GENERIC POINTS IN THE CARTESIAN POWERS OF THE MORSE DYNAMICAL SYSTEM

by Emmanuel Lesigne, Anthony Quas & Máté Wierdl

Abstract. — The symbolic dynamical system associated with the Morse sequence is strictly ergodic. We describe some topological and metrical properties of the Cartesian powers of this system, and some of its other self-joinings. Among other things, we show that non generic points appear in the fourth power of the system, but not in lower powers. We exhibit various examples and counterexamples related to the property of weak disjointness of measure preserving dynamical systems.

Résumé (*Points génériques dans les puissances cartésiennes du système dynamique de Morse*)

Le système dynamique symbolique associé à la suite de Morse est strictement ergodique. Nous décrivons certaines propriétés topologiques et métriques des puissances cartésiennes de ce système, et de certains de ses auto-couplages. Nous montrons en particulier que des points non génériques apparaissent dans la puissance quatrième du système mais n'apparaissent pas dans les puissances inférieures. Nous présentons divers exemples et contre-exemples illustrant la notion de disjonction faible de systèmes dynamiques mesurés.

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Contents

Introduction

In this article, we describe ergodic properties of some self-joinings of the dynamical system associated to the Morse sequence. Some of these properties are relevant to the topological dynamics setting, and some other ones are relevant to the measurable dynamics setting.

The dynamical system associated to the Morse sequence, called the *Morse dynamical system* is a well known and widely studied object. We recall its definition and some of its basic properties in Section 1. In this section the classical notions of generic points in dynamical systems and of strict ergodicity are also recalled.

The Morse dynamical system $\mathcal{M}$ is a simple example of a strictly ergodic dynamical system, probably the simplest example after the ergodic translations of compact abelian groups. It has been a surprise for us to discover that non trivial phenomena appear in the genericity properties of points in the Cartesian powers of $\mathcal{M}$: in the Cartesian square and cube of $\mathcal{M}$ every point is generic (for a measure which of course depends on the point) ; but in the fourth Cartesian power of $\mathcal{M}$, this is no longer true. We will see that, in a certain sense, there are a lot of non generic points in the kth Cartesian power of $\mathcal{M}$, when $k \geq 4$.

The study of generic points in Cartesian products of dynamical systems is linked with the notion of *weak-disjointness of dynamical systems*, that has been introduced in [3] and [4] and that we recall now.

DEFINITION. — *Two probability measure preserving dynamical systems $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ are weakly disjoint if, given any function f in $L^2(\mu)$ and any function g in $L^2(\nu)$, there exist a set A in $\mathcal{A}$ and a set B in $\mathcal{B}$ such that*

- $\mu(A) = \nu(B) = 1$
- *for all $x \in A$ and $y \in B$, the sequence* $\left(\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)\cdot g(S^n y)\right)_{N>0}$ *converges.*

This notion of weak disjointness is an invariant of isomorphism in the category of measurable dynamical systems.

If X and Y are compact metric spaces equipped with their Borel σ-algebras and with Borel probabilities μ and ν, it can be shown ([4]) that the dynamical systems $(X, \mathcal{A}, \mu, T)$ and $(Y, \mathcal{B}, \nu, S)$ are weakly disjoint if and only if the set of generic points in the Cartesian product $(X \times Y, T \times S)$ contains a "rectangle" $A \times B$ of full $\mu \otimes \nu$ measure.

We say that a dynamical system $(X, \mathcal{A}, \mu, T)$ is self-weakly disjoint (of order 2) if it is weakly disjoint from itself. This notion has an obvious k-fold extension, for any integer $k \geq 2$.

From the study of generic points in the Cartesian powers of the Morse dynamical system $\mathcal{M}$, we deduce that this dynamical system is self-weakly disjoint of orders 2 and 3, but not of order ≥ 4. We prove that $\mathcal{M}$ is weakly disjoint from any ergodic k-fold joining of itself (Corollary 2.13), which implies (due to a result of [4]) that *the Morse dynamical system $\mathcal{M}$ is weakly disjoint from any ergodic dynamical system.* We do not know if $\mathcal{M}$ is weakly disjoint from any dynamical system.

On the side of "negative" results we prove that the Cartesian square of $\mathcal{M}$ is not self-weakly disjoint (Corollary 3.6) and we prove that most ergodic self-joinings of $\mathcal{M}$ are not self-weakly disjoint (Theorem 3.7). This provides the simplest known example of a non self-weakly disjoint ergodic dynamical system with zero entropy.

We notice here that the Cartesian square of $\mathcal{M}$ is not self-weakly disjoint although it is weakly disjoint from each of its ergodic components (Corollary 2.15).

We give examples of topological dynamical systems in which every point is generic for some measure and such that this property fails in the Cartesian square. In the last section of this article we describe the construction of a dynamical system in which every point is generic for some measure and such that this property is preserved in the Cartesian square, but not in the Cartesian cube. We claimed that for the Morse dynamical system this property of genericity is preserved for the cube, but not for the fourth power. We do not know how to construct examples of dynamical systems for which this genericity property is preserved exactly till a given order ≥ 4.

The description of $\mathcal{M}$ as a two point extension of the dyadic odometer plays a major role in our study. The convergence results are deduced from the ergodicity of some cocycles defined on the odometer, the main result here being Proposition 2.10. In order to prove divergence results, we exhibit a particular block structure of a family of four points in the odometer (or equivalently four points in the space $\mathcal{M}$) which give strongly oscillating Birkhoff sums, hence diverging ergodic averages (see Section 3.1).

1. Reminder of some classical notions

1.1. The Morse sequence. — The (Prouhet-Thue-)Morse sequence

$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,\ldots$$

is the sequence $u = (u_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}}$ defined by one of the following equivalent rules

- $u_0 = 0, u_{2n} = u_n, u_{2n+1} = 1 - u_n$;
- $u_n = 0$ iff there is an even number of 1's in the base 2 expansion of the integer n ;
- the sequence begins with a 0 and is a fixed point of the substitution $0 \to 01$, $1 \to 10$;
- $(u_{2^n}, u_{2^n+1}, \ldots, u_{2^{n+1}-1}) = (1-u_0, 1-u_1, \ldots, 1-u_{2^n-1})$ and $u_0 = 0$.

This sequence appears independently in various mathematical works.

In 1851, Prouhet [7] showed that any set of 2^n consecutive integer numbers can be divided into two subsets such that, for any integer k between 0 and $n-1$, the sum of the k-th powers of the elements of one subset is equal to the sum of the k-th powers of the elements of the other subset. He noticed that if we denote by A, resp. B, the set of integers i between 0 and $2^n - 1$ such that $u_i = 0$, resp $u_i = 1$, then for all k between 0 and $n-1$, for all integers m,

$$\sum_{i\in A}(m+i)^k = \sum_{i\in B}(m+i)^k \ .$$

At the beginning of the last century, Thue [8] was looking for a sequence "without any cube" and exhibited the sequence u. Indeed, it can be verified that no finite word from u is repeated three times consecutively in u.

Morse [6] introduced the sequence u in his study of recurrence properties of geodesics on some surface with negative curvature. He was interested in the fact that the sequence u is non periodic but minimal : every finite word wich appears once in u appears infinitely often with bounded gaps.

1.2. The Morse dynamical system. — The procedure which associates a dynamical system to the Morse sequence u is standard. We consider the compact space $\{0,1\}^{\mathbb{N}}$ equipped with the shift transformation σ. We denote by K_u the closure of the orbit of the Morse sequence u under σ:

$$K_u := \overline{\{\sigma^n u \,:\, n \in \mathbb{N}\}} \; .$$

K_u is a compact metrizable space, invariant under σ. The Morse dynamical system is (K_u, σ). The ergodic and spectral properties of this dynamical system have been widely studied by numerous authors including Kakutani, Keane, Kwiatkowski, Lemanczyck. A list of references can be found in [5].

It is known that the dynamical system (K_u, σ) is strictly ergodic (that is minimal and uniquely ergodic). At the level of the sequence u this means that every word that appears once in u appears in u with a strictly positive asymptotic frequency. At the level of the dynamical system (K_u, σ) strict ergodicity means that there exists on K_u a unique σ-invariant probability measure whose support is all of K_u. This measure will be denoted by ν. (A proof of unique ergodicity of the Morse dynamical system is given at the end of this section.)

If $v = (v_n)$ is a sequence of 0's and 1's we denote by $\overline{v}$ the conjugate sequence $\overline{v} = (1 - v_n)$. We remark that K_u is preserved by this conjugacy.

There is a well known and very useful description of the Morse dynamical system as a two point extension of the dyadic odometer. In the more general setting of 'q-multiplicative sequences' this description is explained in, for example, [2]. Let us recall it in the Morse case.

We denote by Ω the topological group of 2-adic integers. It is the set $\{0,1\}^{\mathbb{N}}$ of sequences of 0's and 1's equipped with the product topology making it a compact space and with the additive law of adding with carry (on the right). Formally if $\omega = (\omega_i)_{i\leq 0}$ and $\omega' = (\omega'_i)_{i\leq 0}$ are two elements of Ω their sum $\omega + \omega'$ is defined by

$$\sum_{i\geq 0} (\omega + \omega')_i 2^i = \sum_{i\geq 0} \omega_i 2^i + \sum_{i\geq 0} \omega'_i 2^i \; .$$

The Haar probability on the compact Abelian group $(\Omega, +)$ is the uniform product probability $\mu := (1/2, 1/2)^{\otimes\mathbb{N}}$ which makes the coordinate maps on Ω independent Bernoulli random variables with parameter 1/2. The zero of the group Ω is the sequence of all zeros, and is denoted by $\mathbf{0}$. The element $(1, 0, 0, 0, \ldots)$ of Ω will be denoted by $\mathbf{1}$ and the infinite cyclic group generated by $\mathbf{1}$ in Ω will be denoted by $\mathbb{Z}$. Thus $\mathbb{Z}$ is the set of sequences of 0's and 1's which are eventually constant. According to this convention, we denote by $\mathbb{Z}^-$ the set of sequences which are eventually constant equal to 1, and if k is an integer we designate also by k the element $k\mathbf{1}$ of Ω.

We denote by T the translation by $\mathbf{1}$ on the group Ω.

$$T(\omega) := \omega + \mathbf{1} \ (= \omega + (1, 0, 0, 0, \ldots)) \ .$$

Since $\mathbb{Z}$ is dense in Ω, the map T is an ergodic automorphism of the probability space (Ω, μ).

