rangle^\perp$. As in Theorem 3.2, we may assume that $\theta(0) = 0$. Now g may be factorized into the product of a polynomials in $\mathcal{L}_0$ and a collection of collinear factors as in the statement of Theorem 3.4. Applying Theorems 3.4 and 3.2, we see that it is sufficient to verify that if h is collinear, then any code from $\langle h \rangle^\perp$ to $\langle f \rangle^\perp$ is constant. Let h be any collinear Laurent polynomial (with direction $d = (m,n)$ say). Then there exists a k such that $\xi_{(i,j)+kd} = \xi_{i,j}$ for each $\xi \in \langle h \rangle^\perp$. Clearly $\theta(\xi)$ must satisfy the same periodicity condition (i.e. $\theta(\xi)$ must belong to $\langle X^{mk}Y^{nk} - 1 \rangle^\perp$). Since f has no collinear factors, it is coprime to $X^{mk}Y^{nk} - 1$ so by Theorem 2.3, $\langle f \rangle^\perp \cap \langle X^{mk}Y^{nk} - 1 \rangle^\perp$ is a finite set. It follows that $\theta\left(\langle h \rangle^\perp\right)$ is a finite set. We therefore have that θ is a shift-commuting mapping from $\langle h \rangle^\perp$ to a finite set. If we consider any expansive direction (i.e. any direction not parallel to d) for $\langle h \rangle^\perp$, then we see that the shift in this direction is topologically conjugate to a Bernoulli shift. Since a Bernoulli shift is mixing, its factors are also mixing, but the only finite mixing systems are systems with one point, so we see that θ maps $\langle h \rangle^\perp$ to a constant as required. $\square$

Proof of Theorem 3.5. Suppose that $f \in \mathcal{L} \setminus \mathcal{L}_0$. We then construct a map from $\langle f \rangle^\perp$ to $\langle f \rangle^\perp$ which is not affine. If $f \notin \mathcal{L}_0$, then f has a factor g which is a polynomial in a single variable X^mY^n. Write $f = gh$. It may be checked that h maps $\langle f \rangle^\perp$ onto $\langle g \rangle^\perp$. It is therefore sufficient to construct a map from $\langle g \rangle^\perp$ to itself which is not affine. To do this, we work as follows. For simplicity, we make the assumption that g is a polynomial in X. This does not entail a loss of generality as was shown in Theorem 3.2.

The polynomial $g = \sum_{i=0}^n a_i X^i$ corresponds to the one-dimensional recurrence relation $\sum_{i=0}^n a_i \rho_{k+i} = 0$ for all $k \in \mathbb{Z}$. The solutions of this recurrence relation form a finite set $0 = \rho^0, \ldots, \rho^{N-1}$. If $\xi \in \langle g \rangle^\perp$, then the rows of ξ are solutions of this recurrence relation. Conversely given any point ξ whose rows are solutions of the recurrence relation, we have $\xi \in \langle g \rangle^\perp$.

We can therefore define a map from $\langle g \rangle^\perp$ to itself by specifying what happens to the rows as long as the map commutes with the horizontal shift.

We note that $\sigma_{1,0}(\rho_0) = \rho_0$. We then define a map as follows: Suppose the point ξ has rows ρ_{n_j} for $j \in \mathbb{Z}$. Then we define $\phi(\xi)$ to be the point with rows

$\rho_{n'_j}$ where n'_j is defined by

$$n'_j = \begin{cases} 0 & \text{if } n_{j-1} = 0 \text{ or } n_{j+1} = 0 \\ n_j & \text{otherwise.} \end{cases}$$

The point $\phi(\xi)$ is therefore defined by setting the jth row equal to ρ_0 if either the $j-1$st row or the $j+1$st row of ξ is ρ_0 and equal to the jth row of ξ otherwise. One can then see that ϕ does indeed commute with both the horizontal and vertical shifts (that ϕ commutes with the horizontal shift follows from the fact that ρ_0 is fixed by the horizontal shift). Also, $\phi(0) = 0$. We see that ϕ is not a homomorphism as follows: Let the configuration ξ^e have even rows equal to ρ_1 and odd rows are equal to ρ_0 and a second configuration ξ^o have odd rows equal to ρ_1 and even rows equal to ρ_0. Then $\xi^e + \xi^o$ is the point all of whose rows are ρ_1 so $\phi(\xi^e + \xi^o) = \xi^e + \xi^o$, but $\phi(\xi^e) = 0$ and $\phi(\xi^o) = 0$. So setting $\theta(\xi) = \phi(h\xi)$ gives the required shift-commuting mapping which is not a homomorphism. □

