Periodicity and Local Complexity

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1 Introduction

For a bi-infinite word $\xi \in S^{\mathbb{Z}}$ (where S is a finite non-empty set of symbols), the *block complexity function* $p_\xi(n)$ counts the number of distinct subwords of ξ of length n. A celebrated result of Morse and Hedlund [6] establishes a link between the block complexity function $p_\xi(n)$ and periodicity of bi-infinite words:

Theorem 1 (Morse, Hedlund [6]) *Let $\xi \in S^{\mathbb{Z}}$ be a bi-infinite word. Then ξ is periodic (i.e., there exists a positive integer p such that $\xi_k = \xi_{k+p}$ for each $k \in \mathbb{Z}$) if and only if there exists $n_0 \in \mathbb{N}$ for which $p_\xi(n_0) \leq n_0$.*

There have been numerous works aimed at generalizing the Morse-Hedlund theorem, either in terms of other types of complexity functions, or in the context of multi-dimensional words (see [1–5,8–10]). For well written and interesting surveys in this direction, we refer the reader to [3] by Cassaigne and [10] by Tijdeman.

As is the case for 1-dimensional bi-infinite words, there are various ways of measuring the complexity of d-dimensional infinite words. By a d-dimensional infinite word we mean a d-dimensional infinite array of symbols $\xi \in S^{\mathbb{Z}^d}$. A d-dimensional infinite word ξ is periodic if there exists a non-zero vector $\mathbf{v} \in \mathbb{Z}^d$ such that $\xi_{\mathbf{x}} = \xi_{\mathbf{x}+\mathbf{v}}$ for all $\mathbf{x} \in \mathbb{Z}^d$. In this case, we say that ξ is periodic with period $\mathbf{v}$. Note that we make no assumption about minimality of the periodicity vector. Perhaps the most natural extension of the block complexity

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function to d-dimensional words $\xi \in S^{\mathbb{Z}^d}$ is the *rectangular complexity function* $N_\xi(n_1, n_2, \ldots, n_d)$ which counts the number of distinct $n_1 \times n_2 \times \cdots \times n_d$ blocks occurring in ξ. For $d = 1$, this coincides with the usual block complexity function $p_\xi(n)$. For $d = 2$, the following conjecture was made independently by Nivat [7] on one hand, and Berthé and Vuillon [1] on the other.

Conjecture 2 (Nivat) *Let $\xi \in S^{\mathbb{Z}^2}$. If there exist positive integers n_1, n_2 such that $N_\xi(n_1, n_2) \leq n_1 n_2$, then ξ is periodic.*

To the best of our knowledge, the conjecture remains open. If true, the conjecture would turn out to be optimal as there exist aperiodic 2-dimensional infinite words ξ of rectangular complexity $N_\xi(n_1, n_2) = n_1 n_2 + 1$ (see [2]). Sanders and Tijdeman [9] established the conjecture in the special case where $n_1 = 2$, and Epifanio, Koskas and Mignosi showed that the conjecture holds if $N_\xi(n_1, n_2) \leq n_1 n_2$ is replaced by $N_\xi(n_1, n_2) \leq \frac{1}{144} n_1 n_2$.

We remark that unlike in the case of the Morse-Hedlund Theorem, Nivat's conjecture is not an equivalence; in fact there exist periodic 2-dimensional infinite words ξ whose rectangular complexity function does not satisfy $N_\xi(n_1, n_2) \leq n_1 n_2$ for any choice of n_1, n_2 (see [1]). Also, the natural extension of Nivat's Conjecture to dimension 3 is known to be false: In [8] Sanders and Tijdeman exhibit the existence of aperiodic 3-dimensional infinite words ξ with the property that $N_\xi(n, n, n) < n^3$ for all $n \geq 3$.

The main result of this paper is to show that Nivat's Conjecture is true if $N_\xi(n_1, n_2) \leq n_1 n_2$ is replaced by $N_\xi(n_1, n_2) \leq \frac{1}{16} n_1 n_2$. Thus although our result is still a much weaker version of the general Nivat Conjecture, it provides a significant improvement on the constant $1/144$ given in [4] as well as a shorter and somewhat simpler proof. Actually, our result is stated in terms of a different complexity function $N_\xi^{\mathrm{loc}}(n_1, n_2, \ldots, n_d)$ of a d-dimensional infinite word ξ , that we call the *local complexity function.* The local complexity function counts the maximum number of distinct $n_1 \times \ldots \times n_d$ blocks occurring in a $2n_1 \times \ldots \times 2n_d$ block.

Clearly for any d-dimensional infinite word ξ we have

$$N_\xi^{\mathrm{loc}}(n_1, n_2, \ldots, n_d) \leq N_\xi(n_1, n_2, \ldots, n_d)$$

and

$$N_\xi^{\mathrm{loc}}(n_1, n_2, \ldots, n_d) \leq (n_1 + 1)(n_2 + 1) \cdots (n_d + 1).$$

For 1-dimensional infinite words ξ we show that:

Theorem 3 *Let $\xi \in S^{\mathbb{Z}}$ be an aperiodic 1-dimensional bi-infinite word. Then*

$N_{\xi}^{loc}(n) = n + 1$ *for each* $n \in \mathbb{N}$.

