DETERMINISTIC REPRESENTATION FOR POSITION DEPENDENT RANDOM MAPS

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Abstract. We give a deterministic representation for position dependent random maps and describe the structure of the set of its invariant densities. This representation is a generalization of skew products which represent random maps. We prove one-to-one correspondence between absolutely continuous invariant measures (acims) for the position dependent random map and the acims for its deterministic representation.

1. Introduction

A random map is a discrete time dynamical system consisting of a collection of transformations τ_k on a state space X, such that, at each iteration, the selection of τ_k is made randomly according to a probability distribution $\vec{p} =< p_k >$. When the distribution is allowed to depend on the state space, we say that the random dynamical system has position dependent probabilities.

If all the p_k's are constant functions over the state space, (constant probabilites) then it is well-known that the map may be realized as a deterministic map via a skew product construction on an extended state space [4] or [5]. While this forms an important class of random dynamical systems, it does not allow for the possibility of 'feedback' from the transformations or from the state space back to the randomizing process. This may be unrealistic in some physical applications. For example, in the case of a random thermostat model, Ruelle, [6] has observed that without this feedback, while the system feels the effect of the randomizer (in this model, a heat bath), the heat bath does not feel the influence of the state of the system. Consequently, the system may heat up indefinitely. For a more realistic model, therefore, it may be important to allow the randomizer to depend on the position of the orbit in state space. In this way, we are led to study position dependent probabilities.

In [2] it was observed that position dependent random maps do not allow a skew product representation in the sense of [4]. In this note, we provide a deterministic skew-type representation for position dependent random maps. When applied to the case of constant probabilities, the representation reduces to the generalized skew product of [4, 5]. We show that there is a one-to-one correspondence between eigenfunctions for the transfer operator associated to the random transformation

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and eigenfunctions for the Frobenius-Perron operator associated to this deterministic skew-type representation. An immediate consequence is that every absolutely continuous invariant measure (acim) for the skew representation is a product measure of the form $\mu \times \lambda$, where μ is an acim for the random map and λ is Lebesgue measure on the unit interval.

The structure of the set of acims for any skew product depends on the randomizing process. Our randomizing process is an uncountable family of piecewise linear and onto transformations of the unit interval. While this is but one of many possible skew-type constructions which could lead to a deterministic representation, it is particularly simple and geometrically appealing.

In Section 2 we present careful definitions of position dependent random maps and their invariant measures. In Section 3 we construct the deterministic skew-product representation for position dependent random maps and investigate the connection between invariant measures of the random map and product-type invariant measures for its deterministic representation. In Section 4 we consider the case of nonsingular transformations and nonsingular skew products, and in Section 5, under a mild additional condition on the position dependent probabilities we derive the structure theorem on eigenfunctions mentioned above (Theorem 5.4). As in Morita [4] which investigates the case of constant probabilities, the connections between ergodic properties of the random map and the skew representation for position dependent probabilites now follow from this structure theorem. At the end of this article we present a simple example to illustrate the construction.

2. Preliminaries

$(X, \mathfrak{B}(X), \mu)$ will denote a measure space, where $\mathfrak{B}(X)$ is a σ-algebra of subsets of X and μ is a probability measure on $(X, \mathfrak{B})$. In particular, $(I, \mathfrak{B}(I), \lambda)$ will be the unit interval $I = [0, 1)$, with $\mathfrak{B}(I)$ the Borel σ-algebra on I and λ being Lebesgue measure on $(I, \mathfrak{B}(I))$. For $k = 1, \dots, K$, let $\tau_k : X \to X$ be measurable transformations and $p_k : X \to I$ be measurable functions such that $\sum_{k=1}^K p_k(x) = 1$, that is, a measurable partition of unity.

Notation 2.1. We let $T = \{\tau_1, \dots \tau_K; p_1, \dots p_K\}$ denote the associated random map.

To explain the notation, 'iterates' of the random map T are performed as follows: For $\vec{k} \in \{1, \dots, K\}^N$, we write

$$T_{\vec{k}}(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x),$$

and

$$p_{\vec{k}}(x) = p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x).$$

A random map is more precisely a Markov process with transition function

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x) \mathbf{1}_A(\tau_k(x)).$$

Here $(1)_A$ denotes the indicator function of a set A. The standard notion of an invariant measure for a Markov process gives the following definition of a T-invariant measure.

Definition 2.2. Let T be a random map and let μ be a measure on X. Define $(E_T\mu)(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x)$. Then μ is T-invariant if $\mu = E_T\mu$.

When the p_k are constant functions, the random map is said to have **constant probabilities**, an important and well-studied case (see [1, 3, 4, 5]). We are mostly interested in the case of non-constant probabilities, which we call the **position dependent** case. Of course our definition of random map above covers both settings.