We define the *Morse cocycle* to be the map φ from Ω into the group with two elements $\mathbb{Z}_2 = (\{0, 1\}, +)$ defined by

$$\varphi(\omega) = \sum_{i \geq 0} (\omega + \mathbf{1})_i - \omega_i \quad (\text{mod.}2) \ .$$

This definition makes sense at every point of Ω except for the point $-\mathbf{1} := (1, 1, 1, \ldots)$, because, if $\omega \neq -\mathbf{1}$, then, for all large enough i, $(\omega + \mathbf{1})_i = \omega_i$. At the point $-\mathbf{1}$ we can give φ an arbitrary value. Another way to define φ is the following: $\varphi(\omega) = 0$ or 1 according to whether the length of the initial block of 1's in the sequence ω is odd or even (if the first digit of ω is zero, then the length is considered to be even). We will use the classical cocycle notation

$$\varphi^{(n)}(\omega) := \sum_{k=0}^{n-1} \varphi(T^k \omega) = \sum_{k=0}^{n-1} \varphi(\omega + k) \ ,$$

if $\omega \in \Omega$ and $n \in \mathbb{N}$. Notice that, for all ω outside $\mathbb{Z}^-$, we have

$$\varphi^{(n)}(\omega) = \sum_{i \geq 0} (\omega + n)_i - \omega_i \quad (\text{mod.}2) \ .$$

The Morse sequence is given by

$$u_n = \varphi^{(n)}(0) = \sum_{k=0}^{n-1} \varphi(k) \quad (\text{mod.}2) \ ,$$

and the Morse dynamical system is isomorphic (in the metrical sense, and almost in the topological sense, cf [2]), to the following skew-product:

$$T_\varphi : \Omega \times \mathbb{Z}_2 \to \Omega \times \mathbb{Z}_2 \quad (\omega, z) \mapsto (\omega + \mathbf{1}, z + \varphi(\omega)) \ .$$

We can be more precise. Let us denote by K'_u the set of elements of K_u which are not preimages of u or $\overline{u}$ by a power of the shift. We define a map I from $\Omega \times \mathbb{Z}_2$ into $\{0, 1\}^{\mathbb{N}}$ by

$$I(\omega, z) := (z, z + \varphi(\omega), z + \varphi^{(2)}(\omega), z + \varphi^{(3)}(\omega), \ldots) \ .$$

The map I establishes a one to one bicontinuous correspondence between $(\Omega \backslash \mathbb{Z}) \times \mathbb{Z}_2$ and K'_u. The map I conjugates the transformations $(I \circ T_\varphi = \sigma \circ I)$ and it sends the measure ν to the product of μ with the uniform probability on $\mathbb{Z}_2$.

1.3. Generic points. — Let X be a compact metric space and T be a Borel-measurable transformation of this space. A point x in X is *generic* in the dynamical system (X,T) if for all continuous functions f on X, the sequence of averages $\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)$ converges. If the point x is generic, there exists a Borel probability measure λ on X such that, for all continuous functions f,

$$\int_X f \, \mathrm{d}\lambda = \lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \ .$$

In this case we say that the point x is generic for the measure λ. If the transformation T is continuous, any such λ is T-invariant.

In the other direction, if a T-invariant probability measure λ is given on X, then we know, by Birkhoff's ergodic theorem, that λ-almost all points of X are generic. If furthermore the dynamical system (X,T,λ) is ergodic, then λ-almost all points of X are generic for λ. Lastly, under the hypothesis of continuity of T the link between unique ergodicity and genericity can be described as follows: if (X,T) is uniquely ergodic, then all the points of X are generic for the unique invariant probability measure ; if all the points of X are generic for a given measure, then this measure is the unique invariant probability and it is ergodic.

1.4. Uniquely ergodic skew-products. — When we represent the Morse dynamical system as a two point extension of an odometer, the transformation we consider is a skew-product with a discontinuous cocycle. But the cocycle is "Riemann-integrable" and the following proposition gives us a satisfactory description of unique ergodicity in that case. This proposition, which extends to the discontinuous case a classical result from [1], is proved in [2] (Proposition 3.1).

Let $(G,+)$ and $(H,+)$ be two compact metrizable Abelian groups, equipped with their respective Haar probability measures m_G and m_H. Let α be an element of the group G, such that the translation $g \mapsto g+\alpha$ is ergodic on G (hence we know that it is uniquely ergodic). A measurable map ρ from G into H is given, and we suppose that *the set of discontinuity points of ρ has zero measure in G*. We denote by T_ρ the transformation of $G \times H$ defined by $T_\rho(g,h) := (g+\alpha, h+\rho(g))$. This transformation preserves the product measure $m_G \otimes m_H$. In this context, we will call ρ the *cocycle* of the *skew-product* T_ρ.

We denote by $\mathbb{U}$ the multiplicative group of complex numbers of modulus one.

PROPOSITION 1.1. — *The following assertions are equivalent.*

- *The measure preserving dynamical system $(G \times H, m_G \otimes m_H, T_\rho)$ is ergodic.*
- *The dynamical system $(G \times H, T_\rho)$ is uniquely ergodic.*
- *In the dynamical system $(G \times H, T_\rho)$, every point is generic for the product measure $m_G \otimes m_H$.*

– *For any non trivial character χ of the group H, there does not exist any measurable function ψ from G into $\mathbb{U}$ such that, for almost all g,*

$$\chi\rho(g) = \psi(g+\alpha)/\psi(g)\ . \tag{1.1}$$

– *For all non trivial characters χ of the group H, for all characters σ of the group G, for all $g \in G$,*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sigma(\alpha)^n \prod_{k=0}^{n-1} \chi\rho(g+k\alpha) = 0\ .$$

A typical application of this proposition is a proof of unique ergodicity of the Morse dynamical system. Let us consider the two point extension of the odometer defined by the Morse cocycle. The group $\mathbb{Z}_2$ has only one non trivial character and in order to prove that the dynamical system $(\Omega \times \mathbb{Z}_2, T_\varphi)$ is uniquely ergodic, it is enough to prove that the functional equation

$$\varphi(\omega) = \psi(\omega + \mathbf{1}) - \psi(\omega) \tag{1.2}$$

has no measurable solution $\psi\colon \Omega \to \mathbb{Z}_2$. The correspondence we described between the two dynamical systems (K_u, σ) and $(\Omega \times \mathbb{Z}_2, T_\varphi)$ shows that the unique ergodicity of one of them implies the unique ergodicity of the other.

As an introduction to what follows let us show that the functional equation (1.2) has no solution. Let us suppose that there exists a measurable solution ψ. Then, for all $n \geq 0$, we have

$$\varphi^{(2^n)}(\omega) = \psi(\omega + 2^n) - \psi(\omega) \quad \text{mod.2}\ . \tag{1.3}$$

In the group Ω, we have $\lim_{n\to+\infty} 2^n = 0$. Hence (1.3) implies that

$$\lim_{n\to+\infty} \varphi^{(2^n)} = 0 \quad \text{in probability}\ . \tag{1.4}$$

But it is easy to describe the map $\varphi^{(2^n)}$. We have $\varphi^{(2^n)}(\omega) = 0$ iff the block of consecutive 1's in the sequence ω starting at the index n has odd length. In other words,

$$\text{if } (\omega_i)_{i\geq n} = 1^\ell 0 \star\star\star, \text{ with } \ell \geq 0 \quad \text{then } \varphi^{(2^n)}(\omega) = \ell - 1 \text{ mod.2}\ .$$

(We can also write $\varphi^{(2^n)}(\omega_0, \omega_1, \omega_2, \ldots) = \varphi(\omega_n, \omega_{n+1}, \omega_{n+2}, \ldots)$.)
We see that, for all $n \geq 0$,

$$\mu\left\{\omega\ :\ \varphi^{(2^n)}(\omega) = 0\right\} = 1/3\ ,$$

which gives a contradiction to (1.4).

We can summarize the previous argument in a more general proposition.

Proposition 1.2. — *Let ρ be a measurable map from the dyadic odometer Ω into the group $\mathbb{Z}_2$, whose set of discontinuity points has zero measure. If the* *sequence* $\left(\rho^{(2^n)}\right)$ *does not go to zero in probability, then the dynamical system* $(\Omega \times \mathbb{Z}_2, T_\rho)$ *is uniquely ergodic and for all* $\omega \in \Omega$,

$$\lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1} (-1)^{\rho^{(n)}(\omega)} \right) = 0 \ .$$

2. Convergence results

We will denote by $c(x) := (-1)^x$ the non trivial character of the group $\mathbb{Z}_2$. In order to establish convergence results, we will use several times the following principle. Given $\omega^1, \omega^2, \dots, \omega^k$ in the odometer Ω and $z_1, z_2, \dots, z_k$ in the group $\mathbb{Z}_2$, the point $(\omega^j, z_j)_{1\le j\le k}$ is generic in the k-th Cartesian power of the dynamical system $(\Omega \times \mathbb{Z}_2, T_\varphi)$ if and only if, for any subset S of $\{1, 2, \dots, k\}$ and any natural number t, the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} c\left(\sum_{j\in S} \varphi^{(n2^t)}(\omega^j) \right) \tag{2.1}$$

converge.

This statement is a direct consequence of the fact that any continuous function on $(\Omega \times \mathbb{Z}_2)^k$ can be uniformly approximated by linear combinations of characters of this product group.

Moreover, using Proposition 1.1 we see that a sufficient condition to insure the convergence of averages (2.1) is that the two point extension of the odometer defined by the cocycle

$$\Omega \to \mathbb{Z}_2, \quad \omega \mapsto \sum_{j\in S} \varphi(\omega + \omega^j)$$

is ergodic.

2.1. Cartesian square.

— In order to facilitate the understanding of our method, we begin with the study of generic points in the Cartesian square of the Morse dynamical system. We recall that the subshift K_u is equipped with the probability measure ν. We denote by $\overline{Id}$ the conjugacy in K_u.

Theorem 2.1. — *In the Cartesian square of the Morse dynamical system, every point is generic for some measure.*

It is of course impossible for the Cartesian square of a non-trivial dynamical system to be uniquely ergodic. But, using the representation as a skew-product we can bring back the problem to the study of a unique ergodicity property.

If v and w are two elements of K_u, and swapping v and w if necessary, we are in one of the following three cases:

1. There exists $n \ge 0$ such that $w = \sigma^n(v)$.

2. There exists $n \geq 0$ such that $w = \sigma^n(\overline{v})$.
3. Neither v and w, nor $\overline{v}$ and w are on the same trajectory in the dynamical system (K_u, σ).

