GLOBAL PROPERTIES OF A FAMILY OF PIECEWISE ISOMETRIES

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Abstract. We investigate a basic system of a piecewise rotation acting on two halfplanes. We prove that for invertible systems, an arbitrary neighborhood of infinity contains infinitely many periodic points surrounded by periodic cells. In the case where the underlying rotation is rational, we show that all orbits remain bounded, whereas in the case where the underlying rotation is irrational, we show that the map is conservative (satisfies the Poincaré recurrence property). A key part of the proof is the construction of periodic orbits that shadow orbits for certain rational rotations of the plane.

1. Introduction

In dynamical systems, one studies complexity in iterated mappings. It is well known that stretching (hyperbolic behaviour) gives rise to positive entropy and hence highly complex behaviour. On the other hand, isometries are known to have extremely simple dynamical behaviour.

In this paper, we study piecewise isometries in which the space is divided into a finite number of pieces and then a different isometry is applied to each piece. Buzzi [4] showed that in this case, the topological entropy is zero. Nonetheless, experiments indicate that there can be considerable complexity (see Figure 1). Such complexity arises solely from the discontinuity. This paper studies a simple concrete family of mappings, and attempts to describe the complexity that occurs.

Specifically for $\theta \in [0,1)$ and $a, b \in \mathbb{C}$, we define the piecewise isometry $T : \mathbb{C} \to \mathbb{C}$ by

$$T(z) = \begin{cases} e^{2\pi i\theta}(z+a) & \text{if } z \in P_1 = \{z : \operatorname{Im}(z) \geq 0\} \\ e^{2\pi i\theta}(z+b) & \text{if } z \in P_{-1} = \{z : \operatorname{Im}(z) < 0\}. \end{cases} \tag{1}$$

Maps of this type were initially studied by Boshernitzan and Goetz [3]. It is straightforward to see that these maps are surjective if and only if $\operatorname{Im}(a-b) \leq 0$ and injective if and only if $\operatorname{Im}(a-b) \geq 0$ and hence bijective if and only if $a-b$ is real. Boshernitzan and Goetz showed that in the non-injective case, the map is *globally attracting* (there exists $M > 0$ such that for all $z \in \mathbb{C}$, $|T^n z| \leq M$ for all sufficiently large n). They also showed that in the non-surjective case, the map is

Date: July 4, 2007.
1991 *Mathematics Subject Classification.* Primary 58F03; Secondary 22C05.

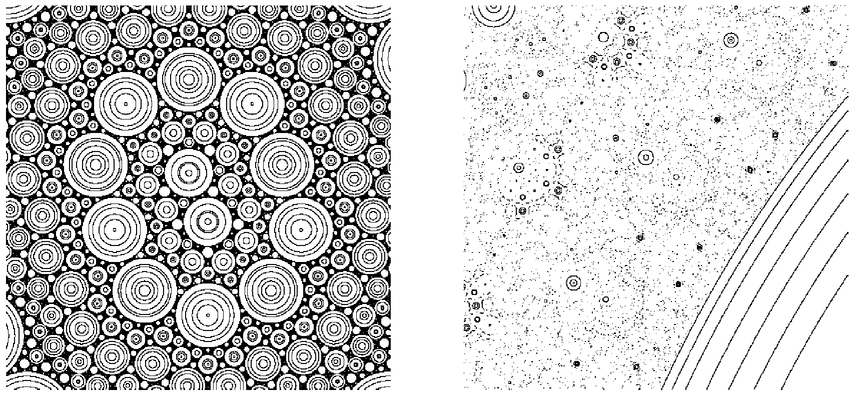

FIGURE 1. A collection of orbits shown alongside a magnification

globally repelling (there exists $M > 0$ such that $\lim_{n\to\infty} |T^n z| = \infty$ for all z satisfying $|z| \geq M$). Their paper left open the apparently more delicate case where T is bijective, and it is that case that we study in this paper. Note that in this case T preserves Lebesgue measure.

Piecewise rotations are examples of Euclidean piecewise isometries. These systems generalize well known and studied interval exchanges to a class of maps of $\mathbb{R}^n$ [4, 5]. However, unlike their one dimensional counterparts, the dynamics of piecewise isometries in higher dimensions often exhibit highly non-ergodic phenomena such as the apparent existence of microscopic periodic domains. While particular systems with rational choices of parameters have been investigated [1, 2, 6, 9, 7, 12], a general local theory of irrational piecewise isometries seems to be lacking.

In particular, one of the most tantalizing questions is the apparent recursive abundance of periodic cells in neighborhoods of a periodic cell. However this article describes at behaviour at a macroscopic scale. Specifically, we show (Theorem 1) that maps in our class have periodic cells in an arbitrary neighbourhood of infinity. These cells form almost-annuli which act as obstructions to orbits crossing the annuli. A consequence (Theorem 2) is that the maps exhibit recurrence even though the invariant measure is infinite.

2. Preliminaries

We refer to the upper and lower half planes as *atoms* of T (i.e. maximal connected sets on which T is continuous). We label the upper atom by $+1$ and the lower atom by -1 and then code the orbit of a point by the sequence of atoms that its iterates lie in. More specifically, we define $s(z) = 1$ if $\mathrm{Im}(z) \geq 0$ and $s(z) = -1$ otherwise. The *itinerary* of a point z is then the sequence $(s(T^n z))_{n\geq 0}$ in $\{\pm 1\}^{\mathbb{Z}^+}$. The equivalence class of points sharing the same itinerary as a point z will be called the *cell* containing z. Since a cell is the intersection of a countable family of half planes, it is convex (but may consist of a single point or possibly a line segment). Clearly T^k acts isometrically on a cell for all k.

Suppose that a point z has the property that $\inf |\operatorname{Im}(T^n z)| = a > 0$. Then if $|y-z| < a$, since y and z lie in the same half plane, $|Ty-Tz| = |y-z| < a$. By assumption Tz lies at least a from the discontinuity so that Ty lies in the same half plane as Tz. Continuing in this way, we see that $T^k y$ lies in the same half plane as $T^k z$ for all k. This shows that given a point z whose orbit remains at least distance a away from the discontinuity, its cell contains a disc of radius a centred at z

In particular, if a periodic point $z = T^q z$ has an orbit which does not intersect the discontinuity, its orbit is bounded away from the discontinuity so that the cell containing z contains a disc. This cell is mapped to the cell containing Tz, and that cell is mapped to the cell containing $T^2 z$ etc, so that the cells are mapped periodically. Such a cell is called a *periodic island*. Recalling the definition of T: $T(z) = e^{2\pi i\theta}(z-a)$ for $z \in P_1$ and $e^{2\pi i\theta}(z-b)$ for $z \in P_{-1}$, the action of T on a periodic island centred at z is to rotate it by $2\pi\theta$ and map it to a periodic island centred at Tz.

In the case where $\theta \notin \mathbb{Q}$, the cell is a closed disk (possibly missing a countable number of boundary points). To see this, note that if z is one of the points on the periodic orbit which lies closest to the discontinuity (say $|\operatorname{Im}(z)| = a$), then if y is any point for which $|y-z| > a$, there will exist (by density of multiples of $2\pi q\theta$ modulo 2π) an n such that rotation of y about z by $2\pi qn\theta$ lies on the opposite side of the discontinuity than z. It then follows that if the itineraries of y and z agree until time $qn-1$, they will disagree at time qn. The conclusion is that any point y with $|y-z| < a$ follows the orbit of z, whereas any point with $|y-z| > a$ has a different itinerary. If $|y-z| = a$, the point has a different itinerary if and only if it lands on the discontinuity at some stage, and this is in a different atom of the map. This establishes that in the case where θ is irrational, the periodic islands are closed discs (possibly with a countable number of boundary points removed). In the rational case, the cell consists of a convex polygon with sides at angles that are multiples of $2\pi/q$ or in the case where q is odd, possibly π/q (this arises as the region around the periodic point can be cut q times by nearby points entering the lower half plane when the periodic point is in the upper half plane; and separately q times by points entering the upper half plane when the periodic point is in the lower half plane).

A central concept that emerges in our work is the idea of periodic orbits whose itineraries match those arising in rational rotations of the circle. More specifically, if $T^q z = z$, we say that z is *p/q-rationally coded* if there exists $y \in \mathbb{C}$ such that $s(T^n z) = s(R^n y)$ and $R^n y \notin \mathbb{R}$ for all $n \geq 0$, where $R(x) = e^{2\pi i \frac{p}{q}} x$. (As usual, we require that p and q have no common factors). It is the existence of such rationally coded

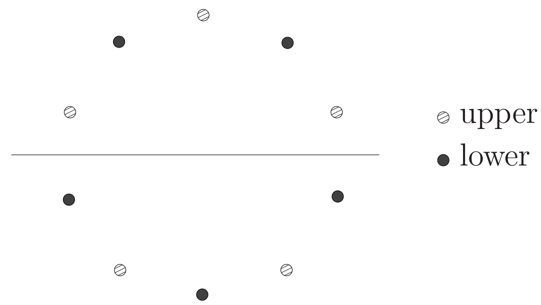

FIGURE 2. Pair of orbits when q is odd

orbits for suitable p/q (approximations of θ from below) that we shall establish in the proof of Theorem 1.