3. A Deterministic Representation for Position Dependent Random Maps

S. Pelikan, in [5], gave a deterministic, skew product representation for constant probability random maps. In fact, he used the term **generalized skew product**, referring to the fact that the fibre maps did not necessarily preserve a common measure. His construction was central to the analysis of random maps that was to follow. Our first task is to extend this construction to the case of position dependent random maps. Given a random map T on X, we will construct a deterministic map S on $X \times I$ that we also call the skew product representation of T. The remainder of the paper will relate the properties of absolutely continuous invariant measures for T to those for S.

We make use of the following simple lemma:

Lemma 3.1. *Let Y and Z be a measurable spaces and let $(J_k)_{k\in\kappa}$ be a finite (or countable), measurable partition of Y. For each $k \in \kappa$, assume that T_k is a measurable map from J_k to Z. Then the piecewise-defined map $T\colon Y \to Z$ defined by $T(x) = T_k(x)$ if $x \in J_k$ is measurable.*

In our construction, $Y = Z = X \times I$ and the set J_k will be given by $J_k = \{(x, \omega) : \sum_{i<k} p_i(x) \le \omega < \sum_{i\le k} p_i(x)\}$. We define maps $\varphi_k\colon J_k \to I$ by

$$\varphi_k(x, \omega) = \frac{1}{p_k(x)}\omega - \frac{\sum_{r=1}^{k-1} p_r(x)}{p_k(x)}$$

The maps T_k are defined on J_k by $T_k(x, \omega) = (\tau_k(x), \varphi_k(x, \omega))$. We also write $\varphi_{k,x}(\omega) = \varphi_k(x, \omega)$. Define the skew product transformation $S : X \times I \to X \times I$ by

$$S(x, \omega) = (\tau_k(x), \varphi_{k,x}(\omega)),$$

for $(x, \omega) \in J_k$. S is then $\mathfrak{B}(X) \times \mathfrak{B}(I)-$ measurable.

Numerous authors have shown the existence of invariant measures for random maps with constant probabilities in a variety of settings. In [2] invariant measures are constructed for case of piecewise expanding maps τ_k on the unit interval and with position dependent probabilities. The construction proceeds directly from Definition 2.2.

In any case, there is a simple relation between invariant measures for T and those of S as follows.

Lemma 3.2. *Let μ be a measure on $(X, \mathfrak{B})$. Then μ is invariant for the random map T if and only if $\mu \times \lambda$ is invariant for the skew product S.*

Proof. Since λ is φ_x invariant, $\int_{\varphi_{k,x}^{-1}(B)} d\lambda(\omega) = p_k(x)\lambda(B)$ for any $B \in \mathfrak{A}$. Let $A \in \mathfrak{B}$ and $B \in \mathfrak{A}$. We have

$$
\begin{aligned}
(\mu \times \lambda)\left(S^{-1}(A \times B)\right) &= \int\int \mathbf{1}_{A\times B}\left(S(x,\omega)\right) d\mu(x) d\lambda(\omega) \\
&= \sum_{k=1}^{K} \int_X \int_{(J_k)_x} \mathbf{1}_A(\tau_k\, x) \cdot \mathbf{1}_B(\varphi_{k,x}\,\omega) d\lambda(\omega) d\mu(x) \\
&= \sum_{k=1}^{K} \int_X \mathbf{1}_A(\tau_k\, x) \int_{(J_k)_x} \mathbf{1}_B(\varphi_{k,x}\,\omega) d\lambda(\omega) d\mu(x) \\
&= \sum_{k=1}^{K} \int_X \mathbf{1}_A(\tau_k(x)) p_k(x) \lambda(B) d\mu(x) \\
&= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x) d\mu(x) \cdot \lambda(B) \\
&= (E_T\mu)(A)\lambda(B),
\end{aligned} \tag{3.1}
$$

where $E_T\mu$ is as in Definition 2.2. If μ is T-invariant, then the above yields $\mu \times \lambda\left(S^{-1}(A \times B)\right) = \mu \times \lambda(A \times B)$ so that $\mu \times \lambda$ is S-invariant.

Conversely, if $\mu \times \lambda$ is S-invariant, then the left side of (3.1) is $\mu \times \lambda(A \times B)$. Set $B = I$ and conclude $\mu(A) = (E_T\mu A)$ so μ is T-invariant. □

Remark 3.3. At this point, a natural question is whether or not there exist additional invariant measures for the skew beyond those of product type as above. We show in the next sections that, in general, this is possible, but under mild conditions on the transformations and probabilities, such non-physical measures are excluded.

4. Nonsingular maps and associated Frobenius-Perron operators

In order to proceed we will assume that X is equipped with a measure ν on $\mathfrak{B}(X)$ such that S is **nonsingular** with respect to $\nu \times \lambda$, that is:

$$(\nu \times \lambda)\, A = 0 \implies (\nu \times \lambda)\, S^{-1}A = 0$$

We denote this by $(\nu \times \lambda) \circ S^{-1} \ll \nu \times \lambda$ and refer to ν or $\nu \times \lambda$ as the **ambient measure** for T or S respectively.