In the first case the unique ergodicity of the Morse dynamical system implies directly that the pair (v, w) is generic for the image of the diagonal measure on $K_u \times K_u$ by the transformation $Id \times \sigma^n$. In the second case the same argument gives that the pair (v, w) is generic for the image of the diagonal measure on $K_u \times K_u$ by the transformation $\overline{Id} \times \sigma^n$.

The proof which follows establishes that, in the third case, the pair (v, w) is generic for a measure which can be described as a relatively independent joining of two copies of (K_u, ν) above their common factor (Ω, μ). Modulo the isomorphism I these joinings are the measures $(\mu \otimes_{\omega^0} \mu) \otimes m \otimes m$ described in Remark 1, which follows the next Proposition.

Proposition 2.2. — *Let ω^1 and ω^2 be two elements of the dyadic odometer Ω such that $\omega^1 - \omega^2 \notin \mathbb{Z}$. The four point extension of the odometer defined by the cocycle*

$$\Omega \to \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \omega \mapsto \big(\varphi(\omega + \omega^1), \varphi(\omega + \omega^2)\big)$$

is ergodic.

Proof. — We use Proposition 1.1. The product group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has three non trivial characters. We have to show that the three following functional equations have no solution.

$$\varphi(\omega + \omega^i) = \psi(\omega + \mathbf{1}) - \psi(\omega) \mod 2 \qquad (i = 1, 2)\ ,$$

$$\varphi(\omega + \omega^1) + \varphi(\omega + \omega^2) = \psi(\omega + \mathbf{1}) - \psi(\omega) \mod 2\ .$$

The two first equations are the same as (1.2), and we know that they don't have any solution. Let us study the third one. We set $\omega^0 = \omega^2 - \omega^1$, we define a new cocycle ρ by $\rho(\omega) := \varphi(\omega) + \varphi(\omega + \omega^0)$ and we study the following equivalent equation :

$$\rho(\omega) = \psi(\omega + \mathbf{1}) - \psi(\omega) \mod 2\ .$$

Following our previous arguments, all that we have to prove is that the sequence of functions $\rho^{(2^n)}(\omega)$ does not go to zero in μ-measure.

Since $\omega^0 \notin \mathbb{Z}$, there are infinitely many positive n such that $\omega^0_{n-1} = 1$ and $\omega^0_n = 0$. For any such n we have the following property : if $\omega \in \Omega$ is such that $\omega_{n-1} = 1$, $\omega_n = 1$ and $\omega_{n+1} = 0$, then $\varphi^{(2^n)}(\omega) = 0$ and $\varphi^{(2^n)}(\omega + \omega^0) = 1$ (because $(\omega + \omega^0)_n = 0$).

Hence there exist infinitely many positive n's such that

$$\mu\{\omega\colon \rho^{(2^n)}(\omega) = 1\} \geq \frac{1}{8}\ .$$

□

Let us denote by m the uniform probability on the two point space $\mathbb{Z}_2$. For each $\omega^0 \in \Omega$, denote by $\mu \otimes_{\omega^0} \mu$ the measure defined on $\Omega \times \Omega$ by the integration formula

$$\int f(\omega,\omega')\ \mathrm{d}(\mu \otimes_{\omega^0} \mu)(\omega,\omega') := \int f(\omega,\omega+\omega^0)\ \mathrm{d}\mu(\omega)\ .$$

Proof of Theorem 2.1. — Let $\omega^1, \omega^2 \in \Omega$ be such that $\omega^1 - \omega^2 \notin \mathbb{Z}$. Propositions 2.2 and 1.1 imply that, for all $(\omega, z_1, z_2) \in \Omega \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and for all continuous functions F on $\Omega \times \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} F(\omega+n, z_1+\varphi^{(n)}(\omega+\omega^1), z_2+\varphi^{(n)}(\omega+\omega^2)) = \int F\ \mathrm{d}(\mu\otimes m\otimes m)\ .$$

Applying this to $\omega = 0$ and to a function F which is the product of a character of the group Ω evaluated at ω with a function g of (z_1, z_2), we obtain the following. For all $z_1, z_2 \in \mathbb{Z}_2$, for all characters χ_1 and χ_2 of Ω, for all functions g,

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_1(\omega^1+n)\cdot\chi_2(\omega^2+n)\cdot g\left(z_1+\varphi^{(n)}(\omega^1), z_2+\varphi^{(n)}(\omega^2)\right) =$$

$$\chi_1(\omega^1)\chi_2(\omega^2) \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} ((\chi_1\chi_2)\otimes g)\left(n, z_1+\varphi^{(n)}(\omega^1), z_2+\varphi^{(n)}(\omega^2)\right) =$$

$$\chi_1(\omega^1)\cdot\chi_2(\omega^2)\cdot \int \chi_1\cdot\chi_2\ \mathrm{d}\mu \cdot \int g\ \mathrm{d}(m\otimes m) =$$

$$\int \chi_1\otimes\chi_2\otimes g\ \mathrm{d}\left((\mu\otimes_{\omega^2-\omega^1}\mu)\otimes m\otimes m\right)\ .$$

By density of linear combinations of characters in the space of continuous functions on Ω equipped with the uniform metric, we obtain that, for all continuous functions G on $\Omega \times \Omega \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} G\left(\omega^1+n, \omega^2+n, z_1+\varphi^{(n)}(\omega^1), z_2+\varphi^{(n)}(\omega^2)\right) =$$

$$\int G\ \ \mathrm{d}\left((\mu\otimes_{\omega^2-\omega^1}\mu)\otimes m\otimes m\right)\ .$$

Notice finally that, via the conjugacy I between the dynamical system $(\Omega \times \mathbb{Z}_2, T_\varphi)$ and the Morse dynamical system, the condition $\omega^1 - \omega^2 \notin \mathbb{Z}$ means excactly that the sequences $v := I(\omega^1, z_1)$ and $w := I(\omega^2, z_2)$ are such that neither v and w, nor $\overline{v}$ and w are on the same trajectory in the Morse dynamical system.

This ends the study of the third case described after the statement of Theorem 2.1 and concludes the proof. □

Remark 1. — The space $\Omega\times\mathbb{Z}_2$ is equiped with the product measure $\mu\otimes m$. The proof of Theorem 2.1 gives us an exhaustive description of the ergodic invariant probability measures on the Cartesian square of the system $(\Omega\times\mathbb{Z}_2,T_\varphi)$. These are the measure $(\mu\otimes_{\omega^0}\mu)\otimes m\otimes m$ for $\omega^0\in\Omega\setminus\mathbb{Z}$ and the images of the diagonal measure on $(\Omega\times\mathbb{Z}_2)^2$ by the maps

$$(\omega,z,\omega',z')\mapsto\Big(\omega,z,\omega'+n,z'+\varphi^{(n)}(\omega)\Big)$$

and

$$(\omega,z,\omega',z')\mapsto\Big(\omega,z,\omega'+n,z'+1+\varphi^{(n)}(\omega)\Big)\ ,$$

for $n\in\mathbb{Z}$.

The formula

$$\mu\otimes\mu\otimes m\otimes m=\int_\Omega(\mu\otimes_{\omega^0}\mu)\otimes m\otimes m\ \mathrm{d}\mu(\omega^0)\ .$$

is an ergodic disintegration of the product measure on $(\Omega\times\mathbb{Z}_2)^2$.

2.2. Cartesian cube. —

Theorem 2.3. — *In the Cartesian cube of the Morse dynamical system, every point is generic for some measure.*

As in the case of the Cartesian square, it is possible to give a precise description of the possible limit measures which appear in the Cartesian cube. If among the three points v,w,x of the Morse dynamical system no pair of them satisfies condition 1 or 2 stated after Theorem 2.1, then the triplet (v,w,x) is generic in the dynamical system $\left(K_u^3,\sigma\times\sigma\times\sigma\right)$ for a relatively independent joining of three copies of (K_u,ν) above their common factor (Ω,μ).

Theorem 2.3 is a consequence of the following proposition.

Proposition 2.4. — *If k is an odd positive number and if $\omega^1,\omega^2,\ldots,\omega^k$ are elements of Ω, then the two point extension of the odometer defined by the cocycle*

$$\Omega\to\mathbb{Z}_2,\quad\omega\mapsto\left(\varphi(\omega+\omega^1)+\varphi(\omega+\omega^2)+\ldots+\varphi(\omega+\omega^k)\right)$$

is ergodic.

How Theorem 2.3 is deduced from Proposition 2.4. — Following the same arguments as in the proof of Theorem 2.1, we just have to prove that, for all $\omega^1,\omega^2,\omega^3\in\Omega$, in the eight point extension of the odometer defined by the cocycle

$$\Omega\to\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,\quad\omega\mapsto\left(\varphi(\omega+\omega^1),\varphi(\omega+\omega^2),\varphi(\omega+\omega^3)\right)$$

every point is generic for some measure.

If we consider three elements $\omega^1,\omega^2,\omega^3$ of Ω such that one of the differences $\omega^a-\omega^b$ (with $a\neq b$) is in $\mathbb{Z}$, then we immediately come back to the previous study of the generic points in the Cartesian square of the Morse system. If none of these differences is in $\mathbb{Z}$, then the eight point extension of the odometer defined by the cocycle

$$\Omega \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \omega \mapsto \big(\varphi(\omega+\omega^1), \varphi(\omega+\omega^2), \varphi(\omega+\omega^3)\big)$$

is ergodic.

Indeed we claim that none of the seven functional equations given by Condition (1.1) has a solution. One of these claims follows from Proposition 2.4 (applied with $k=3$), three of them follow from Proposition 2.2 and the remaining three follow from the ergodicity of the Morse dynamical system (there are no measurable solutions to equation (1.2)). □

The proof of Proposition 2.4 will be described in several steps.

If $n,k \in \mathbb{N}$, with $k>0$, we define the finite set $\Omega_{n,k}$ by

$$\Omega_{n,k} := \{\omega \in \Omega \,:\, \omega_i = 0 \ \text{ if } \ i<n \ \text{ or } \ i \geq n+k\}.$$

Notice that $\Omega_{0,k}$ is the interval of integers $[0,2^k)$ and that $\Omega_{n,k} = 2^n\Omega_{0,k}$.

Lemma 2.5. — *Let $n,k \in \mathbb{N}$, with k odd. For all $\omega^1, \omega^2, \ldots, \omega^k \in \Omega$, there exist $\omega^0, \omega^{0'} \in \Omega_{n,2k}$ such that,*

$$\sum_{\ell=1}^{k} \varphi^{(2^n)}\left(\omega^0+\omega^\ell\right) = 1 \qquad \text{and} \qquad \sum_{\ell=1}^{k} \varphi^{(2^n)}\left(\omega^{0'}+\omega^\ell\right) = 0\,.$$

Proof. — For all $\omega \in \Omega$, we have $\varphi^{(2^n)}(\omega) = \varphi(\omega_n, \omega_{n+1}, \omega_{n+2}, \ldots)$, and this shows that it is sufficient to prove the lemma in the case when $n=0$. We denote $\Omega_k := \Omega_{0,k}$.

