

Weak- L^1 Estimates and Ergodic Theorems

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ABSTRACT. We prove that for any dynamical system (X, Σ, m, T) , the maximal operator defined by $N^*f(x) = \sup_n \frac{1}{n} \#\{1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n}\}$ is almost everywhere finite for f in the Orlicz class $L \log \log L(X)$, extending a result of Assani [2]. As an application, a weighted return times theorem is also proved.

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1. Introduction

Let T be a measure preserving transformation of a probability space (X, Σ, m) . We call (X, Σ, m, T) a dynamical system. The following return times theorem was proved in [4]:

Theorem 1 (Bourgain). *Let $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$. For each dynamical system (X, Σ, m, T) and $f \in L^p(X)$, there is a set $X_0 \subset X$ of full measure, such that for any other dynamical system (Y, \mathcal{F}, μ, S) , $g \in L^q(Y)$ and $x \in X_0$, the limit,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(T^k x) g(S^k y),$$

exists for μ a.e. y .

One of the most interesting unanswered questions that emerges from this result is whether or not the fact that f and g lie in dual spaces is in general necessary in order to have a positive result. Neither of the existing proofs of Theorem 1 gives any indication on this, since each of them relies on Hölder's inequality.

On the other hand, if (gS^k) is replaced with a sequence (ξ_k) of independent identically distributed random variables such that $\mathbb{E}(|\xi_1|) < \infty$, then the following

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criterion of B. Jamison, S. Orey and W. Pruitt [5] proves to be an excellent tool to break the duality.

Theorem 2 (Jamison, Orey and Pruitt). *Let (a_k) be a sequence of positive real numbers and let $N^* = \sup_n \frac{1}{n} \#\{k : a_k / \sum_{i=1}^k a_i \geq 1/n\}$, then the following are equivalent:*

1. $N^* < \infty$;
2. *For any i.i.d. sequence of random variables (ξ_k) such that $\mathbb{E}(|\xi_1|) < \infty$, defining a new sequence (Ξ_n) of random variables by*

$$\Xi_n(\omega) = \sum_{k=1}^n a_k \xi_k(\omega) / \sum_{k=1}^n a_k,$$

the sequence (Ξ_n) converges pointwise almost surely.

Motivated by this criterion, Assani [1] introduced the following maximal function: given $f \in L^1(X)$, consider

$$N^* f(x) = \sup_n \frac{1}{n} \#\{1 \leq i : \frac{f(T^i x)}{i} \geq \frac{1}{n}\}.$$

He proved in [2] for $f \in L \log L(X)$, $N^* f \in L^1$ and in particular $N^* f(x) < \infty$ for a.e. x . Based on this and Theorem 2, the following “duality-breaking” version of Theorem 1 follows almost immediately:

Corollary 3 (Assani). *For each dynamical system (X, Σ, m, T) and function f such that $\int |f| \log^+ |f| dm < \infty$ (i.e. $f \in L \log L(X)$), there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(T^k x) \xi_k(\omega)$$

exists for μ a.e. ω .

Moreover in [1] it is proved that if Theorem 1 is true for $p = q = 1$, then $N^* f(x)$ must be finite almost everywhere for all $f \in L^1(X)$. This connection sheds more light on the importance of the operator N^* and motivates its further study.

In the next section we will prove the finiteness of N^* for functions in the larger class $L \log \log L$. Note that while Assani shows that $N^* f \in L^1$ for $f \in L \log L$, our result establishes that $N^* f \in L^{1,\infty}$ for $f \in L \log \log L$ (i.e. that $\sup_t t m\{x : N^* f(x) > t\} < \infty$) so that while our hypothesis is weaker, so is our conclusion. Note however that since our conclusion implies that $N^* f(x) < \infty$ for almost every x , it is sufficient to imply a corollary like Corollary 3 in the case where $f \in L \log \log L$.

In a preprint that appeared at around the time this paper was submitted, Assani, Buczolich, and Mauldin [3] show that there exists an $f \in L^1(X)$ such that $N^* f(x) = \infty$ almost everywhere.