Lemma 4.1. *If $\tau_k,\ k = 1, 2, \dots K$ are nonsingular with respect to ν then S is nonsingular with respect to $\nu \times \lambda$.*

Proof. For a measurable subset $A \subseteq X \times I$ and $x \in X$ let $(A)_x$ denote the second component section at x . Let $A_1 = \{x \in X : (A)_x \neq \emptyset\}$, $A_1^+ = \{x \in A_1 : \lambda((A)_x) > 0\}$ and $A_1^0 = \{x \in A_1 : \lambda((A)_x) = 0\}$. Then $A_1 = A_1^+ \cup A_1^0$. Now suppose $\nu \times \lambda(A) = 0$ and fix k. Then

$$\nu \times \lambda(T_k^{-1}A) = \int_{\tau_k^{-1}A_1^+} \int_{\varphi_{k,y}^{-1}(A)_{\tau_k y}} d\lambda\, d\nu(y) + \int_{\tau_k^{-1}A_1^0} \int_{\varphi_{k,y}^{-1}(A)_{\tau_k y}} d\lambda\, d\nu(y) = 0;$$

the first integral evaluating to zero because $\nu(A_1^+) = 0$ and τ_k is nonsingular; the second integral being zero because the $\varphi_{k,y}$ are individually nonsingular with respect to λ, for each y. □

From this point on it will therefore be convenient to assume that all tranformations τ_k are nonsingular with respect to ν.

We denote the Frobenius-Perron operator associated to S by P_S. Note that $P_S : L^1 \to L^1$ is a Markov operator, uniquely determined by the property

$$\int_{S^{-1}A} f \, d(\nu \times \lambda) = \int_A P_S f \, d(\nu \times \lambda)$$

for all $f \in L^1$ and measurable $A \subseteq X \times I$. On the other hand, looking at Definition 2.2 we have a natural 'transfer operator' on $L^1(X)$ whose dual is given by the action of T on L^∞. We will call this the Frobenius-Perron operator of the random map T (see, for example [2]):

$$(\mathcal{L}_T f)(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x),$$

where each P_{τ_k} is the Frobenius-Perron operator associated with the transformation τ_k. It is easy to see that a measure $d\mu = f \, d\nu$ is T-invariant if and only if $\mathcal{L}_T f = f$.

The connection between P_S and $\mathcal{L}_T$ is as follows. First, we set some notation. If $g \in L^1(X)$ we define $\check{g}(x, \omega) = g(x)$, the 'lift' of g onto the product space. Denote by $\check{L}^1 = \check{L}^1(X \times I) \subseteq L^1(X \times I)$, the subspace of constant fibre lifts from $L^1(X)$ into $L^1(X \times I)$.

Lemma 4.2. *$P_S : \check{L}^1 \to \check{L}^1$. If $g \in L^1(X)$ then $P_S \check{g} = \check{\mathcal{L}_T g}$ almost everywhere.*

Proof. Let A be measurable and let $\check{g}$ be given.

$$\begin{aligned}
\int_A P_S \check{g} d(\nu \times \lambda) &= \sum_k \int_{T_k^{-1}A} \check{g} d(\nu \times \lambda) \\
&= \sum_k \int_X \int_{(J_k)_x} \check{g}(x, \omega) \mathbf{1}_A(\tau_k(x), \varphi_{k,x}\omega) d\lambda(\omega) d\nu(x) \\
&= \sum_k \int_X \int_{(J_k)_x} g(x) \mathbf{1}_A(\tau_k(x), \varphi_{k,x}\omega) d\lambda(\omega) d\nu(x) \\
&= \sum_k \int_X \int_0^1 g(x) \mathbf{1}_A(\tau_k(x), t) p_k(x) d\lambda(t) d\nu(x) \\
&= \sum_k \int_0^1 \int_X g(x) \mathbf{1}_A(\tau_k(x), t) p_k(x) d\nu(x) d\lambda(t) \\
&= \int_I \int_X \sum_k P_{\tau_k}(p_k g) \mathbf{1}_A(x, t) d\nu(x) d\lambda(t) \\
&= \int_A (\sum_k P_{\tau_k}(p_k g))\check{} d(\nu \times \lambda)
\end{aligned}$$