Proof of Corollary 3.6. From Theorems 3.2 and 3.3, we see that θ must be an affine map between M and M'. As above, setting $\phi(\xi) = \theta(\xi) - \theta(0)$ gives a homomorphism from M to M'. If the characteristic of M is p and the characteristic of M' is q, then there exists a positive integer n such that $n \equiv 1 \pmod p$ and $n \equiv 0 \pmod q$. Then we observe

$$\phi(\xi) = \phi(\overbrace{\xi + \xi + \ldots + \xi}^{n \text{ times}}) = \overbrace{\phi(\xi) + \phi(\xi) + \ldots + \phi(\xi)}^{n \text{ times}} = 0.$$

This completes the proof. □

Proof of Corollary 3.7. Suppose that $f \in \mathcal{L}_0$ over $\mathbb{Z}_p$, and that $\theta : \langle f \rangle^\perp \to M$ is a code. By Theorem 3.2, $\theta = L + c$, where L is homomorphism. By the discussion in Section 2, L is given by a polynomial $r \in \mathcal{L}$. Clearly, L maps $\langle f \rangle^\perp$ into itself, so the image is a subgroup of $\langle f \rangle^\perp$, and the image of θ is a coset. If θ is a topological conjugacy from $\langle f \rangle^\perp$ to $\langle g \rangle^\perp$, then it follows from the previous statement that $\langle g \rangle^\perp \subset \langle f \rangle^\perp$. Since θ^{-1} is a code, we must have $\langle f \rangle^\perp \subset \langle g \rangle^\perp$, so $\langle f \rangle^\perp = \langle g \rangle^\perp$. Therefore $\langle f \rangle = \langle g \rangle$, so $f = g$ up to multiplication by a unit. □

Proof of Theorem 3.8. We first suppose that f is a linear map, so we can assume that f is a Laurent polynomial in one variable, X. Now, f and θ extend to codes on ${\mathbb{Z}_p}^{\mathbb{Z}^2}$, by applying them to each row of a point, and clearly these maps commute with each other. Let $\bar{f} = y - f$, which is a Laurent polynomial in two variables. Since f is not a conjugacy, it has at least two non-zero terms. It follows that any factor of $\bar{f}$ is non-collinear, so $\bar{f} \in \mathcal{L}_0$. Let $\xi \in \langle \bar{f} \rangle^\perp$. Then $f(\xi) = \sigma_{0,1}(\xi)$. Since θ commutes with $\sigma_{(0,1)}$ and f, we see $f(\theta(\xi)) = \theta(f(\xi)) = \theta(\sigma_{0,1}(\xi)) = \sigma_{0,1}(\theta(\xi))$ so $\theta(\xi) \in \langle \bar{f} \rangle^\perp$ and θ maps $\langle \bar{f} \rangle^\perp$ into itself. By Theorem 3.2, θ is affine.

If f is affine, say $f = g + c$ where g is linear, we can reduce to the previous case as follows. If $g(c) = 0$, then one shows that g commutes with $h^{-1}\theta h$, where h sends ξ to $\xi + c$. Therefore $h^{-1}\theta h$ is affine, and so θ is affine. If $g(c) \neq 0$, then it can be shown that $(g+c)^{np} = g^{np}$ where $g^n(c) = c$, and therefore θ commutes with g^{np}, which is linear. The details are left to the reader. □

6. The case of collinear factors

In the case where $\langle f \rangle^\perp \simeq \langle g \rangle^\perp$, but f and g are not assumed to belong to $\mathcal{L}_0$, it is still possible to say a considerable amount about any topological conjugacy, however some questions remain open. In this section, we prove a necessary condition for $\langle f \rangle^\perp$ to be topologically conjugate to $\langle g \rangle^\perp$ and give some examples of systems which are and are not topologically conjugate.