Let us prove by induction on k that, for all $\omega^1, \omega^2, \ldots, \omega^k \in \Omega$, there exist $\omega^0, \omega^{0'} \in \Omega_{2k}$ such that,

$$\sum_{\ell=1}^{k} \varphi\left(\omega^0+\omega^\ell\right) = 1 \qquad \text{and} \qquad \sum_{\ell=1}^{k} \varphi\left(\omega^{0'}+\omega^\ell\right) = 0\,.$$

The case $k=1$ is easy : if ω^1 is given there exists $\omega^0 \in \Omega_1$ such that

$$(\omega^0+\omega^1)_0 = 0, \qquad \text{hence} \quad \varphi(\omega^0+\omega^1) = 1\,,$$

and there exists $\omega^{0'} \in \Omega_2$ such that

$$\Big((\omega^{0'}+\omega^1)_0, (\omega^{0'}+\omega^1)_1\Big) = (1,0), \qquad \text{hence} \quad \varphi(\omega^{0'}+\omega^1) = 0\,.$$

Let now k be an odd number ≥ 3, and $\omega^1, \omega^2, \ldots, \omega^k \in \Omega$. If all the ω_0^ℓ are equal to 0, then we choose $\omega^0 = \mathbf{0}$. If all the ω_0^ℓ are equal to 1, then we choose $\omega^0 = \mathbf{1}$. If there are 0's and 1's in the family $(\omega_0^\ell)_{1\leq\ell\leq k}$ we choose $\alpha = \mathbf{0}$ or $\alpha = \mathbf{1}$ in order to have an odd number k' of 1's in the family $\big((\alpha+\omega^\ell)_0\big)_{1\leq\ell\leq k}$.

Note that $k' < k$. Applying if necessary a permutation to the family $\left(\omega^\ell\right)_{1\le\ell\le k}$ we can suppose that

$$(\alpha+\omega^\ell)_0 = \begin{cases} 1 & \text{if } 1\le \ell\le k'\ , \\ 0 & \text{if } k' < \ell \le k\ . \end{cases}$$

Using the induction hypothesis, we claim that there exists $\beta\in\Omega_{1,1+2k'}$ such that

$$\sum_{\ell=1}^{k'} \varphi^{(2)}\left(\beta+\alpha+\omega^\ell\right) = 0\ .$$

For ℓ between 1 and k' we have $\varphi\left(\beta+\alpha+\omega^\ell\right) = 1+\varphi^{(2)}\left(\beta+\alpha+\omega^\ell\right)$. (Because $(\beta+\alpha+\omega^\ell)_0 = 1$.)
For ℓ larger than k' (there is an even number of such ℓ's), $\varphi\left(\beta+\alpha+\omega^\ell\right) = 1$.
We set $\omega^0 = \beta+\alpha$ and we conclude that

$$\sum_{\ell=1}^{k} \varphi\left(\omega^0+\omega^\ell\right) = 1\ .$$

We notice that $\omega^0\in\Omega_{2k-1}$.

Let us choose $\alpha' = \mathbf{0}$ or $\alpha' = \mathbf{1}$ in order to have an odd number k' of 1's in the family $\left((\alpha'+\omega^\ell)_0\right)_{1\le\ell\le k}$. (Here we can have $k' = k$). Applying if necessary a permutation to the family $\left(\omega^\ell\right)_{1\le\ell\le k}$ we can suppose that

$$(\alpha'+\omega^\ell)_0 = \begin{cases} 1 & \text{if } 1\le \ell\le k'\ , \\ 0 & \text{if } k' < \ell \le k\ . \end{cases}$$

If $k' < k$ we can use the induction hypothesis, and if $k' = k$ we can use the above discussion, in order to claim that there exists $\beta'\in\Omega_{1,2k-1}$ such that

$$\sum_{\ell=1}^{k'} \varphi^{(2)}\left(\beta'+\alpha'+\omega^\ell\right) = 1\ .$$

For ℓ between 1 and k' we have $\varphi\left(\beta'+\alpha'+\omega^\ell\right) = 1+\varphi^{(2)}\left(\beta'+\alpha'+\omega^\ell\right)$.
For ℓ larger than k' (if there exists one) we have $\varphi\left(\beta'+\alpha'+\omega^\ell\right) = 1$.
We set $\omega^{0'} = \beta'+\alpha'$ and we conclude that

$$\sum_{\ell=1}^{k} \varphi\left(\omega^{0'}+\omega^\ell\right) = 0\ .$$

We notice that $\omega^{0'}\in\Omega_{2k}$. Our induction procedure is now complete. □

Lemma 2.6. — *If k is an odd positive number and if $\omega^1, \omega^2, \ldots, \omega^k$ are elements of Ω, then, for all $n \in \mathbb{N}$,*

$$\mu\left\{\omega \in \Omega \,:\, \sum_{\ell=1}^{k} \varphi^{(2^n)}\left(\omega + \omega^\ell\right) = 1\right\} \geq 4^{-k} \,.$$

Proof. — Let us denote $E_n := \left\{\omega \in \Omega \,:\, \sum_{\ell=1}^{k} \varphi^{(2^n)}\left(\omega + \omega^\ell\right) = 1\right\}$. From Lemma 2.5 (applied to the family $\left(\omega + \omega^\ell\right)_{1\leq \ell \leq k}$) we deduce that, for all $\omega \in \Omega$ there exists $\omega^0 \in \Omega_{n,2k}$ such that $\omega^0 + \omega \in E_n$. This implies that $\mu\left(E_n\right) \geq 1/\mathrm{card}\left(\Omega_{n,2k}\right)$. □

Proof of Proposition 2.4. — Let us denote by ρ the cocycle

$$\rho(\omega) := \varphi(\omega + \omega^1) + \varphi(\omega + \omega^2) + \ldots + \varphi(\omega + \omega^k) \,.$$

Lemma 2.6 tells us that, for all $n \in \mathbb{N}$, $\mu\left\{\rho^{(2^n)} = 1\right\} \geq 4^{-k}$. This of course implies that the sequence $\left(\rho^{(2^n)}\right)$ does not go to zero in probability, and we conclude by applying Proposition 1.2. □

2.3. Other joinings. — We describe some conditions on a family $\left(\omega^1, \omega^2, \ldots, \omega^k\right)$ of elements of Ω under which the conclusion of Proposition 2.4 is true, even in the case when k is even. The following lemma will play a similar role to Lemma 2.5.

Lemma 2.7. — *Let $\omega^1, \omega^2, \ldots, \omega^k \in \Omega$ and $n, j \in \mathbb{N}$, with $j > 0$. If among the k finite words $\left(\omega_i^\ell\right)_{n\leq i<n+j}$, $1 \leq \ell \leq k$, one of the possible sequence of 0's and 1's appears an odd number of times, then there exists $\omega^0 \in \Omega_{n,j+2k}$ such that*

$$\sum_{\ell=1}^{k} \varphi^{(2^n)}\left(\omega^0 + \omega^\ell\right) = 1 \,.$$

Proof. — The digits of indices less than n play no role in this statement, and it is sufficient to give a proof for $n = 0$. We suppose that among the k finite words $\left(\omega_i^\ell\right)_{0\leq i<j}$, $1 \leq \ell \leq k$, one sequence appears an odd number of times. We use an element of Ω_j in order to translate this particular word to the word $1^j = (1, 1, \ldots, 1)$. There exists $\alpha \in \Omega_j$ such that the set

$$E := \left\{\ell \in \{1, \ldots, k\} \,:\, \left((\alpha + \omega^\ell)_i\right)_{0\leq i<j} = 1^j\right\}$$

has an odd number of elements. For all $\ell \notin E$ at least one 0 appears in the finite word $\left((\alpha + \omega^\ell)_i\right)_{0\leq i<j}$. We set $\epsilon := \sum_{\ell\notin E} \varphi\left(\alpha + \omega^\ell\right)$. Lemma 2.5 implies that there exists $\beta \in \Omega_{j,2k}$ such that $\sum_{\ell\in E} \varphi\left(\beta + \alpha + \omega^\ell\right) = 1 - \epsilon$. To see this, note that if ($\epsilon = 0$ and j is even) or if ($\epsilon = 1$ and j is odd), we can choose β such that $\sum_{\ell\in E} \varphi^{(2^j)}\left(\beta + \alpha + \omega^\ell\right) = 1$; if ($\epsilon = 0$ and j is odd) or if ($\epsilon = 1$ and j is even), we can choose β such that $\sum_{\ell\in E} \varphi^{(2^j)}\left(\beta + \alpha + \omega^\ell\right) = 0$.

For all ℓ outside E, we have $\varphi(\beta+\alpha+\omega^\ell)=\varphi(\alpha+\omega^\ell)$.
We set $\omega^0 := \alpha+\beta$. We have $\omega^0 \in \Omega_{j+2k}$ and $\sum_{\ell=1}^k \varphi\left(\omega^0+\omega^\ell\right)=1$. □

We define a family of characters on Ω by

$$\chi_n : \Omega \to \mathbb{R}/\mathbb{Z}\,, \qquad \chi_n(\omega)=\sum_{i=0}^{n-1} 2^{i-n}\omega_i .$$

We write d for the distance on the circle $\mathbb{R}/\mathbb{Z}$ and write $d_n(x,y)$ for $d(\chi_n(x),\chi_n(y))$.

Let $k,j\in\mathbb{N}$ and $\left(\omega^1,\omega^2,\dots,\omega^k\right)$ be a family of elements of Ω. Say that the family is *j-separated at the nth stage* if the set of indices $\{1,2,\dots,k\}$ can be divided into two subsets A and B such that A contains an odd number of elements and such that, for all $\ell\in A$ and all $m\in B$, $d\left(\chi_n\left(\omega^\ell\right),\chi_n\left(\omega^m\right)\right) > 2^{-j}$. Notice that if k is odd this condition is trivially satisfied. Notice also that this condition of separation is stable under translation.