Note that in the case where q is odd, there are **two** periodic itineraries of the rotation $R : z \mapsto e^{2\pi ip/q}z$, $(s(e^{2\pi i(np+\frac{1}{4})/q}))_{n\geq 0}$ and $(s(e^{2\pi i(np-\frac{1}{4})/q}))_{n\geq 0}$. The first of these corresponds to an $R-$orbit that is in the upper half plane for $(q+1)/2$ steps per period and the lower half plane for $(q-1)/2$ steps per period, whereas the second corresponds to an R-orbit that is in the lower half plane for $(q+1)/2$ steps per period and the upper half plane for $(q-1)/2$ steps per period. We call the first of these the *upper* p/q-rotational itinerary and the second the *lower* p/q-rotational itinerary (see Figure 2). In the case where q is even, there is a single periodic orbit in $\{\pm 1\}^{\mathbb{Z}^+}$ that is the itinerary the rotation by p/q, namely $(s(e^{2\pi i(np+\frac{1}{2})/q}))_{n\geq 0}$. (Note that the choice of phase: 1/4 in the odd case and 1/2 in the even case is not unique, but these phases maximize the distance of the periodic orbit from the discontinuity. It may also be shown that the periodic orbits arising in the true maps have arguments close to these values).

Of particular importance for us are the points on the periodic orbits that correspond (in terms of itinerary) to the points of the rotation orbits that lie nearest to the discontinuity. More specifically, if q is even, we call the points with itineraries matching the R itineraries of $e^{i\frac{\pi}{q}}$, $e^{i\pi(1-\frac{1}{q})}$, $e^{i\pi(1+\frac{1}{q})}$, $e^{-i\frac{\pi}{q}}$ respectively the *I, II, III and IV quadrant critical points*. If q is odd, I, II, III and IV quadrant critical points have itineraries that match the R-orbits of $e^{i\frac{\pi}{2q}}$, $e^{i\pi(1-\frac{1}{2q})}$, $e^{i\pi(1+\frac{1}{2q})}$ and $e^{-i\frac{\pi}{2q}}$. Note that in the case where q is even, all four of the points lie on a single orbit, whereas when q is odd, the I and II quadrant critical points lie on the upper orbit and the III and IV quadrant critical points lie on the lower orbit. Typically the I and II quadrant critical points on the p/q-rotationally coded orbit(s) are the points nearest the discontinuity in the upper half plane, while the III and IV quadrant critical points are nearest the discontinuity in the lower half plane.

We will use the familiar $O(X)$ notation to denote quantities Y such that $|Y| \leq C|X|$ for some unspecified constant C whenever q is sufficiently large. The constant C may depend on θ and σ (that is the parameters defining the map T) but nothing else.

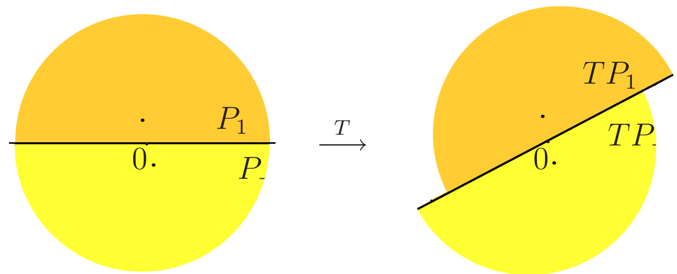

FIGURE 3. The action of the invertible map $T : \mathbb{C} \to \mathbb{C}$. The map rotates the upper and lower halfplanes P_1 and P_{-1} by an angle $2\pi\theta$ and their images are then slid by different amounts.

3. RESULTS

We study the bijective subclass of maps of the type defined in (1). This corresponds to $a - b \in \mathbb{R}$. We may assume that $a \neq b$, as the case $a = b$ is trivial. By conjugating by a translation, we may assume that a and b are themselves real and by conjugating by a dilation, we may assume that $a - b = \pm 2$. If the parameters satisfy $a - b = -2$, we can check by conjugating with the map $R(z) = -\bar{z}$ that T is conjugate to the map with angle $-\theta$ parameters $-a$ and $-b$ (so that $-a - (-b) = 2$). Given $a - b = 2$, we can write $a = 1 + \sigma$ and $b = 1 - \sigma$.

For this reason, we study the piecewise isometries defined below. For $\theta \in [0, 1)$ and $\sigma \in \mathbb{R}$, define the piecewise isometry $T_{\theta,\sigma} : \mathbb{C} \to \mathbb{C}$ by

$$T^{(\theta,\sigma)}z = \begin{cases} T^{(\theta,\sigma)}_{+1} z & \text{if } z \in P_1 = \{z : \mathrm{Im}(z) \geq 0\} \\ T^{(\theta,\sigma)}_{-1} z & \text{if } z \in P_{-1} = \{z : \mathrm{Im}(z) < 0\}. \end{cases} \tag{2}$$

where $T^{(\theta,\sigma)}_{\pm 1}(z) = e^{2\pi i\theta}(z + \sigma \pm 1)$. This means that $T^{(\theta,\sigma)}_{+1}$ and $T^{(\theta,\sigma)}_{-1}$ are two distinct Euclidean rotations through some angle $2\pi\theta$. We let $\mathcal{T}$ denote the class of all $T^{(\theta,\sigma)}$ as θ runs over $[0, 1)$ and σ runs over $\mathbb{R}$. We call σ the *asymmetry* of $T^{(\theta,\sigma)}$. In the case where $\sigma = 0$, the map is called *symmetric* as T is self-conjugate by the map $z \mapsto -z$.

It is useful to subdivide the class $\mathcal{T}$ according to whether the angle θ is rational or irrational. Specifically, we let $\mathcal{T}_{\mathrm{irr}}$ denote the members of the class whose underlying rotation is an irrational multiple of π and $\mathcal{T}_{\mathrm{rat}}$ denote those maps whose underlying rotation is a rational multiple of π.

Theorem 1. *Let $T \in \mathcal{T}$ be an invertible piecewise rotation with angle of rotation $2\pi\theta$.*

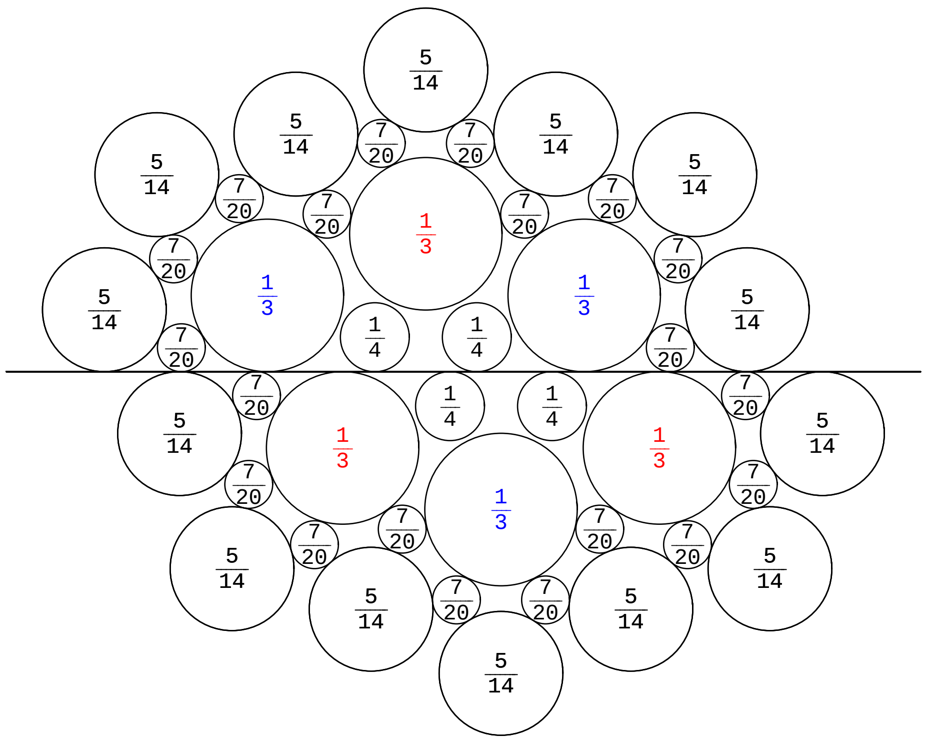

FIGURE 4. Rationally Coded Periodic Islands for $\theta = (3-\sqrt{5})/2$. The islands are labelled by their rational rotation codes. Note that the islands are close, but do not actually touch.

Irrational case: *Let $\theta \notin \mathbb{Q}$ and $\sigma \in \mathbb{R}$. Then T has rotationally coded periodic points in an arbitrary neighbourhood of infinity.*

Rational case: *Let $\theta = c/d \in \mathbb{Q}$ with $d > 2$. Let p_0/q_0 be the largest rational value less than θ with $q_0 < d$ and let $p_k/q_k = (p_0 + kc)/(q_0 + kd)$. Then for sufficiently small σ,*

1. *For each k, T has a p_k/q_k-rotationally coded periodic orbit (or a pair of such orbits in the case where q_k is odd);*
2. *The periodic islands surrounding the points on this orbit (this pair of orbits) touch so that the union forms an invariant 'annulus' surrounding the origin.*
3. *If q_j and q_k have the same parity, then the periodic islands on the p_j/q_j and p_k/q_k orbits are equal up to translation (if the parity is odd, then the periodic islands corresponding to the upper orbit are equal, as are the periodic islands corresponding to the lower orbit).*

As a consequence, all orbits are bounded.