For 2-dimensional infinite words ξ we show that:

Theorem 4 *Let* $\xi \in S^{\mathbb{Z}^2}$ *be an aperiodic 2-dimensional infinite word. Then* $N_{\xi}^{loc}(n_1, n_2) > \frac{1}{16} n_1 n_2$ *for all* n_1 *and* n_2 *in* $\mathbb{N}$.

As a consequence of Theorem 4 we deduce that $N_{\xi}(n_1, n_2) > \frac{1}{16} n_1 n_2$ for all $n_1, n_2 \in \mathbb{N}$.

Before proving Theorems 3 and 4 we introduce some notation. Let ξ be a one or two dimensional infinite word. For each point $\mathbf{x}$ in $\mathbb{Z}$ or $\mathbb{Z}^2$ we denote by $\xi_{\mathbf{x}}$ the value of ξ at $\mathbf{x}$. For a 1-dimensional infinite word ξ and integers $a < b$, we put $\xi[a, b] = \xi_a \xi_{a+1} \cdots \xi_b$. Let $B(m, n) = \{(i, j) : 0 \leq i < m,\ 0 \leq j < n\}$ denote the $m \times n$ rectangle in $\mathbb{Z}^2$. For a 2-dimensional infinite word ξ and $\mathbf{y} = (y_1, y_2) \in \mathbb{Z}^2$, we denote by $\xi[\mathbf{y} + B(m, n)]$ the $m \times n$ word (or block) occurring at a point $\mathbf{y}$. More precisely, $\xi[\mathbf{y} + B(m, n)]$ is the $m \times n$ word defined by $\xi[\mathbf{y} + B(m, n)]_{i,j} = \xi_{y_1+i, y_2+j}$ for $(i, j) \in B(m, n)$.

2 Proof of Theorem 3

We will show that if for some $n \in \mathbb{N}$, each block or window in ξ of size $2n$ has fewer than $n + 1$ distinct subwords of length n, then ξ is periodic. This is clearly true in case $n = 1$. In fact, if each window of size two contains one distinct subword of length one, it follows that ξ is of the form $..., a, a, a, a, ...$ for some $a \in S$.

So let $n_0 > 1$ be the least positive integer such that each window in ξ of size $2n_0$ contains fewer than $n_0 + 1$ distinct subwords of length n_0. We will show that n_0 is a period of ξ, i.e., $\xi_k = \xi_{k+n_0}$ for each $k \in \mathbb{Z}$, and hence it will follow that n_0 is the least period of ξ.

By minimality of n_0 there exists a window of ξ of size $2(n_0 - 1)$ having n_0 distinct subwords of length $n_0 - 1$. Up to a re-indexing of ξ we can suppose without loss of generality that the subword $\xi[1, 2(n_0 - 1)] = \xi_1 \xi_2 \cdots \xi_{2n_0 - 2}$ contains n_0 distinct subwords of length $n_0 - 1$.

Now consider the subword $\xi[0, 2n_0 - 1]$ of length $2n_0$. By our assumption it contains at most n_0 distinct subwords of length n_0, i.e., there exist integers $0 \leq r < s \leq n_0$ such that $\xi[r, r + n_0 - 1] = \xi[s, s + n_0 - 1]$. We claim that $r = 0$ and $s = n_0$, in other words that

$$\xi[0, 2n_0 - 1] = \xi_0 \xi_1 \cdots \xi_{n_0 - 1} \xi_0 \xi_1 \cdots \xi_{n_0 - 1}.$$

In fact, if $r > 0$, then by deleting the last symbol we would have $\xi[r, r+n_0-2] = \xi[s, s+n_0-2]$ contradicting that $\xi[1, 2(n_0-1)]$ contains n_0 distinct subwords of length $n_0 - 1$. Similarly, if $s < n_0$, then by deleting the first symbol we would obtain a word of length $n_0 - 1$ occurring twice in $\xi[1, 2(n_0 - 1)]$, a contradiction.

We now claim that $\xi_k = \xi_{k+n_0}$ for each $k \in \mathbb{Z}$. We prove this only for $k \geq 0$, but the same argument will apply to $k \leq 0$. Thus far we have deduced that $\xi_k = \xi_{k+n_0}$ for $0 \leq k \leq n_0 - 1$. Suppose, contrary to our claim, that for some $k \geq n_0$ we have $\xi_k \neq \xi_{k+n_0}$. Let $k_0 \geq n_0$ be the least such integer with this property. Then we have

$$\xi[k_0 - n_0 + 1, k_0 + n_0] = \xi_{k_0+1} \cdots \xi_{k_0+n_0-1} \xi_{k_0} \xi_{k_0+1} \cdots \xi_{k_0+n_0-1} \xi_{k_0+n_0}$$

with $\xi_{k_0} \neq \xi_{k_0+n_0}$.