Since A is arbitrary, the lemma is established. ∎

Finally, we set notation that we need later in the text. For $x \in X$ and $\vec{k} \in \{1, \ldots, K\}^N$, $B \in \mathfrak{B}(I)$ and $A \in \mathfrak{B}(X)$, we make the following definitions

$$\begin{aligned}
\varphi_{\vec{k},x} &= \varphi_{k_N, \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)} \circ \cdots \circ \varphi_{k_2, \tau_{k_1}(x)} \circ \varphi_{k_1, x} \\
\tau_{\vec{k}}(x) &= \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \tau_{k_1}(x) \\
I^{\vec{k}} &= \{(x, \omega) : \ \omega \in \varphi^{-1}_{\vec{k},x}(I)\} \\
I^{\vec{k}}_x &= \varphi^{-1}_{\vec{k},x}(I) \subseteq I \\
B^{\vec{k}} &= \{(x, \omega) : \ \omega \in \varphi^{-1}_{\vec{k},x}(B)\} \\
B^{\vec{k}}_x &= \varphi^{-1}_{\vec{k},x}(B) \subseteq I^{\vec{k}}_x \\
A_{\vec{k}} &= \{(x, \omega) : \tau_{\vec{k}}(x) \in A\}.
\end{aligned}$$

Remark 4.3. In the case that all constituent maps τ_k are local homeomorphisms on a subset of $\mathbb{R}^n$ and if the p_i are continuous and everywhere non-zero, then the deterministic representation S is a local homeomorphism of $\mathbb{R}^{n+1}$ – a class for which considerable machinery exists for the analysis of ergodic properties.

5. Absolutely Continuous Invariant Measures and Eigenfunctions of the Frobenius-Perron Operators

In this section we establish a one-to-one correspondence between eigenvalues and eigenfunctions for the operators $\mathcal{L}_T$ and those for P_S. As an immediate consequence (Corollary 5.5), we conclude that all absolutely continuous invariant measures for the skew S are of product type $\nu \times \lambda$ where μ is an acim for T on X. This requires one additional mild assumption on the spatially dependent probabilities. We remind the reader that we are assuming that T and S are nonsingular with respect to the ambient measures ν and $\nu \times \lambda$ and we assume

$$p_k(x) < 1 \quad \nu - \text{a.e. } \forall k. \tag{5.1}$$

Remark 5.1. That the correspondence does not hold in general is shown by the following simple example.

Example 5.2.

$$\tau_1(x) = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2} \\ x & \text{for } \frac{1}{2} < x \le 1 \end{cases},$$

$$\tau_2(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} < x \le 1 \end{cases};$$

and

$$p_1(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < x \le 1 \end{cases};$$

$$p_2(x) = 1 - p_1(x).$$

It is easy to see that S is the identity on the unit square which preserves any measure and hence, there can be no identification between eigenfunctions for T and S (in this case, corresponding to eigenvalue $\lambda = 1$).

Remark 5.3. Condition 5.1 is related to our choice of the randomizing process. Other randomizers could lead to different conditions which could be advantageous in certain applications. However, our example above shows that some sort of constraint will be necessary.

We now present a generalization of the main structure theorem from Morita [4] to the position-dependent case.

Theorem 5.4. *Suppose that S is the skew product system arising from a position-dependent, nonsingular random map T satisfying* (5.1). *If $P_S g = \lambda g$ where $|\lambda| = 1$, then g agrees almost everywhere with a function dependent on the first coordinate alone.*

In particular if $d\mu_g = gd(\nu \times \lambda)$ is an acim for S then $g(x, \omega) = \hat{g}(x)$ a.e. (x, ω) where $\hat{g}(x) = \int_0^1 g(x, \omega) d\lambda(\omega)$.

Before proving Theorem 5.4, we will state a corollary. Let

$$F_S = \left\{ g \geq 0, \int gd(\nu \times \lambda) = 1 : P_S g = g \right\} \text{ and } F_T = \left\{ \hat{g} \geq 0, \int \hat{g} d\nu = 1 : P_T \hat{g} = \hat{g} \right\}.$$

Corollary 5.5. Under the condition (5.1), there is a bijective correspondence between absolutely continuous invariant measures for the random map T and absolutely continuous invariant measures for the skew product S given in terms of densities by $g \leftrightarrow g \times \mathbf{1} = \check{g}$. Further, an invariant density g is an extreme point for F_S if and only if $g = \hat{g} \times \mathbf{1}$ and $\hat{g}$ is an extreme point for F_T.

Proof. The theorem above together with Lemma 3.2 establishes that the correspondence $g \leftrightarrow g \times \mathbf{1}$ between invariant densities for T and invariant densities for S is a bijection. This correspondence preserves convex combinations and hence extreme points. □

We need a few lemmas before proving Theorem 5.4. First, we introduce some notation. Write $p(x, \omega) = p_i(x)$, where i is the smallest $r \in \{1, \ldots, K\}$ such that $p_1(x) + \cdots + p_i(x) \geq \omega$.