The theorem is illustrated in Figures 4 and 5.

Theorem 2. *Let $T \in \mathcal{T}_{irr}$. Then for every set A of positive measure in the plane, Lebesgue-almost every point of A visits A infinitely often (i.e. T is conservative as a Lebesgue measure-preserving transformation).*

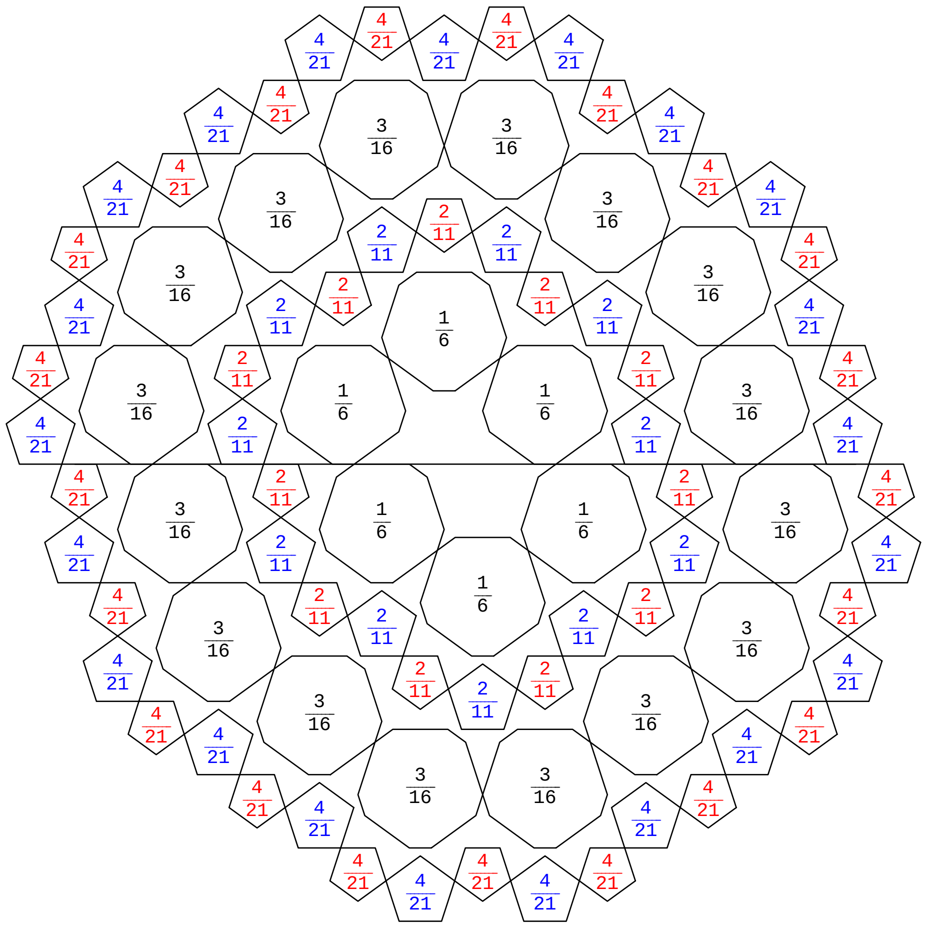

FIGURE 5. Periodic islands for θ=1/5 with $\sigma = 1$. The islands are labelled by their rational rotation codes. The islands touch vertex-to-vertex.

Remark 1. Calculations show that the periodic islands for $T^{(\theta,\sigma)}$ provide some kind of visualization of the rationals that approximate θ from below. If p/q is an approximation from below whose code is realized, then the islands corresponding to the orbit lie roughly around a circle of radius approximately $1/|\theta - p/q|$. The islands themselves have radius approximately $1/|q\theta - p|$.

Remark 2. While writing this paper, we found at least three fairly distinct proofs of the Theorem 1. They can be briefly summarized as follows:

Perturbation of near-returns: Find a point x such that $T^n x \approx x$. Argue that T^n behaves in each atom like a rotation through a known angle so that from x and $T^n x$, you can infer a centre of rotation z. Check that x and z lie in the same half plane for n steps. Then z is forced to be a periodic point.

Taylor Series: If the map is a piecewise rotation with irrational angle θ, find p/q very close to θ (with p/q less than θ). Given a periodic itinerary (in this case, we use the itinerary of a periodic point under rotation by $2\pi p/q$), there is a simple rational expression for the position of a potential periodic point with the given itinerary. The point obtained in this way is a true periodic point with that itinerary if and only if its itinerary agrees with the desired itinerary for a single period. This method consists of controlling the location of the points on the orbit using a Taylor series expansion in powers of $\theta - p/q$ of the rational expression mentioned above.

Elementary Geometric/Algebraic: We are looking for periodic points whose itineraries coincide with fixed points of rational rotations of rotation angle $2\pi p/q$. These itineraries have many Sturmian-type properties (e.g. the itinerary for two adjacent periodic points on a single orbit differs in exactly two places per period). We use simple formal arguments as above to locate the potential periodic orbits. We then exploit the properties of the codings to derive basic geometric properties of the set of points obtained from these formal expressions to simplify the process of checking that they do form a genuine periodic orbit.

In the end, we elected to use the third method on the basis that it could be used both for cases where θ was rational and irrational. Further, the arguments used here are applicable to the proof of Theorem 2. There is a cost to using this approach, and that is a loss of generality. It is of interest to carry out calculations of this type in which the transformation consists of an interval exchange on the angles composed with a piecewise translation. In this case, many of the symmetries exploited in the third method can be expected not to persist (probably those in the second method also), and so it may be necessary to use the first method which has a more generic character.

Corollary 3 (to Theorem 2)**.** *Let $T \in \mathcal{T}_{irr}$. Then T has no wandering domains (where an open set U is a wandering domain if $T^nU \cap U = \emptyset$ for all $n > 0$).*

One helpful heuristic for the action of T is as follows. The translation part of the maps in $\mathcal{T}$ always acts to move points a net amount in the clockwise direction (e.g. in the symmetric case where $\sigma = 0$, it moves points in the upper half plane to the right and points in the lower half plane to the left). The magnitude of the translations is constant across the plane, so that in terms of argument the translations have a greater effect nearer the origin. The perturbations to the argument scale as $C/|z|$. In fact, one can show that $T(z) \approx ze^{2\pi i(\theta - \frac{1}{\pi^2|z|})}$. (If instead of taking $a-b=2$, we take $a-b=2\pi^2$, this becomes $T(z) \approx ze^{2\pi i(\theta - \frac{1}{|z|})}$.) This heuristic successfully predicts the location of the points of period

q: specifically if $\theta = p/q + \eta/q^2$, then taking points of absolute value $\pi^2 q^2/\eta$, we have that $T(z) \approx ze^{2\pi i \frac{p}{q}}$.

Sketch of the conservativity proof. Using Theorem 1, when θ is well-approximated by p/q with q odd, there are two families of periodic islands with period q. Some symmetry calculations show that these islands have only very narrow gaps between them (of size $O(1/q)$). (In the case where the map T is asymmetric (i.e. $\sigma \neq 0$), if q is even the gaps are much larger (of size $O(1)$) and it is for this reason that we work on the odd case). Since the islands are invariant, this suggests that it is hard for mass to escape from 'inside' the island chain to outside. Blowing up the islands slightly gives an almost-invariant region that disconnects the plane with the property that any orbit escaping has to enter a small (in measure) region. This allows us to bound the measure of a wandering set by an arbitrarily small quantity so that there are no wandering sets.

4. Preliminary Lemmas

Let $T = T^{(\theta,\sigma)} \in \mathcal{T}$ be fixed for the remainder of this section. We will suppress the θ and σ in our notation for brevity. We will write $T_{\pm 1}(x) = e^{2\pi i\theta}(x + \sigma \pm 1)$.

In the proof of Theorem 1, we will construct candidate periodic points, and subsequently prove that they are true periodic points. Given a sequence $\varepsilon \in \{\pm 1\}^{\mathbb{Z}^+}$ with $\varepsilon_{n+q} = \varepsilon_n$, we start by constructing the constructing the fixed point z_ε of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$ (such a fixed point necessarily exists provided $q\theta \notin \mathbb{Z}$).

The lemmas in this section exploit combinatorial properties of the itineraries that we use to allow explicit calculations to control the locations of the periodic points.

Lemma 4 (**Criterion for Existence of periodic points**). *Let $T \in \mathcal{T}, T_{\pm 1}$ be as above. Let $\varepsilon \in \{\pm 1\}^{\mathbb{Z}^+}$ satisfy $\varepsilon_{n+q} = \varepsilon_n$. Suppose that z_ε is a fixed point of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$.*

The following hold:

(i) *$T_{\varepsilon_0}(z_\varepsilon)$ is a fixed point of $T_{\varepsilon_0} \circ T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_1}$.*
(ii) *z_ε is a periodic point of T with itinerary ε if and only if*

$$s(T^j(z_\varepsilon)) = \varepsilon_j \text{ for } 0 \le j < q. \tag{3}$$

Proof. The proof of (i) is an immediate calculation. To see (ii), we argue as follows. If z_ε is a periodic point of T with itinerary ε, then condition (3) holds by definition. Conversely, suppose that z_ε is a fixed point of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$ and note that $T(z) = T_{s(z)}(z)$ for all z. It follows inductively that $T^q(z) = T_{s(T^{q-1}z)} \circ T_{s(T^{q-2}z)} \circ \cdots \circ T_{s(z)}(z)$. By assumption we have $s(T^j z_\varepsilon) = \varepsilon_j$ so that $T^q(z_\varepsilon) = T_{\varepsilon_{q-1}} \circ \cdots T_{\varepsilon_0}(z_\varepsilon)$. Since z_ε was assumed to be a fixed point of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$, it follows that

$T^q(z_\varepsilon) = z_\varepsilon$. Finally given $n \geq 0$, write $n = aq+r$ with $0 \leq r < q$. Then $s(T^n z_\varepsilon) = s(T^r z_\varepsilon) = \varepsilon_r = \varepsilon_n$ so that z_ε has itinerary ε as required. □

The next lemma is a simple explicit formula for the fixed point of a composition of T_{+1}'s and T_{-1}'s.