Lemma 2.8. — *If the family $\left(\omega^1,\omega^2,\dots,\omega^k\right)$ is j-separated at the $(n+j)$th stage, then among the k finite words $\left(\omega_i^\ell\right)_{n\le i<n+j}$, $1\le\ell\le k$, one of the possible sequences of 0's and 1's appears an odd number of times.*

Proof. — The parameters k, j and n are fixed. We use the notations A and B given by the definition of our notion of separation. We denote by A' (resp. B') the set of points $\chi_{n+j}\left(\omega^\ell\right)$ for $\ell\in A$ (resp. $\ell\in B$). The circle is represented as the unit interval $[0,1)$. The letter h represents an integer between 0 and 2^j. Let us consider the collection of dyadic intervals $\left[h2^{-j},(h+1)2^{-j}\right)$ containing at least one element of A'. These intervals do not contain any element of B'. Since the cardinality of A is odd, one of these dyadic intervals contains an odd number of points $\chi_{n+j}\left(\omega^\ell\right)$. □

Lemma 2.9. — *If the family $\left(\omega^1,\omega^2,\dots,\omega^k\right)$ is j-separated at the $(n+j)$th stage, then*

$$\mu\left\{\omega\in\Omega\,:\,\sum_{\ell=1}^k \varphi^{(2^n)}\left(\omega+\omega^\ell\right)=1\right\}\ge 2^{-j-2k}\,.$$

Proof. — Let us denote $E_n := \left\{\omega\in\Omega\,:\,\sum_{\ell=1}^k \varphi^{(2^n)}\left(\omega+\omega^\ell\right)=1\right\}$. Suppose that the family $\left(\omega^\ell\right)$ is j-separated at the $(n+j)$th stage. Then, for all $\omega\in\Omega$, the family $\left(\omega+\omega^\ell\right)$ is j-separated at the $(n+j)$th stage and this implies (Lemma 2.8) that among the k finite words $\left((\omega+\omega^\ell)_i\right)_{n\le i<n+j}$, $1\le\ell\le k$, one word appears an odd number of times. From Lemma 2.7, we deduce now that for all $\omega\in\Omega$ there exists $\omega^0\in\Omega_{n,j+2k}$ such that $\omega^0+\omega\in E_n$. We conclude that $\mu\left(E_n\right)\ge 1/\mathrm{card}\left(\Omega_{n,j+2k}\right)$. □

THEOREM 2.10. — *If a finite family $\left(\omega^1, \omega^2, \ldots, \omega^k\right)$ in Ω is j-separated at the nth stage for a fixed j and infinitely many n, then the two point extension of the odometer defined by the cocycle*

$$\Omega \to \mathbb{Z}_2, \quad \omega \mapsto \left(\varphi(\omega+\omega^1)+\varphi(\omega+\omega^2)+\ldots+\varphi(\omega+\omega^k)\right)$$

is ergodic.

Proof. — Let us denote by ρ the cocycle

$$\rho(\omega) := \varphi(\omega+\omega^1)+\varphi(\omega+\omega^2)+\ldots+\varphi(\omega+\omega^k)\ .$$

Lemma 2.9 tells us that, for infinitely many n, $\mu\left\{\rho^{(2^n)}=1\right\} \geq 2^{-j-2k}$. The sequence $\left(\rho^{(2^n)}\right)$ does not go to zero in probability, and we conclude by Proposition 1.2. □

COROLLARY 2.11. — *The Morse dynamical system is weakly disjoint from any of its Cartesian powers.*

The proof of this corollary uses the following simple lemma.

LEMMA 2.12. — *Let $k \geq 2$ and j such that $2^j \geq 2k$ be fixed. For almost all $\left(\omega^1, \omega^2, \ldots, \omega^k\right) \in \Omega^k$, for all $\omega^0 \in \Omega$, the family $\left(\omega^0, \omega^1, \omega^2, \ldots, \omega^k\right)$ is j-separated at the nth stage for infinitely many n.*

Proof. — Consider a collection $(b_1, b_2, \ldots, b_k)$ of distinct blocks of 0's and 1's with length j and beginning with a 0. For almost all $\left(\omega^1, \omega^2, \ldots, \omega^k\right) \in \Omega^k$, there exist infinitely many n's such that, for all ℓ between 1 and k, $(\omega^\ell_i)_{n-j\leq i<n} = b_\ell$. For each such $\left(\omega^1, \omega^2, \ldots, \omega^k\right)$, for each such n, for all $\omega^0 \in \Omega$, the family $\left(\omega^0, \omega^1, \omega^2, \ldots, \omega^k\right)$ is j-separated at the nth stage. □

Proof of Corollary 2.11. — From Theorem 2.10 and Lemma 2.12, we deduce that, for any $k \geq 1$, there exists a set of full measure Ω_k in Ω^k such that, for all $\left(\omega^1, \omega^2, \ldots, \omega^k\right) \in \Omega_k$, for all $\omega^0 \in \Omega$ and for all subset S of $\{1, 2, \ldots, k\}$ with at least two elements, the two point extension of the odometer defined by the cocycle

$$\Omega \to \mathbb{Z}_2, \quad \omega \mapsto \left(\varphi(\omega+\omega^0)+\sum_{\ell\in S}\varphi(\omega+\omega^\ell)\right)$$

is ergodic. If the set S contains only one element this cocycle is not necessary ergodic, but there is no problem of convergence of ergodic averages (cf Section 2.1).

We conclude that for all $\omega^0 \in \Omega$ and all $\omega^{(k)} \in \Omega_k$, the point $(\omega^0, \omega^{(k)})$ is generic in the product of the dynamical system $(\Omega \times \mathbb{Z}_2, T_\varphi)$ with its kth Cartesian power.

□

Corollary 2.13. — *The Morse dynamical system is weakly disjoint from any ergodic k-fold joining of itself.*

The proof of this corollary uses the following lemma.

Lemma 2.14. — *Let* $\left(\omega^1, \omega^2, \ldots, \omega^{2k-1}\right) \in \Omega$. *Let* j *be a positive integer such that* $2^j \geq 8k$. *For all* $n_0 \in \mathbb{N}$, *for all* $\omega \in \Omega$ *but finitely many, there exists* $n \geq n_0$ *such that the family* $\left(\omega, \omega^1, \omega^2, \ldots, \omega^{2k-1}\right)$ *is* j*-separated at the* n*th stage.*

Proof. — Let us fix $n \geq 0$ and denote by $x_1, x_2, \ldots, x_{2k-1}$ the points $\chi_n(\omega^1), \chi_n(\omega^2), \ldots, \chi_n(\omega^{2k-1})$ arranged in increasing order on the circle. Let us define a component to be any maximal subset of $\{x_1, x_2, \ldots, x_{2k-1}\}$ of the form $\{x_\ell, x_{\ell+1}, \ldots, x_{\ell+t}\}$ with $d(x_{\ell+s}, x_{\ell+s+1}) \leq 2^{-j}$ if $0 \leq s < t$. One of these components is such that $t+1$ is odd. We choose one such component that we denote by $\{x_\ell, x_{\ell+1}, \ldots, x_{\ell+t}\}$. If the family $\left(\omega, \omega^1, \omega^2, \ldots, \omega^{2k-1}\right)$ is not j-separated at the nth stage, then the point $\chi_n(\omega)$ is inside the arc $[x_\ell - 2^{-j}, x_{\ell+t} + 2^{-j}]$. The length L of this arc satisfies $L \leq t2^{-j} + 2 \cdot 2^{-j} \leq 2k2^{-j} \leq 1/4$.

We conclude that, for all n, there exists an arc I_n of length $1/4$ such that, if the family $\left(\omega, \omega^1, \omega^2, \ldots, \omega^{2k-1}\right)$ is not j-separated at the nth stage, then the point $\chi_n(\omega)$ belongs to I_n.

Notice that, for all $n > 0$, we have $2\chi_n(\omega) = \chi_{n-1}(\omega)$. Hence, if $\chi_{n-1}(\omega)$ is known and if we know that $\chi_n(\omega) \in I_n$, then the point $\chi_n(\omega)$ is uniquely determined.

By iteration, we obtain the following : if $\chi_{n_0}(\omega)$ is known and if, for all $n > n_0$ the family $\left(\omega, \omega^1, \omega^2, \ldots, \omega^{2k-1}\right)$ is not j-separated at the nth stage, then the sequence $\chi_n(\omega)$ (hence ω itself) is uniquely determined. This gives a proof of Lemma 2.14 because $\chi_{n_0}(\omega)$ can take only finitely many values. □

Proof of Corollary 2.13. — Let us fix $\omega^1, \omega^2, \ldots, \omega^\ell$ in Ω, and j big enough. By Lemma 2.14 we know that, for all ω^0 outside a countable subset C of Ω, the family $\left(\omega^0, \omega^1, \ldots, \omega^\ell\right)$ is j-separated at an infinity of stages. For all $\omega^0 \notin C$, the cocycle

$$\Omega \to \mathbb{Z}_2, \quad \omega \mapsto \left(\varphi(\omega + \omega^0) + \varphi(\omega + \omega^1) + \ldots + \varphi(\omega + \omega^\ell)\right)$$

is ergodic. This is our first claim.

Now fix $\omega^0 \in C$. By the ergodic theorem, there exists a set $Z(\omega^0)$ of zero measure in Ω such that, for all $\omega \notin Z(\omega^0)$, the point $(\omega, 0)$ is generic in the finite group extension of the odometer defined by the cocycle

$$\Omega \to \mathbb{Z}_2^{\ell+1}, \quad \omega \mapsto \left(\varphi(\omega + \omega^0), \varphi(\omega + \omega^1), \ldots, \varphi(\omega + \omega^\ell)\right) .$$

This is our second claim.

Let us consider the full measure subset Ω' of Ω defined by $\Omega \setminus \Omega' := \bigcup_{\omega^0 \in C} Z(\omega^0)$. If $\omega' \in \Omega'$ and $\omega \in \Omega$ then we are in one of the two following cases :

$$\omega - \omega' \notin C \qquad \text{or} \qquad \omega' \notin Z(\omega - \omega') .$$

In these two cases, the averages

$$\frac{1}{N}\sum_{n<N} \chi(n)\cdot c\left(\varphi^{(n)}(\omega) + \sum_{i=1}^{\ell} \varphi^{(n)}(\omega' + \omega^j)\right) \tag{2.2}$$

do converge. (Here χ is any character of the odometer, and $c(x) := (-1)^x$). Indeed, each of these two cases correspond to one of the preceding claims.