But by assumption on n_0, the word $\xi[k_0-n_0+1, k_0+n_0]$ of length $2n_0$ contains fewer that n_0+1 subwords of length n_0. Since the prefix of $\xi[k_0-n_0+1, k_0+n_0]$ of length n_0 is distinct from the suffix of $\xi[k_0 - n_0 + 1, k_0 + n_0]$ of length n_0, it follows that there exists integers $k_0 - n_0 + 1 \leq r < s \leq k_0 + 1$ with $s - r < n_0$ such that $\xi[s, s+n_0-1] = \xi[r, r+n_0-1]$. By deleting the last symbol we have $\xi[s, s+n_0-2] = \xi[r, r+n_0-2]$. But by minimality of k_0 we have that $\xi_t = \xi_{[t]}$ for each $0 \leq t \leq k_0+n_0-1$, where $0 \leq [t] \leq n_0-1$ denotes the equivalence class of t modulo n_0. Thus we deduce that $\xi[[s], [s]+n_0-2] = \xi[[r], [r]+n_0-2]$, and $[r] \neq [s]$. But this implies the existence of a word of length $n_0 - 1$ occurring twice in $\xi[1, 2(n_0 - 1)]$, a contradiction. Hence $\xi_k = \xi_{k+n_0}$ for each $k \in \mathbb{Z}$ implying that ξ is periodic of period n_0. This concludes our proof of Theorem 3.

3 Proof of Theorem 4

We may assume that both n_1 and n_2 are greater than 16; in fact, if $n_1 \leq 16$ we have $N^{\mathrm{loc}}(n_1, n_2) \geq N^{\mathrm{loc}}(1, n_2) > n_2 \geq \frac{n_1 n_2}{16}$, where the second inequality follows from Theorem 3. Set $m_i = \lfloor n_i/4 \rfloor$ and $l_i = n_i - m_i$.

We prove the theorem using a series of lemmas. The proof will be by contradiction. Let $\xi \in S^{\mathbb{Z}^2}$ be a 2-dimensional infinite word and suppose that for some n_1 and n_2, $N_\xi^{\mathrm{loc}}(n_1, n_2) \leq n_1 n_2/16$. We will show that ξ is necessarily periodic. A simple but crucial observation is the following.

Lemma 5 *If $N_\xi^{loc}(n_1, n_2) \leq n_1 n_2/16$, then for $\mathbf{z} = (z_1, z_2) \in \mathbb{Z}^2$, there exists a vector $\mathbf{v} = (v_1, v_2)$ with $|v_i| \leq m_i$ such that $\xi[\mathbf{z} + B(l_1, l_2)] = \xi[\mathbf{z} + \mathbf{v} + B(l_1, l_2)] = \xi[\mathbf{z} - \mathbf{v} + B(l_1, l_2)]$.*

PROOF. Consider the $n_1 \times n_2$ blocks $\xi[\mathbf{y} + B(n_1, n_2)]$ with $\mathbf{y} = (y_1, y_2)$ satisfying $z_i - m_i \leq y_i \leq z_i$. Since there are $(1 + m_1)(1 + m_2) > n_1 n_2/16$ such blocks, there must exist distinct points $\mathbf{y}$ and $\mathbf{y}'$ of this form such that $\xi[\mathbf{y} + B(n_1, n_2)] = \xi[\mathbf{y}' + B(n_1, n_2)]$. Let $\mathbf{v} = \mathbf{y} - \mathbf{y}'$. If $\mathbf{x} \in \mathbf{z} + B(l_1, l_2)$, we see that $\mathbf{x} \in \mathbf{y} + B(n_1, n_2) \cap \mathbf{y}' + B(n_1, n_2)$. Using the first of these and the fact that

$\xi[\mathbf{y} + B(n_1, n_2)] = \xi[\mathbf{y} - \mathbf{v} + B(n_1, n_2)]$, we see that $\xi_{\mathbf{x}-\mathbf{v}} = \xi_{\mathbf{x}}$. Similarly, using the second, we see that $\xi_{\mathbf{x}+\mathbf{v}} = \xi_{\mathbf{x}}$.

We have shown that for each location $\mathbf{z}$ in $\mathbb{Z}^2$, there exists a vector $\mathbf{v}$ with $|v_i| \leq m_i$ such that the $l_1 \times l_2$ word occurring at $\mathbf{z}$ is the same as the $l_1 \times l_2$ word appearing at $\mathbf{z} \pm \mathbf{v}$. In this case, we call the vector $\mathbf{v}$ an *actual translation vector* of the $l_1 \times l_2$ word occurring at $\mathbf{z}$.

Now let W be any $l_1 \times l_2$ word occurring in ξ. Since W occurs at some location $\mathbf{z} \in \mathbb{Z}^2$, there is an actual translation vector $\mathbf{v}$ for the given occurrence of W at $\mathbf{z}$. It follows that if $\mathbf{x}$ and $\mathbf{x} + \mathbf{v}$ lie in $B(l_1, l_2)$, then $W_{\mathbf{x}} = W_{\mathbf{x}+\mathbf{v}}$. Such a vector $\mathbf{v}$ (with $|v_i| \leq m_i$) is called a *potential translation vector* of W.

Thus each $l_1 \times l_2$ word W that appears in ξ has at least one potential translation vector. In addition, for each occurrence of the word, a non-empty subset of the potential translation vectors of the word are actual translation vectors of the specific occurrence of the word.

We divide the proof of the Theorem 4 into two main cases:

Case 1 There exists an $l_1 \times l_2$ word in ξ all of whose potential translation vectors are collinear.

Case 2 Every $l_1 \times l_2$ word in ξ has two non-collinear potential translation vectors.