Lemma 5 **(Fixed point formula).** *Let $\varepsilon_0, \dots, \varepsilon_{q-1}$ be any sequence of $+1$'s and -1's. Then provided $q\theta \notin \mathbb{Z}$, the (unique) fixed point of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$ is*

$$\begin{aligned} &Fix(T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}) = \\ &\frac{\sigma e^{2\pi i\theta}}{1-e^{2\pi i\theta}} + \frac{e^{2\pi iq\theta}}{1-e^{2\pi iq\theta}}(\varepsilon_0 + e^{-2\pi i\theta}\varepsilon_1 + \ldots + e^{-2\pi i(q-1)\theta}\varepsilon_{q-1}). \end{aligned} \tag{4}$$

The conclusion of Lemma 5 follows immediately from the inductive observation

$$T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}(z) = e^{2\pi iq\theta}z + e^{2\pi i\theta}(\varepsilon_{q-1}+\sigma) + \ldots + e^{2\pi iq\theta}(\varepsilon_0+\sigma).$$

The above two lemmas form the crux of our strategy for producing periodic orbits of T. We first guess suitable itineraries of periodic orbits, construct the corresponding z_ε and verify (3).

Given a point $\varepsilon \in \{\pm 1\}^{\mathbb{Z}^+}$ with $\varepsilon_{n+q} = \varepsilon_n$ for each n, we call the fixed point z_ε of $T_{\varepsilon_{q-1}} \circ \cdots \circ T_{\varepsilon_0}$ a *potential periodic point* for T with code ε. From Lemma 4, if (3) is satisfied, then z_ε is a true periodic point for T.

As mentioned above, the itineraries that we work with are the p/q-rationally coded orbits for p/q a rational approximation of θ from below. More specifically, given p/q, we define a family of itineraries corresponding to orbits under rotations by $2\pi p/q$. If q is even, we define $\varepsilon_j^{(n)} = s(e^{2\pi i(pj+n+\frac{1}{2})/q})$ for $n \in \{0, 1, \dots, q-1\}$, whereas if q is odd, we define $\varepsilon^{(n)}$ by $\varepsilon_j^{(n)} = s(e^{2\pi i(pj+n+1/4)/q})$ for $n \in \{0, 1/2, 1, 3/2, \dots q - 1/2\}$.

Note that $S(\varepsilon^{(n)}) = \varepsilon^{(n+p \bmod q)}$ where S is the shift map. In the case where q is even, there is a single periodic orbit, whereas when q is odd, there are a pair of periodic orbits. We let $z^{(n)}$ denote the fixed point of $T_{\varepsilon_{q-1}^{(n)}} \circ \cdots \circ T_{\varepsilon_0^{(n)}}$ and note from Lemma 4 (part (i)) that $T_{\varepsilon_0^{(n)}}(z^{(n)}) = z^{(n+p \bmod q)}$.

If q is even, we call $z^{(0)}$, $z^{(q/2-1)}$, $z^{(q/2)}$ and $z^{(q-1)}$ the *I, II, III and IV quadrant potential periodic points* respectively. These are the points whose codes correspond to points on the rotation orbit closest to the real axis in their respective quadrants.

If q is odd, the I, II, III and IV quadrant potential periodic points are $z^{(0)}$, $z^{((q-1)/2)}$, $z^{(q/2)}$ and $z^{(q-1/2)}$.

We now exploit certain symmetries of the sequences $\varepsilon^{(n)}$ to get more precise information about the locations of the potential periodic points $(z^{(n)})$.

Lemma 6 (**Periodic points above the discontinuity**). *Let y and z be I and II quadrant potential p/q-coded periodic points. Then* $\mathrm{Im}(y) = \mathrm{Im}(z)$.

A similar result holds for III and IV quadrant potential periodic points.

Proof. Let ζ be the itinerary of y and η be the itinerary of z. We check that $\eta_j = \zeta_{q-j}$ for $0 \le j < q$. To see this in the case where q is odd, notice that $\eta_{q-j} = s(e^{2\pi i(\frac{1}{2}-\frac{1}{4q}-\frac{jp}{q})})$ and $\zeta_j = s(e^{2\pi i(\frac{1}{4q}+\frac{jp}{q})})$. Since the two points are reflections in the imaginary axis, their imaginary parts have the same sign. An almost identical proof works when q is even.

We then have from Lemma 5 that

$$\begin{aligned} y &= \frac{\sigma e^{2\pi i\theta}}{1-e^{2\pi i\theta}} + \frac{1}{1-e^{2\pi iq\theta}}(\zeta_0 e^{2\pi iq\theta} + \zeta_1 e^{2\pi i(q-1)\theta} \ldots + \zeta_{q-1}e^{2\pi i\theta}) \\ z &= \frac{\sigma e^{2\pi i\theta}}{1-e^{2\pi i\theta}} + \frac{1}{1-e^{2\pi iq\theta}}(\zeta_0 e^{2\pi iq\theta} + \zeta_{q-1} e^{2\pi i(q-1)\theta} + \ldots \zeta_1 e^{2\pi i\theta}). \end{aligned}$$

Subtracting, we get

$$\begin{aligned} y - z &= \frac{1}{1-e^{2\pi iq\theta}}(\zeta_1[e^{2\pi i(q-1)\theta} - e^{2\pi i\theta}] + \ldots + \zeta_{q-1}[e^{2\pi i\theta} - e^{2\pi i(q-1)\theta}]) \\ &= \frac{1}{e^{-\pi iq\theta} - e^{\pi iq\theta}}(\zeta_1[e^{2\pi i(q/2-1)\theta} - e^{-2\pi i(q/2-1)\theta}] + \ldots \\ &\qquad\qquad + \zeta_{q-1}[e^{-2\pi i(q/2-1)\theta} - e^{2\pi i(q/2-1)\theta}]). \end{aligned}$$

Since all the terms in the outer parentheses of this last expression are imaginary as is the denominator, it follows that $y - z$ is real, so that y and z have the same imaginary parts as required. □

Lemma 7 (**Separation between periodic points**). *Let $\theta = p/q + h$ and let the $z^{(n)}$ be the points constructed above.*

If q is even, there is a collection of points $y^{(n)}$ (for $n = 0, 1, \ldots, q-1$ lying on a circle centred at the origin with Arg$(y^{(n)}) = 2\pi(n/q + 1/(2q))$ such that

$$z^{(n)} - z^{(n-1)} = e^{2\pi irh}(y^{(n)} - y^{(n-1)}), \tag{5}$$

where $|r| \le q/4$.

If q is odd, there is a collection of points $y^{(n)}$ (for $n = 0, 1/2, 1, \ldots, q - 1/2$) lying on a circle centred at the origin with Arg$(y^{(n)}) = 2\pi(n/q + 1/(4q))$ such that

$$z^{(n)} - z^{(n-1/2)} = e^{2\pi irh}(y^{(n)} - y^{(n-1/2)}), \tag{6}$$

where $|r| \le q/2$.

Proof. We deal with the (slightly harder) even case indicating the differences with the odd case. Let η and ζ be the codes corresponding to $z^{(n)}$ and $z^{(n-1)}$ respectively. Let $kp \bmod q = 1$. Then amongst the first q steps, η and ζ differ only in two places, specifically at the $k(q/2-n) \bmod q$ position (where η has a -1 while ζ has a $+1$) and $k(q-n) \bmod q$ position (when η has a $+1$ and ζ has a -1). We then use Lemma 5 to control $z^{(n)} - z^{(n-1)}$. We will make use of the relation $e^{2\pi i (kq/2)\frac{p}{q}} = -1$.