This is enough to prove the Corollary since every k-fold joining of the Morse dynamical system is isomorphic to an extension of the odometer by a finite product $\mathbb{Z}_2^m$ defined by a cocycle of the form

$$\omega \mapsto \big(\varphi(\omega+\omega^1), \varphi(\omega+\omega^2), \dots, \varphi(\omega+\omega^m)\big) .$$

Thus if we want to prove the weak disjointness with the Morse dynamical system, we just have to test the convergence on characters of the products of groups, and we find expressions of the form (2.2). □

Corollary 2.15. — *The Cartesian square of the Morse dynamical system is weakly disjoint from all its ergodic components. More precisely, for any ergodic invariant probability measure λ on the dynamical system $(K_u \times K_u, \sigma\times\sigma)$ the measure preserving systems $(K_u\times K_u, \nu\otimes\nu, \sigma\times\sigma)$ and $(K_u\times K_u, \lambda, \sigma\times\sigma)$ are weakly disjoint.*

Proof. — As before we use the desciption of the Morse dynamical system as a two point extension of the odometer. The description of the ergodic invariant measures λ on the Cartesian square is given in Remark 1. Either the system $\big((\Omega\times\mathbb{Z}_2)^2, \lambda, T_\varphi\times T_\varphi\big)$ is isomorphic to the Morse dynamical system, or the measure λ is of the form $(\mu\otimes_{\omega^0}\mu)\otimes m\otimes m$, with $\omega^0\in\Omega$. In the first situation, the weak disjointness property is already known. In order to prove this property in the second situation, it is sufficient to prove that for all $\omega^0\in\Omega$, there exists a subset Ω_2 of $\Omega\times\Omega$ of full measure, such that, for all $\omega^1\in\Omega$ and all $(\omega^2,\omega^3)\in\Omega_2$, the cocycle

$$\Omega\to\mathbb{Z}_2, \quad \omega\mapsto \big(\varphi(\omega+\omega^1)+\varphi(\omega+\omega^1+\omega^0)+\varphi(\omega+\omega^2)+\varphi(\omega+\omega^3)\big)$$

is ergodic. We claim that, for all $\omega^0\in\Omega$ there exists a subset Ω_2 of $\Omega\times\Omega$ of full measure such that for all $\omega^1\in\Omega$ and all $(\omega^2,\omega^3)\in\Omega_2$, the family $(\omega^1, \omega^1+\omega^0, \omega^2, \omega^3)$ is infinitely often 6-separated. The result then follows by Theorem 2.10.

Let us prove the claim. We will distinguish two cases. Let us suppose first that the sequence ω^0 contains infinitely many blocks of 4 consecutive 0's. Let (n_j) be an increasing sequence of integers such that $\omega^0_i = 0$ if $n_j - 4 \le i < n_j$. For all ω and all j, we have

$$d\left(\chi_{n_j}(\omega), \chi_{n_j}(\omega+\omega^0)\right) < 1/8.$$

Now fix $n > 1$. If $\omega^2 = (\omega_i^2)_{i\geq 0}$ and $\omega^3 = (\omega_i^3)_{i\geq 0}$ are such that $(\omega_{n-2}^2, \omega_{n-1}^2) = (0,1)$ and $(\omega_{n-2}^3, \omega_{n-1}^3) = (0,0)$, then $d\left(\chi_n(\omega^2), \chi_n(\omega^3)\right) > 1/4$. We call Ω_2 the set of pairs (ω^2, ω^3) such that, for infinitely many j, we have

$$d\left(\chi_{n_j}(\omega^2), \chi_{n_j}(\omega^3)\right) > 1/4.$$

This set Ω_2 has full $\mu \otimes \mu$-measure. If $\omega^1 \in \Omega$ and if $(\omega^2, \omega^3) \in \Omega_2$, then, for infinitely many n, we have $d\left(\chi_n(\omega), \chi_n(\omega + \omega^0)\right) < 1/8$, and $d\left(\chi_n(\omega^2), \chi_n(\omega^3)\right) > 1/4$, which implies that the family $(\omega^1, \omega^1 + \omega^0, \omega^2, \omega^3)$ is 4-separated at the nth stage. This proves our claim in the first case.

Let us consider the other case. We have $d(\chi_n(\omega^0), \chi_n(0)) > 1/16$ eventually. We call now Ω_2 the set of all pairs (ω^2, ω^3) such that, for infinitely many n's, $d(\chi_n(\omega^2), \chi_n(\omega^3)) < 1/32$. This set Ω_2 has full $\mu \otimes \mu$-measure. Moreover if $\omega^1 \in \Omega$ and if $(\omega^2, \omega^3) \in \Omega_2$, then the family $(\omega^1, \omega^1 + \omega^0, \omega^2, \omega^3)$ is infinitely often 6-separated. This proves our claim in the second case. □

3. Divergence results

3.1. Construction of diverging averages. — We want to show now that in the Cartesian product of four copies of the Morse dynamical system some points are not generic. We need to consider some families of elements of Ω which do not satisfy the separation property described in the previous section.

We define, for each positive integer a, four words on the letters 0 and 1:

$$X_1(a) := 0^a 1^a 000^a ,$$

$$X_2(a) := 0^a 0^a 100^a ,$$

$$X_3(a) := 0^a 1^a 100^a ,$$

$$X_4(a) := 0^a 0^a 010^a .$$

REMARK 2. — (This remark is not used in the sequel of the article but it is useful to understand the link with the notion of separation.) Denote by $b := 3a + 2$ the length of the words X_i. We see that, when a is large the points $\chi_b(X_1)$ and $\chi_b(X_2)$ are near on the circle and the same goes for the points $\chi_b(X_3)$ and $\chi_b(X_4)$. More precisely, if W_ℓ, $1 \leq \ell \leq 4$, are finite words (on the letters 0 and 1) of the same length b' and if ω^ℓ, $1 \leq \ell \leq 4$, are elements of Ω such that $\omega^\ell = W_\ell X_\ell \ldots$, then the family $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ is not $2a+1$-separated at the $(b + b')$th stage.

We define also words $Y_\ell(a)$, $\ell = 1, 2, 3, 4$, by

$$Y_\ell(a) := X_\ell(a) 1^{a+2} .$$

These words Y_ℓ have been chosen in such a way that they cannot overlap (without coincidence) in an infinite sequence of 0's and 1's. That is to say, if

$i, j \in \{1,2,3,4\}$, it is not possible to find three words m, m' and m'', of positive length, such that

$$Y_i(a) = mm' \quad \text{and} \quad Y_j(a) = m'm'' \ .$$

Let $(a_k)_{k\geq 1}$ be a fixed sequence of positive integers. Almost every element ω of Ω can be written in a unique way in the form

$$\omega = W_1 Y_{\ell_1}(a_1) W_2 Y_{\ell_2}(a_2) \ldots W_k Y_{\ell_k}(a_k) \ldots \tag{3.1}$$

where indices ℓ_k are in $\{1,2,3,4\}$ and the W_k's are words of 0's and 1's, of minimal length. We'll call such a writing the *standard description* of ω. Because of the non overlapping property, we have the following claim.

Claim 1. — *If* (3.1) *is the standard description of an element of* Ω*, and if* (ℓ'_k) *is any sequence of indices in* $\{1,2,3,4\}$*, then*

$$W_1 Y_{\ell'_1}(a_1) W_2 Y_{\ell'_2}(a_2) \ldots W_k Y_{\ell'_k}(a_k) \ldots$$

is the standard description of an element of Ω*.*

As a consequence of this claim we have the following lemma.

Let σ be a permutation of $\{1,2,3,4\}$. We define a transformation U of Ω by the following rule: if (3.1) is the standard description of ω then

$$U(\omega) := W_1 Y_{\sigma(\ell_1)}(a_1) W_2 Y_{\sigma(\ell_2)}(a_2) \ldots W_k Y_{\sigma(\ell_k)}(a_k) \ldots \ .$$

Lemma 3.1. — *The transformation* U*, defined on a measurable set of full measure, is bijective, bimeasurable and measure preserving.*

Let us consider now four elements of Ω which can be written

$$\omega^1 := W_{1,1} X_{\ell(1,1)}(a_1) W_{1,2} X_{\ell(1,2)}(a_2) \ldots W_{1,k} X_{\ell(1,k)}(a_k) \ldots \ ,$$

$$\omega^2 := W_{2,1} X_{\ell(2,1)}(a_1) W_{2,2} X_{\ell(2,2)}(a_2) \ldots W_{2,k} X_{\ell(2,k)}(a_k) \ldots \ ,$$

$$\omega^3 := W_{3,1} X_{\ell(3,1)}(a_1) W_{3,2} X_{\ell(3,2)}(a_2) \ldots W_{3,k} X_{\ell(3,k)}(a_k) \ldots \ ,$$

$$\omega^4 := W_{4,1} X_{\ell(4,1)}(a_1) W_{4,2} X_{\ell(4,2)}(a_2) \ldots W_{4,k} X_{\ell(4,k)}(a_k) \ldots \ .$$

We suppose that the length m_k of the word $W_{\ell,k}$ does not depend on ℓ, and that, for each k, the four words $W_{\ell,k}$, $\ell = 1,2,3,4$, are two pairs of identical words (or four identical words). We suppose also that, for each k, the four indices $\ell(j,k)$, $j = 1,2,3,4$, are pairwise distinct, so that they describe exactly the set $\{1,2,3,4\}$. Finally, we suppose that

$$\sum_{k\geq 1} 2^{-a_k} < \frac{1}{8} \ .$$

Let us denote by c the non trivial character of $\mathbb{Z}_2$, $c(x) = (-1)^x$.

Lemma 3.2. — *The sequence*

$$\frac{1}{N}\sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega^1)+\varphi^{(n)}(\omega^2)+\varphi^{(n)}(\omega^3)+\varphi^{(n)}(\omega^4)\right)$$

does not converge.

Proof. — A sketch of the proof is the following : if we look at the words $(X_\ell(a))$ as integer numbers written in base 2 and if we denote by $S(n)$ the sum of the digits of n written in base 2, then we can see the number $S(X_1(a)) + S(X_2(a)) + S(X_3(a)) + S(X_4(a))$ is odd, but for the great majority of $n's$, the number $S(n+X_1(a)) + S(n+X_2(a)) + S(n+X_3(a)) + S(n+X_4(a))$ is even. The fact that the words $(X_\ell(a))$ begin and finish with long sequences of 0's implies that this "local" information (the position of the blocks $(X_\ell(a))$ in ω^ℓ) is preserved when we add infinite sequences.

Let us go into more detail. We set

$$c_k := m_k + \sum_{j=1}^{k-1}(m_j+3a_j+2), \quad d_k := c_k + a_k, \quad e_k := d_k + a_k \ ,$$

and

$$N_k := \sum_{j=1}^{k}(m_j+3a_j+2) \ .$$

(In the sequences ω^ℓ, the first "a-block" of the kth word X lies between indices c_k and d_k-1, the second "a-block" of the kth word X lies between indices d_k and e_k-1, the last "a-block" lies between indices e_k+2 and N_k-1.)

