A SEMI-INVERTIBLE OSELEDETS THEOREM WITH APPLICATIONS TO TRANSFER OPERATOR COCYCLES

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Abstract. Oseledets' celebrated Multiplicative Ergodic Theorem (MET) [V.I. Oseledec, *A multiplicative ergodic theorem. Characteristic Ljapunov, exponents of dynamical systems*, Trudy Moskov. Mat. Obšč. **19** (1968), 179–210.] is concerned with the exponential growth rates of vectors under the action of a linear cocycle on $\mathbb{R}^d$. When the linear actions are invertible, the MET guarantees an almost-everywhere pointwise *splitting* of $\mathbb{R}^d$ into subspaces of distinct exponential growth rates (called Lyapunov exponents). When the linear actions are non-invertible, Oseledets' MET only yields the existence of a *filtration* of subspaces, the elements of which contain all vectors that grow no faster than exponential rates given by the Lyapunov exponents. The authors recently demonstrated [G. Froyland, S. Lloyd, and A. Quas, *Coherent structures and exceptional spectrum for Perron–Frobenius cocycles*, Ergodic Theory and Dynam. Systems (to appear).] that a splitting over $\mathbb{R}^d$ is guaranteed *without* the invertibility assumption on the linear actions. Motivated by applications of the MET to cocycles of (non-invertible) transfer operators arising from random dynamical systems, we demonstrate the existence of an Oseledets splitting for cocycles of quasi-compact non-invertible linear operators on Banach spaces.

1. Introduction

Oseledets-type ergodic theorems deal with dynamical systems $\sigma\colon \Omega \to \Omega$ where for each $\omega \in \Omega$ there is an operator (or in the original Oseledets case a matrix) $\mathcal{L}_\omega$ acting on a linear space X. One then studies the properties of the operator $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_\omega$, giving an ω-dependent decomposition of X into subspaces with a hierarchy of expansion properties.

Prior to the previous work of the current authors, [10], all of the Oseledets-type theorems in the literature split into two cases according to the hypotheses: the invertible and non-invertible cases.

Invertible case: In this case the base dynamical system σ is assumed to be invertible and the operators $\mathcal{L}_\omega$ are assumed to be invertible (or in some cases just injective). Integrability conditions may be imposed on $\|\mathcal{L}_\omega^{-1}\|$.

The conclusion here is that the space X admits an invariant *splitting* $E_1(\omega) \oplus E_2(\omega) \oplus \cdots$, finite or countable, possibly with a 'remainder' in the infinite-dimensional case. Non-zero vectors in $E_i(\omega)$ expand exactly at rate λ_i.

Non-invertible case: In the non-invertible case no assumptions are made about invertibility of the base nor about injectivity of the operators. The weaker conclusion here is that X admits an invariant *filtration* $V_1(\omega) \supset V_2(\omega) \supset \cdots$ such that vectors in $V_i(\omega) \setminus V_{i+1}(\omega)$ expand at rate λ_i.

The conclusion in the invertible case may be seen to be much stronger as one obtains an invariant family of complements to $V_{i+1}(\omega)$ in $V_i(\omega)$. These are in general finite-dimensional so that one 'sees the vectors responsible for λ_i expansion'. This is of considerable importance in applications.

Our principal contribution here is to focus on the *semi-invertible* case. Here assumptions are made about the invertibility of the base transformation, but there are no assumptions about invertibility or injectivity of the operators $\mathcal{L}_\omega$. In spite of this we are able to show that one can obtain an invariant splitting rather than the weaker invariant filtration, for the setting investigated by Thieullen [27] where the random compositions have some quasi-compactness properties. In [10] we obtained an analagous result for the original Oseledets setting of matrices acting on $\mathbb{R}^d$.

1.1. **Set-up.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X, \|\cdot\|)$ a Banach space. A *random dynamical system* is a tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$, where σ is an invertible measure-preserving transformation of $(\Omega, \mathcal{F}, \mathbb{P})$, called the *base* transformation, and $\mathcal{L} : \Omega \to L(X, X)$ is a family of bounded linear maps of X, called the *generator*. We will later impose suitable measurability conditions on $\mathcal{L}$.