Theorem 6.1. *Suppose $f, f' \in \mathcal{L}$ over $\mathbb{Z}_p$. If θ is a topological conjugacy from $\langle f \rangle^\perp$ to $\langle f' \rangle^\perp$, then $f = f_0 g$ and $f' = f_0 g'$ where $f_0 \in \mathcal{L}_0$ and g, g' have only collinear factors. Let the factorization of g into collinear factors with distinct directions be given by $g = g_1 \cdots g_n$ with directions $d_1, \ldots, d_n$. We then have the following conclusions:*

1. *the factorization of g' into collinear factors with distinct directions is given by $g' = g'_1 \cdots g'_n$ where g'_i is collinear with direction d_i and $C(g_i)$ is equal to $C(g'_i)$ up to translation;*
2. *$\langle f_0 \rangle^\perp \cap \langle g_i \rangle^\perp = \langle f_0 \rangle^\perp \cap \langle g'_i \rangle^\perp$ for each i;*
3. *For any $1 \le i_1 < i_2 < \ldots < i_k \le n$, we have*

$$\theta\left(\langle g_{i_1} \rangle^\perp \cap \langle g_{i_2} \rangle^\perp \cap \ldots \cap \langle g_{i_k} \rangle^\perp\right) = \langle g'_{i_1} \rangle^\perp \cap \langle g'_{i_2} \rangle^\perp \cap \ldots \cap \langle g'_{i_k} \rangle^\perp.$$

We can re-express the conclusion of the above theorem as a set of necessary conditions for two group shifts $\langle f \rangle^\perp$ and $\langle f' \rangle^\perp$ to be topologically conjugate. They are Conditions 1, 2 and 3, where the first two conditions are just Conclusions 1 and 2 of Theorem 6.1 and the other condition is the following:

3. For any $1 \le i_1 < i_2 < \ldots < i_k \le n$, we have

$$\langle g_{i_1} \rangle^\perp \cap \langle g_{i_2} \rangle^\perp \cap \ldots \cap \langle g_{i_k} \rangle^\perp \simeq \langle g'_{i_1} \rangle^\perp \cap \langle g'_{i_2} \rangle^\perp \cap \ldots \cap \langle g'_{i_k} \rangle^\perp,$$

where $A \simeq B$ means that there is a shift-commuting bijection between A and B.

Note that in the case $k = 1$, this condition amounts to the requirement that for each i, the solutions of the recurrence relation corresponding to g_i have the same set of periods as the solutions of the recurrence relation corresponding to g'_i. For $k > 1$, the sets in question are finite by Theorem 2.3, so in this case, the requirement is that the finite sets may be put in bijection so that related points have the same set of periods. In either case, we see that Condition 3 is finitely checkable. Clearly this also applies to the other conditions. Unfortunately, we do not know whether these necessary conditions are also sufficient.

Proof of Theorem 6.1. As previously, we make the assumption (without loss of generality) that $\theta(0) = 0$. From [7], we know that the directions d_i are precisely the directions in which $\langle f \rangle^\perp$ is not directionally mixing. Since these directions are preserved under topological conjugacy, we see that g' has a factorization as claimed. To show that $C(g_i)$ is equal to $C(g_i')$ up to translation, we use the one-dimensional topological entropy of a subset of $\langle f\rangle^\perp$. Let v be a complementary direction to d_i. Since points of $\langle g_i\rangle^\perp$ satisfy a one-dimensional recurrence relation in the direction d_i, there exists a period t such that if $\xi \in \langle g_i\rangle^\perp$, then $\sigma_{td_i}(\xi) = \xi$. Let $A = \{\xi \in \langle f\rangle^\perp : \sigma_{td_i}(\xi) = \xi\}$ and $B = \{\eta \in \langle f'\rangle^\perp : \sigma_{td_i}(\eta) = \eta\}$. Then since g_i and g_i' are the only factors of f and f' in direction d_i, it follows that A and B are finite extensions of $\langle g_i\rangle^\perp$ and $\langle g_i'\rangle^\perp$. It is clear that $\theta(A) = B$ since θ is a topological conjugacy. The directional topological entropies of A and B in the v direction are given by $\mathrm{ht}_v(g_i)\log|K|$ and $\mathrm{ht}_v(g_i')\log|K|$ respectively. Since these are equal, conclusion 1 follows.

To show conclusions 2 and 3, we first demonstrate that $\theta(\langle g_i\rangle^\perp) \subseteq \langle g_i'\rangle^\perp$. Let t' be such that $\eta \in \langle g_i'\rangle^\perp$ implies $\sigma_{t'd_i}(\eta) = \eta$. Let T be the least common multiple of t and t'. Let $h' = f'/g_i'$ be the product of all the factors of f' apart from g_i'. It follows that h' has no collinear factor in the direction d_i. Write $d_i = (m_i, n_i)$.