Step 1. — *Let n be an integer, with $0\le n<2^{N_k}$, whose dyadic expansion is $n=\sum_{i=0}^{N_k-1} n_i 2^i$. If, for all j between 1 and k, we have*

$$(n_i)_{c_j\le i<d_j} \neq 1^a \ , \tag{3.2}$$

$$(n_i)_{d_j\le i<e_j} \neq 1^a \text{ and } (n_i)_{d_j\le i<e_j} \neq 0^a \ , \tag{3.3}$$

and

$$(n_i)_{e_j+2\le i<N_j} \neq 1^a \ , \tag{3.4}$$

then

$$\sum_{\ell=1}^{4}\varphi^{(n)}(\omega^\ell) = \sum_{i=0}^{+\infty}\sum_{\ell=1}^{4}(\omega^\ell+n)_i-\omega_i^\ell = k \quad \text{mod.}2 \ .$$

This statement can be verified along the following lines. First notice that, since $n < 2^{N_k}$ and since $\omega^\ell_{N_k+m_{k+1}} = 0$, we have $(\omega^\ell + n)_i = \omega^\ell_i$ if $i > N_k + m_{k+1}$. Then, we see directly from the definition of the ω^ℓ's that

$$\sum_{i=0}^{N_k+m_{k+1}} \sum_{\ell=1}^{4} \omega^\ell_i = k \quad \text{mod.}2 \ .$$

(Indeed, for all j, the words $W_{\ell,j}$, $1 \leq \ell \leq 4$ are pairwise identical and $\sum_{\ell=1}^{4} S(X_\ell(a_j))$ is odd.)

It then remains to verify that

$$\sum_{i=0}^{N_k+m_{k+1}} \sum_{\ell=1}^{4} (\omega^\ell + n)_i \text{ is even .}$$

We begin by studying the case $k = 1$. Permuting the ω^ℓ if necessary, we can suppose that $\ell(j, 1) = j$, $1 \leq j \leq 4$.

Because the initial words of length $d_1 = m_1 + a_1$ of the sequences ω^ℓ are pairwise identical, we have

$$\sum_{i=0}^{d_1-1} \sum_{\ell=1}^{4} (\omega^\ell + n)_i \text{ is even .}$$

By condition (3.2), there is no carry at rank d_1 in the addition of ω^ℓ with n. Hence, if $d_1 \leq i < e_1 = m_1 + 2a_1$ we have

$$(\omega^1 + n)_i = (\omega^3 + n)_i \text{ and } (\omega^2 + n)_i = (\omega^4 + n)_i \ .$$

Due to condition (3.3), there is a carry at rank e_1 in the addition of ω^1 or ω^3 with n, but no carry at rank e_1 in the addition of ω^2 or ω^4 with n.

Looking at the words X_1 and X_2, and using the fact that there is a carry at rank e_1 in the addition of ω^1 with n and no carry at rank e_1 in the addition of ω^2 with n, we observe that

$$(\omega^1 + n)_i = (\omega^2 + n)_i \quad \text{if} \quad e_1 \leq i < N_1 = e_1 + a_1 + 2 \ .$$

Looking at the words X_3 and X_4, and using the fact that there is a carry at rank e_1 in the addition of ω^3 with n and no carry at rank e_1 in the addition of ω^4 with n, we observe that

$$(\omega^3 + n)_i = (\omega^4 + n)_i \quad \text{if} \quad e_1 \leq i < N_1 = e_1 + a_1 + 2 \ .$$

We have proved that

$$\sum_{i=0}^{N_1-1} \sum_{\ell=1}^{4} (\omega^\ell + n)_i \text{ is even .}$$

Furthermore, by condition (3.4), we know that there is no carry at rank N_1 in the addition of ω^ℓ with n, $1 \le \ell \le 4$. This means that when we begin the addition of ω^ℓ with n at the level of the block $W_{\ell,2}$, there is no trace of the previous calculations and the story described along the first N_1 digits repeats identically for the digits between indices N_1 and $N_2 - 1$. And so on, and so forth... This concludes the first step.

Step 2. — *If* $\sum_{k=1}^{+\infty} 2^{-a_k} < \frac{1}{8}$, *then the sequence*

$$2^{-N_k} \sum_{n=0}^{2^{N_k}-1} c\left(\sum_{t=1}^{4} \varphi^{(n)}(\omega^t)\right)$$

does not converge.

The proportion of numbers n between 0 and $2^{N_k} - 1$ which do not satisfy one of the conditions (3.2), (3.3) and (3.4) is less than $\sum_{j=1}^{k} 4 \cdot 2^{-a_j}$. Hence we deduce from Step 1 that

$$\left| \frac{1}{2^{N_k}} \sum_{n<2^{N_k}} c\left(\sum_{\ell=1}^{4} \varphi^{(n)}(\omega^\ell)\right) - (-1)^k \right| \le 2 \sum_{j=1}^{k} 4 \cdot 2^{-a_j} .$$

This is enough to justify Step 2, and Lemma 3.2 is proved. □

3.2. Cartesian fourth power. —

Theorem 3.3. — *Let A be a measurable subset of the subshift K_u with positive ν-measure. There exist four elements v^1, v^2, v^3, v^4 of A such that the sequence*

$$\frac{1}{N} \sum_{n=0}^{N-1} c\left(v_n^1 + v_n^2 + v_n^3 + v_n^4\right)$$

does not converge.

Corollary 3.4. — *Let $k \ge 4$. The Morse dynamical system is not self-weakly disjoint of order k. In particular, in the kth Cartesian power of the Morse dynamical system, there exist non generic points.*

Proof of Theorem 3.3. — Using our representation of the Morse dynamical system as a two point extension of the odometer $(\Omega, +)$, it is sufficient to prove that, for all measurable subsets B of Ω, with positive μ-measure, there exist four elements $\omega^1, \omega^2, \omega^3, \omega^4$ of B such that the sequence

$$\frac{1}{N} \sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega^1) + \varphi^{(n)}(\omega^2) + \varphi^{(n)}(\omega^3) + \varphi^{(n)}(\omega^4)\right)$$

does not converge.

Let $B \subset \Omega$ be given, with positive measure. If W is a finite word on the letters 0 and 1, we denote by C_W the cylinder set of all sequences in Ω which begin by the word W. Since $\mu(B) > 0$ there exist a finite word W such that $\mu(B \cap C_W)/\mu(C_W) > 3/4$. We will use Lemma 3.1 restricted to C_W. Replacing B by $B \cap C_W$, we suppose now that $B \subset C_W$.

Let us fix a sequence (a_k) such that $8\sum_{k\geq 1} 2^{-a_k} < 1$. Almost every element ω of C_W can be written in a unique way in the form

$$\omega = WW_1Y_{\ell_1}(a_1)W_2Y_{\ell_2}(a_2)\dots W_kY_{\ell_k}(a_k)\dots$$

where indices ℓ_k are in $\{1,2,3,4\}$ and the W_k's are words of 0's and 1's, of minimal length. We fix a circular permutation σ of $\{1,2,3,4\}$ and we consider the transformation U of C_W defined by

$$U(WW_1Y_{\ell_1}(a_1)\dots W_kY_{\ell_k}(a_k)\dots) = WW_1Y_{\sigma(\ell_1)}(a_1)\dots W_kY_{\sigma(\ell_k)}(a_k)\dots .$$

The transformation U preserves the conditional probability $\mu\left(\cdot|C_W\right)$, and since $\mu\left(B|C_W\right) > 3/4$, we have $B \cap U^{-1}(B) \cap U^{-2}(B) \cap U^{-3}(B) \neq \emptyset$. But Lemma 3.2 tells us that, for almost all $\omega \in C_W$ the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega) + \varphi^{(n)}(U(\omega)) + \varphi^{(n)}(U^2(\omega)) + \varphi^{(n)}(U^3(\omega))\right)$$

does not converge.

This concludes the proof of the Theorem. □

3.3. Cartesian square of the Cartesian square. — The following result is a reinforcement of Theorem 3.3.

Theorem 3.5. — *Let A be a measurable subset of $K_u \times K_u$ with positive $\nu\otimes\nu$-measure. There exist two elements (v^1, v^2) and (v^3, v^4) of A such that the sequence*

$$\frac{1}{N}\sum_{n=0}^{N-1} c\left(v_n^1 + v_n^2 + v_n^3 + v_n^4\right)$$

does not converge.

Corollary 3.6. — *The Cartesian square of the Morse dynamical system is not self-weakly disjoint.*

Proof of Theorem 3.5. — Using our representation of the Morse dynamical system as a two point extension of the odometer $(\Omega, +)$, it is sufficient to prove that, for all measurable subsets B of $\Omega\times\Omega$, with positive $\mu\otimes\mu$-measure, there exist two elements (ω^1, ω^2) and (ω^3, ω^4) of B such that the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega^1) + \varphi^{(n)}(\omega^2) + \varphi^{(n)}(\omega^3) + \varphi^{(n)}(\omega^4)\right)$$

does not converge.

Let $B \subset \Omega \times \Omega$ be given, with positive measure. If W is a finite word on the letters 0 and 1, we denote by C_W the cylinder set of all sequences in

Ω which begin with the word W. Since $\mu \otimes \mu(B) > 0$ there exist two finite words W_1 and W_2 of the same length L such that $\mu \otimes \mu\,(B \cap (C_{W_1} \times C_{W_2})) > \frac{1}{2}\mu \otimes \mu\,(C_{W_1} \times C_{W_2})$. We fix such words W_1 and W_2, and replacing B by $B \cap (C_{W_1} \times C_{W_2})$, we suppose now that $B \subset C_{W_1} \times C_{W_2}$.

With respect to the product measure $\mu \otimes \mu|_{C_{W_1} \times C_{W_2}}$, the coordinate maps

$$C_{W_1} \times C_{W_2} \to \{0,1\}^2 \ , \quad (\omega^1, \omega^2) \mapsto (\omega^1_i, \omega^2_i) \ , \qquad i > L \ ,$$

are independent random variables. Thus for all positive integers a, for all pairs of numbers ℓ and ℓ' between 1 and 4 and for almost every element (ω^1, ω^2) of $C_{W_1} \times C_{W_2}$ the word $(Y_\ell(a), Y_{\ell'}(a))$ appears infinitely often in (ω^1, ω^2). (Here we consider a finite block from the sequence (ω^1, ω^2) as a word on four letters.) We are going to mark the simultaneous appearance of distinct words Y_ℓ in the sequences ω^1 and ω^2.

Let us fix a sequence (a_k) such that $8\sum_{k\geq 1} 2^{-a_k} < 1$.