For notational convenience, we write $\mathcal{L}(\omega)$ as $\mathcal{L}_\omega$. A random dynamical system defines a *cocycle* $\mathbb{N} \times \Omega \to L(X, X)$:

$$(n, \omega) \mapsto \mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega. \tag{1}$$

We define the *Lyapunov exponent in direction* v, $\lambda(\omega, v)$, by

$$\lambda(\omega, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} v\|, \quad \omega \in \Omega, \quad v \in X. \tag{2}$$

Lyapunov exponents have the following well-known properties. For all $\omega \in \Omega$, $u, v \in X$ and $\alpha \neq 0$:

- (i) $\lambda(\omega, 0) = -\infty$;
- (ii) $\lambda(\omega, \alpha v) = \lambda(\omega, v)$;
- (iii) $\lambda(\omega, u + v) \le \max\{\lambda(\omega, u), \lambda(\omega, v)\}$ with equality if $\lambda(\omega, u) \neq \lambda(\omega, v)$;
- (iv) $\lambda(\sigma\omega, \mathcal{L}_\omega v) = \lambda(\omega, v)$.

We call the set $\Lambda(\omega) = \{\lambda(\omega, v) : v \in X\}$ the *Lyapunov spectrum*. For $\alpha \in \mathbb{R}$, the set $\mathcal{V}_\alpha(\omega) := \{v \in X : \lambda(\omega, v) \le \alpha\}$ is a linear subspace of X, $\mathcal{L}_\omega \mathcal{V}_\alpha(\omega) \subset \mathcal{V}_\alpha(\sigma\omega)$ and if $\alpha' < \alpha$, then $\mathcal{V}_{\alpha'}(\omega) \subset \mathcal{V}_\alpha(\omega)$. For each $\omega \in \Omega$, the quantity $\lambda(\omega)$ is defined by

$$\lambda(\omega) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|. \tag{3}$$

Definition 1. Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a random dynamical system.

- We say that $\mathcal{R}$ is *quasi-compact* if for almost every ω there is an $\alpha < \lambda(\omega)$ such that $\mathcal{V}_\alpha(\omega)$ is finite co-dimensional. Of particular interest is the infimal α with this property. We call this quantity $\alpha(\omega)$.
- For each isolated Lyapunov exponent $r \in \Lambda(\omega)$, let $\epsilon_r > 0$ be small enough that $\Lambda(\omega) \cap (r - \epsilon_r, r) = \emptyset$. If the codimension d of $\mathcal{V}_{r-\epsilon_r}(\omega)$ in $\mathcal{V}_r(\omega)$ is finite, then we say r is a Lyapunov exponent of *multiplicity* d.
- The Lyapunov exponents greater than $\alpha(\omega)$ are said to be *exceptional*. As they are isolated, the exceptional Lyapunov exponents $\{\lambda_i(\omega)\}_{i=1}^{p(\omega)}$ are either finite in number ($p(\omega) < \infty$) or else they are countably infinite ($p(\omega) = \infty$), accumulating only at $\alpha(\omega)$. We shall always enumerate the

exceptional Lyapunov exponents in decreasing order $\lambda_1(\omega) > \lambda_2(\omega) > \cdots$. The *exceptional Lyapunov spectrum*, $\mathrm{EX}(\mathcal{R})(\omega) = \{(\lambda_i(\omega), d_i(\omega))\}_{i=1}^{p(\omega)}$, consists of all exceptional Lyapunov exponents paired with their multiplicities.

In the setting where $\mathbb{P}$ is ergodic and the generator $\mathcal{L}$ satisfies suitable measurability conditions, $\lambda(\omega) = \lambda^*$, $\alpha(\omega) = \alpha^*$ and the exceptional Lyapunov spectrum will be independent of ω $\mathbb{P}$-a.e.

If X is a finite dimensional space, then $\alpha(\omega) = -\infty$ for each ω and so all Lyapunov exponents are exceptional. Since the sets $\mathcal{V}_\alpha(\omega)$ are subspaces for each $\alpha \in \mathbb{R}$, the number of Lyapunov exponents counted with multiplicity is bounded by the dimension of X, and so each is isolated and of finite multiplicity.

We are interested in Banach space analogues of the multiplicative ergodic theorem. In order to make sense of this it will be necessary to put a topology on suitable collections of subspaces of Banach spaces. The Grassmannian $\mathcal{G}(X)$ of a Banach space X is defined to be the set of *complemented* closed subspaces E of X (that is, those for which there is a second closed subspace F with the property that $X = E \oplus F$). Since every finite dimensional subspace of X is closed and complemented, the collection of d-dimensional subspaces of X forms a subset of $\mathcal{G}(X)$, which we denote by $\mathcal{G}_d(X)$. Also, finite codimensional subspaces are necessarily complemented, so the collection of *closed* c-codimensional subspaces of X forms a subset of $\mathcal{G}(X)$, which we denote by respectively $\mathcal{G}^c(X)$. More details on the Grassmannian are given in Section 2 along with proofs of some basic theorems concerning Grassmannians that we shall need later.