Let $\xi \in \langle g_i\rangle^\perp$. Then $\theta(\xi) \in \langle f'\rangle^\perp = \langle h'\rangle^\perp + \langle g_i'\rangle^\perp$. Write $\theta(\xi)$ as $\eta^1 + \eta^2$ where $\eta^1 \in \langle h'\rangle^\perp$ and $\eta^2 \in \langle g_i'\rangle^\perp$. Since $\sigma_{Td_i}(\xi) = \xi$, it follows that $\sigma_{Td_i}(\theta(\xi)) = \theta(\xi)$. We have also $\sigma_{Td_i}(\eta^2) = \eta^2$ so it follows that $\sigma_{Td_i}(\eta^1) = \eta^1$. In particular $\eta^1 \in \langle X^{Tm_i}Y^{Tn_i} - 1\rangle^\perp \cap \langle h'\rangle^\perp$. By Theorem 2.3, this is a finite set consisting of points $\pi_1, \ldots \pi_S$ for some $S > 0$. We can therefore write for a general ζ, $\theta(\zeta) = \pi_{n(\zeta)} + \phi(\zeta)$ where $\phi(\zeta) \in \langle g_i'\rangle^\perp$. We recall that a point belongs to $\langle g_i\rangle^\perp$ if and only if its diagonals in the direction d_i are solutions of the recurrence relation corresponding to g_i. We can therefore define new points ξ_k in $\langle g_i\rangle^\perp$ by requiring that ξ_k is equal to ξ on the diagonals which pass within k of the origin and equal to 0 outside. Then we have $\theta(\xi_k) = \pi_{n_k} + \phi(\xi_k)$. We know that the π_j are periodic in the v direction: there exists an R such that $\sigma_{Rv}(\pi_j) = \pi_j$ for all j. As θ is defined by a block map and $\theta(0)$ was assumed to be 0, we see that $\lim_{n\to\infty} \sigma_{nRv}(\pi_{n_k} + \phi(\xi_k))$ is the 0 configuration. It follows that $\pi_{n_k} = -\lim_{n\to\infty}\sigma_{nRv}(\phi(\xi_k))$. Since $\phi(\xi_k) \in \langle g_i'\rangle^\perp$, we see that $\pi_{n_k} \in \langle g_i'\rangle^\perp$. It follows that $\theta(\xi_k) \in \langle g_i'\rangle^\perp$ and since the limit of these is $\theta(\xi)$, it follows that $\theta(\xi) \in \langle g_i'\rangle^\perp$.

Conclusion 3 now follows easily: Given $\xi \in \langle g_{i_1}\rangle^\perp \cap \ldots \cap \langle g_{i_k}\rangle^\perp$, $\theta(\xi)$ is seen to be a member of each of $\langle g_{i_1}'\rangle^\perp, \ldots, \langle g_{i_k}'\rangle^\perp$. In particular,

$$\theta\left(\langle g_{i_1}\rangle^\perp \cap \langle g_{i_2}\rangle^\perp \cap \ldots \cap \langle g_{i_k}\rangle^\perp\right) \subseteq \langle g_{i_1}'\rangle^\perp \cap \langle g_{i_2}'\rangle^\perp \cap \ldots \cap \langle g_{i_k}'\rangle^\perp.$$

Applying the argument for θ^{-1} gives the containment in the opposite direction.

The same argument shows that $\theta\left(\langle f_0\rangle^\perp \cap \langle g_i\rangle^\perp\right) = \langle f\rangle^\perp \cap \langle g_i'\rangle^\perp$. We showed in Theorem 3.2 that the restriction of θ to $\langle f_0\rangle^\perp$ is a homomorphism so in particular if $\xi \in \langle f_0\rangle^\perp \cap \langle g_i\rangle^\perp$, then $\theta(\xi) \in \langle f_0\rangle^\perp \cap \langle g_i\rangle^\perp$. It follows that $\langle f_0\rangle^\perp \cap \langle g_i'\rangle^\perp \supseteq \langle f_0\rangle^\perp \cap \langle g_i\rangle^\perp$. As before, the reverse conclusion also holds showing that conclusion 2 holds.

□

We continue this section with the study of a number of examples with collinear factors and examine the existence or non-existence of topological conjugacies.

We will consider examples in which f has a factor which is a polynomial in X and possibly an irreducible part also.

