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Coherent structures and isolated spectrum for Perron–Frobenius cocycles

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Abstract. We present an analysis of one-dimensional models of dynamical systems that possess "coherent structures"; global structures that disperse more slowly than local trajectory separation. We study cocycles generated by expanding interval maps and the rates of decay for functions of bounded variation under the action of the associated Perron–Frobenius cocycles.

We prove that when the generators are piecewise affine and share a common Markov partition, the Lyapunov spectrum of the Perron–Frobenius cocycle has at most finitely many isolated points. Moreover, we develop a strengthened version of the Multiplicative Ergodic Theorem for non-invertible matrices and construct an invariant splitting into Oseledets subspaces.

We detail examples of cocycles of expanding maps with isolated Lyapunov spectrum and calculate the Oseledets subspaces, which lead to an identification of the underlying coherent structures.

Our constructions generalise the notions of almost-invariant and almost-cyclic sets to non-autonomous dynamical systems and provide a new ensemble-based formalism for coherent structures in one-dimensional non-autonomous dynamics.

1. *Introduction*

Transport and mixing processes play an important role in many natural phenomena and their mathematical analysis has received considerable attention in the last two decades. The *geometric approach* to transport includes the study of invariant manifolds, which may act as barriers to particle transport and inhibit mixing. So-called *Lagrangian coherent structures* were introduced ([**HY00, H01**]) as finite-time proxies for invariant manifolds in non-autonomous settings. The *ergodic-theoretic approach* to transport includes the study of relaxation of initial ensemble densities to an invariant density, with a special focus on initial densities that relax more slowly than suggested by the rate of local trajectory separation. Such slowly decaying ensembles have been studied as "strange eigenmodes" ([**LH04, PP03, PPE07**] in fluids and have been used to identify almost-invariant sets [**DJ99, F05, FP08, F08**]). Until now, a suitable framework for the ergodic-theoretic approach that deals with truly non-autonomous dynamics has been lacking. The main aim of this work is to develop the fundamental structures and results that will support a non-autonomous theory for an ensemble-based approach to coherent structures.

We study non-autonomous one-dimensional dynamical systems that are given by compositions of expanding interval maps, and their action on ensembles represented by probability densities. The time evolution of a density is given by the Perron–Frobenius operator. For a single piecewise C^2 expanding map that is topologically mixing these densities converge to a unique equilibrium distribution which is absolutely continuous (see [**B00**]). Thus the equilibrium distribution is an eigenfunction of the Perron–Frobenius operator with eigenvalue 1. The exponential rate of convergence to equilibrium is governed by the spectrum of the Perron–Frobenius operator. When restricted to the space of functions of bounded variation (BV), the Perron–Frobenius operator is quasicompact (see [**HK82**]), meaning that each point in the spectrum of modulus greater than the essential spectral radius is an isolated eigenvalue of finite multiplicity. It is known that in the BV setting the essential spectral radius equals the long-term rate of separation of nearby trajectories, which we denote by θ (see Section 2.1). We will say an eigenvalue is *exceptional* if it is different from 1 and has modulus greater than θ. Eigenfunctions corresponding to exceptional eigenvalues relax more slowly to equilibrium than suggested by the local separation of trajectories, and their existence has been attributed to the presence of "almost-invariant sets" (see [**DJ99, DFS00, F07**]).

Exceptional eigenvalues have previously been found by considering piecewise-affine expanding maps with a Markov partition ([**B96**], [**DFS00**], [**KR04**]). When restricted to the space of step-functions constant on the Markov partition intervals, the associated Perron–Frobenius operator becomes a finite dimensional operator. In the present work we extend these results to the non-autonomous setting. Instead of iterating a single map, we consider a cocycle of maps and its associated Perron–Frobenius cocycle. The appropriate way to describe exponential rate of convergence to equilibrium is via the *Lyapunov spectrum* of the Perron–Frobenius cocycle. As the Perron–Frobenius operator is a Markov operator, the Lyapunov spectrum is contained in the interval $[-\infty, 0]$. In analogy with the autonomous case, we look for *exceptional* Lyapunov exponents, namely those negative exponents greater than the long-term *exponential* rate of separation of nearby trajectories, which we denote by ϑ (see Section 2.2).

We obtain a Lyapunov spectral decomposition for the Perron–Frobenius cocycle into invariant subspaces with given Lyapunov exponents (see Corollary 4.1). This relies on a new version of the Multiplicative Ergodic Theorem (see Theorem 4.1), which provides an invariant splitting into Oseledets spaces *even when the generators* *are non-invertible.* Our new version strengthens the standard Multiplicative Ergodic Theorem (see, for example [**A98**, Theorem 3.4.1]) where only an invariant flag of nested subspaces is supplied.

We demonstrate the existence of slow-mixing coherent structures by constructing periodic (see Theorem 5.1) and non-periodic (see Theorem 6.1) examples of Lebesgue measure-preserving one-sided cocycles with exceptional Lyapunov exponents. In each case, we calculate algebraically the Oseledets subspaces associated with the largest exceptional exponent and verify that the second largest Oseledets space captures the coherent structures.

Finally, we present an algorithm for approximating the Oseledets splitting, which is based on a computational approach suggested by the proof of Theorem 4.1. We demonstrate the effectiveness of the algorithm, by approximating some Oseledets subspaces numerically.

2. *Preliminaries*

We study the Perron–Frobenius operator for compositions of expanding maps. We first introduce the necessary notation and relevant results for autonomous systems, and then extend these to the non-autonomous case.

2.1. *Autonomous systems*

We say that $T : I \to I$, where $I = [0, 1]$ or $I = S^1 = [0, 1]/(0 \sim 1)$, is an *expanding map* if there exist points $0 = a_0 < a_1 < \ldots < a_m = 1$ such that, for each $i = 1, \ldots, m$, $T|_{(a_{i-1},a_i)}$ is continuous and extends to a C^2 map on $[a_{i-1}, a_i]$ satisfying $|DT|_{(a_{i-1},a_i)}| \geq \gamma$, for some $\gamma > 1$.

The *Perron–Frobenius operator* for an expanding map $T : I \to I$ is defined, for an L^1 function $f : I \to \mathbb{R}$, by

$$\mathcal{P}f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|DT(y)|}. \quad (1)$$

In [**LY73**], the Perron–Frobenius operator is used to prove that expanding maps have an absolutely continuous invariant probability measure. The key step of their proof is to show that the Perron–Frobenius operator contracts the norm on a suitable space of functions: the functions of bounded variation.

The *variation* of a function $f : I \to \mathbb{R}$ on a subinterval $A \subset I$ is defined by

$$\operatorname*{var}_A f := \operatorname*{var}_{x \in A} f(x) = \sup \sum_{i=1}^{k} |f(p_i) - f(p'_i)|,$$

where the supremum is taken over all finite collections $\{p_i, p'_i\}_{i=1}^k$ such that p_i and p'_i are the endpoints of an interval $I_i \subset A$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. Given $f \in L^\infty \subset L^1$, the variation is defined by $\operatorname{var}_I f = \inf\{\operatorname{var}_I g : f = g \ a.e.\}$. We denote by $\mathrm{BV} = \mathrm{BV}(I)$ the Banach space

$$\mathrm{BV} = \left\{ f \in L^\infty : \operatorname*{var}_I f < \infty \right\},$$

equipped with the norm $\|f\| = \max\{\|f\|_{L^1}, \mathrm{var}_I f\}$. We denote Lebesgue measure on I by m, and $f \in \mathrm{BV}$ is called a *(probability) density* if $f \geq 0$ on I (and $\|f\|_{L^1} = 1$).

The Perron–Frobenius operator is *Markov*: that is, if $f \in L^1$ is a density, then $\mathcal{P}f$ is also a density and $\|\mathcal{P}f\|_{L^1} = \|f\|_{L^1}$. A probability density f^* satisfying $\mathcal{P}f^* = f^*$ is an invariant probability density for T.

Keller [**K84**] shows that the Perron–Frobenius operator of an expanding map has at most countably many exceptional eigenvalues.

Theorem *(Keller, 1984). Given an expanding interval map $T : I \to I$, its Perron–Frobenius operator $\mathcal{P}$ acting on* BV *has essential spectral radius*

$$\theta := \lim_{n\to\infty} \sup_{x\in I} \left(\frac{1}{|D(T^n)(x)|} \right)^{1/n}, \tag{2}$$

and there are at most countably many points in the spectrum of modulus greater than θ, each an isolated eigenvalue of finite multiplicity.

Exceptional eigenvalues have a distinguished dynamical significance as their eigenfunctions are associated with relaxation to equilibrium at exponential rates slower than the rate suggested by the average local separation of trajectories θ. For example, if $\mathcal{P}g = \lambda g$ with $\theta < |\lambda| < 1$ then an initial density $f^* + \alpha g$, $\alpha \neq 0$ will relax to f^* at a rate slower than θ.

Dellnitz and Junge [**DJ99**] suggested that positive real Perron–Frobenius eigenvalues near to 1 correspond to *almost-invariant sets*; more precisely, they suggested the sets $A^+ := \{g > 0\}$ and $A^- := \{g \leq 0\}$ formed an almost-invariant partition of the state space. Dellnitz *et al.* [**DFS00**] showed the converse, presenting a class of interval maps with almost-invariant sets and proving the existence of exceptional eigenvalues. Froyland [**F07**] constructed a two-dimensional hyperbolic map with almost-invariant sets and proved the existence of an exceptional eigenvalue. Numerical methods have been developed ([**DJ99, F05, F08**]) for the computation of exceptional eigenfunctions and almost-invariant sets; these have been applied successfully in molecular dynamics ([**SHD99**]), astrodynamics ([**D+05**]), and ocean circulation ([**F+07**]).

Our intent in the present work is to generalise the notion of almost-invariant sets in autonomous systems to that of coherent structures in non-autonomous systems. The latter will represent structures that are perhaps quite mobile, but disperse at rates slower than suggested by local trajectory separation.

2.2. *Non-autonomous systems* We will examine exceptional spectral points in the non-autonomous case, and study compositions of expanding maps taken from a finite collection, and composed in order according to given sequences.

Let s be an invertible transformation of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that preserves the probability $\mathbb{P}$. Given a measurable/topological/vector space X, a *(one-sided) cocycle* over s is a function $H : \mathbb{Z}^+ \times \Omega \times X \to X$ with the properties that for all $\omega \in \Omega$ and $x \in X$:

- $H(0, \omega, x) = x$;
- for all $m, n \in \mathbb{Z}^+$, $H(m + n, \omega, x) = H(m, s^n\omega, H(n, \omega, x))$.

We sometimes write $H^{(n)}(\omega)(x)$ for $H(n, \omega, x)$, and $H(\omega)(x)$ for $H(1, \omega, x)$. The *generator* of a cocycle H is the mapping $\tilde{H} : \Omega \to \mathrm{End}(X)$ given by $\tilde{H}(\omega) = H(\omega)$. Since the cocycle is uniquely determined by $\tilde{H}$, we occasionally refer to $\tilde{H}$ itself as the cocycle when no confusion can occur.

In the sequel, our probability space will frequently be a (bi-infinite) sequence space $(\Sigma, \mathcal{H}, p)$ on K symbols $\{1, \dots, K\}$, and our invertible transformation a (left) shift σ, defined by $(\sigma\omega)_i = \omega_{i+1}$, $i \in \mathbb{Z}$, where $\Sigma \subset \{1, \dots, K\}^{\mathbb{Z}}$ is invariant under σ.

Definition. Let $\{T_i\}_{i=1}^K$ be a set of expanding maps of I, and let $\mathcal{P}_i : \mathrm{BV} \to \mathrm{BV}$ be the Perron–Frobenius operator associated to $T_i : I \to I$. The *map cocycle generated by* $\{T_i\}_{i=1}^K$, denoted by $\Phi : \mathbb{Z}^+ \times \Sigma \times I \to I$, is defined to be the one-sided cocycle with generator $\tilde{\Phi}(\omega) = T_{\omega_0} \in \{T_i\}_{i=1}^K$. Associated to Φ is the *Perron–Frobenius cocycle* $\mathcal{P} : \mathbb{Z}^+ \times \Sigma \times \mathrm{BV} \to \mathrm{BV}$, which is defined to be the one-sided cocycle with generator $\tilde{\mathcal{P}}(\omega) = \mathcal{P}_{\omega_0} \in \{\mathcal{P}_i\}_{i=1}^K$ for $\omega \in \Sigma$.

Notice that even though we use a two-sided shift space, we only form one-sided cocycles not two-sided cocycles. This is because expanding maps are not invertible, and nor are their Perron–Frobenius operators.

We say a cocycle is *periodic* if the underlying sequence space Σ is generated by a single element: that is, there exists $R \in \mathbb{N}$, called the *period*, and $\omega \in \Sigma$ such that $\Sigma = \{\omega, \sigma\omega, \dots, \sigma^{R-1}\omega\}$; we say a cocycle is *autonomous* if Σ contains a single element.

3. *Quasicompactness of the Perron–Frobenius cocycle*

Information about the exponential decay rates of the Perron–Frobenius cocycle is given by its Lyapunov spectrum.