Lemma 5.6. *Let R be a nonsingular mapping of a probability space $(Z, \mathcal{F}, \rho)$ and let $\mathcal{L}$ be the associated Frobenius-Perron operator. If $g \in L^1$ is an eigenfunction of $\mathcal{L}$ with eigenvalue λ satisfying $|\lambda| = 1$, then the following are true:*

1. $\mathcal{L}(|g|) = |g|$;
2. *There exist $h \in L^\infty$ such that $g = h|g|$ and h satisfies $h \circ T = \bar{\lambda} h$ almost everywhere.*

Proof. To see (1), note that $\mathcal{L}$ is a Markov operator so that $|g| = |\mathcal{L}g| \leq \mathcal{L}(|g|)$. Since $\mathcal{L}$ preserves integrals, we see $\mathcal{L}(|g|) = |g|$ a.e.

For (2), set $h_0(x) = g(x)/|g(x)|$ if $g(x) \neq 0$ and 0 otherwise. We first claim that $\int |g||h_0 \circ T - \bar{\lambda} h_0|^2 \, d\rho = 0$. To see this, we argue as follows:

$$\begin{aligned} \int |g||h_0 \circ T - \bar{\lambda} h_0|^2 \, d\rho &= \int \left(|g||h_0|^2 \circ T - \bar{\lambda}(|g|h_0)\bar{h}_0 \circ T - \lambda(|g|\bar{h}_0)h_0 \circ T + |g||h_0|^2 \right) d\rho \\ &= \int \left(|g||h_0|^2 \circ T - \bar{\lambda} g(\bar{h}_0 \circ T) - \lambda \bar{g}(h_0 \circ T) + |g||h_0|^2 \right) d\rho. \end{aligned}$$

Applying the definition of the Perron-Frobenius operator and using $\mathcal{L}(|g|) = |g|$, $\mathcal{L}(g) = \lambda g$ and $\mathcal{L}(\bar{g}) = \bar{\lambda}\bar{g}$ shows that the integral is 0, so that on $C = \{x \colon g(x) \neq 0\}$,

$h_0 \circ T = \bar{\lambda} h_0$. We then define $h(x) = \lim_{n\to\infty} \lambda^n h_0(T^n x)$. Since $\{x\colon g(x) \neq 0\}$ is forward-invariant (as $\mathcal{L}(|g|) = |g|$), we see that if $T^n(x) \in C$, then $\lambda^{n+1} h_0(T^{n+1}x) = \lambda^n h_0(T^n x)$. This shows that the limit exists for x in $\bigcup T^{-n}C$. For x in the complement of this set, $\lambda^n h_0(T^n x) = 0$ for all n so the limit exists for all x. Given that the limit exists, it follows immediately from the algebra of limits that $h \circ T = \bar{\lambda} h$. Since $h = h_0$ on C, we have that $g = |g|h$ as required. □

Lemma 5.7. *Suppose that S is the skew product system arising from a position dependent random map satisfying (5.1). If $d\mu_{|g|} = |g|\, d(\nu \times \lambda)$ is an acim for S then for all $\varepsilon > 0$ and $\delta > 0$, there exists an $N > 0$ and a measurable set $G \subseteq X \times I$, such that:*

1. *For $(x, \omega) \in G$, $p(x, \omega) \cdot p(S(x, \omega)) \cdots p(S^{N-1}(x, \omega)) < \delta$;*
2. *$\mu_{|g|}(G) \geq 1 - \varepsilon$;*
3. *The set G is a union of sets of the form $\{x\} \times I_x^{\vec{k}}$ for $\vec{k}$ of length N.*

Remark 5.8. Condition (3) can be stated as a measurability condition: $G \in \mathcal{F}_N$, where $\mathcal{F}_N = \bigvee_{n=0}^{N-1} S^{-n}\mathcal{P} \vee \mathcal{B}(X)$, and $\mathcal{P} = \{J_1, \ldots, J_K\}$.

Proof. Let $G(\delta, N) = \{(x, \omega)\colon p(x, \omega)p(S(x, \omega)) \cdots p(S^{N-1}(x, \omega)) < \delta\}$. Since the function $p(x, \omega)p(S(x, \omega)) \cdots p(S^{N-1}(x, \omega))$ is constant on sets of the form $\{x\} \times I_x^{\vec{k}}$ for $\vec{k}$ of length N, we see that condition (3) is always satisfied. Condition (1) is also satisfied by definition.