Definition 2. Consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ with ergodic base, and suppose $\mathcal{R}$ is quasi-compact with exceptional spectrum $\{(\lambda_i, d_i)\}_{i=1}^{p}$ (where $1 \le p \le \infty$). A *Lyapunov filtration* for $\mathcal{R}$ is a collection of maps $(V_i : \Omega \to \mathcal{G}^{c_i}(X))_{i=1}^{p}$, such that for all ω in a full measure σ-invariant subset $\Omega' \subset \Omega$ and for each $i = 1, \dots, p$:

(1) $X = V_1(\omega) \supset \cdots \supset V_i(\omega) \supset V_{i+1}(\omega)$;
(2) $\mathcal{V}_{\alpha(\omega)}(\omega) \subseteq \bigcap_{i=1}^{p} V_i(\omega)$ with equality if and only if $p = \infty$;
(3) $\mathcal{L}_\omega V_i(\omega) = V_i(\sigma\omega)$;
(4) $\lambda(\omega, v) = \lim_{n\to\infty} (1/n) \log \|\mathcal{L}_\omega^{(n)} v\| = \lambda_i$ if and only if $v \in V_i(\omega) \backslash V_{i+1}(\omega)$,

where we set $V_{p+1}(\omega) := \mathcal{V}_{\alpha(\omega)}(\omega)$ if $p < \infty$. An *Oseledets splitting* for $\mathcal{R}$ is a Lyapunov filtration $(V_i : \Omega \to \mathcal{G}^{c_i}(X))_{i=1}^{p}$ together with an additional collection of maps $(E_i : \Omega \to \mathcal{G}_{d_i}(X))_{i=1}^{p}$ (with $d_i = c_{i+1} - c_i$), called *Oseledets subspaces*, such that for all ω in a full measure σ-invariant subset $\Omega' \subset \Omega$ and for each $i = 1, \dots, p$:

(5) $V_i(\omega) = E_i(\omega) \oplus V_{i+1}(\omega)$;
(6) $\mathcal{L}_\omega E_i(\omega) = E_i(\sigma\omega)$;
(7) $\lambda(\omega, v) = \lim_{n\to\infty} (1/n) \log \|\mathcal{L}_\omega^{(n)} v\| = \lambda_i$ if $v \in E_i(\omega) \backslash \{0\}$.

We say a Lyapunov filtration is measurable if the maps $V_i : \Omega \to \mathcal{G}^{c_i}(X)$ are measurable with respect to the Borel σ-algebra on $\mathcal{G}(X)$ for each $1 \le i \le p$, where the topology will be defined in the next section. We say an Oseledets splitting is measurable if, in addition, the maps $E_i : \Omega \to \mathcal{G}_{d_i}(X)$ are measurable.

1.2. **Multiplicative Ergodic Theorems.** The first result on the existence of Lyapunov filtrations and Oseledets splittings in the finite dimensional setting is the Multiplicative Ergodic Theorem of Oseledets. Throughout, we define $\log^+(x) = \max\{0, \log x\}$.

Theorem 3 (Oseledets [22])**.** *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathbb{R}^d, \mathcal{L})$ be a random dynamical system with ergodic base, and suppose that the generator $\mathcal{L}$ is measurable and $\int \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P} < +\infty$. Then $\mathcal{R}$ admits a measurable Lyapunov filtration.*

Moreover, if the base is invertible, $\mathcal{L}_\omega$ is invertible a. e. and $\int \log^+ \|\mathcal{L}_\omega^{\pm 1}\| \, d\mathbb{P} < +\infty$, then $\mathcal{R}$ admits a measurable Oseledets splitting.

This situation may be summarized by saying that Oseledets splittings can be found when the base is invertible and the linear actions in the cocycle are invertible with bounded inverses, whereas in the non-invertible linear action cases the theorem only guarantees a Lyapunov filtration. This situation persisted in all subsequent versions [24, 19, 27] and extensions of the Oseledets theorem, to our knowledge, until the result stated below by the current authors which obtained a Oseledets splitting in the *semi-invertible* case where the base is invertible but the operators themselves are not assumed to be invertible (or they are invertible but there is no bound on the logarithmic norms of their inverses).