Definition. We denote by $\lambda(\omega, f)$ the *(forward) Lyapunov exponent* of $f \in \mathrm{BV}$ at $\omega \in \Sigma$, defined by

$$\lambda(\omega, f) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}^{(n)}(\omega) f\|.$$

We define the *Lyapunov spectrum* $\Lambda(\mathcal{P}(\omega)) \subset \mathbb{R}$ of the Perron–Frobenius cocycle at $\omega \in \Sigma$ to be the set

$$\Lambda(\mathcal{P}(\omega)) := \{\lambda(\omega, f) : f \in \mathrm{BV}\}.$$

The exponential rate of decay that can be expected purely from the local expansion at $\omega \in \Sigma$ is

$$\vartheta(\omega) := \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in I} \frac{1}{|D(\Phi^{(n)}(\omega))(x)|}, \tag{3}$$

If $\sigma : \Sigma \to \Sigma$ is ergodic, then $(1/n) \log \sup_{x\in I}(1/|D(\Phi^{(n)}(\omega))(x)|)$ converges to $\vartheta(\omega)$ by Kingman's subadditive ergodic theorem (see [**A98**, Theorem 3.3.2]) and, since ϑ is σ-invariant, $\vartheta(\omega) = \vartheta$ p-almost everywhere.

If Φ is a composition of expanding maps, then $\vartheta(\omega) < 0$. Comparing with (2), we have $\vartheta = \log \theta$ for autonomous systems.

Points in $\Lambda(\mathcal{P}(\omega))$ that are greater than $\vartheta(\omega)$ indicate the presence of large-scale structures that reduce the rate of mixing of the system, except for the maximal Lyapunov exponent, 0, which is associated with an invariant density. We refer to Lyapunov exponents in the interval $(\vartheta(\omega), 0)$ as *exceptional*.

In order to find systems with exceptional Lyapunov spectrum, we restrict our attention to map cocycles generated by piecewise-affine maps with a Markov partition. We say an expanding map $T : I \to I$ is *piecewise-affine* if there exists a partition $\mathcal{A} = \{A_i\}_{i=1}^m$ of I into intervals such that T has constant derivative on each interval A_i. A refinement $\mathcal{B} = \{B_i\}_{i=1}^M$ of the partition $\mathcal{A}$ into intervals is called a *Markov partition* if for each pair $1 \leq i, j \leq M$ such that $\mathrm{int}(B_i) \cap T(B_j) \neq \emptyset$, we have $\mathrm{int}(B_i) \subset T(B_j)$. Associated to T is a *transition matrix* $\Gamma = (\gamma_{i,j})_{1 \leq i,j \leq M}$, where $\gamma_{i,j} = 1$ if $T(B_j) \supset \mathrm{int}(B_i)$ and 0 otherwise.

For a Markov partition $\mathcal{B}$, we let $\chi(\mathcal{B})$ denote the space of step-functions $I \to \mathbb{R}$ that are constant on the intervals of $\mathcal{B}$. We say $\mathcal{B}$ is a *common Markov partition* for a collection of maps $\{T_i\}_{i=1}^K$ if it is a Markov partition for each map T_i.

We now fix a collection $\mathcal{T} := \{T_i\}_{i=1}^K$ of piecewise-affine expanding maps and denote by Φ the associated map cocycle and by $\mathcal{P}$ the Perron–Frobenius cocycle acting on BV. Suppose the generators $\mathcal{T}$ have a common Markov partition $\mathcal{B} = \{B_1, \ldots, B_M\}$, where $\overline{B_i} = [b_{i-1}, b_i]$. Then $F = \chi(\mathcal{B}) \subset \mathrm{BV}$ is an invariant subspace for $\mathcal{P}$. We may consider the quotient space $Q = Q(I) = \mathrm{BV}(I)/F$, which is a Banach space with the quotient norm $\|f + F\|_Q := \inf_{s \in F} \|f - s\|$.

Given $f \in \mathrm{BV}$ we construct a step function $\pi(f) \in F$ such that:

1. for $i = 1, \ldots, M-1$, $(f - \pi(f))(b_i-) = (f - \pi(f))(b_i+)$;
2. $(f - \pi)(0+) = 0$,

where we denote $h(x+) = \lim_{y \downarrow x} h(y)$ and $h(x-) = \lim_{y \uparrow x} h(y)$. Condition (1) ensures that $\mathrm{var}_I(f - \pi(f)) = \inf_{s \in F} \mathrm{var}_I(f - s)$. Additionally, using condition (2) we have that $\|f - \pi(f)\|_{L^1} \leq \mathrm{var}_I(f - \pi(f))$. An explicit formula for $\pi : \mathrm{BV} \to F$ is

$$\pi(f) = \sum_{i=1}^{M} h_i(f) \chi_{B_i}, \quad h_i(f) = f(b_{i-1}+) - \sum_{j=1}^{i-1} \left(f(b_j-) - f(b_{j-1}+) \right).$$

Thus the map π is linear, and is a projection onto F since $\pi(\chi_{B_i}) = \chi_{B_i}$, $i = 1, \ldots, M$. We define $\tau = \mathrm{Id} - \pi$. Since $\pi^2 = \pi$, we see $\tau^2 = \mathrm{Id} - 2\pi + \pi = \tau$, and so τ is also a projection. Thus

$$\|f + F\|_Q = \|\tau f\| = \operatorname*{var}_I \tau f, \quad f \in \mathrm{BV}. \tag{4}$$

Lemma 3.1. *The projection τ satisfies the following identity for any $\omega \in \Sigma$ and any $n \in \mathbb{N}$:*

$$\tau \mathcal{P}^{(n)}(\omega) = \tau \mathcal{P}^{(n)}(\omega) \tau = \tau (\mathcal{P}\tau)^{(n)}(\omega), \tag{5}$$

where we denote $(\mathcal{P}\tau)^{(n)}(\omega) = \mathcal{P}(\sigma^{n-1}\omega)\tau \cdots \mathcal{P}(\sigma\omega)\tau\mathcal{P}(\omega)\tau$.

Proof. We have, for any $n \in \mathbb{N}$,

$$\tau\mathcal{P}^{(n)}(\omega) \quad = \tau\mathcal{P}^{(n)}(\omega)(\tau+\pi) = \tau\mathcal{P}^{(n)}(\omega)\tau + \tau\mathcal{P}^{(n)}(\omega)\pi = \tau\mathcal{P}^{(n)}(\omega)\tau \qquad (6)$$

since $\tau F = \{0\}$, giving the left identity. In particular $\tau\mathcal{P}(\omega) = \tau\mathcal{P}(\omega)\tau$, and so by induction

$$\begin{aligned}
\tau(\mathcal{P}\tau)^{(n)}(\omega) &= \tau(\mathcal{P}\tau)^{(n-1)}(\sigma\omega)(\mathcal{P}(\omega)\tau) \\
&= \tau\mathcal{P}^{(n-1)}(\sigma\omega)(\mathcal{P}(\omega)\tau), \quad \text{assuming } \tau(\mathcal{P}\tau)^{(n-1)}(\omega) = \tau\mathcal{P}^{(n-1)}(\omega) \\
&= \tau\mathcal{P}^{(n)}(\omega)\tau \\
&= \tau\mathcal{P}^{(n)}(\omega), \quad \text{by (6).}
\end{aligned}$$

□

For the rest of this section, we consider functions of $I = [0,1]$. By (4) we have

$$\|f + F\|_Q = \sum_{i=1}^{M} \operatorname*{var}_{B_i} f, \quad f \in \mathrm{BV}(I). \qquad (7)$$

We denote by $\mathcal{B}^n(\omega)$ the refinement $\bigvee_{i=0}^{n-1}[\Phi^{(i)}(\omega)]^{-1}\mathcal{B}$. We show that ϑ is an upper bound for the Lyapunov spectrum of the quotient cocycle.

Lemma 3.2. *For each $\omega \in \Sigma$,*

$$\Lambda(\mathcal{P}_Q(\omega)) \subset [-\infty, \vartheta(\omega)], \qquad (8)$$

where $\mathcal{P}_Q$ is the quotient cocycle on the space $Q = \mathrm{BV}([0,1])/F$.

Proof. For $A \in \mathcal{B}^{(n)}(\omega)$, the support of $\mathcal{P}^{(n)}(\omega)(f\chi_A)$ is contained in the interval $\Phi^{(n)}(\omega)A$, which is equal to the closure of a union of elements of $\mathcal{B}$. Thus by (7), for $f \in \mathrm{BV}([0,1])$ we have

$$\begin{aligned}
\|\mathcal{P}^{(n)}(\omega)f + F\|_Q &= \sum_{B\in\mathcal{B}} \operatorname*{var}_{B} \mathcal{P}^{(n)}(\omega)f \\
&\le \sum_{A\in\mathcal{B}^{(n)}(\omega)} \operatorname*{var}_{\Phi^{(n)}(\omega)A} \mathcal{P}^{(n)}(\omega)(f\chi_A) \\
&= \sum_{A\in\mathcal{B}^{(n)}(\omega)} \operatorname*{var}_{A} \frac{f}{|D\Phi^{(n)}(\omega)|} \\
&\le \max_{A\in\mathcal{B}^{(n)}(\omega)} \left(\frac{1}{|D\Phi^{(n)}(\omega)|_A|}\right) \sum_{A\in\mathcal{B}^{(n)}(\omega)} \operatorname*{var}_{A} f,
\end{aligned}$$

since $D\Phi^{(n)}(\omega)|_A$ is constant for each $A \in \mathcal{B}^{(n)}(\omega)$. So

$$\begin{aligned}
\|\mathcal{P}^{(n)}(\omega)f + F\|_Q &\le \sup_I \left(\frac{1}{|D\Phi^{(n)}(\omega)|}\right) \sum_{B\in\mathcal{B}} \operatorname*{var}_{B} f \\
&= \sup_I \left(\frac{1}{|D\Phi^{(n)}(\omega)|}\right) \|f + F\|_Q.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}^{(n)}(\omega)\|_Q \le \limsup_{n\to\infty} \frac{1}{n} \log \sup_I \left(\frac{1}{|D\Phi^{(n)}(\omega)|}\right) = \vartheta(\omega), \qquad (9)$$

as required. □

Remark. Quasicompactness of the autonomous Perron–Frobenius operator has been proven for a variety of Banach spaces (e.g. [**R89**], [**BKL02**], [**GL06**]). It would be natural to consider bounds analogous to Lemma 3.2 for these other spaces in the cocycle setting.

We now prove that the exceptional Lyapunov spectrum of the cocycle $\mathcal{P}(\omega)$ is contained in $\Lambda(\mathcal{P}|_{\chi(\mathcal{B})}(\omega))$. For the autonomous case, see for example [**BK98**, Lemma 3.1].

PROPOSITION 3.1. *Let σ be an invertible ergodic measure-preserving shift of the sequence space $(\Sigma, \mathcal{H}, p)$, and $\mathcal{P} : \mathbb{Z}^+ \times \Sigma \times \mathrm{BV}([0,1]) \to \mathrm{BV}([0,1])$ be the Perron–Frobenius cocycle associated to a map cocycle over σ generated by piecewise-affine expanding maps with a common Markov partition $\mathcal{B}$. Let $F = \chi(\mathcal{B})$ be the finite dimensional subspace spanned by $\{\chi_B : B \in \mathcal{B}\}$. Then, for almost every $\omega \in \Sigma$,*

$$\Lambda(\mathcal{P}(\omega)) \cap (\vartheta, 0] \subset \Lambda(\mathcal{P}|_F(\omega)),$$

where $\mathcal{P}|_F$ is the finite dimensional cocycle induced on F. In particular, $\mathcal{P}(\omega)$ has at most $\dim F = \#\mathcal{B}$ exceptional Lyapunov exponents for almost every $\omega \in \Sigma$.

Proof. For each $\omega \in \Sigma$ and $f \in \mathrm{BV}([0,1])$, since $f = \pi f + \tau f$, we have

$$\lambda(\omega, f) \leq \max\{\lambda(\omega, \pi f), \lambda(\omega, \tau f)\} \tag{10}$$

with equality if $\lambda(\omega, \pi f) \neq \lambda(\omega, \tau f)$. So either $\lambda(\omega, f) = \lambda(\omega, \pi f) \in \Lambda(\mathcal{P}|_F(\omega))$ or else $\lambda(\omega, f) \leq \lambda(\omega, \tau f)$. In the latter case, applying (10) to $\lambda(\sigma\omega, \mathcal{P}(\omega)\tau f)$ we have that either $\lambda(\omega, f) = \lambda(\sigma\omega, \pi\mathcal{P}(\omega)\tau f) \in \Lambda(\mathcal{P}|_F(\sigma\omega))$ or

$$\lambda(\omega, f) = \lambda(\sigma\omega, \mathcal{P}(\omega) f) \leq \lambda(\sigma\omega, \mathcal{P}(\omega)\tau f) \leq \lambda(\sigma\omega, \tau\mathcal{P}(\omega)\tau f).$$

Inductively, we have that either $\lambda(\omega, f) \in \bigcup_{n=0}^{\infty} \Lambda(\mathcal{P}|_F(\sigma^n\omega))$ or else

$$\begin{aligned}
\lambda(\omega, f) &\leq \inf_{n\geq 0} \lambda(\sigma^n\omega, \tau(\mathcal{P}\tau)^{(n)}(\omega) f) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \|\tau(\mathcal{P}\tau)^{(n)}(\omega) f\| \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \|\tau\mathcal{P}^{(n)}(\omega) f\|, \quad \text{by Lemma 3.1} \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}^{(n)}(\omega) f + F\|_Q \\
&\leq \vartheta(\omega), \quad \text{by Lemma 3.2.}
\end{aligned}$$

Clearly $\bigcup_{n=0}^{\infty} \Lambda(\mathcal{P}|_F(\sigma^n\omega)) = \Lambda(\mathcal{P}|_F(\omega))$ for almost every ω. Thus by the standard MET (see [**A98**, Theorem 3.4.1]), we have that for almost every ω the number of exceptional exponents of $\mathcal{P}(\omega)$ is no greater than $\#\Lambda(\mathcal{P}|_F(\omega)) \leq \#\mathcal{B}$. □

Remark. For $f \in \mathrm{BV}(S^1)$, let $\tilde{f} \in \mathrm{BV}([0,1])$ be the map obtained by considering f as a function on $[0,1]$. Notice that for $f \in \mathrm{BV}(S^1)$,

$$\operatorname*{var}_{[0,1]} \tilde{f} \leq \operatorname*{var}_{S^1} f = \operatorname*{var}_{[0,1]} \tilde{f} + |\tilde{f}(1-) - \tilde{f}(0+)| \leq 2 \operatorname*{var}_{[0,1]} \tilde{f}. \tag{11}$$

Thus, we have two equivalent norms on $Q(S^1) = \mathrm{BV}(S^1)/F$: $\|f + F\|_{Q(S^1)}$ and $\|f+F\|'_{Q(S^1)} := \|\tilde{f}+F\|_{Q([0,1])}$, for $f \in \mathrm{BV}(S^1)$. Therefore, we see that Lemma 3.2 and Proposition 3.1 hold for the case $f \in \mathrm{BV}(S^1)$ by following the proofs applied to $\tilde{f} \in \mathrm{BV}([0,1])$ and then using the norm equivalence.