We deal first with Case 1. In this case we shall show that ξ contains a bi-infinite strip in which there is periodicity. We will derive a contradiction by showing that the boundary of the periodic region cannot exist.

Let W be an $l_1 \times l_2$ word all of whose potential translation vectors are collinear. Let $\mathbf{h}$ be a potential translation vector. We will assume without loss of generality that the coordinates of $\mathbf{h}$ are positive and $h_1/l_1 \geq h_2/l_2$. In fact, if both h_1 and h_2 are negative, we replace $\mathbf{h}$ by $-\mathbf{h}$. If h_1 and h_2 are of opposite signs, then we obtain the desired $\mathbf{h}$ by reflecting ξ across the appropriate coordinate axis to obtain a 2-dimensional word ξ' which is periodic if and only if ξ is periodic. Finally if $h_2/l_2 \geq h_1/l_1$ then we obtain the desired $\mathbf{h}$ by reflecting ξ across the line $y = x$.

We now fix an occurrence of the word W at some point $\mathbf{x} \in \mathbb{Z}^2$, and write $W = \xi[B]$ for some $l_1 \times l_2$ rectangular block $B \subset \mathbb{Z}^2$. Then for some positive rational t, we have that $t\mathbf{h}$ is the actual translation vector for the occurrence of W at $\mathbf{x}$ so that W occurs at $\mathbf{x} \pm t\mathbf{h}$. Similarly, the actual translation vector for W at $\mathbf{x}+t\mathbf{h}$ is a rational multiple of $\mathbf{h}$ so that W occurs at $\mathbf{x}+(t+s)\mathbf{h}$ for some s. Inductively, we see that the word W is repeated, possibly irregularly spaced, along the bi-infinite line $\{\mathbf{x} + t\mathbf{h} \colon t \in \mathbb{R}\}$ as shown in Figure 1.

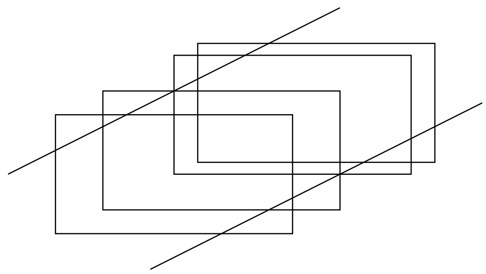

Fig. 1. Copies of W along a line

For $\mathbf{v} \in \mathbb{Z}^2$ we define the line $L_\mathbf{v} = \{\xi_{\mathbf{v}+t\mathbf{h}} : t \in \mathbb{R}\} \cap \mathbb{Z}^2$. Consider the 'band region' $W' = \xi[B']$ of W as illustrated in Figure 2 where $B' \subset B$ consists of those points $\mathbf{v} \in B$ such that the line $L_\mathbf{v}$ does not hit the lower edge of B further than $l_1/3$ from the lower left corner or the upper edge of B further than $l_1/3$ from the upper right corner. Note that since $h_1/l_1 \geq h_2/l_2$, the 'upper boundary line' (resp. 'lower boundary line') of W' intersects the left (resp. right) edge of W more than $l_2/3 \geq m_2$ away from the lower left (resp. upper right) corner of W. In particular if $\xi_\mathbf{z}$ and $\xi_{\mathbf{z}+\mathbf{v}}$ are on opposite sides of the band region W', then $|v_1| > m_1$ or $|v_2| > m_2$.

Since the width of W is $l_1 - 1$, it follows that for each $\mathbf{v} \in B'$, the horizontal projection of the intersection $L_\mathbf{v} \cap W'$ has length at least $2l_1/3 - 1 \geq 2m_1 - 1$.

Since the copies of W are spaced out at horizontal intervals of at most m_1, it follows that for $\mathbf{v} \in B'$ and $\mathbf{x} \in L_\mathbf{v}$, there is a copy of W containing $\mathbf{x}$ and $\mathbf{x}+\mathbf{h}$. Since $\mathbf{h}$ is a potential translation vector of W, it follows that $\xi_\mathbf{x} = \xi_{\mathbf{x}+\mathbf{h}}$.

Thus the union of all lines $L_\mathbf{v}$ with $\mathbf{v} \in B'$ defines an infinite band region (or strip) $\xi[\Omega_W]$ parallel to $\mathbf{h}$ on which ξ is periodic with period $\mathbf{h}$, and such that if $\xi_\mathbf{z}$ and $\xi_{\mathbf{z}+\mathbf{v}}$ are on opposite sides of the strip, then $|v_1| > m_1$ or $|v_2| > m_2$.