2. Main results

Throughout this section we will denote the natural logarithm of x by $\log x$ and the weak- L^1 norm of f by

$$\|f\|_{1,\infty} = \sup_{\lambda > 0} \lambda m\{x : |f(x)| > \lambda\}.$$

Also, as usual, we denote the ergodic maximal function by f^* , where $f^*(x) = \sup_n \left| \frac{1}{n} \sum_{k=1}^n f(T^k x) \right|$. The maximal ergodic theorem asserts that $\|f^*\|_{1,\infty} \leq \|f\|_1$ for all $f \in L^1(X)$. The following inequality from [7] turns out to be extremely useful to our investigation:

Lemma 4. *Suppose that for $i = 1, 2, \dots$, $g_i(x)$ is a nonnegative function on a measure space with $\sum \|g_i\|_{1,\infty} \leq 1$. Set $K = \sum_{i=1}^{\infty} \|g_i\|_{1,\infty} \log(1/\|g_i\|_{1,\infty})$, the entropy of the sequence $(\|g_i\|_{1,\infty})$. Then $\|\sum_{i=1}^{\infty} g_i\|_{1,\infty} \leq 2(K + 2)$.*

We can now prove our main result.

Theorem 5. *For each dynamical system (X, Σ, m, T) and each $f \in L \log \log L(X)$ (that is f satisfying $\int |f| \log^+ \log^+ |f| dm < \infty$), $N^* f(x) < \infty$ for a.e. x .*

Proof. It is enough to consider f positive. Making use of the fact that $f(x) \leq \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$, where $A_i = \{x : 2^{i-1} < f(x) \leq 2^i\}$ for $i \geq 2$ and $A_1 = \{x : f(x) \leq 2\}$, it easily follows that for each n ,

$$\frac{1}{n} \#\{1 \leq k : \frac{f(T^k x)}{k} \geq 1/n\} \leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{k=1}^{n2^i} \chi_{A_i}(T^k x) \leq \sum_{i=1}^{\infty} 2^i (\chi_{A_i})^*(x).$$

We will show that the last term in the above inequality is finite a.e. by proving that its $L_{1,\infty}$ norm is finite. Let $M = \sum 2^n m(A_n) \leq 2\|f\|_1$ and let $g_i(x) = 2^i (\chi_{A_i})^*(x)/M$. By the maximal ergodic theorem, we see

$$\|g_i\|_{1,\infty} \leq \frac{2^i m(A_i)}{M} = \frac{2^i m(A_i)}{\sum_n 2^n m(A_n)}$$

so that $\sum_i \|g_i\|_{1,\infty} \leq 1$. Based on Lemma 4, it will be sufficient to prove that $\sum_{i=1}^{\infty} \|g_i\|_{1,\infty} \log(1/\|g_i\|_{1,\infty}) < \infty$. It is quickly seen that this condition is equivalent to establishing that $\sum_{i=1}^{\infty} 2^i m(A_i) \log(1/(2^i m(A_i))) < \infty$.

Here and in the future the summation only runs over the indices i for which $m(A_i) > 0$. Consider now $S_1 = \{i : 2^i m(A_i) \leq 1/i^2\}$ and $S_2 = \{i : 2^i m(A_i) > 1/i^2\}$. Now

$$\sum_{i \in S_1} 2^i \|\chi_{A_i}\|_1 \log \left(\frac{1}{2^i \|\chi_{A_i}\|_1} \right) \leq \sum_{j=2}^{\infty} \log(j^2)/j^2 < \infty$$

since $\psi(t) = t \log(\frac{1}{t})$ is increasing on $[0, 1/e]$. On the other hand

$$\sum_{i \in S_2} 2^i \|\chi_{A_i}\|_1 \log \left(\frac{1}{2^i \|\chi_{A_i}\|_1} \right) < 2 \sum_{i \in S_2} 2^i \|\chi_{A_i}\|_1 \log i < \infty$$

since $f \in L \log \log L(X)$. This ends the proof. \square

Corollary 6. *For each dynamical system (X, Σ, m, T) and non-negative function $f \in L \log \log L(X)$, there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(T^k x) \xi_k(\omega)$$

exists for μ a.e. ω .