4. *A stronger Multiplicative Ergodic Theorem for non-invertible matrices*

By Proposition 3.1, for almost every $\omega \in \Sigma$, all exceptional Lyapunov exponents of $\mathcal{P}(\omega)$ are contained in the Lyapunov spectrum $\Lambda(\mathcal{P}|_{\chi(\mathcal{B})}(\omega))$. We now represent $\mathcal{P}(\omega)|_{\chi(\mathcal{B})}$ as a matrix cocycle.

The set $\{\chi_{B_i}\}_{i=1}^M$ forms a basis for $\chi(\mathcal{B})$, and thus each $f \in \chi(\mathcal{B})$ may be written as $f = \sum_{i=1}^M v_i \chi_{B_i}$ in a unique way. Similarly, given $v \in \mathbb{R}^M$, we write $\langle v \rangle := \sum_{i=1}^M v_i \chi_{B_i}$ for the corresponding function in BV.

For $T \in \mathcal{T}$, the matrix $P = (p_{i,j})_{1 \leq i,j \leq M}$, where

$$p_{i,j} = \frac{\gamma_{j,i}}{|DT_{|B_j}|} = \frac{m(T^{-1}(B_i) \cap B_j)}{m(B_j)}, \quad 1 \leq i,j \leq M,$$

represents the Perron–Frobenius operator for T with respect to the basis $\{\chi(B_i)\}_{i=1}^M$ of $\chi(\mathcal{B})$ (see, for example, [**BG97**, p.176]). That is, for each $v \in \mathbb{R}^M$ we have

$$\mathcal{P} \langle v \rangle = \langle Pv \rangle .$$

Let P_i denote the matrix representing the restricted Perron–Frobenius operator $\mathcal{P}_i|_{\chi(\mathcal{B})}$ with respect to the basis $\{\chi_{B_i}\}_{i=1}^M$ of $\chi(\mathcal{B})$. The matrix cocycle $A : \mathbb{Z}^+ \times \Sigma \times \mathbb{R}^M \to \mathbb{R}^M$ is the one-sided cocycle with generator $\tilde{A}(\omega) = P_{\omega_0}$.

Thus for almost every $\omega \in \Sigma$, all exceptional Lyapunov exponents of $\mathcal{P}(\omega)$ are captured by the Lyapunov spectrum of the cocycle $\Lambda(A(\omega)) = \Lambda(\mathcal{P}|_{\chi(\mathcal{B})}(\omega))$.

The Multiplicative Ergodic Theorem for one-sided matrix cocycles (see, for example, [**A98**, Theorem 3.4.1]) provides us with a description of the asymptotic behaviour of the cocycle $A(\omega)$. It reveals that the Lyapunov spectra $0 = \lambda_1 > \lambda_2 > \cdots > \lambda_\ell \geq -\infty$ of the $A(\omega)$ coincide for all ω in a σ-invariant $\tilde{\Sigma} \subset \Sigma$ of full p-measure. Moreover, it states that for each $\omega \in \tilde{\Sigma}$, a Lyapunov exponent $\lambda(\omega, v)$ of $v \in \mathbb{R}^M$ for $A(\omega)$ is determined by the position of v within a *flag* of nested subspaces $\{0\} = V_\ell(\omega) \subset \cdots \subset V_2(\omega) \subset V_1(\omega) = \chi(\mathcal{B})$. Specifically, for each $i = 1, \ldots, \ell$,

$$\lambda(\omega, v) = \lambda_i \iff v \in V_i(\omega) \setminus V_{i+1}(\omega). \tag{12}$$

In addition, the flag of subspaces is preserved by the action of the cocycle: for $i = 1, \ldots, \ell$,

$$A(\omega)V_i(\omega) \subset V_i(\sigma\omega).$$

For two-sided matrix cocycles (see, for example, [**A98**, Theorem 3.4.11]), by intersecting the corresponding subspaces of the flags for the cocycle and for its inverse, one obtains an *Oseledets splitting*: that is, for each $\omega \in \Sigma$ we have a decomposition $\mathbb{R}^M = \bigoplus_{i=1}^\ell W_i(\omega)$ such that for $i = 1, \ldots, \ell$,

$$v \in W_i(\omega) \setminus \{0\} \iff \lim_{n \to \pm\infty} \frac{1}{n} \log \|A^{(n)}(\omega)v\| = \lambda_i,$$

where $A^{(-n)}(\omega) := A^{-1}(\sigma^{-n}\omega) \cdots A^{-1}(\sigma^{-1}\omega)$ for $n > 0$, and

$$A(\omega)W_i(\omega) = W_i(\sigma\omega).$$

Our cocycle $A(\omega)$ sits between these two extremes: the shift σ is invertible, but the matrices $\{P_i\}_{i=1}^K$ generating $A(\omega)$ are not. Because of the non-invertibility of the cocycle, we cannot use the standard approach described above to define an Oseledets splitting.

The following new result relies on a push-forward approach to prove the existence of an Oseledets splitting even when the generators are non-invertible. We state and prove the theorem for an arbitrary matrix cocycle over an invertible ergodic measure-preserving transformation of a probability space. Afterwards, we apply the theorem to the special case of a Perron-Frobenius cocycle over a shift of a sequence space.

THEOREM 4.1. Let s be an invertible ergodic measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $A\colon \Omega \to M_d(\mathbb{R})$ be a measurable family of matrices satisfying

$$\int \log^+ \|A(\omega)\| \, d\mathbb{P}(\omega) < \infty.$$

Then there exist $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell \geq -\infty$ and dimensions $m_1, \ldots, m_\ell$, with $m_1 + \cdots + m_\ell = d$, and a measurable family of subspaces $W_i(\omega) \subseteq \mathbb{R}^d$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$ the following hold:

1. $\dim W_i(\omega) = m_i$;
2. $\mathbb{R}^d = \bigoplus_{i=1}^{\ell} W_i(\omega)$;
3. $A(\omega)W_i(\omega) \subseteq W_i(s\omega)$ (with equality if $\lambda_i > -\infty$);
4. for all $v \in W_i(\omega) \setminus \{0\}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \log \|A(s^{n-1}\omega) \cdots A(\omega)v\| = \lambda_i.$$

Proof. See Section 8. □

Remarks.

1. It follows from part (iv) of Theorem 4.1 that for almost every $\omega \in \Omega$, we can determine the Lyapunov exponent for any vector $v \in \mathbb{R}^d \setminus \{0\}$ by

$$\lambda(\omega, v) = \lambda_i \iff v \in \bigoplus_{k=i}^{\ell+1} W_k(\omega) \setminus \bigoplus_{k=i+1}^{\ell+1} W_k(\omega),$$

   where we set $W_{\ell+1}(\omega) = \{0\}$ for all $\omega \in \Omega$.

2. If the family $A : \Omega \to M_d(\mathbb{R})$ is non-invertible on a set of positive measure, then we can obtain a lower bound for the dimension of the subspace $W_\ell(\omega)$ corresponding to the lowest exponent $\lambda_\ell = -\infty$ using the nullities of the generator as follows. Let $\Omega_c = \{\omega \in \Omega : \dim\ker A(s^n\omega) \geq c \text{ for some } n \in \mathbb{N}\}$. Since $s^{-1}\Omega_c \subset \Omega_c$, we have that $\mathbb{P}(\Omega_c) = 0$ or 1 by ergodicity. Choose the maximal $c \in \mathbb{N}$ for which $\mathbb{P}(\Omega_c) = 1$. Then, by Theorem 4.1 $\dim W_\ell(\omega)$ is constant almost everywhere, and so $\dim W_\ell(\omega) \geq c$ for almost every $\omega \in \Omega$.

3. Let $\gamma(u, v)$ denote the angle between non-zero vectors $u, v \in \mathbb{R}^d$. As $\mathbb{R}^d = \sum_{i=1}^{\ell} W_i(\omega)$, we may decompose u, v as $u = \sum_{i=1}^{\ell} u_i(\omega)$ and $v = \sum_{i=1}^{\ell} v_i(\omega)$, where $u_i(\omega), v_i(\omega) \in W_i(\omega)$. Let $i(\omega, u) := \min\{i : u_i(\omega) \neq 0\}$. Since γ is bounded, we have $\limsup_{n\to\infty}(1/n)\log\gamma(A^{(n)}(\omega)u, A^{(n)}(\omega)v) \leq 0$. In fact $\lim_{n\to\infty}(1/n)\log\gamma(A^{(n)}(\omega)u, A^{(n)}(\omega)v)$ exists, and it is negative if and only if $u_{i(\omega,u)}(\omega)$ and $v_{i(\omega,v)}(\omega)$ are linearly dependent (see [**A98**, Corollary 5.3.7]).

We now apply Theorem 4.1 to our cocycle $A(\omega)$ induced by $\mathcal{P}(\omega)|_{\chi(\mathcal{B})}$. Consider the part of the Lyapunov spectrum of $A(\omega)$ that is greater than ϑ. Let $1 \leq r \leq \ell$ satisfy $\lambda_{r+1} \leq \vartheta < \lambda_r$. Thus, the part of $\Lambda(\mathcal{P}|_{\chi(\mathcal{B})}(\omega))$ strictly greater than ϑ is precisely $\lambda_1 > \lambda_2 > \cdots > \lambda_r$. It follows from Proposition 3.1 that the exceptional Lyapunov spectrum of $\mathcal{P}(\omega)$ is precisely $\lambda_2, \ldots, \lambda_r$. By defining $\mathcal{W}_i(\omega) = \{\langle v\rangle : v \in W_i(\omega)\}$ for $1 \leq i \leq r$, we transfer the splitting of $\mathbb{R}^M$ obtained from Theorem 4.1 into a splitting of $\chi(\mathcal{B})$ and obtain the following result:

Corollary 4.1. *Let σ be an invertible ergodic measure-preserving shift of the sequence space $(\Sigma, \mathcal{H}, p)$, and $\mathcal{P} : \mathbb{Z}^+ \times \Sigma \times \mathrm{BV} \to \mathrm{BV}$ be the Perron–Frobenius cocycle associated to a map cocycle over σ generated by piecewise-affine expanding maps with a common Markov partition $\mathcal{B}$. Let $\emptyset \neq \{\lambda_i\}_{i=1}^r = \Lambda(\mathcal{P}(\omega)) \cap (\vartheta, 0]$ be the Lyapunov exponents of $\mathcal{P}(\omega)$ greater than ϑ, where $0 = \lambda_1 > \cdots > \lambda_r > \vartheta$.*

Then there exists a forward invariant full p-measure subset $\tilde{\Sigma} \subset \Sigma$ and $m_1, \ldots, m_r \in \mathbb{N}$, satisfying $m_1 + \cdots + m_r \leq \#\mathcal{B}$, such that for all $\omega \in \tilde{\Sigma}$:

1. *there exist subspaces $\mathcal{W}_i(\omega) \subset \chi(\mathcal{B})$, $\dim \mathcal{W}_i(\omega) = m_i$;*
2. *$\mathcal{P}(\omega)\mathcal{W}_i(\omega) = \mathcal{W}_i(\sigma\omega)$;*
3. *$f \in \mathcal{W}_i(\omega) \setminus \{0\} \Longrightarrow \lambda(\omega, f) = \lambda_i$.*

5. *Construction of periodic cocycles with exceptional Lyapunov spectrum*

In this section we build a periodic map cocycle for which the Perron–Frobenius cocycle has an exceptional Lyapunov spectrum.

In [**DFS00**] individual maps are constructed for which the Perron–Frobenius operator has exceptional eigenvalues. The construction uses so-called ‘almost-invariant’ sets. Given a map $T : I \to I$ with an absolutely continuous invariant probability measure μ, a subset $U \subset I$ is almost-invariant if

$$\frac{\mu(U \cap T^{-1}U)}{\mu(U)} \approx 1.$$

For a map with an almost-invariant set U, the transfer of mass between U and $I \setminus U$ is low, and so we expect to find that a mean-zero function positive on U and negative on $I \setminus U$ decays to zero slowly. It is shown that for piecewise-affine Markov maps, one often obtains an almost-invariant set from the support of either the positive or negative part of the eigenfunction associated to the second largest eigenvalue of the Perron–Frobenius operator.