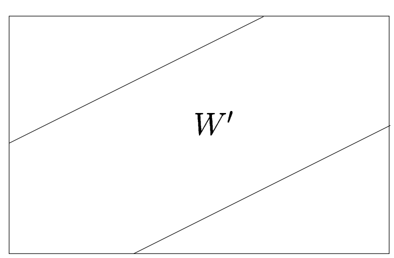

Fig. 2. 'Band region' of the word W

We note that although we were assuming that all potential translation vectors of W are parallel to $\mathbf{h}$, the existence of the band Ω_W as above follows from the weaker assumption that there exists an occurrence of W at some point $\mathbf{v} \in \mathbb{Z}^2$ and that all actual translation vectors of W for each occurrence of W at a point $\mathbf{z} \in L_{\mathbf{v}}$ are parallel to $\mathbf{h}$. We will use this fact in the proof of the following lemma:

Lemma 6 *Let $\Omega \subset \mathbb{Z}^2$ be an infinite strip (possibly unbordered) parallel to $\mathbf{h}$ with the property that the restriction of ξ to Ω is periodic with period $\mathbf{h}$ and such that if $\xi_{\mathbf{z}}$ and $\xi_{\mathbf{z}+\mathbf{v}}$ are on opposite sides of Ω, then $|v_1| > m_1$ or $|v_2| > m_2$. Then ξ is periodic with period $M\mathbf{h}$ for some M.*

PROOF. If $\Omega = \mathbb{Z}^2$, then ξ is periodic with period $\mathbf{h}$. Otherwise, there exists a closest line L' to Ω, parallel to $\mathbf{h}$ and not contained in Ω. Now if the restriction of ξ to L' is periodic with period $\mathbf{h}$, then we can adjoin L' to Ω to form a wider band parallel to $\mathbf{h}$ satisfying the hypothesis of the lemma. Otherwise, if the restriction of ξ to L' is not periodic with period $\mathbf{h}$, it follows that there exists some $\mathbf{z}' \in L'$ such that $\xi_{\mathbf{z}'} \neq \xi_{\mathbf{z}'+\mathbf{h}}$. In this case, let V be the $l_1 \times l_2$ word occurring at $\mathbf{z}'$ and hence containing both $\xi_{\mathbf{z}'}$ and $\xi_{\mathbf{z}'+\mathbf{h}}$, and let $\mathbf{v}' = (v_1', v_2')$ be an actual translation vector for this occurrence of V. It follows that $\mathbf{v}'$ is parallel to $\mathbf{h}$. In fact, if $\mathbf{v}'$ is not parallel to $\mathbf{h}$ then the condition on the width of the band implies that either $\mathbf{z}' + \mathbf{v}'$ and $\mathbf{z}' + \mathbf{h} + \mathbf{v}'$ are both in Ω, or $\mathbf{z}' - \mathbf{v}'$ and $\mathbf{z}' + \mathbf{h} - \mathbf{v}'$ are both in Ω. Either way this contradicts our assumption that $\xi_{\mathbf{z}'} \neq \xi_{\mathbf{z}'+\mathbf{h}}$. Moreover this argument applies to each translate of V along the boundary of Ω, that is for each occurrence of V along the boundary of Ω, all actual translation vectors of V are parallel to $\mathbf{h}$.

Thus as to W we associated an infinite strip Ω_W satisfying the hypothesis of the lemma, we associate to V an infinite strip Ω_V parallel to $\mathbf{h}$ overlapping Ω and satisfying the hypothesis of the lemma. By a repetition of this construction we see that each line $L_{\mathbf{v}}$ is eventually contained in such a strip, that is for each $\mathbf{v} \in \mathbb{Z}^2$, there exists a multiple $t\mathbf{h}$ of $\mathbf{h}$ with integer coefficients bounded by m_1 and m_2 such that the restriction of ξ to $L_{\mathbf{v}}$ is periodic with period $t\mathbf{h}$. Since there are finitely many such t, we may take the least common multiple. Hence, ξ is periodic with period $M\mathbf{h}$ for some M.

Our proof of Theorem 4 in **Case 1** now follows from Lemma 6.

We next consider **Case 2** in which each $l_1 \times l_2$ block has two non-collinear potential translation vectors. In this case we will show that for each $l_1 \times l_2$ word, there is a suitable subword which is the restriction of a doubly periodic configuration of the entire lattice. We then study the way that these fit together to deduce that all of ξ is doubly periodic.

We start with some definitions. Let Λ be a non-empty subset of $\mathbb{Z}^2$, and P a non-empty set of vectors in $\mathbb{Z}^2$. In general we assume that P contains at least two non-collinear vectors, although unless explicitly stated, some of what follows does not require this assumption. We denote the integer linear span of P by $\text{lin}(P)$. Let W denote $\xi[\Lambda]$. We say W is *locally P-periodic* if whenever $\mathbf{x}$ and $\mathbf{y}$ belong to Λ and differ by a vector $\mathbf{v}$ belonging to P, then $\xi_{\mathbf{x}} = \xi_{\mathbf{y}}$. We say W is *globally P-periodic* if whenever $\mathbf{x}$ and $\mathbf{y}$ belong Λ and differ by a vector $\mathbf{v}$ belonging to $\text{lin}(P)$, then $\xi_{\mathbf{x}} = \xi_{\mathbf{y}}$.

We make use of the following lemma several times.