There does not seem to be better way of exploiting Lemma 4 in order to extend even more the class of functions for which N^*f is almost everywhere finite. Moreover, as we show in the following proposition, the inequality in Lemma 4 is sharp up to a constant. We note that a more general version of this proposition appears in work of Kalton [6]

Proposition 7. *Given positive numbers a_1, \dots, a_n which sum to 1, there exist functions g_1, \dots, g_n with $\|g_i\|_{1,\infty} = a_i$ such that $\|g_1 + \dots + g_n\|_{1,\infty} \geq \frac{1}{6}(2+K) \sum \|g_i\|_{1,\infty}$, where K is the entropy of the sequence (a_i) .*

Proof. For each i , let ξ_i be a random variable taking the value $1/n$ with probability $(1 - a_i)^{n-1} a_i$. Moreover, the ξ_i 's will be chosen to be independent. One can then check that $\mathbb{P}(\xi_i > \lambda) \leq a_i/\lambda$ while $\mathbb{P}(\xi_i \geq 1 - \epsilon) = a_i$ for ϵ small enough, so that $\|\xi_i\|_{1,\infty} = a_i$.

We see that

$$\begin{aligned} \mathbb{E}(\xi_i) &= \sum_{n=1}^{\infty} a_i \frac{(1 - a_i)^{n-1}}{n} \\ &= -\frac{a_i}{1 - a_i} \log a_i \geq -a_i \log a_i. \end{aligned}$$

Similarly, we see that

$$\mathbb{E}(\xi_i^2) = a_i \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - a_i)^{n-1} \leq a_i \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2a_i.$$

In particular, setting $\Xi = \xi_1 + \dots + \xi_n$, we see that $\mathbb{E}(\Xi) \geq K$ but $\text{Var}(\Xi) \leq 2$.

Using Tchebychev's inequality, we see that

$$\mathbb{P}(\Xi \geq K - 2) \geq \mathbb{P}(|\Xi - \mathbb{E}(\Xi)| \leq 2) \geq 1 - \frac{\text{Var}(\Xi)}{2^2} \geq \frac{1}{2}.$$

If $K > 4$, we have $\mathbb{P}(\Xi \geq K/2) \geq \frac{1}{2}$ so that the weak- L^1 norm exceeds $K/4$, which in turn exceeds $(K+2)/6$. If $K \leq 4$, take f to be any function of weak L^1 norm 1 and let $f_n = a_n f$, so that $\|f_n\|_{1,\infty} = a_n$. Then $\sum f_i = f$, so that $\|\sum f_i\|_{1,\infty} = 1 \geq \frac{1}{6}(K+2) \sum \|f_i\|_{1,\infty}$. This completes the proof of the proposition. \square

Remark 8. Note that although $f \in L \log \log L$ is sufficient to guarantee that $N^*f < \infty$ almost everywhere, there are functions f outside $L \log \log L(X)$, for which $N^*f(x) < \infty$ for a.e. x . In particular, it is easy to construct functions outside $L \log \log L$ for which the entropy computed in Theorem 5.

Further, if we are willing to restrict the system, we see that no condition on the distribution of f can guarantee the divergence of $N^*f(x)$. Specifically, Lemma 1 of [1] guarantees that whenever $T^k f$ are independent random variables (take for

example (X, Σ, m, T) to be a Bernoulli shift and f to depend only on the first coordinate), then $N^* f(x) < \infty$ for a.e. x .

Another consequence of Theorem 5 is the following weighted version of Theorem 3.

Theorem 9. *For each dynamical system (X, Σ, m, T) and $f \in L^1(X)$, there is a set $X_0 \subset X$ of full measure, such that for any sequence (ξ_k) of i.i.d. random variables on the probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi_1 \in L^1(\Omega)$ and any $x \in X_0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n \log \log n} \sum_{k=1}^n f(T^k x) \xi_k(\omega) = 0$$

for μ a.e. ω .