For this first example, we construct a cocycle over a periodic shift space of period 3 that has a cyclic coherent structure. More precisely, we take three maps, each having a distinct interval from the partition $\mathcal{J} = \{[0, 1/3], [1/3, 2/3], [2/3, 1]\}$ of S^1 as an almost-invariant set. Post-composing these maps with the rotation by 1/3, we form three new maps which we apply in sequence repeatedly, thus forming a periodic map cocycle Φ. In this way, each generator $\tilde{\Phi}(\omega)$ of the map cocycle moves the majority of the mass of one distinguished interval $J(\omega) \in \mathcal{J}$ into another interval $J(\sigma\omega) \in \mathcal{J}$ with some small dissipation. Thus a map $J : \Sigma \to \mathcal{J}$ specifies the location of our coherent structure.

Theorem 5.1. *There exists a collection of three piecewise-affine expanding maps $T_1, T_2, T_3 : S^1 \to S^1$ with a common Markov partition $\mathcal{B}$ that generates a map cocycle $\Phi : \mathbb{Z}^+ \times \Sigma \times S^1 \to S^1$ over the shift σ on the periodic sequence space $\Sigma \subset \{1, 2, 3\}^{\mathbb{Z}}$ generated by $\alpha = \overline{123}$ with the following properties for $i = 1, 2, 3$:*

1. *each map T_i preserves Lebesgue measure;*
2. *$\vartheta = \log 1/3$;*
3. *each finite dimensional restriction $\mathcal{P}_i|_{\chi(\mathcal{B})}$ of the Perron–Frobenius operator of T_i has no exceptional eigenvalues;*
4. *$\mathcal{P}(\omega)$ has an exceptional Lyapunov spectrum that is independent of ω and satisfies*
$$\Lambda(\mathcal{P}) \cap (\vartheta, 0) \supset \left\{ \log \left(\frac{\sqrt[3]{8 \pm 2\sqrt{11}}}{3} \right) \right\}.$$
5. *the Oseledets subspace $\mathcal{W}_2(\omega)$ corresponding to the largest exceptional Lyapunov exponent exists for all $\omega \in \Sigma$, and depends only on ω_0.*

For periodic map cocycles, one can find Lyapunov spectral points from the eigenvalues of the cyclic composition of Perron–Frobenius operators.

Lemma 5.1. *Consider a periodic map cocycle $\Phi : \mathbb{Z}^+ \times \Sigma \times I \to \Omega \times I$ of period R. If η is an eigenvalue of the Perron–Frobenius operator $\mathcal{P}^{(R)}(\omega)$, then*
$$\frac{\log \eta}{R} \in \Lambda(\mathcal{P}(\omega)).$$

Proof. There exists a function $0 \neq f \in \mathrm{BV}$ such that $\mathcal{P}^{(R)}(\omega) f = \eta f$. Hence for any $k \in \mathbb{N}$ and $0 \leq r < R$,
$$\min_{0 \leq i < R} \{\|\mathcal{P}^{(i)}(\omega) f\|\} \leq \frac{\|\mathcal{P}^{(kR+r)}(\omega) f\|}{\eta^k} \leq \max_{0 \leq i < R} \{\|\mathcal{P}^{(i)}(\omega) f\|\},$$
and the result follows. □

Proof. [Proof of Theorem 5.1] Consider the partition $\mathcal{J} = \{J_1, J_2, J_3\}$ of S^1 into the subintervals $J_i = [(i-1)/3, i/3]$. Let $\Phi : \mathbb{Z}^+ \times \Sigma \times S^1 \to S^1$ be the map cocycle with generator $\tilde{\Phi}(\omega) = T_{\omega_0}$, where the maps $\mathcal{T} = \{T_1, T_2, T_3\}$ are given by

$$T_i(x) = 3x - \frac{j}{3} + \frac{G_{i,j}}{9} \pmod 1, \quad x \in B_j = \left[\frac{j-1}{9}, \frac{j}{9}\right), \quad j = 1, \ldots, 9,$$

where

$$G = \begin{pmatrix} 6 & 7 & 6 & 1 & 3 & 0 & 4 & 3 & 0 \\ 3 & 6 & 5 & 0 & 0 & 8 & 3 & 6 & 2 \\ 0 & 6 & 7 & 1 & 0 & 6 & 3 & 3 & 4 \end{pmatrix}.$$

The graphs of T_1, T_2, T_3 are shown in Figure 1: note that, by construction, each map T_i largely maps the interval J_i into the interval J_{i+1}, taking indices modulo 3: in fact, for $i \in 1, 2, 3$,

$$\frac{m(J_i \cap T_i^{-1} J_{i+1})}{m(J_i)} = \frac{8}{9}.$$

Thus we have a coherent structure built around the family of intervals $J : \Sigma \to \mathcal{J}$ given by $J(\omega) = J_{\omega_0}$.

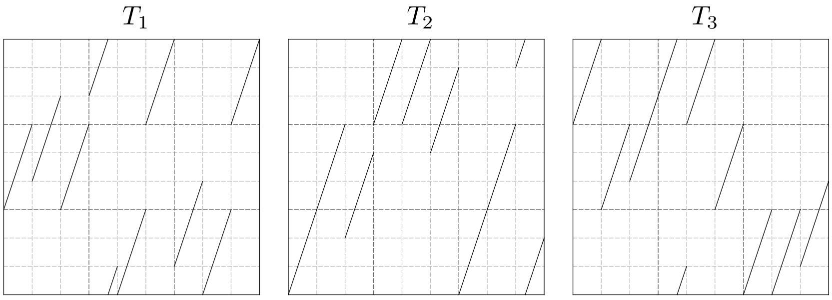

FIGURE 1. Graphs of T_1, T_2, T_3.

Note also that each map T_i is piecewise-affine expanding and there is a common Markov partition for $\mathcal{T}$ given by $\mathcal{B} = \{B_i : i = 1, \ldots, 9\}$. Notice that for each map $T \in \mathcal{T}$ and interval $B \in \mathcal{B}$, $DT|_B = 3$, and so $\vartheta(\omega) = \log 1/3$ for each $\omega \in \Sigma$.

Moreover, for each map $T \in \mathcal{T}$ and interval $B \in \mathcal{B}$, the preimage $T^{-1}B$ has precisely three components, each of one third of the length of B. Thus each $T \in \mathcal{T}$ preserves Lebesgue measure, and hence each $\Phi^{(n)}(\omega)$, $\omega \in \Sigma$ and $n \in \mathbb{N}$, does also.

As before, let P_i denote the matrix of the restriction $\mathcal{P}_i|_{\chi(\mathcal{B})}$ with respect to the basis $\{\chi(B)\}_{i=1}^9$. Here $P_i = \Gamma_{T_i}/3$ is the one third scaling of the transition matrix Γ_{T_i}, which is itself easily observed from the graph of T_i: the (p, q)th entry of the 0-1 matrix Γ_i is 1 if and only if the graph of T_i intersects the (p, q)th square of $\mathcal{B} \times \mathcal{B}$. Each matrix P_i has a simple eigenvalue 1, and all other non-zero eigenvalues lie on the circle of radius 1/3:

$$\begin{aligned} \mathrm{spec}(P_1) &= (1, -1/3, -1/3, 0, \ldots, 0) \\ \mathrm{spec}(P_2) &= (1, 1/3, 0, \ldots, 0) \\ \mathrm{spec}(P_3) &= (1, -1/3, -1/6 \pm \mathrm{i}\sqrt{3}/6, 0, \ldots, 0). \end{aligned}$$

Unlike in Theorem 6.1 in the following section, the maps used here cannot be expressed as different rotations of a single map.

We can find slowly decaying functions by examining the triple composition $\Phi^{(3)}(\alpha) = T_3 \circ T_2 \circ T_1$. The Perron–Frobenius operator $\mathcal{P}^{(3)}(\alpha)$, when restricted to the space $\chi(\mathcal{B})$, can be represented by the matrix $A^{(3)}(\alpha) = P_3 P_2 P_1$. We have

$$\mathrm{spec}(A^{(3)}(\alpha)) = \left(1, \frac{2}{27}(4 \pm 2\sqrt{11}), 0, \ldots, 0\right)$$

Since the cocycle is periodic, we find that the spectrum of $A^{(3)}(\omega)$ is independent of $\omega \in \Sigma$. Applying Lemma 5.1 we have that $\Lambda(\mathcal{P})$ has the two exceptional elements with approximate values

$$\lambda_2 \approx \log 0.8153, \quad \lambda_3 \approx \log 0.3699.$$

Moreover, these Lyapunov exponents are achieved by the corresponding eigenvectors of $A^{(3)}(\omega)$. For $\omega = \alpha$, the space $W_2(\alpha)$ is spanned by the second eigenvector w_2 of the matrix $A^{(3)}(\alpha) = P_3 P_2 P_1$, with approximate entries

$$w_2 = (0.105, 0.193, 0.193, 0.008, -0.059, -0.059, -0.113, -0.134, -0.134),$$

and the graph of $\langle w_2 \rangle \in \chi(\mathcal{B})$, which spans $\mathcal{W}_2(\alpha)$, is shown in Figure 2. For

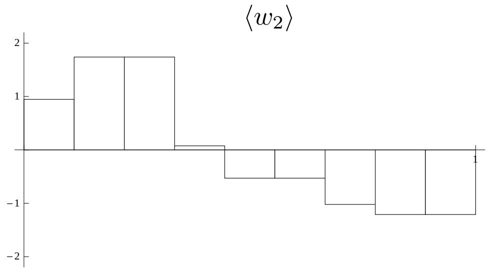

FIGURE 2. The graph of $\langle w_2 \rangle \in \chi(\mathcal{B})$ for Theorem 1.

$i = 1, 2$, $\mathcal{W}_2(\sigma^i \alpha)$ is spanned by $\langle w_2 \rangle \circ \rho^{-i}$, where $\rho : S^1 \to S^1$ is the rotation $\rho(x) = x + 1/3 \,(\mathrm{mod}\, 1)$. □

Evidence of the cyclic coherent structure is visible in the second eigenfunction of the Perron–Frobenius operator. Note that $J(\alpha) = [0, 1/3]$ supports the majority of the mass of the positive part of $\langle w_2 \rangle$. Similarly, the distinguished interval $J(\sigma^i \alpha) = [(i-1)/3, i/3]$, $i = 1, 2$, is picked up by $\langle w_2 \rangle \circ \rho^{-i}$.

6. *Construction of non-periodic cocycles with exceptional Lyapunov spectrum*

We now construct a non-periodic map cocycle with exceptional Lyapunov spectrum. The map cocycle is generated by six maps, including T_1 used in the previous example. The shift space is taken to be a subshift of finite type that has the Bernoulli shift on two symbols as a factor.

Let $\Theta \subset \{1, \dots, 6\}^{\mathbb{Z}}$ be the subshift of finite type

$$\Theta := \{\omega \in \{1, \dots, 6\}^{\mathbb{Z}} : \forall k \in \mathbb{Z}, E_{\omega_k, \omega_{k+1}} = 1\},$$

with transition matrix

$$E = (E_{i,j})_{1 \le i,j \le 6} = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right).$$

We let $\sigma : \Theta \to \Theta$ be the left shift, and p the uniform measure on Θ. As an indication of the complexity, notice that the full two-shift $\zeta : \{1, 2\}^{\mathbb{Z}} \to \{1, 2\}^{\mathbb{Z}}$ is a factor of $\sigma : \Theta \to \Theta$ via the mapping

$$h(\omega)_i = \begin{cases} 1, & \omega_i \in \{1, 2, 3\}, \\ 2, & \omega_i \in \{4, 5, 6\}. \end{cases}$$

The six maps $\{S_i\}_{i=1}^{6}$ are obtained from T_1 by rotations, and constructed so that

$$\frac{m(J_i \cap S_i^{-1} J_{i+1})}{m(J_i)} = \frac{8}{9} \quad \text{for } i = 1, 2, 3, \tag{13}$$

$$\frac{m(J_i \cap S_i^{-1} J_{i-1})}{m(J_i)} = \frac{8}{9} \quad \text{for } i = 4, 5, 6. \tag{14}$$

From these maps we construct a map cocycle with a non-periodic coherent structure that is responsible for the slow decay.