Lemma 7 *Let $\Lambda \subset \mathbb{Z}^2$, and put $W = \xi[\Lambda]$. Let P be a non-empty set of vectors in $\mathbb{Z}^2$ and $Q \subset P$. Suppose that W is locally P-periodic and globally Q-periodic. Suppose further that for all $\mathbf{v} \in P \setminus Q$ and $\mathbf{x} \in \Lambda$, there exists $\mathbf{x}' \in \mathbf{x} + lin(Q)$ such that both $\mathbf{x}'$ and $\mathbf{x}' + \mathbf{v}$ are in Λ. Then W is globally P-periodic.*

PROOF. We need to show that for all $\mathbf{x}, \mathbf{y} \in \Lambda$ such that

$$\mathbf{y} = \mathbf{x} + \epsilon_1 \mathbf{v_1} + \epsilon_2 \mathbf{v_2} + \ldots + \epsilon_k \mathbf{v_k} + \mathbf{w},$$

where $\epsilon_i \in \{1, -1\}$, $\mathbf{v_i} \in P \setminus Q$ and $\mathbf{w} \in \text{lin}(Q)$, that $\xi_{\mathbf{x}} = \xi_{\mathbf{y}}$. We prove this by induction on k. The result is clear if $k = 0$, that is in case $\mathbf{y} = \mathbf{x} + \mathbf{w}$, since W is assumed to be globally Q-periodic.

Now suppose $k \geq 1$ and the result holds for $k - 1$. Suppose $\mathbf{x}, \mathbf{y} \in \Lambda$ as above. Without loss of generality we can assume that $\epsilon_k = +1$; in fact, if $\epsilon_k = -1$, then we consider the equation $\mathbf{x} = \mathbf{y} - \epsilon_1 \mathbf{v_1} - \epsilon_2 \mathbf{v_2} - \ldots - \epsilon_k \mathbf{v_k} - \mathbf{w}$.

By hypothesis, there exists $\mathbf{x}' \in \mathbf{x} + \text{lin}(Q)$ such that both $\mathbf{x}'$ and $\mathbf{x}' + \mathbf{v_k}$ are in Λ. Hence

$$\xi_{\mathbf{x}} = \xi_{\mathbf{x}'} = \xi_{\mathbf{x}' + \mathbf{v_k}}$$

since W is globally Q-periodic and locally P-periodic. Since $\mathbf{x}$ and $\mathbf{x}'$ differ by a vector in $\text{lin}(Q)$, we can write

$$\mathbf{y} = (\mathbf{x}' + \mathbf{v_k}) + \epsilon_1 \mathbf{v_1} + \cdots + \epsilon_{k-1} \mathbf{v_{k-1}} + \mathbf{w}'$$

for some $\mathbf{w}' \in \text{lin}(Q)$ and apply our induction hypothesis to deduce that $\xi_{\mathbf{y}} = \xi_{\mathbf{x}' + \mathbf{v_k}}$. Hence $\xi_{\mathbf{y}} = \xi_{\mathbf{x}}$ as required.

Henceforth we will assume that P consists of a finite collection of vectors $\mathbf{v} = (v_1, v_2)$ with $|v_i| \leq m_i$, of which at least two are non-collinear.

Lemma 8 *Suppose the $r_1 \times r_2$ word $W = \xi[\Lambda]$ (where $r_1 \geq l_1$ and $r_2 \geq l_2 - 1$ or $r_1 \geq l_1 - 1$ and $r_2 \geq l_2$) is locally P-periodic, where P contains two non-collinear vectors $\mathbf{u}, \mathbf{v}$ from two adjacent quadrants (where a vector parallel to a coordinate axis is considered to belong to both of the quadrants that it separates). Then*

(1) W contains a fundamental domain for any lattice generated by a non-collinear subset of P;
(2) W is globally P-periodic;

PROOF.

To see the first claim, note that for any non-collinear pair of vectors in P, the lattice that they generate has a fundamental domain of width at most $2m_1 - 1$ and height at most $2m_2$. They also have a (possibly different) fundamental domain of width at most $2m_1$ and height at most $2m_2 - 1$. Any superset of this collection of vectors generates a finer lattice, whose fundamental domain is therefore a subset of this fundamental domain.

Let P, $\mathbf{u}$ and $\mathbf{v}$ be as in the statement of the lemma and set $Q = \{\mathbf{u}, \mathbf{v}\}$. We assume without loss of generality that $\mathbf{u}$ and $\mathbf{v}$ are in the first and second quadrants. We further assume that $r_1 \geq 3m_1 - 1$ and $r_2 \geq 3m_2$

We will show below that W is globally Q-periodic. If $|P| = 2$, we will be done. Otherwise, we recall that any $2m_1 - 1 \times 2m_2$ subword of W contains a fundamental domain for the lattice generated by Q. Hence given $\mathbf{w} \in P \setminus Q$, one can choose a subword Λ' so that $\Lambda' \cup \Lambda' + \mathbf{w} \subset \Lambda$ and Λ' contains a fundamental domain for the lattice generated by Q. This will allow us to conclude the proof of the lemma by applying Lemma 7.

To see that W is Q-globally periodic, we argue as follows. For each $\mathbf{x} \in \Lambda$, we define a graph on the vertex set $V_{\mathbf{x}} = (\mathbf{x} + \text{lin}(Q)) \cap \Lambda$ where two vertices are joined by an edge if they differ by $\mathbf{u}$ or $\mathbf{v}$. We need to show that these graphs are connected. Fix an $\mathbf{x} \in \Lambda$. We define the *rung* through $\mathbf{y} \in V_{\mathbf{x}}$ to be the set $\{\mathbf{y} + n\mathbf{v} \colon n \in \mathbb{Z}, \mathbf{y} + n\mathbf{v} \in \Lambda\}$. By convexity of Λ, the rungs are connected. To show that the graph is connected it is sufficient to show that the rungs all lie in the same component. Since the rungs are linearly ordered and there can only be edges between consecutive rungs, it is therefore sufficient to show that if a rung is not connected to any rung below itself, then it is the bottom rung and similarly for the top rungs.