We have

$$\begin{aligned}
&z^{(n)} - z^{(n-1)} \\
&= \frac{2e^{2\pi i q(\frac{p}{q}+h)}}{1-e^{2\pi i q(\frac{p}{q}+h)}} \cdot \left(e^{-2\pi i[(k(q-n)) \bmod q]\theta} - e^{-2\pi i[(k(\frac{q}{2}-n)) \bmod q]\theta}\right) \\
&= \frac{2e^{2\pi i q h} e^{2\pi i \frac{nkp}{q}}}{1-e^{2\pi i q h}} \left(e^{-2\pi i[(-kn) \bmod q]h} + e^{-2\pi i[(\frac{q}{2}-kn) \bmod q]h}\right) \\
&= 2e^{2\pi i \frac{n}{q}} \frac{e^{2\pi i q h}(1+e^{-\pi i q h})}{1-e^{2\pi i q h}} e^{-2\pi i[(-kn) \bmod \frac{q}{2}]h} \\
&= 2e^{2\pi i \frac{n}{q}} \frac{1}{e^{-\pi i h \frac{q}{2}} - e^{\pi i h \frac{q}{2}}} e^{2\pi i h(\frac{q}{4}-[(-kn) \bmod \frac{q}{2}])} \\
&= R\, e^{2\pi i r h}(e^{2\pi i(\frac{n}{q}+\frac{1}{2q})} - e^{2\pi i(\frac{n}{q}-\frac{1}{2q})}),
\end{aligned}$$

where $r = q/4 - [(-kn) \bmod (q/2)]$ satisfies $|r| \le q/4$ and

$$R = \frac{2}{(e^{i\frac{\pi}{q}} - e^{-\frac{i\pi}{q}})(e^{-i\pi\frac{hq}{2}} - e^{\pi i \frac{hq}{2}})}$$

so that the denominator appearing in R is the product of a positive and negative imaginary term so that $R > 0$.

To finish, set $y^{(n)} = R\, e^{2\pi i(n+\frac{1}{2})/q}$.

The odd case works similarly, except that in this case, adjacent itineraries differ in a single location only. If n is an integer, then $\varepsilon^{(n)}$ and $\varepsilon^{(n-1/2)}$ differ in the $[k(q-n) \bmod q]$ position where $\varepsilon^{(n)}$ has a 1 and $\varepsilon^{(n-1/2)}$ has a -1. Also, $\varepsilon^{(n+1/2)}$ and $\varepsilon^{(n)}$ differ in the $[k((q-1)/2-n) \bmod q]$ position, where $\varepsilon^{(n+1/2)}$ has a -1 and $\varepsilon^{(n)}$ has a 1. □

Corollary 8 (**Equidistance between periodic points**). *All adjacent pairs of rotation coded potential periodic points are equally separated.*

Proof. This follows immediately from Lemma 7. □

Lemma 9 (**Points above and below the real axis**). *Let y and z denote potential I and IV quadrant critical p/q-coded points. Then $Re(z-y) = 1$.*

Also, if q is odd then $\mathrm{Im}(y-z) = \cot(\pi q\theta)$*, whereas if q is even then* $\mathrm{Im}(y-z) = -\tan(\pi q\theta/2)$.

Similarly if y and z are II and III quadrant critical p/q-coded points, then $\mathrm{Re}(z-y)=1$.

Proof. In the case where q is odd, the two points have the same itinerary for the first q steps except that they differ at time 0, where y is coded by $+1$ and z is coded by -1. It follows that using Lemma 5 that $z-y=-2e^{2\pi iq\theta}/(1-e^{2\pi iq\theta})=1-i\cot(\pi q\theta)$.

If q is even, the two points differ at time 0 (when y is coded by $+1$ and z by -1) and time $q/2$ (when y is coded by -1 and z by $+1$). From Lemma 5, we see that $z-y=-2(e^{2\pi iq\theta}-e^{\pi iq\theta})/(1-e^{2\pi iq\theta})=2e^{\pi iq\theta}/(1+e^{\pi iq\theta})=1+i\tan(\pi q\theta/2)$. □

Lemma 10. *Let z and z^* be potential periodic points coded by strings s and $-s$ (where by $-s$, we mean that $+1$'s and -1's are reversed). Then $z+z^*=2\sigma e^{2\pi i\theta}/(1-e^{2\pi i\theta})=-\sigma+i\sigma\cot(\pi\theta)$*

The result is a simple computation using Lemma 5.

We finish this section with a geometric lemma describing the shape of the periodic islands in the case where the map has a rational rotation angle.

Lemma 11. *Let $\theta=c/d$ and let z be a periodic point of period q with $\gcd(q,d)=1$. Let $a=\min\{|\,\mathrm{Im}(T^nz)|\colon \mathrm{Im}(T^nz)\geq 0\}$ and $b=\min\{|\,\mathrm{Im}(T^nz)|\colon \mathrm{Im}(T^nz)<0\}$.*

If d is even, the periodic islands surrounding the points on the orbit of z are regular d-gons all in the same orientation with tops (and therefore bottoms) parallel to the real axis at a distance $\min(a,b)$ from the central periodic point.

If d is odd then there are three cases depending on the ratio of a and b:

- *If $a/b\leq\cos(\pi/d)$ then the islands are regular d-gons with flat bottoms at a distance a from the centre.*
- *If $a/b\geq\sec(\pi/d)$ then the islands are regular d-gons with flat tops at a distance b from the centre.*
- *If $\cos(\pi/d)<a/b<\sec(\pi/d)$, then the islands are semi-regular $2d$-gons with all exterior angles given by π/d and face lengths alternating between $2(a-b\cos(\pi/d))/\sin(\pi/d)$ and $2(b-a\cos(\pi/d))/\sin(\pi/d)$, with the top face of the former length and the bottom face of the latter length. All of the islands are translates of one another.*

Proof. We deal first with the case where d is odd. Let $z'=T^rz$ and $z''=T^sz$ be the points on the orbit closest to the real axis from above and below so that $\mathrm{Im}(z')=a$ and $\mathrm{Im}(z'')=-b$. Let $P=\{y\colon \mathrm{Im}(e^{2\pi i\frac{kc}{d}}(y-z))\in[-a,b)\text{ for all }k\}$.

We claim that if $y\in P$, then it has the same itinerary as z. We show this inductively. Suppose y and z have the same itinerary for $k-1$ steps. Then $T^ky-T^kz=e^{2\pi i\frac{kc}{d}}(y-z)$. The imaginary part of this is in $[-a,b)$ so that since T^kz has an imaginary part in $(-\infty,-b]\cup[a,\infty)$,

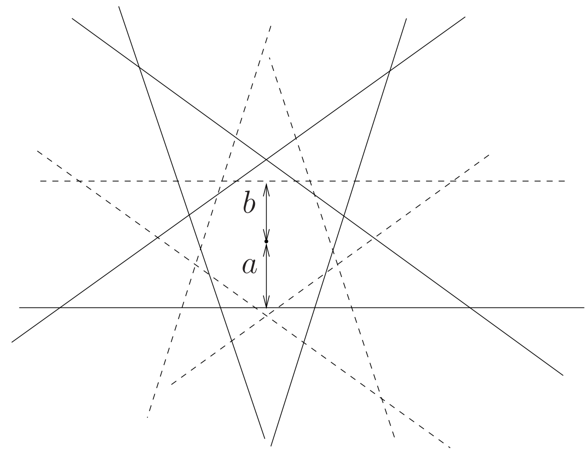

FIGURE 6. Odd Case

it follows that the imaginary part of $T^k y$ has the same sign as the imaginary part of $T^k z$ (considering the sign of 0 to be positive), so that the itineraries agree also at the kth step.

Conversely, suppose that $y \notin P$. Then there exists an integer k such that $\mathrm{Im}(e^{2\pi i \frac{kc}{d}}(y-z))$ is either less than $-a$ or greater than or equal to b. Since q and d are coprime, by adding multiples of d, k may be chosen to be congruent to r modulo q in the former case or congruent to s modulo q in the latter case. Now we check that if y and z have itineraries which agree up to time $k-1$, then $T^k y - T^k z = e^{2\pi i \frac{kc}{d}}(y-z)$ so that they must disagree at the kth step.

This shows that P is precisely the periodic island around z. From the definition, we see that P is the intersection of two sets of q halfplanes containing z: q (closed) halfplanes at a distance a from z with nearest point in direction $-ie^{2\pi i \frac{kc}{d}}$ for $k = 0, \ldots, d-1$; and q (open) halfplanes at a distance b from z with nearest point in direction $ie^{2\pi i \frac{kc}{d}}\ z$. If a and b are close in size, this gives rise to a semi-regular $2d$-gon as illustrated in Figure 6.

If a is considerably smaller than b, one obtains a d-gon, as the a-polygon lies entirely inside the halfplanes at distance b. Similarly if b is considerably smaller than a. The critical case is when the vertices of the a polygon are exactly at a distance b from the periodic point (see Figure 7) or vice versa. Elementary trigonometry shows these cases occur when $a = b\cos(\pi/d)$ or $b = a\cos(\pi/d)$ as required.

In the even case, since the a and the b polygons have the same face directions, one d-gon is nested inside the other so that one obtains simply a d-gon with faces at a distance $\min(a, b)$ from the periodic point at angles $2\pi kc/d$ from the horizontal.