THEOREM 6.1. *There exists a collection $\mathcal{S}$ of six piecewise-affine expanding maps $S_1, \dots, S_6 : S^1 \to S^1$ with a common Markov partition $\mathcal{B}$ that generate a map cocycle $\Phi : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ over the shift $\sigma : \Theta \to \Theta$ with the following properties for $i = 1, \dots, 6$:*

1. *each map S_i preserves Lebesgue measure;*
2. *$\vartheta = \log 1/3$;*
3. *the restricted Perron–Frobenius operator $\mathcal{P}_i|_{\chi(\mathcal{B})}$ has no exceptional eigenvalues;*
4. *for each $\omega \in \Theta$, $\Lambda(\mathcal{P}(\omega))$ contains a unique exceptional exponent*

$$\log \frac{1 + \sqrt{2}}{3}.$$

5. *there exists an Oseledets decomposition for all $\omega \in \Theta$, and the Oseledets subspace $\mathcal{W}_2(\omega)$ depends only on ω_0.*

Proof. Let $\rho : S^1 \to S^1$ be the rotation $x \mapsto x + 1/3 \pmod 1$ and let $S : S^1 \to S^1$ be the map given by

$$S(x) = 3x - \frac{j}{3} + \frac{g_j}{9} \pmod 1, \quad x \in B_j = \left[\frac{j-1}{9}, \frac{j}{9}\right), \quad j = 1, \dots, 9,$$

where $g = (3, 4, 3, 7, 0, 6, 1, 0, 6)$. The interval $J_1 = [0, 1/3]$ is an almost-invariant subset of S^1, with $m(J_1 \cap S^{-1}J_1)/m(J_1) = 8/9$. Let P_S be the matrix of $\mathcal{P}_S|_{\chi(\mathcal{B})}$ with respect to the basis $\chi(\mathcal{B})$. The spectrum of P_S is

$$\mathrm{spec}(P_S) = \left(1, \frac{1 \pm \sqrt{2}}{3}, 0, \dots, 0\right).$$

We define the collection of maps $\mathcal{S} = \{S_i\}_{i=1}^6$ in terms of S and ρ:

$$S_1 = \rho \circ S \qquad S_4 = \rho^2 \circ S$$
$$S_2 = \rho^2 \circ S \circ \rho^2 \qquad S_5 = S \circ \rho^2$$
$$S_3 = S \circ \rho \qquad S_6 = \rho \circ S \circ \rho.$$

The graphs of $S_1, \dots, S_6$ are shown in Figure 3. Note that the graph of S_1 is the

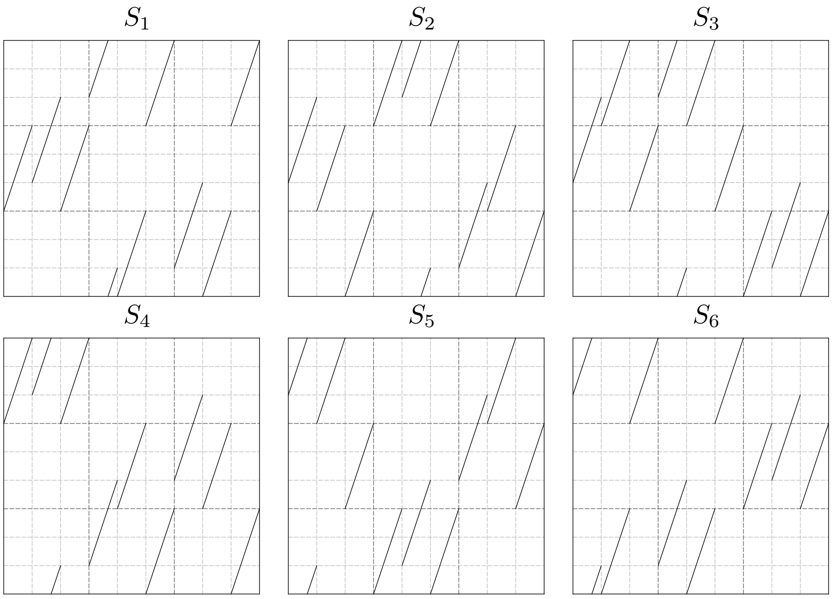

FIGURE 3. Graphs of $S_1, \dots, S_6$.

same as that of T_1 shown in Figure 1.

Let $\Phi : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ be the map cocycle with generator $\tilde{\Phi}(\omega) = S_{\omega_0} \in \mathcal{S}$. Let $\mathcal{J} = \{J_i\}_{i=1}^3$, where $J_i = [(i-1)/3, i/3]$. As a consequence of (13) and (14), we have a coherent structure built around the family of intervals $J : \Theta \to \mathcal{J}$, where

$$J(\omega) = \begin{cases} J_{\omega_0}, & \text{if } \omega_0 \le 3; \\ J_{\omega_0 - 3}, & \text{if } \omega_0 > 3. \end{cases}$$

Let $\mathcal{P}_i$ be the Perron–Frobenius operator of S_i. Let $\mathcal{P} : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ the Perron–Frobenius cocycle associated to Φ. Let P_i be the matrix representing $\mathcal{P}_i|_{\chi(\mathcal{B})}$ with respect to the basis $\chi(\mathcal{B})$ and let $A : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ be the matrix cocycle with generator $\tilde{A}(\omega) = P_{\omega_0}$. Let R denote the matrix with $R_{i,j} = 1$ if $i - j = 3 \,(\mathrm{mod}\, 9)$ and 0 otherwise. Note that R^3 is the identity matrix. For $i = 1, \ldots, 6$, the formula for P_i is obtained directly from the formula for S_i by replacing ρ by R and replacing S by P_S. Thus, for $i = 1, \ldots, 6$, we may write $P_i = R^{l_i} P_S R^{r_i}$, where $l = (1, 2, 0, 2, 0, 1)$ and $r = (0, 2, 1, 0, 2, 1)$.

One may confirm that

$$\mathrm{spec}(P_i) = \begin{cases} (1, -1/3, -1/3, 0, \ldots, 0), & \text{if } i \leq 3; \\ (1, 0, \ldots, 0), & \text{if } i > 3, \end{cases}$$

and so no map in $\mathcal{S}$ has exceptional eigenvalues.

Note that whenever $E_{i,j} = 1$, we find $l_i + r_j = 0 \,(\mathrm{mod}\, 3)$. Hence for any $\omega \in \Theta$, we have that

$$A^{(n)}(\omega) = R^{l_{\omega_{n-1}}} \, (P_S)^n \, R^{r_{\omega_0}},$$

with all inner R factors cancelling.

Hence for any $v \in \mathbb{R}^M$,

$$\begin{aligned} \|A^{(n)}(\omega)v\| &= \|R^{l_{\omega_{n-1}}} (P_S)^n R^{r_{\omega_0}} v\| \\ &= \|(P_S)^n R^{r(\omega_0)} v\| \\ &= \|(P_S)^n v'\|, \end{aligned}$$

where $v' = R^{r(\omega_0)} v$. So $\Lambda(A)$ is precisely the set of logarithms of the eigenvalues of P_S, and in particular, is independent of ω. Thus, $\Lambda(\mathcal{P})$ has a unique exceptional exponent $\log(1 + \sqrt{2})/3$ with approximate value $\log 0.8047$ for every $\omega \in \Theta$.

Let w_2 be an eigenvector of P_S corresponding to the second largest eigenvalue $(1 + \sqrt{2})/3$. The graph of $\langle w_2 \rangle \in \chi(\mathcal{B})$, which spans $\mathcal{W}_2(\alpha)$, is shown in Figure 4. Moreover, we have an Oseledets splitting for every $\omega \in \Theta$: for each

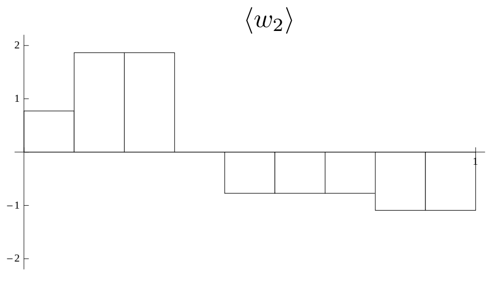

FIGURE 4. The graph of $\langle w_2 \rangle \in \chi(\mathcal{B})$ for Theorem 2.

$\omega \in \Theta$, the function $\langle R^{-r(\omega_0)} w_2 \rangle$ spans the Oseledets subspace $\mathcal{W}_2(\omega)$ associated to $\log(1 + \sqrt{2})/3$ and thus $\mathcal{W}_2(\omega)$ depends only on ω_0. □

As in the periodic example, the coherent structure responsible for the slow decay is detected by the second eigenfunction of the Perron–Frobenius operator. When $\omega_0 = 1$, $J(\omega) = [0, 1/3]$ is the distinguished interval for $\Phi(\omega)$, and this interval supports the majority of the mass of the positive part of the function $\langle w_2 \rangle$ spanning $\mathcal{W}_2(\omega)$. More generally, for $\omega \in \Theta$, the positive part of $\langle w_2 \rangle \circ \rho^{-r(\omega_0)} = \langle R^{-r(\omega_0)} w_2 \rangle$ is supported approximately on the interval $J(\omega)$.

7. *Numerical approximation of Oseledets subspaces*

In this section we outline a numerical algorithm to approximate the $W_i(\omega)$ subspaces. The Oseledets splittings for the cocycles in Theorem 5.1 and Theorem 6.1 were explicitly constructed as eigenvectors. In general, the Oseledets splittings are difficult to compute. The algorithm is based on the push-forward limit argument developed in the proof of Theorem 4.1. After stating the algorithm for an arbitrary matrix cocycle, we apply it to an example of a finite dimensional Perron–Frobenius cocycle over a non-periodic shift space that has Oseledets subspaces which cannot be found algebraically.

Algorithm. [Approximation of the Oseledets subspaces $W_i(\omega)$ at $\omega \in \Omega$.]

Let $A : \mathbb{Z}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a matrix cocycle over an invertible ergodic measure-preserving transformation s of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. Choose $M, N > 0$ and form

$$\Psi^{(M)}(s^{-N}\omega) := (A^{(M)}(s^{-N}\omega)^{\mathrm{T}} A^{(M)}(s^{-N}\omega))^{1/2M}$$

as an approximation to the standard limiting matrix

$$B(s^{-N}\omega) := \lim_{M\to\infty} \left(A^{(M)}(s^{-N}\omega)^{\mathrm{T}} A^{(M)}(s^{-N}\omega) \right)^{1/2M}$$

appearing in the Multiplicative Ergodic Theorem.

2. Calculate the orthonormal eigenspace decomposition of $\Psi^{(M)}(s^{-N}\omega)$, denoted by $U_i^{(M)}(s^{-N}\omega)$, $i = 1, \ldots, \ell$.

3. Define $W_i^{(M,N)}(\omega) := A^{(N)}(s^{-N}\omega) U_i^{(M)}(s^{-N}\omega)$ via the push forward under the matrix cocycle.

4. $W_i^{(M,N)}(\omega)$ is our numerical approximation to $W_i(\omega)$.

Remarks.

1. For fixed $N \in \mathbb{Z}$, the limit $W_i^{(\infty,N)}(\omega) := \lim_{M\to\infty} W_i^{(M,N)}(\omega)$ exists by the standard MET (eg. [**A98**, Theorem 3.4.1]). Theorem 4.1 states that $W_i^{(\infty,N)}(\omega) \to W_i(\omega)$ as $N \to \infty$.

2. This algorithm also provides an efficient numerical method for calculating the Oseledets subspaces for two-sided linear cocycles.

3. There is freedom in the choice of relative sizes of M and N: in order to sample equal numbers of positive and negative terms of ω, we take $M = 2N$.

The numerical approximation of the Oseledets subspaces has been considered by a variety of authors in the context of (usually invertible) nonlinear differentiable dynamical systems, where the linear cocycle is generated by Jacobian matrices concatenated along trajectories of the nonlinear system. Froyland *et al.* [**FJM95**] approximate the Oseledets subspaces in invertible two-dimensional systems by multiplying a randomly chosen vector by $A^{(N)}(s^{-N}\omega)$ (pushing forward) or $A^{(-N)}(s^N\omega)$ (pulling back). Trevisan and Pancotti [**TP98**] calculate eigenvectors of $\Psi^{(M)}(\omega)$ for the three-dimensional Lorenz flow, increasing M until numerical convergence of the eigenvectors is observed. Ershov and Potapov [**EP98**] use an approach similar to ours, combining eigenvectors of a $\Psi^{(M)}$ with pushing forward under $A^{(N)}$. Ginelli *et al.* [**G+07**] embed the approach of [**FJM95**] in a *QR*-decomposition methodology to estimate the Oseledets vectors in higher dimensions.

In the numerical experiments we describe next, we have found our approach to work very well, with fast convergence in terms of both M and N.

Example. To illustrate this technique, we calculate the Oseledets subspaces $W_2(\omega)$, $\omega \in \Theta$, for a non-periodic map cocycle, created from the maps of Theorem 5.1 and the sequence space Θ of Theorem 6.1. Unlike the example of Theorem 2, this example does not have a simple structure that makes it possible to relate the Oseledets subspaces to those of a single autonomous transformation.

Let $\mathcal{T} = \{T_i\}_{i=1}^6$ denote the collection of piecewise-affine expanding maps of the circle consisting of the three maps T_1, T_2, T_3 defined in Theorem 5.1 and the three maps $T_4 = \rho \circ T_1$, $T_5 = \rho \circ T_2$ and $T_6 = \rho \circ T_3$, where $\rho : S^1 \to S^1$ is the rotation $\rho(x) = x + 1/3 \pmod 1$ as before. The graphs of the maps in $\mathcal{T}$ are shown in Figures 1 and 5. Let $\Phi : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ be the map cocycle over $\sigma : \Theta \to \Theta$ generated by $\mathcal{T}$. The collection $\mathcal{T}$ has a common Markov partition $\mathcal{B} = \{[(i-1)/9, i/9) : i = 1, \ldots, 9\}$. We expect to find an exceptional Lyapunov spectrum since the cocycle has a coherent structure similar to that of Theorem 2, built around the family of intervals $J : \Theta \to \mathcal{J}$ given by

$$J(\omega) = \begin{cases} J_{\omega_0}, & \text{if } \omega_0 \le 3; \\ J_{\omega_0 - 3}, & \text{if } \omega_0 > 3. \end{cases}$$

We generate a test sequence in Θ as follows. Let $\hat{\alpha}^* \in \{0,1\}^{\mathbb{N}}$ be the fractional part of the binary expansion of π:

$$\hat{\alpha}^* = (0,0,1,0,0,1,0,0,0,0,1,1,1,1,1,1,0,1,1,0,1,0,1,0,1,0,0,\ldots),$$

and extend it to a two-sided sequence $\alpha^* \in \{0,1\}^{\mathbb{Z}}$ by defining $\alpha_i^* = 0$ for $i < 0$. We define $\omega^* = h^{-1}(\sigma^{120}\alpha^*)$, where h is the 3-to-1 factor defined in Section 6, and we take the inverse branch with $\omega_0^* = 1$. Note that $\omega^* \in \Theta$ has the form

$$\omega^* = (\ldots,1,2,3,1,2,3,4,\ldots,5,4,6,2,3,1,5,4,3,\underline{1},5,1,5,4,6,2,6,5,1,\ldots),$$

where the zeroth term is underlined.