Define $L^N(x, \omega) = \sum_{n=0}^{N-1} \log p(S^n(x, \omega))$. Since $\log p \leq 0$ (and by (5.1), <0 almost everywhere), for fixed (x, ω), $L^N(x, \omega)$ is a decreasing sequence. It is sufficient to show that for a set of full $\mu_{|g|}-$ measure, $\lim_{N\to\infty} L^N(x, \omega) < \log \delta$. By the Birkhoff ergodic theorem, we see that for almost all (x, ω), $(1/N)L^N(x, \omega)$ is convergent (Note that in the case that $\log p$ is not integrable, the assertion still holds since $\log p$ is non-positive). Since the set $\mathcal{Z}$ of points (x, ω) for which this limit is 0 is an invariant set, the integral of the limit over $\mathcal{Z}$ will agree with the integral of $\log p$ over $\mathcal{Z}$. Since the integral of the limit over $\mathcal{Z}$ is 0, it follows that the integral over $\mathcal{Z}$ of $\log p$ is 0. However, since $\log p$ is strictly negative $(\nu \times \lambda)-$almost everywhere, this implies that $\mathcal{Z}$ is of measure 0. We conclude that for almost every (x, ω), $(1/N)L^N(x, \omega)$ converges to a strictly negative quantity (possibly $-\infty$). It follows that for almost every (x, ω), $L^N(x, \omega) \to -\infty$ so that the increasing union of the $G(\delta, N)$ over N is of measure 1. In particular, there exists an $N > 0$ such that $\mu_{|g|}(G(\delta, N)) > 1 - \varepsilon$ satisfying conclusion (2).

□

Lemma 5.9. *Let $\vec{k}$, $\vec{k}' \in \{1, \ldots, K\}^N$, $\vec{k} \neq \vec{k}'$. Then $I_x^{\vec{k}} \cap I_x^{\vec{k}'} = \emptyset$.*

Proof. First note that if $\vec{w}$ is a string of length n whose first m terms are a string $\vec{v}$, then $I_x^{\vec{v}} \supset I_x^{\vec{w}}$. If $\vec{k}$ and $\vec{k}'$ first differ in the mth term, then let $\vec{v}$ and $\vec{v}'$ be the truncated strings of length m. Since $I_x^{\vec{v}} \supset I_x^{\vec{k}}$ and $I_x^{\vec{v}'} \supset I_x^{\vec{k}'}$, it is sufficient to show that $I_x^{\vec{v}}$ and $I_x^{\vec{v}'}$ are disjoint. Accordingly, we deal with the case where $\vec{k}$

and $\vec{k}'$ differ only in the last term. In this case, we have

$$I_x^{\vec{k}} = \varphi_{k_1,x}^{-1}\varphi_{k_2,\tau_{k_1}x}^{-1}\cdots\varphi_{k_{n-1},\tau_{k_{n-2}}\cdots\tau_{k_1}x}^{-1}\left(\varphi_{k_n,\tau_{k_{n-1}}\cdots\tau_{k_1}x}^{-1}(I)\right)$$
$$I_x^{\vec{k}'} = \varphi_{k_1,x}^{-1}\varphi_{k_2,\tau_{k_1}x}^{-1}\cdots\varphi_{k_{n-1},\tau_{k_{n-2}}\cdots\tau_{k_1}x}^{-1}\left(\varphi_{k_n',\tau_{k_{n-1}}\cdots\tau_{k_1}x}^{-1}(I)\right)$$

We see that the final two intervals are disjoint, whereas all the preceding inverse maps are the same. Since inverse maps preserve disjointness, the lemma follows. □

Corollary 5.10. For measurable sets $A \subseteq X$, $B \subseteq I$, we have

$$S^{-N}(A \times B) = \bigcup_{\vec{k}\in\{1,\dots,K\}^N} (A_{\vec{k}} \cap B^{\vec{k}}), \tag{5.2}$$

where (5.2) is a mutually disjoint union.

Proof. The decomposition is a straightforward calculation. To see that the subsets are disjoint, observe $B_x^{\vec{k}} \subset I_x^{\vec{k}}$ and $B_x^{\vec{k}'} \subset I_x^{\vec{k}'}$. Thus, the corollary follows from Lemma 5.9. □

We will also need to make use of the following standard fact.

Lemma 5.11. *Let g be an L^1 function on $X \times I$ and let $\varepsilon > 0$. Then there exists $h \in L^1(\nu \times \lambda)$ and a $\delta > 0$ such that:*

(1) $\|g - h\| < \varepsilon$;
(2) *for* $|\omega - \omega'| < \delta$, $|h(x,\omega) - h(x,\omega')| < \varepsilon$.

Proof of Theorem 5.4. Let $P_S g = \lambda g$ where $g \in L^1$ and $|\lambda| = 1$. We may assume that $\|g\|_1 = 1$. Let $\varepsilon > 0$ be given. We will show that for any measurable sets $A \subseteq X$ and $B \subseteq I$ we have

$$\left|\int_{A\times B} (g - \hat{g})\, d(\nu \times \lambda)\right| < 6\varepsilon.$$

From this it follows that $g = \hat{g}$ $(\nu \times \lambda)$-almost everywhere.

First, applying Lemma 5.11, we find $\delta > 0$ and h satisfying the conclusions of the lemma for our chosen ε. Next, let $N > 0$ and G be as given Lemma 5.7 for our chosen ε and the δ coming from above. We then argue as follows.