All that remains is to calculate the side lengths in the case where d is odd and the island is a $(2d)$-gon. We calculate using elementary trigonometry as shown in Figure 8

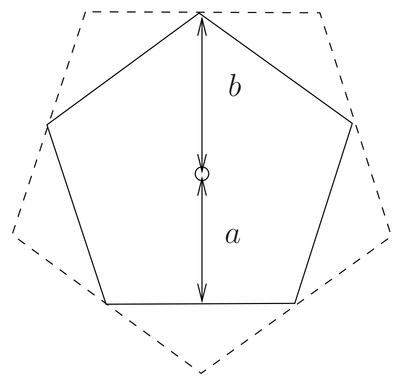

FIGURE 7. Critical Case (d odd)

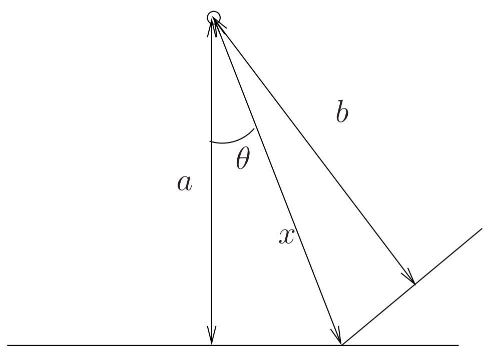

FIGURE 8. Calculating the side lengths of a $(2d)$-gon for d odd

We see from the figure that $x\cos\theta = a$ and $x\cos(\pi/d - \theta) = b$. Substituting for x in the second equation, we get $a\tan\theta = (b - a\cos(\pi/d))/\sin(\pi/d)$. Similarly working in the other triangle, we get $b\tan(\pi/d - \theta) = (a - b\cos(\pi/d))/\sin(\pi/d)$. These are the two half-edge lengths in the semi-regular polygon giving the required edge lengths. □

5. Proof of Theorem 1

Proof of Theorem 1. Fix p/q with $h = \theta - p/q < 1/(4q)$. (Note that there are always such p/q when provided that θ is not a rational with denominator 4 or less). Let $\varepsilon^{(n)}$ be the sequence of itineraries constructed in Section 4 (indexed by $\{0, 1, \ldots, q-1\}$ if q is even or $\{0, 1/2, 1, \ldots, q - 1/2\}$ if q is odd) and let $z^{(n)}$ be the corresponding potential periodic points.

From Lemma 7, we see that $z^{(n)}$ lies in the cone between with vertex at $z^{(0)}$ and sides $e^{-i\pi qh}(y^{(n)} - y^{(0)})$ and $e^{\pi iqh}(y^{(n)} - y^{(0)})$. By the choice of h, we see that if $n \le q/4$, then $z^{(n)}$ has a greater imaginary part than $z^{(0)}$. A similar argument based at the II quadrant potential periodic point together with an application of Lemma 6 establishes that for $0 \le n < q/2$, $z^{(n)}$ has an imaginary part that is at least as large as that of $z^{(0)}$. Similarly for $q/2 \le n < q$, $z^{(n)}$ has an imaginary part that is no bigger than the imaginary part of $z^{(q/2)}$.

From criterion (3) of Lemma 4, we see that in order for the potential periodic orbit with code p/q to be realized, it is therefore sufficient that the I quadrant potential periodic orbit lie in the upper half plane and the IV quadrant potential periodic orbit lie in the lower half plane.

We can use Lemmas 9 and 10 to calculate the imaginary parts of the I and IV quadrant potential periodic points.

Let a be the imaginary part of the I quadrant potential periodic orbit and b be the imaginary part of the IV quadrant potential periodic point (which is the same by Lemma 6 as the imaginary part of the III quadrant potential periodic point).

From Lemma 10 (applied to the I and III potential periodic points) we see that $a+b = \mathrm{Im}(2\sigma e^{2\pi i\theta}/(1-e^{2\pi i\theta})) = \sigma\cot(\pi\theta)$.

From Lemma 9, we have that $a-b = \cot(\pi q\theta)$ if q is odd and $a-b = -\tan(\pi q\theta/2)$ if q is even.

The condition (3) therefore boils down to $\sigma\cot(\pi\theta)+\cot(\pi q\theta) > 0$ and $\sigma\cot(\pi\theta)-\cot(\pi q\theta) < 0$ if q is odd; and $\sigma\cot(\pi\theta)-\tan(\pi q\theta/2) > 0$ and $\sigma\cot(\pi\theta)+\tan(\pi q\theta/2) < 0$ if q is even.

More straightforwardly, the conditions are:

$$|\sigma| < |\tan\pi\theta|\cot(\pi q\theta) \text{ if } q \text{ is odd; and} \tag{7}$$

$$|\sigma| < -|\tan\pi\theta|\tan(\pi q\theta/2) \text{ if } q \text{ is even.} \tag{8}$$

If θ is irrational, then taking p/q a close approximation to θ from below, $q\theta$ can be made to exceed an integer p by an arbitrarily small amount. If q is odd, then $\pi q\theta$ is slightly greater than a multiple of π so its cotangent is arbitrarily large. If q is even, then p is odd and $\pi q\theta/2$ is larger than an odd multiple of $\pi/2$ by an arbitrarily small amount so that $-\tan(\pi q\theta/2)$ is arbitrarily large and positive. We therefore see that there are infinitely many solutions. Further, we see that the a and b become arbitrarily large. Since the I, II, III and IV potential periodic points are the closest points to the discontinuity on the potential periodic orbit, it follows that the periodic islands become arbitrarily large. Since they have different itineraries they must be disjoint and it follows that there are periodic orbits in an arbitrary neighbourhood of infinity.

If θ is rational, let $\theta = c/d$. Let p_0/q_0 be the largest rational below c/d with denominator less than d. Then $c/d - p_0/q_0 = 1/(dq_0)$ (so that $cq_0 - dp_0 = 1$). We will study the periodic islands with code $p_k/q_k = (p_0+kc)/(q_0+kd)$. Note that $c/d - p_k/q_k = 1/(dq_k)$.

We will consider two cases based on the parity of q_k.

Case 1: q_k odd

In this case, there are two p_k/q_k rotation itineraries. From (7), the condition for these both to be realized is $|\sigma| < |\tan\pi\theta|\cot(\pi q_k c/d) =$

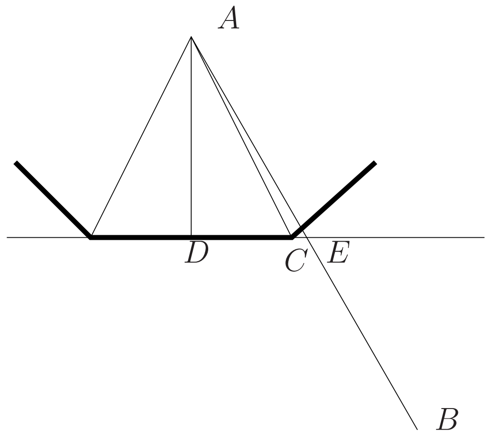

FIGURE 9

$|\tan \pi\theta| \cot(\pi/d)$. Since this is independent of k, the entire family of itineraries are present or absent simultaneously. Also, from Lemma 11, the geometry of the polygons does not depend on k. In the case where d is even, the periodic points are surrounded by periodic islands that are regular d-gons.

In the case where d is odd, we need to establish that the criteria in Lemma 11 for the islands to be regular d-gons are satisfied. These criteria are that for both periodic orbits, the ratio of the closest point on the orbit to the axis from above to the closest point on the orbit from below should lie outside the interval $(\cos(\pi/d), \sec(\pi/d))$. Thus we need to establish that

$$|\operatorname{Im} z_{q_k-1}|/|\operatorname{Im} z_0| > \sec(\pi/d); \text{ and} \\ |\operatorname{Im} z_{1/2}|/|\operatorname{Im} z_{q_k-1/2}| > \sec(\pi/d). \tag{9}$$

We verify that these hold when $\sigma = 0$ and by continuity, the inequalities will persist for small values of σ. When $\sigma = 0$, as above we have $\operatorname{Im}(z_0) = -\operatorname{Im}(z_{q_k-1/2}) = \cos(\pi/d)/(2\sin(\pi/d))$. On the other hand, $\operatorname{Im}(z_{1/2}) = -\operatorname{Im}(z_{q_k-1}) = (\cos(\pi/d) + 2\cos(\pi/q_k \pm \pi/d)/(2\sin(\pi/d))$ from Lemma 9, where the $\pm\pi/d$ corresponds to a term less than π/d in absolute value. The desired condition (9) is satisfied if $\cos(\pi/d) + 2\cos(\pi/q_k + \pi/d) > 1$. Since d and q_k are distinct and odd, one can check that this inequality holds for all $d > 1$ and $q_k \geq 7$.

Let A be the I quadrant periodic point, B be the IV quadrant periodic point, C be the rightmost point of the base of the d-gon about A, D be the midpoint of the base of the d-gon and E be the point on the discontinuity line that lies on the line joining A and B. This is illustrated in Figure 9.

Note that C and E lie on the discontinuity line. We aim to show that $C = E$. To see this, note that since the d-gon is regular, $DAC = \pi/d$. By Lemma 9, we have $\vec{BA} = -1 + i\cot(\pi q_k c/d) = -1 + i\cot(\pi/d)$, so that $DAE = DAB = \pi/d$. Since B and C both lie on the discontinuity line and make the same angle, they must coincide. The same calculation shows that E coincides with the left vertex of the top of the d-gon centred at B.

An identical argument shows that the II and III quadrant periodic points have touching islands. Now given any adjacent pair of periodic points p and p' in the collection, suppose that their itineraries first differ at time n (so that $T^n p$ and $T^n p'$ are on opposite sides of the discontinuity, but $T^j p$ and $T^j p'$ are on the same side of the discontinuity for $j < n$). This means that the restriction of T^n to the periodic islands about p and p' is an isometry, mapping the islands onto a pair of adjacent islands on opposite sides of the discontinuity (they are only mapped by different atoms in the following step). Since we have just established that these islands are touching, their inverse image under T^n, the islands about p and p' must also be touching. It follows that the periodic islands form an annulus as claimed.