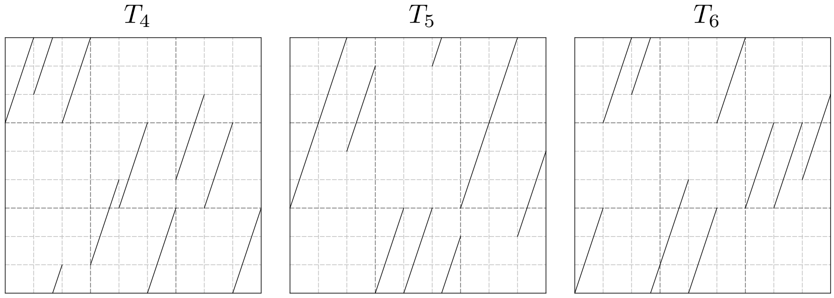

FIGURE 5. Graphs of T_4, T_5 and T_6.

As before, we denote by P_i the matrix representation of the Perron–Frobenius operator $\mathcal{P}_i|_{\chi(\mathcal{B})}$ of T_i, $i = 1, \ldots, 6$, with respect the basis $\chi(\mathcal{B})$, and denote by $A : \mathbb{Z}^+ \times \Theta \times S^1 \to S^1$ the matrix cocycle with the generator $\tilde{A}(\omega) = P_{\omega_0}$. The Multiplicative Ergodic Theorem states that for almost every ω, $\Psi^{(M)}(\omega)$ converges to a limit $B(\omega)$ as $M \to \infty$, and moreover $\Lambda(A) = \log \operatorname{spec}(B)$.

Calculating $\Psi^{(M)}(\omega^*)$ for $M = 40$, we find that $\Psi^{(M)}(\omega^*)$ has a simple eigenvalue $\lambda_2 \approx 0.81$, suggesting that $\mathcal{P}$ has exceptional Lyapunov exponent approximately equal to $\log 0.81$.

In order to approximate the Oseledets subspace $W_2(\omega^*)$ numerically, we set $M = 2N = 40$, form the matrix $\Psi^{(2N)}(\sigma^{-N}\omega^*)$ and denote by $u_2^{(2N)}(\sigma^{-N}\omega^*)$ the eigenvector corresponding to the eigenvalue λ_2. We then calculate

$$A^{(N)}(\sigma^{-N}\omega^*)u_2^{(2N)}(\sigma^{-N}\omega^*)$$

and normalize to give the vector $w_2^{(2N,N)}(\omega^*)$. The unit vector $w_2^{(2N,N)}(\omega^*)$ is our approximation to a unit vector spanning the subspace $W_2(\omega^*)$.

Although Theorem 4.1 holds only for a full p-measure subset of Θ, and so can tell us nothing about a particular sequence such as ω^*, we can still check whether its conclusions hold in this case. Taking $N = 20$, we calculate for $k = 0, \ldots, 7$, a vector $w_2^{(2N,N)}(\sigma^k\omega^*)$ spanning $W_2^{(2N,N)}(\sigma^k\omega^*)$ (see Figure 6).

Recall that $\{\omega_k^*\}_{k=0}^7 = \{1, 5, 1, 5, 4, 6, 2, 6\}$. For $k = 0, \ldots, 7$, by examining Figure 6, and comparing with the list $(J(\sigma^k\omega^*))_{k=0}^7$ given by

$$([0, 1/3], [1/3, 2/3], [0, 1/3], [1/3, 2/3], [0, 1/3], [2/3, 1], [1/3, 2/3], [2/3, 1]),$$

we see that the interval $J(\sigma^k\omega^*)$ is approximately picked up by the support of the positive part of $w_2^{(2N,N)}(\sigma^k\omega^*)$.

In order to check property (iii) of Theorem 4.1, that is, whether $A(\omega^*)W_2^{(2N,N)}(\omega^*)$ is close to $W_2^{(2N,N)}(\sigma\omega^*)$, we calculate the quantity

$$\Delta^{(2N,N)}(\omega^*) := \min\left\{ \left\| \left\langle w_2^{(2N,N)}(\sigma\omega^*) \pm \frac{A(\omega^*)w_2^{(2N,N)}(\omega^*)}{\|\langle A(\omega^*)w_2^{(2N,N)}(\omega^*)\rangle\|_{L^1}} \right\rangle \right\|_{L^1} \right\},$$

for $N = 1, \ldots, 20$ (see Figure 7).

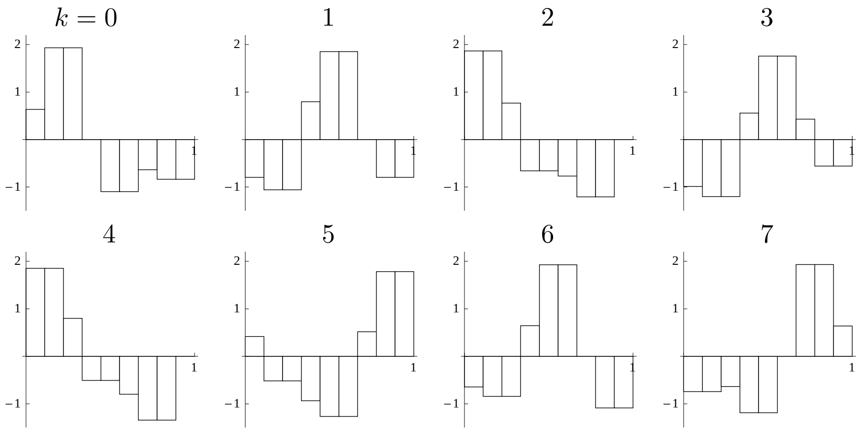

FIGURE 6. The graph of $\langle w_2^{(2N,N)}(\sigma^k\omega^*)\rangle$ for $k = 0, \ldots, 7$.

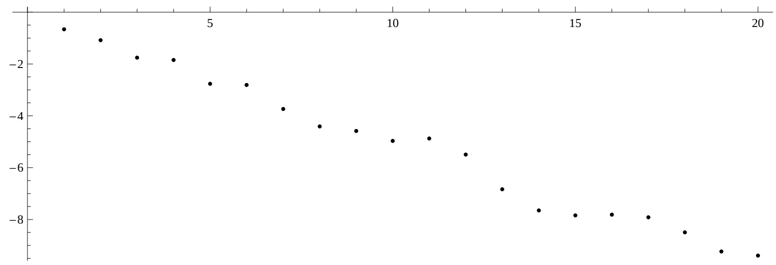

FIGURE 7. Graph showing $\log_{10} \Delta^{(2N,N)}(\omega^*)$ against N for $N = 1, \ldots, 20$.

Thus for $N = 20$, there are unit L^1-norm functions spanning the $\mathcal{W}_2^{(2N,N)}(\sigma\omega^*)$ and $\mathcal{P}(\omega^*)\mathcal{W}_2^{(2N,N)}(\omega^*)$ subspaces whose difference in L^1-norm is less than 10^{-8}.

Recall that for the cocycle in Theorem 2, the Oseledets subspace $W_2(\omega)$ is in fact independent of ω_i for $i \neq 0$. This contrasts with the current example: to see that here the Oseledets spaces $W_2(\omega)$ do not depend only on ω_0, it is enough to observe, for example, that $\omega_0 = \omega_2 = 1$ but $w_2^{(2N,N)}(\omega^*)$ and $w_2^{(2N,N)}(\sigma^2\omega^*)$ are markedly dissimilar.

8. *Proof of the Multiplicative Ergodic Theorem for non-invertible matrices*

In this section we present a strengthened version of the Multiplicative Ergodic Theorem (MET) for the case of non-invertible matrices. Let s be an invertible measure-preserving transformation of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a linear cocycle $P : \mathbb{Z}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$. Note that even though the matrices may be non-invertible, the invertibility of s is crucial to the argument. If the matrices are invertible then the two-sided cocycle is naturally defined as a map $P\colon \mathbb{Z} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$.

Recall that in the case of a one-sided linear cocycle (satisfying certain integrability conditions), the MET provides an invariant flag of subspaces of $\mathbb{R}^d$ characterising the exponential growth rates of all vectors. For a two-sided cocycle, one obtains an invariant splitting of $\mathbb{R}^d$ into *Oseledets spaces* by considering the intersection of each subspace in the flag of the forward cocycle with the corresponding subspace of the flag of the backward cocycle. Non-zero vectors v in the jth Oseledets space $W_j(\omega)$ satisfy $\lim_{n\to\pm\infty}(1/n)\log\|P(n,\omega,v)\| \to \lambda_j$.

In the case of a one-sided cocycle it clearly makes no sense to consider the limit $\lim_{n\to-\infty}(1/n)\log\|P(n,\omega,v)\|$. Nevertheless one may still hope for an invariant splitting of $\mathbb{R}^d$ rather than an invariant flag. This distinction is important if one is interested in the *vector* corresponding to the one of the top characteristic exponents: the flag would only provide an invariant family of *high-dimensional subspaces* with the property that most vectors in the space have the correct expansion rate, whereas a splitting would provide an invariant family of *low-dimensional subspaces*, whose vectors are responsible for all expansion at the chosen rate.

In this section present the proof of Theorem 4.1: we obtain a decomposition into Oseledets subspaces for a one-sided forward cocycle over an invertible transformation by means of a push-forward limit argument. Let $\|\cdot\|$ denote the matrix operator norm with respect to the Euclidean norm on $\mathbb{R}^d$.

LEMMA 8.1. *Let $B\colon \Omega \to M_d(\mathbb{R})$ be a measurable mapping into the space of symmetric matrices such that for almost all ω, $B(\omega)$ has real eigenvalues $\mu_1 > \cdots > \mu_\ell$ with multiplicities $m_1, \cdots, m_\ell$. Then there exists a measurable family $\left(e_i^j(\omega)\right)_{1\le j\le \ell,\ 1\le i\le m_j}$ of vectors such that the $\left(e_i^j(\omega)\right)$ form an orthonormal basis of $\mathbb{R}^d$ and $e_i^j(\omega)$ lies in the μ_j eigenspace of $B(\omega)$.*

Proof. Consider the map R that takes a matrix and applies a single step of a row-reduction algorithm (e.g. find the first column that is not in row-reduced echelon form; transpose rows to put a non-zero entry in the correct place; divide so the leading coefficient is 1; subtract multiples of that row from all of the others; repeat) or does nothing in the case that the matrix is already in row-reduced echelon form. The domains of the pieces are measurable and therefore R is measurable. For all matrices A, $R^n(A)$ is a convergent sequence so the limit $RRE(A)$ is a measurable function of the matrix.

A collection of vectors spanning the kernel of a row-reduced matrix may be obtained in a measurable way. These vectors may then be measurably converted to an orthonormal set by applying the Gram–Schmidt orthogonalization algorithm.

We apply this by taking a symmetric matrix B with eigenvalues $\mu_1, \ldots, \mu_\ell$ with multiplicities $m_1, \ldots, m_\ell$. We find an orthogonal set of vectors with each of the eigenvalues by applying the above procedures to $B - \mu_j I$. Since all operations are measurable the proof is complete. □

LEMMA 8.2. *Let $s\colon \Omega \to \Omega$ be an invertible ergodic measure-preserving transformation and let $(f_n)_{n=1}^\infty$ be a subadditive sequence of functions (that is a sequence such that for every $\omega \in \Omega$ and each m and n, $f_{n+m}(\omega) \le f_n(\omega) + f_m(s^n\omega)$).*

Assume further that $\max(f_1, 0)$ *is an* L^1 *function. Then there is a* $C \in [-\infty, \infty)$ *such that for almost every* ω *one has* $f_n(\omega)/n \to C$ *and* $f_n(s^{-n}\omega)/n \to C$.

Proof. The fact that there is a C such that $f_n/n \to C$ is Kingman's subadditive ergodic theorem. Letting $g_n(\omega) = f_n(s^{-n}\omega)$, we see that $g_{n+m}(\omega) \leq g_n(\omega) + g_m(s^{-n}\omega)$ so that the subadditive ergodic theorem applies to g_n also (with the measure-preserving transformation being s^{-1}) and there is a constant D such that $g_n(\omega)/n \to D$ for almost all ω.

Since f_n/n converges pointwise to C it also converges to C in measure. Similarly g_n/n converges in measure to D. Since f_n/n and g_n/n have the same distribution, the constants to which they converge in measure must be equal. □

We say a tuple $A = (a_1, \ldots, a_n)$ is *decreasing* if $a_i \geq a_{i+1}$ for $1 \leq i < n$.