Example 1 Working over ${\mathbb{Z}_2}^{\mathbb{Z}^2}$, let $f_1 = 1 + X + X^3$ and $f_2 = 1 + X^2 + X^3$. As noted above, points in $\langle f_1 \rangle^\perp$ are solutions of the recurrence relation $\xi_{i,j} + \xi_{i+1,j} + \xi_{i+3,j} = 0$. Each row is then a solution of the recurrence relation $x_n = x_{n-2} + x_{n-3}$ and the rows may be chosen independently. We are then able to analyze the structure of the set of solutions of this recurrence relation: As it is a third order recurrence relation, the solution is determined by x_0, x_1 and x_2. If these are taken all to be 0, then it is clear that the solution is $x_n = 0$ for all n. If they are taken to be 1,0 and 0, then we see the solution is

$$x_0 x_1 x_2 \ldots = 10010111001011100\ldots$$

This solution is periodic with period 7 and clearly may be extended to negative values of n also. Translating the sequence gives all 7 possible configurations of 1s and 0s (apart from 0,0,0) on x_0, x_1 and x_2. We see therefore that points of $\langle f_1 \rangle^\perp$ are made up of arbitrary configurations of eight possible rows: $\rho_0 = \ldots 000\ldots$, $\rho_1 = \ldots 100 \ldots$, and $\rho_n = \sigma_{n-1}\rho_1$ for n between 2 and 7, where the crucial point about these rows is that $\sigma(\rho_0) = \rho_0$ and the other seven rows form a single periodic orbit under σ.

Points of $\langle f_2 \rangle^\perp$ are also made up of arbitrary configurations of 8 possible rows: $\rho'_0, \ldots, \rho'_7$, where again $\rho'_1, \ldots \rho'_7$ form a periodic orbit and it is assumed that they are numbered in such a way that $\sigma(\rho'_i) = \rho'_{i+1}$ for $i < 7$ and $\sigma(\rho'_7) = \rho'_1$. There is then a topological conjugacy between $\langle f_1 \rangle^\perp$ and $\langle f_2 \rangle^\perp$ given by sending the row ρ_i to ρ'_i. This conjugacy is clearly not a homomorphism (as otherwise $\langle f_1 \rangle^\perp$ and $\langle f_2 \rangle^\perp$ would be equal).

What was important in establishing the topological conjugacy in the above example was showing that the periodic orbit structure of the sets of solutions of the two recurrence relations corresponding to f_1 and f_2 were the same. It may be asked whether any two irreducible polynomials of the same degree necessarily have the same periodic orbit structure. The following example shows that this is not the case.

Example 2 Let $g_1 = 1 + X^2$ and $g_2 = 2 + X + X^2$ over $\mathbb{Z}_3$. Then solutions of the recurrence relation corresponding to g_1 form 3 periodic orbits: one of period 1 and two of period 4; while solutions of the recurrence relation corresponding to g_2 form 2 periodic orbits: one of period 1 and one of period 8. It is then clear that no topological conjugacy can exist between $\langle g_1 \rangle^\perp$ and $\langle g_2 \rangle^\perp$ as $\langle g_1 \rangle^\perp$ has points with horizontal periods 1 and 4, whereas $\langle g_2 \rangle^\perp$ has points with horizontal periods 1 and 8.

We now give a more complicated example based on Example 1 of polynomials g_1 and g_2 which have a common factor in $\mathcal{L}_0$ such that $\langle g_1 \rangle^\perp$ and $\langle g_2 \rangle^\perp$ are topologically conjugate.

Example 3 Let f_1 and f_2 be as in Example 1. Let $h = f_1 f_2 + Y$, $g_1 = h f_1$ and $g_2 = h f_2$. We will then establish that $\langle g_1 \rangle^\perp$ and $\langle g_2 \rangle^\perp$ are topologically conjugate. We first note that h is irreducible: h is of height 1. If h is a product, then it is a product of a polynomial of height 1 with a polynomial of height 0. Clearly h has no factors which are polynomials in X alone so it is irreducible.

We also see that $\mathrm{hcf}(h, f_1) = \mathrm{hcf}(h, f_2) = 1$. The additional feature which will be of importance for us is that 1 can be written as a linear combination of h and f_1 or of h and f_2 as follows: $1 = Y^{-1}h - Y^{-1}f_1 f_2$. We established in Theorem 2.4 that $\langle g_1\rangle^\perp = \langle f_1\rangle^\perp + \langle h\rangle^\perp$ and $\langle g_2\rangle^\perp = \langle f_2\rangle^\perp + \langle h\rangle^\perp$. The above fact allows us to deduce that these sums are direct sums as follows: If $\xi \in \langle f_1\rangle^\perp \cap \langle h\rangle^\perp$, then $h\xi = 0$ and $f_1\xi = 0$, so any $\mathcal{L}$-linear combination of h and f_1 annihilates ξ also. In particular $1\xi = 0$.