Theorem 4 (Froyland, Lloyd and Quas [10])**.** *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathbb{R}^d, \mathcal{L})$ be a random dynamical system with an invertible ergodic base, and suppose $\mathcal{L}$ is measurable and $\int \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P} < +\infty$. Then $\mathcal{R}$ admits a measurable Oseledets splitting.*

Remark 5. It is natural to ask whether one can obtain an invariant splitting in the absence of invertibility of either the base or the operators. In section 3.3 we show that if the base is non-invertible then even in the case where the operators are invertible one cannot in general obtain an invariant splitting.

The result of Oseledets has been extended by many authors. Of particular relevance to us are the result of Ruelle [24] dealing with the case where X is a Hilbert space and the result of Mañé [19] on random dynamical systems of compact operators in Banach spaces. This was subsequently extended to the quasi-compact case by Thieullen [27]. Thieullen's result will be stated precisely in Section 3. A key requirement for Thieullen's extension is that the dependence of the operator $\mathcal{L}_\omega$ on ω is required to be $\mathbb{P}$-continuous (the definition follows in Section 3) and it is upon this that we build. It should be pointed out that this is a significant limitation as many natural random dynamical systems fail to satisfy this condition (e.g. if T_ω is a family of Lasota–Yorke maps, it is almost never the case that their Perron–Frobenius operators depend in a $\mathbb{P}$-continuous way on ω). A parallel approach was taken in the recent thesis of Lian [17] where the measurability condition is relaxed to the weaker 'strongly measurable' condition (meaning that for each fixed $x \in X$, the map $\omega \mapsto \mathcal{L}_\omega x$ is measurable). The cost (which is again heavy from the point of view of applications) is that in order to obtain suitable measurability Lian imposes the condition that the Banach space X be separable.

Our main theorem is related to Thieullen's theorem in exactly the way that our theorem from [10] is related to Oseledets' Theorem: it provides an Oseledets splitting for the category of a quasi-compact linear action in the semi-invertible case where the base is invertible without any invertibility assumptions on the operators. We include the statement here, but defer some relevant definitions to section 3.

Main Theorem (Theorem 17)**.** *Let Ω be a Borel subset of a separable complete metric space, $\mathcal{F}$ the Borel sigma-algebra and $\mathbb{P}$ a Borel probabilty. Let X be a Banach space and consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ with base transformation $\sigma : \Omega \to \Omega$ an ergodic homeomorphism, and suppose that the* *generator $\mathcal{L} : \Omega \to L(X, X)$ is $\mathbb{P}$-continuous and satisfies*

$$\int \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P} < +\infty.$$

If $\kappa(\omega) < \lambda(\omega)$ (where κ is the "index of compactness" of $\mathcal{L}$) for almost every ω, then $\mathcal{R}$ is quasi-compact and admits a unique $\mathbb{P}$-continuous Oseledets splitting.

1.3. **Overview.** An outline of the paper is as follows. In Section 2 we prove some basic results concerning Grassmanians. In Section 3 we describe the result of Thieullen [27] and introduce the key notions of $\mathbb{P}$-continuity and index of compactness from that work. We then prove our main result. Section 4 describes two applications of our main result: Perron–Frobenius cocycles generated by random "Rychlik" maps (generalisations of Lasota–Yorke maps), and transfer operator cocycles generated by subshifts of finite type with random weight functions.

2. The Grassmannian of a Banach Space

Let X be a Banach space and suppose that $E, F \subset X$ are subspaces forming a direct (algebraic) sum: that is, $E + F = X$ and $E \cap F = \{0\}$. This decomposition specifies a linear map $\mathrm{Pr}_{F/\!\!/E}(e + f) = f$ with range F and kernel E, called the *projection onto F along E*. Conversely, any *projection* $P : X \to X$ (that is, a linear map satisfying $P^2 = P$) determines a decomposition $X = \ker(P) + \mathrm{ran}(P)$, where $\ker(P) \cap \mathrm{ran}(P) = \{0\}$.