Since g is an eigenfunction of P_S, we have $g = \lambda^{-N} P_S^N g$ and

$$\begin{aligned}
&\left|\int_{A\times B} (g - \hat{g})\, d\nu \times \lambda\right| \\
&= \left|\int \mathbf{1}_{A\times B} P_S^N g\, d(\nu\times\lambda) - \lambda(B)\int \mathbf{1}_{A\times I} P_S^N g\, d(\nu\times\lambda)\right| \\
&= \left|\int \mathbf{1}_{S^{-N}(A\times B)} g\, d(\nu\times\lambda) - \lambda(B)\int \mathbf{1}_{S^{-N}(A\times I)} g\, d(\nu\times\lambda)\right| \\
&\le 2\varepsilon + \left|\int_{S^{-N}(A\times B)} h\, d(\nu\times\lambda) - \lambda(B)\int_{S^{-N}(A\times I)} h\, d(\nu\times\lambda)\right|,
\end{aligned} \tag{5.3}$$

where for the inequality we used the fact that $\|g - h\|_{L^1(\nu\times\lambda)} < \varepsilon$. We now estimate the remaining term.

From Corollary 5.10

$$\int_{S^{-N}(A\times B)} h\,d\nu\times\lambda = \sum_{\vec{k}\in\{1,\dots,K\}^N}\int_{A_{\vec{k}}\cap B^{\vec{k}}} h\,d\nu\times\lambda \quad\text{and}$$

$$\int_{S^{-N}(A\times I)} h\,d\nu\times\lambda = \sum_{\vec{k}\in\{1,\dots,K\}^N}\int_{A_{\vec{k}}\cap I^{\vec{k}}} h\,d\nu\times\lambda.$$

We see that

$$\begin{aligned}&\left|\int_{S^{-N}(A\times B)} h\,d(\nu\times\lambda) - \lambda(B)\int_{S^{-N}(A\times I)} h\,d(\nu\times\lambda)\right|\\ &\le \sum_{\vec{k}\in\{1,\dots,K\}^N}\left|\int_{A_{\vec{k}}\cap B^{\vec{k}}} h\,d\nu\times\lambda - \lambda(B)\int_{A_{\vec{k}}\cap I^{\vec{k}}} h\,d\nu\times\lambda\right|.\end{aligned} \tag{5.4}$$

We now estimate for a fixed $\vec{k}\in\{1,\dots,K\}^N$, the term appearing in the sum. Notice that $\lambda(B_x^{\vec{k}}) = \lambda(I_x^{\vec{k}})\lambda(B)$ since $\varphi_{\vec{k},x}$ is an affine map on $I_x^{\vec{k}}$.

Let $h_{\vec{k},x}$ be the average value of h on the vertical interval $I_x^{\vec{k}}$, i.e.

$$h_{\vec{k},x} = \frac{1}{\lambda(I_x^{\vec{k}})}\int_{I_x^{\vec{k}}} h(x,\omega)d\lambda(\omega);$$

Let π_1 be the projection to the x component. Using condition (3) of Lemma 5.7, we have

$$\begin{aligned}\int_{A_{\vec{k}}\cap B^{\vec{k}}} h\,d\nu\times\lambda &= \int_{\pi_1(A_{\vec{k}})}\int_{B_x^{\vec{k}}} h(x,\omega)d\lambda d\nu\\ &= \int_{\pi_1(A'_{\vec{k}})}\int_{B_x^{\vec{k}}} h(x,\omega)d\lambda d\nu + \int_{\pi_1(A''_{\vec{k}})}\int_{B_x^{\vec{k}}} h(x,\omega)d\lambda d\nu\end{aligned}$$

where $\pi_1(A'_{\vec{k}}) = \{x\in\pi_1(A_{\vec{k}}) : \lambda(I_x^{\vec{k}}) < \delta\}$ and $\pi_1(A''_{\vec{k}}) = \{x\in\pi_1(A_{\vec{k}}) : \lambda(I_x^{\vec{k}}) \ge \delta\}$

We then estimate

$$\begin{aligned}&\left|\int_{A_{\vec{k}}\cap B^{\vec{k}}} h\,d\nu\times\lambda - \lambda(B)\int_{A_{\vec{k}}\cap I^{\vec{k}}} h\,d\nu\times\lambda\right|\\ &\le\left|\int_{\pi_1(A'_{\vec{k}})}\int_{B_x^{\vec{k}}}(h(x,\omega)-h_{\vec{k},x})d\lambda d\nu + \int_{A''_{\vec{k}}\cap B^{\vec{k}}} h\,d\nu\times\lambda - \lambda(B)\int_{A''_{\vec{k}}\cap I^{\vec{k}}} h\,d\nu\times\lambda\right|\\ &\le\int_{A'_{\vec{k}}\cap B^{\vec{k}}}\varepsilon\,d\nu\times\lambda + \int_{G^c\cap A_{\vec{k}}\cap I^{\vec{k}}}|h|\,d\nu\times\lambda,\end{aligned}$$

where we used conclusions (2) of Lemma 5.7 and (2) of Lemma 5.11 to get the third line. Combining this inequality with (5.4) and (5.3), we see

$$\left|\int(g-\hat{g})\,d\nu\times\lambda\right| \le 3\varepsilon + \int_{G^c}(|g| + |h-g|)\,d\nu\times\lambda \le 5\varepsilon.$$

□

Finally, we give an illustrative example.