Let R be a rung which we assume is not connected to any rung below. Let $\bar{\Lambda}$ denote the convex hull of Λ and $\bar{R}$ be the part of the line in $\mathbb{R}^2$ through R parallel to $\mathbf{v}$ that is contained within the boundaries of $\bar{\Lambda}$. Set $T = \{\mathbf{z} \in \bar{\Lambda} \colon \mathbf{z} - \mathbf{u} \in \bar{\Lambda}\}$. The part of $\bar{R}$ lying in T must be shorter than $\mathbf{v}$ (otherwise

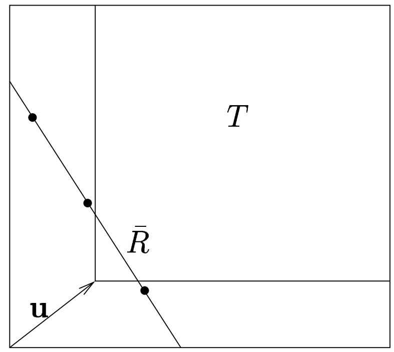

Fig. 3. The bottom rung

there would necessarily be a point in $R \cap T$, which would be connected to a lower rung). This is illustrated in Figure 3. It follows that points in $\bar{R} \cap T$ have x-coordinates less than $u_1 + |v_1| \leq u_1 + m_1$ and y-coordinates less than $u_2 + v_2 \leq u_2 + m_2$. In particular, points in $\bar{R} - k\mathbf{u} \cap \bar{\Lambda}$ for $k \geq 1$ have coordinates bounded above by m_1 and m_2, so that given any point in $R - k\mathbf{u} \cap \Lambda$, it is possible to repeatedly add $\mathbf{u}$ until it hits R. This contradicts the assumption that R was not connected to any rung below and completes the proof.

Lemma 9 *Suppose the $r_1 \times r_2$ word W (where $r_1 \geq l_1 - 1$ and $r_2 \geq l_2 - 1$) is locally P-periodic, where all vectors in P lie in a pair of opposite quadrants. Then there is a subword W' of W such that*

(1) W' contains a fundamental domain for any lattice generated by a non-collinear subset of P;
(2) W' is globally P-periodic;
(3) W' contains the $(\lfloor r_1/2 \rfloor, \lfloor r_2/2 \rfloor)$ entry of W.

PROOF. By symmetry, we may assume that all vectors in P lie in the first or third quadrants. Since the definitions are unaffected if the vectors are negated, we may further assume that all vectors in P lie in the first quadrant.

If P contains two vectors with the same x-coordinate, denote them by $\mathbf{u}$ and $\mathbf{v}$ and set $Q = \{\mathbf{u}, \mathbf{v}\}$. Assume that $\mathbf{u}$ is steeper than $\mathbf{v}$. Remove the top left and bottom right $m_1 \times (m_2 - 1)$ subregions of Λ to form a region Λ'. Since the y coordinates of $\mathbf{u}$ and $\mathbf{v}$ are positive, their difference h is at most $m_2 - 1$. If $\mathbf{x}, \mathbf{y} \in \Lambda'$ where $\mathbf{y}$ is a distance h vertically above $\mathbf{x}$, then we see that either $\mathbf{x} - \mathbf{v}$ or $\mathbf{x} + \mathbf{u}$ lies in Λ'. Call this point $\mathbf{z}$ so that $\xi_{\mathbf{x}} = \xi_{\mathbf{z}} = \xi_{\mathbf{y}}$. We then see that $W' = \xi[\Lambda']$ is globally Q-periodic. This situation is illustrated in Figure 4.

We see that $[m_1 - 1, 2m_1 - 1) \times [m_2 - 1, 2m_2 - 1) \cap \mathbb{Z}^2$ contains a fundamental domain for the lattice so that given $\mathbf{x} \in \Lambda'$, there exists $\mathbf{w} \in \mathrm{lin}(Q)$ such that $\mathbf{x}' = \mathbf{x} + \mathbf{w} \in [m_1 - 1, 2m_1 - 1) \times [m_2 - 1, 2m_2 - 1) \cap \mathbb{Z}^2$. For any vector $\mathbf{z} \in P$,

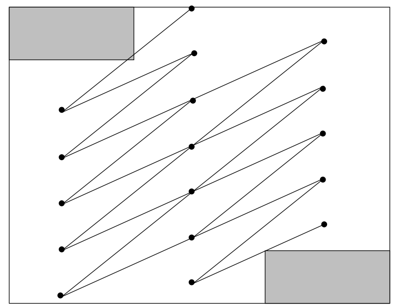

Fig. 4. Connectedness if two vectors agree on a coordinate

we can check that $\mathbf{x}' + \mathbf{z} \in \Lambda'$, hence the lemma follows from Lemma 7. We can deal with the case where $\mathbf{u}$ and $\mathbf{v}$ have the same y-coordinate similarly. Thus we have proved the lemma in the case that any two vectors in P have the same x or y coordinates.