We have shown that $\langle g_1\rangle^\perp = \langle f_1\rangle^\perp \oplus \langle h\rangle^\perp$ and $\langle g_2\rangle^\perp = \langle f_2\rangle^\perp \oplus \langle h\rangle^\perp$. Since we showed in Example 1 that $\langle f_1\rangle^\perp$ and $\langle f_2\rangle^\perp$ are topologically conjugate, it follows that $\langle g_1\rangle^\perp$ and $\langle g_2\rangle^\perp$ are topologically conjugate.

Our next example shows that even if $\langle f_1\rangle^\perp$ and $\langle f_2\rangle^\perp$ are topologically conjugate, it need not follow that $\langle hf_1\rangle^\perp$ and $\langle hf_2\rangle^\perp$ are topologically conjugate.

Example 4 Let f_1 and f_2 be as in Example 1. Let $h = 1 + X + X^3 + Y$. As in the previous example, we see that $\langle f_1\rangle^\perp \cap \langle h\rangle^\perp = \{0\}$. The same does not hold for $\langle f_2 h\rangle^\perp$. The reader may check that in addition to the point consisting of all 0s, there are seven additional points in $\langle f_2\rangle^\perp \cap \langle h\rangle^\perp$. These points are translates of the following point:

$$\begin{matrix} \dots & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & \dots \\ \dots & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \dots \\ \dots & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & \dots \\ \dots & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \end{matrix}$$

By Theorem 6.1, we see that $\langle hf_1\rangle^\perp$ and $\langle hf_2\rangle^\perp$ are not topologically conjugate.

Example 5 Let $h_1 = (1 + X + X^3)(1 + Y + Y^3)$ and $h_2 = (1 + X^2 + X^3)(1 + Y^2 + Y^3)$. We check that the necessary conditions for $\langle h_1\rangle^\perp$ and $\langle h_2\rangle^\perp$ to be topologically conjugate are satisfied as follows: Condition 1 is clearly satisfied in this case. Condition 2 is vacuous as there is no non-collinear part. To see that Condition 3 is satisfied, we note that the periodic data for the two systems is the same as the map $F(\xi)_{i,j} = \xi_{-i,-j}$ is a bijection between the two systems (not shift-commuting) which leaves all periods of periodic points unaffected. (It may in fact be verified that each of $\langle 1 + X + X^3\rangle^\perp \cap \langle 1 + Y + Y^3\rangle^\perp$ and $\langle 1 + X^2 + X^3\rangle^\perp \cap \langle 1 + Y^2 + Y^3\rangle^\perp$ have 10 periodic orbits with a 7×7 fundamental domain, 3 periodic orbits with a 7×1 fundamental domain and 1 fixed point.)

We are unable to determine as yet whether $\langle h_1\rangle^\perp$ and $\langle h_2\rangle^\perp$ are topologically conjugate. One strategy would be to find a shift-commuting bijection between the periodic points (the existence of such a bijection is guaranteed by the above observations) and to attempt to extend this to a conjugacy of the whole system. It seems however that this approach may not succeed as a result of Kim and

Roush [5] gave an example of a permutation of a finite set of periodic points of a full shift which cannot be extended to a shift-commuting bijection.

7. Open problems and future directions

We mention here three areas in the paper which appear to merit further attention.

- We considered in Section 6 the case where $\langle f\rangle^{\perp}$ is not mixing. We found necessary conditions for two such group shifts to be isomorphic. Are they also sufficient? If not, is it possible to give a set of necessary and sufficient conditions? Is it decidable whether two non-mixing group shifts $\langle f\rangle^{\perp}$ and $\langle f'\rangle^{\perp}$ are isomorphic? For a more concrete question, are $\langle h_1\rangle^{\perp}$ and $\langle h_2\rangle^{\perp}$ in Example 5 topologically conjugate?
- In this paper, we have been working only with the two-dimensional case of subgroups of $K^{\mathbb{Z}^2}$. We note that Theorem 2.3 depends crucially on the dimension. In the case where f is an irreducible polynomial in $\mathcal{L}_0$, this means that the only proper translation-invariant subgroups of $\langle f\rangle^{\perp}$ are finite. This condition is the irreducibility condition in [8] and it fails in higher-dimensional systems. As a result, there is an essential difference both in our results and in the results of Kitchens and Schmidt between the case of two dimensions and the case of higher dimensions. We hope to address the question of group shifts in dimensions greater than 2 in a later paper.
- It would be interesting to know about the structure of the automorphism group of a group shift in general: in particular, is it finitely generated?

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