For almost every element (ω^1, ω^2) of $C_{W_1} \times C_{W_2}$ there exists a description of the form

$$\omega^1 = W_1 W_{1,1} Y_{\ell(1,1)}(a_1) W_{1,2} Y_{\ell(1,2)}(a_2) \ldots W_{1,k} Y_{\ell(1,k)}(a_k) \ldots \ , \tag{3.5}$$

$$\omega^2 = W_2 W_{2,1} Y_{\ell(2,1)}(a_1) W_{2,2} Y_{\ell(2,2)}(a_2) \ldots W_{2,k} Y_{\ell(2,k)}(a_k) \ldots \ , \tag{3.6}$$

where, for each $k \geq 1$, the two indices $\ell(j,k)$ are distinct in $\{1,2,3,4\}$ and the words $W_{1,k}$ and $W_{2,k}$ have the same length, which is chosen to be minimal.

Denote by P the set of pairs (ℓ_1, ℓ_2) of two distinct numbers between 1 and 4, and fix a one to one map σ from P into itself such that if $(\ell_3, \ell_4) = \sigma(\ell_1, \ell_2)$, then the four numbers ℓ_j, $1 \leq j \leq 4$, describe the set $\{1,2,3,4\}$.

We define a transformation U of $C_{W_1} \times C_{W_2}$, by the following rule : if a pair $(\omega^1, \omega^2) \in C_{W_1} \times C_{W_2}$ is given by the description (3.5) and (3.6) we set, for each $k \geq 1$,

$$(\ell(3,k), \ell(4,k)) = \sigma(\ell(1,k), \ell(2,k)) \ ,$$

and

$$U(\omega^1, \omega^2) = (\omega^3, \omega^4) \ ,$$

where

$$\omega^3 = W_1 W_{1,1} Y_{\ell(3,1)}(a_1) W_{1,2} Y_{\ell(3,2)}(a_2) \ldots W_{1,k} Y_{\ell(3,k)}(a_k) \ldots \ ,$$

$$\omega^4 = W_2 W_{2,1} Y_{\ell(4,1)}(a_1) W_{2,2} Y_{\ell(4,2)}(a_2) \ldots W_{2,k} Y_{\ell(4,k)}(a_k) \ldots \ .$$

The transformation U preserves the conditional probability $\mu \otimes \mu\,(\cdot | C_{W_1} \times C_{W_2})$ and, since $\mu \otimes \mu\,(B | C_{W_1} \times C_{W_2}) > 1/2$, we have $B \cap U^{-1}(B) \neq \emptyset$. But Lemma 3.2 tells us that, for almost all $(\omega^1, \omega^2) \in C_{W_1} \times C_{W_2}$, if $(\omega^3, \omega^4) = U(\omega^1, \omega^2)$, then the sequence

$$\frac{1}{N} \sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega^1) + \varphi^{(n)}(\omega^2) + \varphi^{(n)}(\omega^3) + \varphi^{(n)}(\omega^4)\right)$$

does not converge.

We have proved that this is possible with $(\omega^1, \omega^2), (\omega^3, \omega^4) \in B$, and this concludes the proof of the Theorem. □

3.4. Ergodic components of the Cartesian square. — We know (Corollary 2.15), that the Cartesian square of the Morse dynamical system is weakly disjoint from its ergodic components. This implies that almost all pairs of these ergodic components are weakly disjoint. We will see now that almost all of these ergodic components are not self-weakly disjoint. It will provide us the simplest known example of an ergodic dynamical system which is not self-weakly disjoint : a four point extension of the odometer.

Recall (from Remark 1) that the ergodic components of the Cartesian square of the Morse dynamical system can be described in the following way. An element δ of $\Omega\backslash\mathbb{Z}$ is fixed and plays the role of a parameter. The space is $\Omega \times \mathbb{Z}_2 \times \mathbb{Z}_2$ equipped with the product of the uniform probability measures, and the transformation $T_{f,\delta}$ is defined by

$$(\omega, z, z') \mapsto (\omega + \mathbf{1}, z + \varphi(\omega), z' + \varphi(\omega + \delta)) \ .$$

THEOREM 3.7. — *For any $\delta \in \Omega\backslash\mathbb{Z}$ such that $\liminf d_n(\delta, 0) = 0$ and any subset $B \subset \Omega$ of positive measure, there exist ω and ω' in B such that the sequence*

$$\frac{1}{N}\sum_{n=0}^{N-1} c\left(\varphi^{(n)}(\omega) + \varphi^{(n)}(\omega+\delta) + \varphi^{(n)}(\omega') + \varphi^{(n)}(\omega'+\delta)\right)$$

does not converge.

For such a δ, the dynamical system $(\Omega \times \mathbb{Z}_2 \times \mathbb{Z}_2, T_{f,\delta})$ is not self-weakly disjoint.

Proof. — Let δ be as in the statement of the theorem. Since $\liminf d_n(\delta, 0) = 0$, there exist n_i such that $\chi_{n_i}(\delta)$ tends to 0. We can extract a subsequence for which the convergence is monotonic. We will assume that the convergence is from the right as if not, $-\delta$ defines the same system and does indeed have convergence from the right.

Given this, it follows that in the sequence δ, we can find arbitrarily long strings of the form $1000\ldots0$. Fix $n_i \to \infty$ such that $\sum_i 2^{-n_i} < 1/32$. Inductively choose an increasing sequence of integers

$$a_{1,1}, a_{1,2}, \ldots, a_{1,2^{2n_1}}, a_{2,1}, \ldots, a_{2,2^{2n_2}}, \ldots$$

such that the block in δ of length $2n_i$ starting from $a_{i,j}$ is of the form $1000\ldots0$.

We call the block of length $2n_i$ starting from $a_{i,j}$ in a string ω an *i-block*. If an i-block in a point ω is of the form n_i 1's followed by n_i 0's, we say it is of type 1, while if it is of the form $n_i + 1$ 1's followed by $n_i - 1$ 0's, it is of type

2. Note that since there are 2^{2n_i} i-blocks in a string ω, it follows that almost every ω contains i-blocks of type 1 or 2 for infinitely many i.

Define as before a measure-preserving involution θ_n from Ω to itself, which given a point ω, for each $i \geq n$ finds the first i-block of type 1 or 2 (if there is one) and replaces it by the i-block of the other type.

If $B \subset \Omega$ is a set of positive measure, as before we can find $n \geq 1$ and $\omega \in B$ such that $\omega' = \theta_n(\omega)$ also belongs to B with infinitely many blocks changed. Let these blocks start at a_i and be of length $2n_i$ (this sequence of lengths will in general be a subsequence of the original sequence n_i).

We claim that for these ω and ω', the averages in the statement of the theorem do not converge. The proof is essentially the same as that in Lemma 3.2.

We note that the four points $\omega, \omega', \omega + \delta$ and $\omega' + \delta$ have lots of blocks of the form below and are pairwise identical outside these blocks:

```
1 1 1 1 1 1 1 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 0 0 0 0 0 0
? 0 0 0 0 0 0 1 0 0 0 0 0 0
? 0 0 0 0 0 0 0 1 0 0 0 0 0
```

These blocks have the property that when a number n is added to the points, for most configurations of n within the block (a proportion $1 - 6(2^{-n_i})$), the rotated block has an even number of 1's and is pairwise identical outside, whereas it started with an odd number of 1's. A typical picture after a number n large enough to perturb within the block has been added is:

```
1 0 1 1 0 0 0 1 1 0 1 0 1 1
1 0 1 1 0 0 0 0 0 1 1 0 1 1
1 1 1 1 0 0 0 1 1 0 1 0 1 1
1 1 1 1 0 0 0 0 0 1 1 0 1 1
```

Here the number n was of the form 01110000101011 inside the block illustrated. The effect of this, exactly as in Lemma 3.2, is that for 'most' n up to $2^{a_i+2n_i}$, the value of the summand in the statement of the theorem is $(-1)^i$. This completes the proof of the theorem.

□

3.5. Final Remarks and Questions. — As a consequence of the results in this section, we see that there exist uniquely ergodic dynamical systems with the property that their squares contain non-generic points. Specifically, we have constructed uniquely ergodic dynamical systems (X, T) such that there exist points (x, y) in $X \times X$ and continuous functions $f\colon X \times X \to \mathbb{R}$ such that $1/n(f(x, y) + \ldots + f(T^{n-1}x, T^{n-1}y))$ is non-convergent.

Define a uniquely ergodic dynamical system (X, T) to be *non-generic of order n* if in the nth power, there exist non-generic points, but in all smaller powers, every point is generic for some measure. In this language, we have shown that almost all ergodic self-joinings of two copies of the Morse system are non-generic of order 2. The Morse system itself is non-generic of order 4.

It is possible to give an example based on the above that is non-generic of order 3. Specifically, let $(\omega^1, \omega^2, \omega^3, \omega^4)$ be a non-generic point in the fourth power of the Morse system. Write $-1/3$ for the point $101010\ldots$ in the odometer and $-2/3$ for $010101\ldots$ (these points are $(-1)(3)^{-1}$ and $(-2)(3)^{-1}$ in the set of 2-adic integers respectively). Let ζ be any point of the odometer such that $d(\chi_n(\zeta + i/3), \chi_n(\omega^j))$ is uniformly bounded away from 0 for all n, $|i| \leq 3$, and $j \leq 4$. Then form a two point extension of the odometer by the cocycle $\psi(\omega) = \varphi(\omega+\omega^1)+\varphi(\omega+\omega^2)+\varphi(\omega+\omega^3)+\varphi(\omega+\omega^4)+\varphi(\omega+\zeta)+\varphi(\omega+\zeta-1/3)$.

We can show as before that this extension is uniquely ergodic. Further it is not hard to show that each point in the square of the system is generic. To see this, one has to consider the vector $V = (\omega^1, \omega^2, \omega^3, \omega^4, \zeta, \zeta - 1/3)$ and one of its translates $V' = (\omega^1 + \eta, \omega^2 + \eta, \omega^3 + \eta, \omega^4 + \eta, \zeta + \eta, \zeta - 1/3 + \eta)$. Provided that $\eta \notin \mathbb{Z}$, there exists j such that the family of 12 points of the odometer defined by V and V' is j-separated at infinitely many stages (see section 2.1).

However, one can check that $((0, 0), (-1/3, 0), (-2/3, 0)) \in (\Omega \times \mathbb{Z}_2)^3$ is a non-generic point in the cube of the system by the same arguments as presented previously.

It seems that it is impossible using the specific constructions in this paper to find systems that are non-generic of any order above 4. We ask for a construction of a system that is non-generic of any given order.

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