Next, assume all vectors in P have different x and y coordinates, let $Q = \{\mathbf{u}, \mathbf{v}\}$ where the slope of $\mathbf{u}$ is greater than the slope of $\mathbf{v}$. Define L_1 to be the line through $(r_1 - u_1 - v_1, -1)$ and $(r_1 - u_1, v_2 - 1)$; L_2 through $(r_1 - u_1, v_2 - 1)$ and $(r_1, v_1 + v_2 - 1)$; L_3 through $(-1, r_2 - u_2 - v_2)$ and $(u_1 - 1, r_2 - v_2)$; and L_4 through $(u_1 - 1, r_2 - v_2)$ and $(u_1 + u_2 - 1, r_2)$. Define Λ' to be the subregion of Λ formed by removing points lying on or below L_1 and L_2 and points lying on or above L_3 and L_4. This is illustrated in Figure 5.

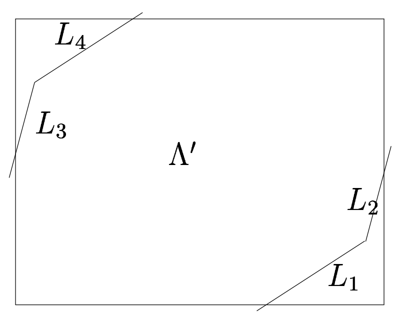

Fig. 5. The region Λ'

To see that $\xi[\Lambda']$ is globally Q-periodic, we argue as in Lemma 8 by considering rungs in lattices. Let L be the line lying a displacement $\mathbf{u}$ above L_1. Suppose that R is a non-empty rung lying above L containing a point of Λ'. This is illustrated in Figure 6. By construction, we see that there is an element $\mathbf{x}$ of R such that $\mathbf{x} - \mathbf{u} \in \Lambda'$ so that R is connected to the rung below.

Let T be the parallelogram with vertices $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ and let $\mathbf{w} \in P \setminus Q$. Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ have distinct x and y coordinates, we have $u_1 + v_1 + w_1 \leq 3m_1 - 2 < r_1$ and similarly $u_2 + v_2 + w_2 < r_2$. We note from this that $\mathbf{v} + \mathbf{w}$ is strictly above and to the left of the intersection of L_1 and L_2. Similarly, $\mathbf{u} + \mathbf{w}$ is below and to the right of the intersection of L_3 and L_4. Accordingly, we conclude that all four vertices of $T + \mathbf{w}$ lie in Λ'. Since T is a fundamental domain for the lattice generated by Q, we see that the conditions of Lemma 7

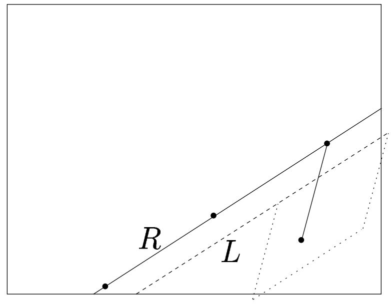

Fig. 6. A rung above L is connected to the rung below

are satisfied and the restriction of ξ to Λ' is globally P-periodic. This completes the proof.

Applying Lemmas 8 and 9, we see that for any $l_1 \times l_2$ word W occurring on a region Λ in ξ, there is a subregion Λ' containing the central point of Λ and a globally doubly periodic $\eta_\Lambda \in S^{\mathbb{Z}^2}$ and such that $\xi[\Lambda'] = \eta_\Lambda[\Lambda']$.

Next, consider two adjacent $l_1 \times l_2$ blocks, Λ_1 and Λ_2. Let P_i denote the set of potential translation vectors of $\xi[\Lambda_i]$ and let Λ_i' be the subregion of Λ_i on which ξ is globally P_i-periodic. Let η_i be the infinite globally P_i-periodic configuration determined by the restriction of ξ to Λ_i'. Finally, write Λ for $\Lambda_1 \cap \Lambda_2$ and note that on Λ, ξ is locally $P_1 \cup P_2$-periodic. By the lemmas, there is a subregion Λ' of Λ on which ξ is globally $P_1 \cup P_2$-periodic. Let η be the infinite globally $P_1 \cup P_2$-periodic word agreeing with ξ on Λ'. Since $\Lambda' \cap \Lambda_1$ necessarily contains a fundamental domain for the lattice generated by P_1, we see that $\eta_1 = \eta$ and similarly, $\eta_2 = \eta$. We conclude that the η_Λ is the same for all $l_1 \times l_2$ regions in $\mathbb{Z}^2$. We denote this unique doubly periodic word by η.

For each $l_1 \times l_2$ region Λ in $\mathbb{Z}^2$, there is a subregion Λ' such that $\xi[\Lambda'] = \eta[\Lambda']$. Given $\mathbf{x} \in \mathbb{Z}^2$, let Λ be an $l_1 \times l_2$ region centered at $\mathbf{x}$. Since $\mathbf{x} \in \Lambda'$, it follows that $\xi_\mathbf{x} = \eta_\mathbf{x}$. Since $\mathbf{x}$ was arbitrary, it follows that $\xi = \eta$ so that ξ is doubly periodic as required.

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