LEMMA 8.3. *Let* $s\colon \Omega \to \Omega$ *be an invertible ergodic measure-preserving transformation and let* $A\colon \Omega \to M_d(\mathbb{R})$ *be a measurable family of matrices satisfying*

$$\int \log^+ \|A(\omega)\| \, \mathrm{d}p(\omega) < \infty.$$

Let S *be the decreasing* d*-tuple of Lyapunov exponents counting multiplicities. Given* $\omega \in \Omega$*, let* $SV^{(n)}(\omega)$ *be the decreasing* d*-tuple of logarithms of the* n*th roots of the singular values of* $A^{(n)}(s^{-n}\omega)$*. Then for almost every* ω*,* $SV^{(n)}(\omega) \to S$ *elementwise.*

Proof. Consider the family $\omega \mapsto A^{\mathrm{T}}(\omega)$ with respect to the dynamical system s^{-1}. Let the Lyapunov exponents be the decreasing d-tuple S'. This means that letting $SV'^{(n)}(\omega)$ be the decreasing d-tuple of logarithms of nth roots of singular values of $A^{\mathrm{T}}(s^{-n}\omega) \cdots A^{\mathrm{T}}(s^{-1}\omega)$, one has $SV'^{(n)}(\omega) \to S'$ for almost every $\omega \in \Omega$ by the standard one-sided MET, discussed in Section 4. Since singular values are preserved by taking transposes we see that $SV'^{(n)}(\omega) = SV^{(n)}(\omega)$. Thus it suffices to prove $S = S'$. To see this, note that $(1/n) \log \| \bigwedge^k A^{(n)}(\omega)\|$ converges to the sum of the first (that is, largest) k members of S, and $(1/n) \log \| \bigwedge^k A^{\mathrm{T}}(s^{-n}\omega) \cdots A^{\mathrm{T}}(s^{-1}\omega)\|$ converges to the sum of the first k members of S', but these limits are equal by Lemma 8.2. □

Proof. [Proof of Theorem 4.1] In the course of the proof we shall repeatedly use the symbol C to denote various constants depending only on ω.

We write $A^{(n)}(\omega)$ for the matrix product $A(s^{n-1}\omega) \cdots A(\omega)$. From standard proofs of the MET, we have that $[A^{(n)}(\omega)^{\mathrm{T}} A^{(n)}(\omega)]^{1/(2n)}$ is convergent to a positive semi-definite matrix $B(\omega)$, for almost all ω, with eigenvalues $e^{\lambda_1} > \cdots > e^{\lambda_\ell}$ with the correct multiplicities. We therefore let $(e_i^j(\omega))$ be as in Lemma 8.1 and let $U_j(\omega)$ be the subspace of $\mathbb{R}^d$ spanned by $\{e_i^j(\omega)\colon 1 \leq i \leq m_j\}$. The standard proofs of the MET show that if one lets $V_j(\omega) = \bigoplus_{i=j}^{\ell} U_i(\omega)$ then the vector spaces $V_j(\omega)$ satisfy:

1. $A(\omega)V_j(\omega) \subseteq V_j(s\omega)$;
2. For all $v \in V_j(\omega) \setminus V_{j+1}(\omega)$, $\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(\omega)v\| \to \lambda_j$;

For $j < \ell$, let $W_j^{(n)}(\omega) = A^{(n)}(s^{-n}\omega)U_j(s^{-n}\omega)$ and let $W_\ell(\omega) = U_\ell(\omega)$. Then we claim the following:

1. For $j < \ell$, $W_j^{(n)}(\omega)$ converges to an m_j-dimensional subspace $W_j(\omega)$;
2. $A(\omega)W_j(\omega) \subseteq W_j(s\omega)$;
3. If $x \in W_j(\omega) \setminus \{0\}$, then $\frac{1}{n}\log\|A^{(n)}(\omega)x\| \to \lambda_j$.
4. $V_{j+1}(\omega) \oplus W_j(\omega) = V_j(\omega)$.

Notice that $W_j^{(n+1)}(s\omega) = A(\omega)W_j^{(n)}(\omega)$ so that in the case $j < \ell$, (2) follows from (1) and the definition. For $j = \ell$, (2) and (3) follow from the standard MET proofs.

Fix a $j < \ell$ and consider a basis $B_0(\omega) = \{e_k^i(\omega)\colon k > j,\ i \le m_k\}$ for $V_{j+1}(\omega)$ and a basis $B_1(\omega) = \{e_j^i(\omega)\colon i \le m_j\}$ for $U_j(\omega)$. The union of $B_0(\omega)$ and $B_1(\omega)$ gives an orthonormal basis for $V_j(\omega)$. Since $A(\omega)V_{j+1}(\omega) \subset V_{j+1}(s\omega)$ and $A(\omega)V_j(\omega) \subset V_j(s\omega)$, it follows that if we express the linear transformation represented by $A(\omega)$ with respect to the bases $B_1(\omega) \cup B_0(\omega)$ and $B_1(s\omega) \cup B_0(s\omega)$, the matrix is of the form

$$L(\omega) = \begin{pmatrix} A_{11}(\omega) & 0 \\ A_{10}(\omega) & A_{00}(\omega) \end{pmatrix},$$

where if $V_{j+1}(\omega)$ is of dimension $q = m_{j+1} + \cdots + m_\ell$, the matrices $A_{11}(\omega)$, $A_{10}(\omega)$ and $A_{00}(\omega)$ have dimensions $m_j \times m_j$, $q \times m_j$ and $q \times q$ respectively. Notice that since the dimension of $V^i(\omega)$ is almost surely independent of $\omega \in \Omega$, the matrix $A_{11}(\omega)$ is almost everywhere invertible.

By definition, $L^{(n)}(\omega) = L(s^{n-1}\omega)\cdots L(\omega)$. By analogy with the above we name the components of this matrix as follows:

$$L^{(n)}(\omega) = \begin{pmatrix} A_{11}^{(n)}(\omega) & 0 \\ A_{10}^{(n)}(\omega) & A_{00}^{(n)}(\omega) \end{pmatrix}.$$

We will need the following matrix identities:

Claim A. With $A_{ij}^{(n)}$ defined as above we have

$$A_{11}^{(n)}(\omega) = A_{11}(s^{n-1}\omega)\cdots A_{11}(\omega) \tag{15}$$

$$A_{00}^{(n)}(\omega) = A_{00}(s^{n-1}\omega)\cdots A_{00}(\omega) \tag{16}$$

$$A_{10}^{(n)}(\omega) = \sum_{k=0}^{n-1} A_{00}^{(k)}(s^{n-k}\omega)A_{10}(s^{n-k-1}\omega)A_{11}^{(n-k-1)}(\omega). \tag{17}$$

Proof. The first two equalities are immediate and the third follows by induction on n. □

Claim B. For almost every $\omega \in \Omega$, $(1/n)\log\|A_{00}^{(n)}(\omega)\| \to \lambda_{j+1}$ as $n \to \infty$.

Proof. One has for each $i > j$ and $1 \le k \le m_i$, $(1/n)\log\|A^{(n)}(\omega)e^i_k\| \to \lambda_i$, by the MET. It follows that considering $A^{(n)}(\omega)$ as a linear map on $V_{j+1}(\omega)$, $(1/n)\log\|A^{(n)}(\omega)|_{V_{j+1}(\omega)}\| \to \lambda_{j+1}$. Thus $(1/n)\log\|A^{(n)}_{00}(\omega)\| \to \lambda_{j+1}$. □

Claim C. For every $\epsilon > 0$ and for almost every $\omega \in \Omega$, there is $D_1(\omega)$ such that $\|A^{(n)}_{00}(s^{-n}\omega)\| \le D_1(\omega)e^{n(\lambda_{j+1}+\epsilon)}$ for all $n \ge 0$.

Proof. Let $f_n(\omega) = \log\|A^{(n)}_{00}(\omega)\|$. This is a sub-additive sequence of functions and $f_n(\omega)/n \to \lambda_{j+1}$ for almost every ω by Claim B. Applying Lemma 8.2 we see that $f_n(s^{-n}\omega)/n \to \lambda_{j+1}$ for almost every ω. The claim follows. □

Claim D. For every $\epsilon > 0$ and for almost every $\omega \in \Omega$, there is a $D_2(\omega) < \infty$ such that for all $n \ge 0$ one has $\|A_{10}(s^{-n}\omega)\| \le D_2(\omega)e^{\epsilon n}$.

Proof. By hypothesis $\log\|A(\omega)\|$ is an integrable function and hence by a standard corollary of Birkhoff's theorem one has $(1/n)\log\|A(s^{-n}\omega)\| \to 0$. It follows that $\|A(s^{-n}\omega)\| \le D_2(\omega)e^{\epsilon n}$ for a suitable $D_2(\omega)$. Since $\|A_{10}(\omega)\| \le \|A(\omega)\|$ the result follows. □

Claim E. Under the above conditions, $\left(A^{(n)}_{11}(\omega)^{\mathrm{T}} A^{(n)}_{11}(\omega)\right)^{1/(2n)} \longrightarrow e^{\lambda_j} I_{m_j}$.

Proof. To see this it is sufficient to show that every non-zero vector in $U_j(\omega)$ has growth rate λ_j. Let $u \in U_j(\omega)$ have expansion $u = \sum_{i \le m_j} v_i e^j_i(\omega)$.

First we show that $A^{(n)}_{10}(\omega)v$ doesn't grow any faster than $A^{(n)}_{11}(\omega)v$. Note that $\|A^{(n)}(\omega)u\|^2 = \|A^{(n)}_{11}(\omega)v\|^2 + \|A^{(n)}_{10}(\omega)v\|^2$ so that we have $\|A^{(n)}_{10}(\omega)v\| + \|A^{(n)}_{11}(\omega)v\|$ grows at rate λ_j. Applying the MET to $A^{(n)}_{11}(\omega)$, we see that $A^{(n)}_{11}(\omega)v$ grows at some rate Λ. We will show that $A^{(n)}_{10}(\omega)v$ grows at a rate no greater than $\max(\Lambda, \lambda_{j+1})$. It will follow that $\Lambda = \lambda_j$.

Equality (17) gives

$$\|A^{(n)}_{10}(\omega)v\| \le \sum_{k=0}^{n-1} \|A^{(k)}_{00}(s^{n-k}\omega)\|\|A_{10}(s^{n-k-1}\omega)\|\|A^{(n-k-1)}_{11}(\omega)v\|$$

Fix an arbitrary $\epsilon > 0$. Claim C shows that $\|A^{(k)}_{00}(s^{n-k}\omega)\| \le D_1(s^n\omega)e^{k(\lambda_{j+1}+\epsilon)}$. Using Claim D also, we see

$$\|A^{(n)}_{10}(\omega)v\| \le D_1(s^n\omega)D_2(\omega)\sum_{k=0}^{n-1} e^{k(\lambda_{j+1}+\epsilon)}e^{\epsilon(n-k-1)}e^{(\Lambda+\epsilon)(n-k-1)}.$$

There exists M such that $D_1(\omega) < M$ on a positive measure subset of Ω. By the ergodicity of s, there are infinitely many n for which $D_1(s^n\omega) < M$. For these n, the right hand side of the inequality is bounded above by $D_2(\omega)Mne^{n(\max(\lambda_{j+1},\Lambda)+2\epsilon)}$. It follows that $\liminf \log\|A^{(n)}_{10}(\omega)v\|^{1/n} \le \max(\lambda_{j+1}, \Lambda)$ and thus

$$\liminf \log\|A^{(n)}(\omega)u\|^{1/n} \le \max(\lambda_{j+1}, \Lambda).$$

Since on the other hand $\lim \log\|A^{(n)}(\omega)u\|^{1/n} = \lambda_j$, we conclude that $\Lambda \ge \lambda_j$. Since $\|A^{(n)}(\omega)u\| \ge \|A^{(n)}_{11}(\omega)v\|$ we have $\lambda_j \ge \Lambda$ so that $\Lambda = \lambda_j$ as required. □

We now estimate

$$g_n(\omega) = \max_{v \in S_1} \frac{\|A_{10}^{(n)}(s^{-n}\omega)v\|}{\|A_{11}^{(n)}(s^{-n}\omega)v\|},$$

where S_1 denotes the unit sphere in $\mathbb{R}^{m_j}$. Note that by scale-invariance one could equivalently define g_n by taking the maximum over $\mathbb{R}^{m_j} \setminus \{0\}$.

We have

$$\begin{aligned}
g_n(\omega) &= \max_{v \in S_1} \frac{\left\| \sum_{k=0}^{n-1} A_{00}^{(k)}(s^{-k}\omega) A_{10}(s^{-(k+1)}\omega) A_{11}^{(n-k-1)}(s^{-n}\omega)v \right\|}{\left\| A_{11}^{(n)}(s^{-n}\omega)v \right\|} \\
&\leq \sum_{k=0}^{n-1} \max_{v \in S_1} \frac{\left\| A_{00}^{(k)}(s^{-k}\omega) A_{10}(s^{-(k+1)}\omega) A_{11}^{(n-k-1)}(s^{-n}\omega)v \right\|}{\left\| A_{11}^{(k+1)}(s^{-(k+1)}\omega) A_{11}^{(n-k-1)}(s^{-n}\omega)v \right\|} \\
&= \sum_{k=0}^{n-1} \max_{u \in S_1} \frac{\left\| A_{00}^{(k)}(s^{-k}\omega) A_{10}(s^{-(k+1)}\omega)u \right\|}{\left\| A_{11}^{(k+1)}(s^{-(k+1)}\omega)u \right\|} \\
&\leq \sum_{k=0}^{n-1} \frac{\max_{u \in S_1} \left\| A_{00}^{(k)}(s^{-k}\omega) A_{10}(s^{-(k+1)}\omega)u \right\|}{\min_{u \in S_1} \left\| A_{11}^{(k+1)}(s^{-(k+1)}\omega)u \right\|}.
\end{aligned}$$

Note that in the third line we are making use of the fact that $A_{11}^{(n-k-1)}(s^{-n}\omega)$ is invertible.