Case 2: q_k even but d odd

Since q_k is even, there is a single p_k/q_k-coded itinerary. By the above, there is a periodic orbit with this coding. We want to establish that for small values of σ, the periodic islands about points on the orbit are touching $(2d)$-gons. Further, we will show that the sizes of the islands and the range of values of σ for which this situation persists is the same for all even q_k.

We appeal to Lemma 11 to verify that the islands are $2d$-gons. To do this, we need to calculate a/b, the ratio of the distances of the closest periodic points above and below the discontinuity. From Lemma 9, we have $a + b = -\tan(\pi q_k c/(2d))$. Since $q_k = q_0 + kd$ with k odd

If q_0 is odd, since we require q_k even, k must also be odd. If $k = 2r+1$, we have $q_k c/(2d) = (q_0 + (2r + 1)d)c/(2d) = rc + (p_0 d + cd + 1)/(2d)$. Since $q_0 c = p_0 d + 1$, and q_0 and d are odd, we see reducing modulo 2 that c and p_0 have opposite parities, so that $p_0 d + cd$ is an odd multiple of d. It follows that the fractional part of $q_k c/(2d)$ is $1/2 + 1/(2d)$ so that $a + b = \cot(\pi/(2d))$ (independently of k for k odd).

If on the other hand, q_0 is even, k must be even ($2r$ say) so that $q_k c/(2d) = (q_0 + 2rd)c/(2d) = rc + (p_0 d + 1)/(2d)$. Since q_0 is even, p_0 must be odd, so that again $q_k c/(2d)$ has fractional part $1/2 + 1/(2d)$ and again $a + b = \cot(\pi/(2d))$ (again independently of k for k even).

From Lemmas 6 and 10, we have $a - b = \sigma \cot(\pi c/d)$. Since $a/b = ((a+b)+(a-b))/((a+b)-(a-b))$, we see that the quantity a/b does not depend on the particular value of k (provided that it has the right parity to make q_k even). We also see that when $\sigma = 0$, the ratio is 1 so that there is an open interval of values of σ for which the inequality $\cos(\pi/d) < a/b < \sec(\pi/d)$ holds. For σ in this range, for each k of the correct parity, the periodic islands are $(2d)$-gons. It remains to show that the islands touch.

To see this, we will show that the sum of the two edge lengths is 2. Since their midpoints are 1 unit apart, this will be sufficient to establish that the two polygons touch on the real axis. From Lemma 11, we see that the sum of the two edge lengths is $2(a+b)(1-\cos(\pi/d))/\sin(\pi/d) = 2(a+b)\tan(\pi/(2d))$. Since we already established $a+b = \cot(\pi/(2d))$. This ensures that the sum of the side lengths is 2 as required.

As in the case where d is odd, this argument establishes that the I and IV islands touch; and also that the II and III islands touch. As in the odd case, this implies that the family of islands forms an annulus.

To see that the families of annuli are indeed nested, note that by Lemma 10, for each annulus, the I quadrant periodic point and the III quadrant periodic point have real parts that average to $\sigma/2$. Since the rings don't intersect (as points of intersection would be forced to have two different itineraries), the nesting is established.

The above establishes that if B_n is the closure of the union of the polygons forming the nth annulus, that $\mathbb{C} \setminus B_n$ is disconnected having one bounded and one unbounded component. Let the bounded component be I and decompose $I \setminus \mathbb{R}$ as $I_+ \cup I_-$ where $I_+ = \{x \in I \colon \operatorname{Im}(x) > 0\}$ and $I_- = \{x \in I \colon \operatorname{Im}(x) < 0\}$. Let U_n be the interior of B_n so that $T(U_n) = U_n$. It follows that $T(I_\pm)$ are open sets that are disjoint from U_n so that they must also be disjoint from B_n. Since $T(I_\pm)$ are connected, and contain points near the origin, it follows that $T(I_\pm) \subset I$. It may be seen that there is a segment of $I \cap \mathbb{R}$ (an interval lying to the left of the I quadrant periodic island) which is mapped to a segment of the inner boundary of polygons forming B_n. This segment follows the inner boundary of B_n until it is mapped to the top of the III quadrant periodic island, whereupon it reenters I.

This proves the claim that all points have bounded orbits. □

6. Proof of Theorem 2

The outline of the proof is as follows. We will deduce from the lemmas making up Theorem 1 that the periodic islands with rotational coding p/q form a near barrier making it hard for points inside the ring of periodic islands to escape to the outside. More specifically, for p/q a close rational approximation to θ with odd denominator, we construct a 'necklace' consisting of tangential discs that are slight enlargements of the islands surrounding period q points. The complement of the necklace splits the plane into a bounded and an unbounded component. Any orbit traversing from the bounded to the unbounded component must do so by entering a small region consisting of a pair of segments of the discs. By showing that these regions can be made arbitrarily small, it will follow that almost all orbits are recurrent.

Lemma 12 (**Odd denominator approximations to** θ)**.** *Let* $\theta \in (0,1) \setminus \mathbb{Q}$. *There exist infinitely many rationals* p/q *with odd denominator satisfying* $\theta < p/q < \theta + 2/q^2$.

Proof. Let p_n/q_n be the convergents to θ arising from the continued fraction algorithm. These satisfy $p_{2n-1}/q_{2n-1} < \theta < p_{2n}/q_{2n}$ and also $|\theta - p_n/q_n| < 1/q_n^2$. We need two more properties of continued fractions. Namely we have $|p_n q_{n+1} - q_n p_{n+1}| = 1$ for each n. Secondly, we have that $(p_{2n+1} + p_{2n})/(q_{2n+1} + q_{2n}) > \theta > p_{2n+1}/q_{2n+1}$. (This follows as we have $p_{2n+1} = Ap_{2n} + p_{2n-1}$ and $q_{2n+1} = Aq_{2n} + q_{2n-1}$ where A is the largest integer such that p_{2n+1}/q_{2n+1} lies on the same side of θ as p_{2n-1}/q_{2n-1}).

We now complete the proof as follows. If there are infinitely many n such that q_{2n} is odd, we are done as these satisfy $\theta < p_{2n}/q_{2n} < \theta + 1/q_{2n}^2$. Otherwise for all sufficiently large n, q_{2n} is even. Since $|p_{2n}q_{2n+1} - q_{2n}p_{2n+1}| = 1$, q_{2n+1} must be odd. Also, as noted above, we have that $(p_{2n+1} + p_{2n})/(q_{2n+1} + q_{2n}) > \theta > p_{2n+1}/q_{2n+1}$.

We note that $q_{2n+1} + q_{2n}$ is odd. Finally, we have

$$\begin{aligned}\left|\theta - \frac{p_{2n+1} + p_{2n}}{q_{2n+1} + q_{2n}}\right| &\le \left|\frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n+1} + p_{2n}}{q_{2n+1} + q_{2n}}\right| \\ &= \left|\frac{p_{2n+1}q_{2n} - q_{2n+1}p_{2n}}{q_{2n+1}(q_{2n+1} + q_{2n})}\right| \le \frac{2}{(q_{2n+1} + q_{2n})^2}.\end{aligned}$$

□

Lemma 13 (**Near-invariant set Criterion for Conservativity**). *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $T\colon X \to X$ be an invertible measure-preserving transformation. Suppose that there is a sequence of sets $(A_n)_{n\in\mathbb{N}}$ satisfying the following properties:*

- $A_1 \subseteq A_2 \subseteq A_3 \ldots$;
- $\mu(A_n) < \infty$ *for all* n
- $\bigcup A_n = X$
- $\mu(A_n \cap T^{-1}(A_n^c)) \to 0$ *as* $n \to \infty$.

Then T is a conservative transformation: for every set B of positive measure, almost every point of B returns to B infinitely often.

Proof. We start by recalling that for every invertible measure-preserving transformation, there is a partition of X into a conservative part C and a dissipative part D. The conservative part C has the property described above: for every subset of positive measure, almost every point returns to the subset infinitely often. The dissipative part D may be expressed as a disjoint union $D = \bigcup_{n\in\mathbb{Z}} T^n(D_0))$(see Krengel's book [8] for details). Here we will show that $\mu(D_0 \cap A_n) < \mu(A_n \cap T^{-1}(A_n^c))$. Then taking limits, the left side converges to $\mu(D_0)$, whereas the right side converges to 0.

We start by observing that for almost every point x of $D_0 \cap A_n$, there is a $k > 0$ such that $T^k x \in A_n \cap T^{-1}A_n^c$. To see this, we consider the set of points that never escape: $S = D_0 \cap \bigcap_{k\ge 0} T^{-k}A_n$. Since $S \subset D_0$, the T^kS are disjoint and of measure equal to S. However, they all lie in the set A_n of finite measure so that $\mu(S) = 0$ as claimed.

Now given $x \in D_0 \cap A_n$, let $k(x) = \min\{k \colon T^k x \in A_n \cap T^{-1} A_n^c\}$. We now decompose $D_0 \cap A_n$ according to the values of the function k: set $C_j = \{x \in D_0 \cap A_n \colon k(x) = j\}$. By the above observation, $(C_j)_{j \geq 0}$ is a partition up to a set of measure 0 of $D_0 \cap A_n$. Notice also that $T^j C_j$ are subsets of $T^j D_0$ and hence are disjoint. Further $T^j C_j \subset A_n \cap T^{-1} A_n^c$. We now have

$$\mu(D_0 \cap A_n) = \sum_j \mu(C_j) = \sum_j \mu(T^j C_j) \leq \mu(A_n \cap T^{-1} A_n^c).$$

This proves the claim and hence the lemma. □

We are now ready to complete the proof of Theorem 2

Proof of Theorem 2. Fix a $T \in \mathcal{T}_{\text{irr}}$. We start by fixing p/q satisfying the following conditions:

(1) q is odd;
(2) $\theta < p/q < \theta + 2/q^2$; and
(3) there is a pair of periodic orbits of period q of T whose orbits have the same codes as periodic orbits under rotation by $2\pi p/q$.