Example 5.12. Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$ where

$$\tau_1(x) = \begin{cases} 2x & \text{for } 0 \le x \le \frac{1}{2} \\ x & \text{for } \frac{1}{2} < x \le 1 \end{cases}, \tag{5.5}$$

$$\tau_2(x) = \begin{cases} x + \frac{1}{2} & \text{for } 0 \le x \le \frac{1}{2} \\ 2x - 1 & \text{for } \frac{1}{2} < x \le 1 \end{cases}; \tag{5.6}$$

and

$$p_1(x) = \begin{cases} \frac{2}{3} & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{1}{3} & \text{for } \frac{1}{2} < x \le 1 \end{cases}, \tag{5.7}$$

$$p_2(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \le x \le \frac{1}{2} \\ \frac{2}{3} & \text{for } \frac{1}{2} < x \le 1 \end{cases}. \tag{5.8}$$

Then, $S(x, \omega)$ is given by:

$$S(x,\omega) = \begin{cases} (2x, \frac{3}{2}\omega) & \text{for } (x,\omega) \in [0, \frac{1}{2}] \times [0, \frac{2}{3}] \\ (x, 3\omega) & \text{for } (x,\omega) \in (\frac{1}{2}, 1] \times [0, \frac{1}{3}] \\ (2x-1, \frac{3}{2}\omega - \frac{1}{2}) & \text{for } (x,\omega) \in (\frac{1}{2}, 1] \times (\frac{1}{3}, 1] \\ (x + \frac{1}{2}, 3\omega - 2) & \text{for } (x,\omega) \in [0, \frac{1}{2}] \times (\frac{2}{3}, 1] \end{cases}. \tag{5.9}$$

Notice that S is piecewise linear Markov transformation with respect to the partition $\{[0, \frac{1}{2}] \times [0, \frac{2}{3}], (\frac{1}{2}, 1] \times [0, \frac{1}{3}], (\frac{1}{2}, 1] \times (\frac{1}{3}, 1], [0, \frac{1}{2}] \times (\frac{2}{3}, 1]\}$. Therefore, its Frobenius-Perron operator reduces to a matrix:

$$P_S = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{5.10}$$

If the invariant density of S is $g = [g_1, g_2, g_3, g_4]$, normalized by $2g_1 + g_2 + 2g_3 + g_4 = 6$ and satisfying equation $gP_S = g$, then $g = [\frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{2}{3}]$. Observe that $P_S^2 > 0$. Thus, μ_g is ergodic. By Theorem 5.4 μ is a simple lift of a T-acim. Namely, g is a simple lift of $\hat{g} = [\frac{2}{3}, \frac{4}{3}]$ which is T-invariant. To verify this fact by direct calculation, notice that τ_1, τ_2 are piecewise linear Markov transformations defined on the same Markov partition $\mathcal{P} : \{[0, \frac{1}{2}], (\frac{1}{2}, 1]\}$. The corresponding Frobenius-Perron operator matrices are:

$$P_{\tau_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}, \quad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{5.11}$$

Since $p_1(x)$ and $p_2(x)$ are piecewise constant on the same partition, the Frobenius-Perron operator of the random map T is given by:

$$P_T = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

If the invariant density of T is $\hat{g} = [\hat{g}_1, \hat{g}_2]$, normalized by $\hat{g}_1 + \hat{g}_2 = 2$ and satisfying the equation $\hat{g}P_T = \hat{g}$, then $\hat{g} = [\frac{2}{3}, \frac{4}{3}]$.

6. Additional Results on Ergodic Properties of T and S

As in Morita [4] we can use Theorem 5.4 to establish a correspondence between ergodic properties of T and S with respect to invariant measures μ and $\mu \times \lambda$. In particular, all these results hold for the absolutely continuous invariant measures for S since we have shown them to be of product type. Notions of ergodicity, weakly mixing, strongly mixing and uniformly mixing for T are as defined in [4]. (but perhaps they should be given here as well??).

Theorem 6.1. *Assume that T is a random transformation with position dependent probabilities satisfying Condition 5.1 and that μ is invariant for T. Then*

(1) *μ is ergodic for T if and only if $\mu \times \lambda$ is ergodic for S.*
(2) *μ is weakly mixing for T if and only if $\mu \times \lambda$ is weakly mixing for S.*
(3) *μ is strongly mixing for T if and only if $\mu \times \lambda$ is strongly mixing for S*
(4) *μ is uniformly mixing for T if and only if $\mu \times \lambda$ is exact for S.*

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