Let $\epsilon < (\lambda_j - \lambda_{j+1})/4$ be fixed for the remainder of the proof. By Lemma 8.3 and Claim E the kth roots of the singular values of $A_{11}^{(k)}(s^{-k}\omega)$ all converge to e^{λ_j}. It follows that there is a $C > 0$ depending on ω such that for all k,

$$\min_{u \in S_1} \left\| A_{11}^{(k+1)}(s^{-(k+1)}\omega)u \right\| > Ce^{k(\lambda_j - \epsilon)}. \tag{18}$$

We remark that similar uniform lower bounds appear in the paper of Barreira and Silva [**BS05**]. Using Claim C and Claim D there exists a C' depending on ω such that for all k,

$$\max_{u \in S_1} \left\| A_{00}^{(k)}(s^{-k}\omega) A_{10}(s^{-(k+1)}\omega)u \right\| \leq C' e^{k(\lambda_{j+1} + \epsilon)} e^{\epsilon k}.$$

Combining the estimates we see

$$g_n(\omega) \leq \frac{C'}{C} \sum_{k=0}^{n-1} e^{k(\lambda_{j+1} - \lambda_j + 3\epsilon)}.$$

Since $3\epsilon < \lambda_j - \lambda_{j+1}$ it follows that defining $M(\omega) = \sup_n g_n(\omega)$, one has $M(\omega) < \infty$ for almost all ω.

We define a distance D between two subspaces of $\mathbb{R}^d$ of the same dimension by the Hausdorff distance of their intersections with the unit ball B_1 in $\mathbb{R}^d$. We now estimate $D\left(W_j^{(n)}(\omega), W_j^{(m)}(\omega)\right)$ for $m > n$.

Let x belong to the unit sphere of $W_j^{(n)}(\omega)$ (the distance is always maximized by points on the boundary). Then $x = A^{(n)}(s^{-n}\omega)u$ for some $u \in U_j(s^{-n}\omega)$. Since for almost all ω, the matrix $A_{11}^{(m-n)}(s^{-m}\omega)$ is invertible, there exists almost surely a $u' \in U_j(s^{-m}\omega)$ such that $A^{(m-n)}(s^{-m}\omega)u' = u + z$ where $z \in V_{j+1}(s^{-n}\omega)$. Let v' be the coordinates of u' with respect to the basis $B_1(s^{-m}\omega)$. Then $\|A_{10}^{(m-n)}(s^{-m}\omega)v'\| = \|z\|$ and $\|A_{11}^{(m-n)}(s^{-m}\omega)v'\| = \|u\|$. It follows that $\|z\| \leq M(s^{-n}\omega)\|u\|$. Let $y = A^{(m)}(s^{-m}\omega)u'$ so that $y \in W_j^{(m)}(\omega)$. We then have $y = x + A^{(n)}(s^{-n}\omega)z$. By Claim C we have

$$\begin{aligned}\|A^{(n)}(s^{-n}\omega)z\| &\leq Ce^{(\lambda_{j+1}+\epsilon)n}\|z\| \\ &\leq Ce^{(\lambda_{j+1}+\epsilon)n}M(s^{-n}\omega)\|u\| \qquad (19)\end{aligned}$$

for a C depending only on ω. On the other hand, (18) implies that

$$1 = \|x\| = \|A^{(n)}(s^{-n}\omega)v'\| \geq C'e^{(\lambda_j-\epsilon)n}\|u\| \qquad (20)$$

for another C' depending just on ω. Let $K = C/C'$ and $\alpha = \lambda_j - \lambda_{j+1} - 2\epsilon > 0$. Dividing (19) by (20) we see

$$\|y - x\| = \|A^{(n)}(s^{-n}\omega)z\| \leq Ke^{-\alpha n}M(s^{-n}\omega).$$

The closest point of $W_j^{(m)}(\omega) \cap B_1$ to x is just the orthogonal projection of x onto $W_j^{(m)}(\omega)$ (which lies in B_1) so that the distance from x to $W_j^{(m)}(\omega) \cap B_1$ is bounded above by $\|y - x\|$ which in turn is bounded above by $Ke^{-\alpha n}M(s^{-n}\omega)$.

Conversely let $y \in B_1 \cap W_j^{(m)}(\omega)$. Then we have $y = A^{(m)}(s^{-m}\omega)u'$ for some $u' \in U_j(s^{-m}\omega)$. Let $A^{(m-n)}(s^{-m}\omega)u'$ be decomposed into $u+z$ with $u \in U_j(s^{-n}\omega)$ and $z \in V_{j+1}(s^{-n}\omega)$. Let $x = A^{(n)}(s^{-n}\omega)u$. Since $\sup_n g_n(\omega) = M(\omega)$, we have $\|z\| \leq M(s^{-n}\omega)\|u\|$. So using (18) again, we get

$$\|A^{(n)}(s^{-n}\omega)z\| \leq KM(s^{-n}\omega)e^{-\alpha n}\|A^{(n)}(s^{-n}\omega)u\|. \qquad (21)$$

We also have

$$\begin{aligned}\|A^{(n)}(s^{-n}\omega)u\| &\leq \|A^{(n)}(s^{-n}\omega)(u + z)\| + \|A^{(n)}(s^{-n}\omega)z\| \\ &\leq 1 + KM(s^{-n}\omega)e^{-\alpha n}\|A^{(n)}(s^{-n}\omega)u\|.\end{aligned}$$

So $\|A^{(n)}(s^{-n}\omega)u\| \leq 1/(1 - KM(s^{-n}\omega)e^{-\alpha n})$, provided $KM(s^{-n}\omega)e^{-\alpha n} < 1$. Combining this estimate with (21) gives

$$\|x - y\| = \|A^{(n)}(s^{-n}\omega)z\| \leq \frac{KM(s^{-n}\omega)e^{-\alpha n}}{1 - KM(s^{-n}\omega)e^{-\alpha n}}.$$

As before it follows that the closest point of $W_j^{(n)}(\omega) \cap B_1$ to y is at a distance at most $KM(s^{-n}\omega)e^{-\alpha n}/(1 - KM(s^{-n}\omega)e^{-\alpha n})$. In particular, provided that $KM(s^{-n}\omega)e^{-\alpha n} < 1$, we have

$$D\left(W_j^{(n)}(\omega), W_j^{(m)}(\omega)\right) \leq \frac{KM(s^{-n}\omega)e^{-\alpha n}}{1 - KM(s^{-n}\omega)e^{-\alpha n}}.$$

Obviously for $m, m' > n$ one then has

$$D\left(W_j^{(m)}(\omega), W_j^{(m')}(\omega)\right) \leq \frac{2KM(s^{-n}\omega)e^{-\alpha n}}{1 - KM(s^{-n}\omega)e^{-\alpha n}}.$$

Since $M(\omega)$ is measurable and s is ergodic, there exist for almost all ω arbitrarily large values of n such that $M(s^{-n}\omega) < A$ for some fixed $A > 0$. It follows that the sequence of subspaces is Cauchy and hence convergent to a subspace $W_j(\omega)$.

Let x belong to the unit sphere of $W_j^{(n)}(\omega)$. Then $x = A^{(n)}(s^{-n}\omega)u$. As before, writing x as $y + z$ with $y \in U_j(\omega)$ and $z \in V_{j+1}(\omega)$, we have $\|z\| \leq M(\omega)\|y\|$. Since $\|y\|^2 + \|z\|^2 = 1$, we have $\|y\|^2(1 + M(\omega)^2) \geq 1$ so that $\|y\| \geq 1/\sqrt{1 + M(\omega)^2} = B$. Thus each point of the unit sphere of $W_j(\omega)$ has a component in $U_j(\omega)$ of norm at least B. It follows that $V_j(\omega) = V_{j+1}(\omega) \oplus W_j(\omega)$, which completes the proof. □

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References

[A98] Arnold, L. *Random dynamical systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[B96] Baladi, V. Personal communication, 1996.

[B00] Baladi, V. *Positive transfer operators and decay of correlations.* Advanced Series in Nonlinear Dynamics, 16. World Scientific, Singapore, 2000.

[BS05] Barreira, L. & Silva, C. Lyapunov exponents for continuous transformations and dimension theory *Discrete Contin. Dyn. Syst.* **13** (2005), 469–490.

[BK98] Blank, M. & Keller, G. Random perturbations of chaotic dynamical systems: stability of the spectrum. *Nonlinearity* **11** (1998), no. 5, 1351–1364.

[BKL02] Blank, M., Keller, G. & Liverani, C. Ruelle-Perron-Frobenius spectrum for Anosov maps. *Nonlinearity* **15** (2002), no. 6, 1905–1973.

[BG97] Boyarsky, A. & Góra, P. *Laws of chaos. Invariant measures and dynamical systems in one dimension.* Probability and its Applications. Birkhuser Boston, Inc., Boston, MA, 1997.

[DJ99] Dellnitz, M. & Junge, O. On the approximation of complicated dynamical behavior. *SIAM J. Numer. Anal.* **36** (1999), no. 2, 491–515.

[DFS00] Dellnitz, M., Froyland, G. & Sertl, S. On the isolated spectrum of the Perron–Frobenius operator. *Nonlinearity* **13** (2000), no. 4, 1171–1188.

[D+05] Dellnitz, M., Junge, O., Koon, W., Lekien, F., Lo, M., Marsden, J., Padberg, K., Preis, R., Ross, S. & Thiere, B. Transport in dynamical astronomy and multibody problems. *Intern. J. of Bifurcation and Chaos* **15**, (2005) 699–727.

[EP98] Ershov, S. & Potapov, A. On the concept of stationary Lyapunov basis. *Phys. D* **118**, (1998) 167–198.

[F05] Froyland, G. Statistically optimal almost-invariant sets. *Phys. D* **200** (2005), no. 3-4, 205–219.

[F07] Froyland, G. On Ulam approximation of the isolated spectrum and eigenfunctions of hyperbolic maps. *Discrete Contin. Dyn. Syst.* **17** (2007), 671–689.

[F08] Froyland, G. Unwrapping eigenfunctions to discover the geometry of almost-invariant sets in hyperbolic maps. *Phys. D.* To appear.

[FJM95] Froyland, G., Judd, K. & Mees, A. Estimation of Lyapunov exponents of dynamical systems using a spatial average. *Phys. Rev. E* **51** (1995), 2844–2855.

[FP08] Froyland, G. & Padberg, K. Almost-invariant sets and invariant manifolds — connecting probabilistic and geometric descriptions of coherent structures in flow. Submitted.

[F+07] Froyland, G., Padberg, K., England, M., & Treguier, A. Detecting coherent oceanic structures via transfer operators. *Phys. Rev. Lett.*, **98** (2007).

[G+07] Ginelli, F., Poggi, P., Turchi, A., Chaté, H., Livi, R. & Politi, A. Characterizing Dynamics with Covariant Lyapunov Vectors. *Phys. Rev. Lett.* **99** (2007)

[GL06] Gouzel, S. & Liverani, C. Banach spaces adapted to Anosov systems. *Ergod. Th. Dynam. Sys.* **26** (2006), no. 1, 189–217.

[H01] Haller, G. Distinguished material surfaces and coherent structures in three-dimensional fluid flows. *Phys. D* **149** (2001), no. 4, 248–277.

[HY00] Haller, G. & Yuan, G. Lagrangian coherent structures and mixing in two-dimensional turbulence. *Phys. D* **147** (2000), no. 3-4, 352–370.

[HK82] Hofbauer, F. & Keller, G. Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.* **180** (1982), no. 1, 119–140.

[K84] Keller, G. On the rate of convergence to equilibrium in one-dimensional systems. *Comm. Math. Phys.* **96** (1984), no. 2, 181–193.

[KR04] Keller, G. & Rugh, H. Eigenfunctions for smooth expanding circle maps. *Nonlinearity* **17** (2004), 1723-1730.

[LH04] Liu, W. & Haller, G. Strange eigenmodes and decay of variance in the mixing of diffusive tracers. *Phys. D* **188** (2004), no. 1-2, 1–39.

[LY73] Lasota, A. & Yorke, J. On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.* **186** (1973), 481–488.

[PP03] Pikovsky, A. & Popovych, O. Persistent patterns in deterministic mixing flows. *Europhys. Lett.* **61** (2003), no. 5, 625-631.

[PPE07] Popovych, O., Pikovsky, A. & Eckhardt, B. Abnormal mixing of passive scalars in chaotic flows *Phys. E* **75** (2007).

[R89] Ruelle, D. The thermodynamic formalism for expanding maps. *Comm. Math. Phys.* **125** (1989), no. 2, 239–262.

[SHD99] Schütte, C., Huisinga, W. & Deuflhard, P. Transfer operator approach to conformational dynamics in biomolecular systems. *Preprint SC-99-36*, Konrad-Zuse-Zentrum, Berlin, 1999. Appeared in *Ergodic Theory, Analysis, and efficient simulation of dynamical systems*, Ed.: B. Fiedler, Springer-Berlin 2001.

[TP98] Trevisan, A. & Pancotti, F. Periodic Orbits, Lyapunov Vectors, and Singular Vectors in the Lorenz System. *J. Atmos. Sci.* **55** (1998), 390–399.

[W93] Walters, P. A dynamical proof of the multiplicative ergodic theorem. *Trans. Amer. Math. Soc.* **335** (1993), no. 1, 245–257.