The proof will be based on the following relative of Theorem 5. Define $L^* f(x) = \sup_n \frac{1}{n} \# \{1 \leq i : \frac{f(T^i x)}{i \log \log i} \geq \frac{1}{n}\}$

Lemma 10. *For each dynamical system (X, Σ, m, T) and each $f \in L_1(X)$, $L^* f(x) < \infty$ for a.e. x .*

Proof. As usual, we can assume f is positive. Fix an $n \in \mathbb{N}$. Using the fact that $f(x) \leq \sum_{i=1}^{\infty} 2^i \chi_{A_i}(x)$ we get that

$$\frac{1}{n} \# \{1 \leq k : \frac{f(T^k x)}{k \log \log k} \geq 1/n\} \leq \frac{1}{n} \sum_{i=1}^{\infty} \sum_{k=1}^{p_i} \chi_{A_i}(T^k x) \leq \frac{1}{n} \sum_{i=1}^{\infty} p_i (\chi_{A_i})^*(x)$$

where p_i is the largest integer such that $p_i (\log \log p_i) \leq n 2^i$. Letting $\phi : (1, \infty) \rightarrow \mathbb{R}$ be the increasing function $\phi(x) = x \log \log x$, we see that $p_i \leq \phi^{-1}(n 2^i)$. We claim that there exists a $C > 0$ such that $p_i \leq C \frac{2^i n}{\log(i+1)}$ for all $i, n \in \mathbb{N}$. To see this, we check the existence of a C such that $\phi^{-1}(2^x) \leq C \frac{2^x}{\log(x+1)}$ or equivalently $2^x \leq \phi(C \frac{2^x}{\log(x+1)})$ for all $x \geq 0$. Hence

$$\sup_n \frac{1}{n} \# \{1 \leq i : \frac{f(T^i x)}{i \log \log i} \geq \frac{1}{n}\} \leq C \sum_{i=1}^{\infty} \left(\frac{2^i}{\log(i+1)} \right) (\chi_{A_i})^*(x).$$

Based on Lemma 4 and on the maximal ergodic theorem, it suffices to prove that $\sum_{i=1}^{\infty} \left(\frac{2^i \|\chi_{A_i}\|_1}{\log(i+1)} \right) \log \left(\frac{2^i \|\chi_{A_i}\|_1}{\log(i+1)} \right) < \infty$. By splitting the sum in two parts depending on whether or not $\frac{2^i \|\chi_{A_i}\|_1}{\log(i+1)} < \frac{1}{i^2}$ and reasoning like in the proof Theorem 5, it easily follows that the sum from above is finite. \square

Proof of Theorem 9. It suffices to assume that both f and ξ_1 are positive. According to the previous lemma, let X_0 the subset of full measure of X containing all the points x for which $L^* f(x) < \infty$. For a fixed $x \in X_0$ denote $w_k := f(T^k x)$ and also $W_k := k \log \log k$. The argument of Jamison, Orey and Pruitt from [5] can be extended with really no essential changes to this case, to conclude that since

$$\sup_n \frac{1}{n} \# \{1 \leq i : \frac{w_i}{W_i} \geq \frac{1}{n}\} < \infty,$$

$$\lim_{n \rightarrow \infty} \frac{1}{W_n} \sum_{k=1}^n w_k \xi_k(\omega) = 0$$

for μ a.e. ω . □

Remark 11. It is not known whether in Theorem 9 the weight $n \log \log n$ can be replaced with a smaller one, like $n \log \log \log n$, much less with n . Any improvement on this weight will necessarily have behind it an extension of the result of Theorem 5 to a larger Orlicz class.

Remark 12. It would be interesting to find the largest Orlicz class that would guarantee that $N^*f(x) < \infty$ almost everywhere. The above establishes that such an Orlicz class would contain $L \log \log L$ and the recent preprint

A careful examination of the proof of [3] demonstrates that in any Orlicz class with an essentially smaller weight than the class $L \log \log \log L$, there exists a function f such that $N^*f(x) = \infty$ almost everywhere.

In particular, these two results demonstrate that the largest Orlicz class that would guarantee that $N^*f(x) < \infty$ almost everywhere lies between $L \log \log L$ and $L \log \log \log L$.

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