The above lemma guarantees the existence of infinitely many q satisfying the first two conditions. Theorem 1 part (a) ensures that all sufficiently large q satisfying the first two conditions also satisfy the third.

Let p/q be as above. From Theorem 1, T has two interwoven orbits of period q. For the remainder of the proof, we will let y be the I quadrant periodic point and z be the IV quadrant periodic point. Let $\eta = \theta - p/q$.

We now define the *necklace* generated by the periodic orbit. Let w be the point on the line joining x and y that lies on the real axis and set $r = |y - w|$ and $r^* = |z - w|$. We observe that $r + r^* = |y - z|$. From Lemma 9, we have $z - y = 1 - i\cot(\pi\eta/q)$, while from Lemma 10, we have $\text{Im}(z + y) = i\sigma\cot(\pi\theta)$. It follows that for large q, we have $r, r' \approx q\eta/(2\pi)$. Let y' be the periodic point diagonally opposite z, z' be the periodic point diagonally opposite y. (These are respectively the periodic points lying above and below the negative real axis). We note that y and y' lie on the same periodic orbit; as do z and z'. Let w' be the point on the line joining y' and z' that lies on the real axis.

It follows from the proof of Corollary 8 that $y' - z' = y - z$. From Lemma 6, we have $\text{Im}(y) = \text{Im}(y')$ and $\text{Im}(z) = \text{Im}(z')$ (i.e. the line segment joining y' and z' is a horizontal translate of the line segment joining y and z). It follows that $|y' - w'| = r$ and $|z' - w'| = r^*$, so that in particular the tangency of the discs of radii r and r^* about y' and z' occurs at w' just as the tangency of the discs of radii r and r^* about y and z occurs at w.

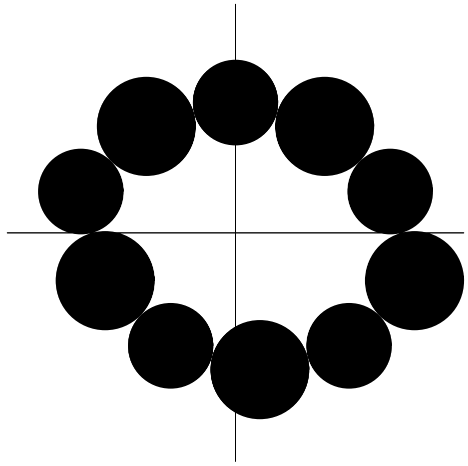

FIGURE 10. An actual Necklace with $q = 5$.

We set $N_q = \bigcup_{j<q} \overline{B}_r(T^j y) \cup \bigcup_{j<q} \overline{B}_{r^*}(T^j z)$. It follows from Corollary 8 (the equidistance between consecutive periodic points) that each adjacent pair of discs is tangent. As in the rational case, the complement of N_q has exactly two path-connected components (see Figure 10). We call the bounded component A_q and the unbounded component U_q.

We observe that since by Lemma 9 the real parts of y and z differ by 1, the segments of the discs about y and z have a combined intersection with the real axis of length 2. We will write A_q^+ for the intersection of A_q with the upper half plane and A_q^- for the intersection with the lower half plane.

We start by calculating $T^{-1}(N_q) \cap A_q$. In particular, we suppose that point x lies in $T^{-1}(N_q) \cap A_q^+$. In order that $T(x)$ lies in N_q, it is necessary that $T(x)$ lies within distance r of one of the points on the orbit of y or r^* of one of the points on the orbit of z. It will be convenient to write the point approached by $T(x)$ as $T(T^j y)$ for some j or $T(T^j z)$ for some j (depending whether $T(x)$ was close to a point on the orbit of y or of z). If the chosen periodic point lies in the same half plane as x, then we derive a contradiction as T preserves the distance between the periodic point and x, so that x must also have belonged to N_q. It follows that we must have that $T(x)$ lies within r of some $T(T^j y)$ or r^* of some $T(T^j z)$, where the $T^j y$ or $T^j z$ lies in the lower half plane. Since the action of T is to perform a relative shear of the two half planes by 2 and then to rotate, it is a necessary and sufficient condition that $x + 2$ lies within one of discs corresponding to a periodic point in the lower half plane. By the geometric observations above, this is only possible if x lies in the segment of the circle congruent to the segment of z's disc lying in the upper half plane shifted 2 to the left. Denote this segment by S_q^+ (see Figure 11). Now $A_q^+ \setminus S_q^+$ is path-connected and lies in a single atom of T. Since its image does not intersect N_q, and is path-connected, it must lie in A_q.

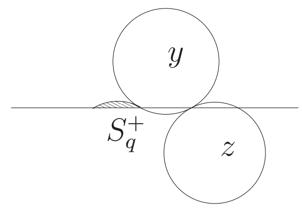

FIGURE 11. The segment S_q^+

Similarly, there is a segment S_q^- (which is a translate 2 to the right of the segment of y''s disc lying in the lower half plane) such that $T(A_q^- \setminus (S_q^-)) \subset A_q$. We let $S_q = S_q^+ \cup S_q^-$.

We crudely estimate the measure of those sets using the intersecting chords theorem: Let δ and δ' be the height of the segments of the discs corresponding to z and y'. We have $\delta(2r^* - \delta) < 1$ and $\delta'(2r - \delta') < 1$ so that $m(A_q \cap T^{-1}A_q^c) \leq m(S_q) = O(\eta/q)$.

To complete the argument, we appeal to Lemma 13. The A_q for q's satisfying the above criteria form a nested family of bounded sets whose union is the whole plane which are closer and closer to being invariant. This proves the conservativity as required.

□

7. Open Questions

In the irrational case, the existence of periodic cells in an arbitrary neighborhood of infinity implies there are many points whose orbits do not escape to infinity. The following proposition (which was already known and appears in [5]) uses elementary topology to show that there are points not lying in any periodic cell.

Proposition 14. *Let T be a piecewise isometry with irrational angle. Then there are points that do not lie in any periodic cell.*

Proof. To see this, we argue by contradiction. Suppose the plane is covered by a (countable) disjoint union of discs possibly missing some boundary points. There is an uncountable family of lines in the plane, so we can pick a line that does not go through any tangency point between a pair of discs. We claim that every disc that intersects this line must do so in a closed interval; if not then the endpoint would have to belong to another disc and that would give rise to a tangency. We therefore need to show that the line cannot be covered by a collection of disjoint bounded closed intervals. To see this, note that their endpoints must form a perfect set, which must therefore be uncountable. □

A version of the following proposition characterizing the points with irrational itinerary appears in [10] and [5].

Proposition 15. *Let $T \in \mathcal{T}$. The set of points whose itinerary is not periodic coincides up to a set of measure with the closure of the set of preimages of the discontinuity.*

Proof. From Theorems 1 and 2, T is a conservative area-preserving map of the plane. If z is a point with an aperiodic itinerary, then let C denote the cell of z. Since the itinerary is aperiodic, it is impossible for C to intersect itself (if $x \in C$ and $y = T^n x \in C$, then since $y \in C$, y has the same itinerary as x; but on the other hand since $y = T^n x$, y's itinerary is an n-times shifted version of x's itinerary which is a contradiction). It follows that every aperiodically coded cell is of measure 0. On the other hand, we already established that if a point x has an orbit that remains bounded a distance δ away from the discontinuity, then its cell contains a disc of radius δ about x. It follows that the orbit of x approaches the discontinuity arbitrarily closely. It follows that x is in the closure of the preimages of the discontinuity.

Conversely, if x is a point in the interior of a periodic island, then points on its orbit have a distance from the discontinuity is bounded below by a positive quantity. It follows that the closure of the preimages of the discontinuity are contained in the set of aperiodically coded points together with the boundaries of the periodic islands. Since there are countably many periodic islands, all with boundaries of measure 0, the statement of the proposition follows.

□

There are many questions that we have been unable to resolve concerning this family of maps. A partial list (with some interdependencies) is the following.

Question 1. *Does there exist a point that is the accumulation of periodic points?*

Question 2. *Is it true that for $T \in \mathcal{T}$, the periodic islands are dense in the plane? How about the measure of the set of points with non-periodic itinerary? From Proposition 15 above, the set of points with non-periodic itinerary agrees up to a set of Hausdorff dimension 1 with the closure of the union of the preimages of the discontinuity. What is the Hausdorff dimension of this set?*

Question 3. *Are there maps in $\mathcal{T}$ for which some points have unbounded orbits? This question has recently been answered in the affirmative by Schwartz* [11] *for the class of outer billiards. If there are points with unbounded orbits, what can be said about the rate of escape of an orbit?*

Question 4. *Can the analysis presented here be generalized to wider classes of bijective piecewise isometries of the plane? For example, one could consider an interval exchange in the angular direction composed with a translation of the plane or half-plane. What about higher-dimensional piecewise isometries?*

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