Unlike in finite dimensions, not all projections in infinite-dimensional Banach spaces are continuous. A necessary (and sufficient) condition for a projection to be continuous is that it has a closed range. Since every continuous linear map has a closed kernel, it follows that every *continuous* projection determines a *topological direct sum*: a direct sum decomposition into complementary *closed* subspaces. We denote the topological direct sum of subspaces $E, F \subset X$ by $E \oplus F$. Conversely, it follows from the Closed Graph Theorem that if $E \subset X$ is a closed subspace with a closed complementary subspace $F \subset X$, then $\mathrm{Pr}_{F/\!\!/E}$ is continuous.

As mentioned before the Grassmannian of X, denoted $\mathcal{G}(X)$, is the collection of complemented closed subspaces of X. The set $\mathcal{G}(X)$ admits a Banach manifold structure as follows. Given $E_0 \in \mathcal{G}(X)$, fix any $F_0 \in \mathcal{G}(X)$ for which $E_0 \oplus F_0 = X$. We can use F_0 to define a neighbourhood of E_0: we set $U_{F_0} = \{E \in \mathcal{G}(X) : E \oplus F_0 = X\}$. We then define an isomorphism ϕ_{E_0,F_0} from U_{F_0} to the Banach space $L(E_0, F_0)$ by $\phi_{E_0,F_0}(E) = \mathrm{Pr}_{F_0/\!\!/E}|E_0$. The triples $\{U_{F_0}, \phi_{E_0,F_0}, L(E_0, F_0)\}$ form an atlas for $\mathcal{G}(X)$ showing that near E_0, $\mathcal{G}(X)$ is locally modelled on $L(E_0, F_0)$. The (E_0, F_0)-*local norm* on U_{F_0} is defined by

$$\|E\|_{(E_0,F_0)} := \|\mathrm{Pr}_{F_0/\!\!/E}|_{E_0}\|. \tag{4}$$

We now prove some basic properties of the Grassmannian $\mathcal{G}(X)$.

Lemma 6. *Let X be a Banach space and let Ω be a topological space. Suppose that for each $\omega \in \Omega$ there are closed subspaces $V(\omega)$ and $W(\omega)$ whose topological direct sum is X. Suppose further that $V(\omega)$ and $W(\omega)$ depend continuously on ω.*

Let $R(\omega) = \mathrm{Pr}_{V(\omega)/\!\!/W(\omega)}$ be the projection of X onto $V(\omega)$ along $W(\omega)$. Then the mapping $\omega \mapsto R(\omega)$ is continuous (with respect to the operator norm on $L(X, X)$).

Proof. Let $\omega_0 \in \Omega$. Since $V(\omega)$ and $W(\omega)$ are continuous families of subspaces, there exists a neighbourhood N_1 of ω_0 such that for all $\omega \in N_1$, $V(\omega) \oplus W(\omega_0) = X$ and $V(\omega_0) \oplus W(\omega) = X$.

Since $X = V(\omega_0) \oplus W(\omega_0)$, both $\|\mathrm{Pr}_{V(\omega_0)/\!\!/W(\omega_0)}\|$ and $\|\mathrm{Pr}_{W(\omega_0)/\!\!/V(\omega_0)}\|$ are finite. Let C be the greater of the two.

Given any $\epsilon > 0$, since $V(\omega)$ and $W(\omega)$ are continuous there is a neighbourhood N_2 of ω_0 contained in N_1 such that for $\omega \in N_2$ one has

$$\|\mathrm{Pr}_{W(\omega_0)/\!\!/V(\omega)}|_{V(\omega_0)}\| < \epsilon; \text{ and}$$
$$\|\mathrm{Pr}_{V(\omega_0)/\!\!/W(\omega)}|_{W(\omega_0)}\| < \epsilon.$$

Let x be in X. We now have $x = \mathrm{Pr}_{V(\omega)/\!\!/W(\omega)}(x) + \mathrm{Pr}_{W(\omega)/\!\!/V(\omega)}(x)$. Write the right side as $x_1 + x_2$. Now we split x_1 and x_2 into parts lying in $V(\omega_0)$ and $W(\omega_0)$ as $x_1 = x_{11} + x_{12}$ and $x_2 = x_{21} + x_{22}$ so that $x = x_{11} + x_{12} + x_{21} + x_{22} = (x_{11} + x_{21}) + (x_{12} + x_{22})$. We have $\mathrm{Pr}_{V(\omega)/\!\!/W(\omega)}(x) = x_1 = x_{11} + x_{12}$ and $\mathrm{Pr}_{V(\omega_0)/\!\!/W(\omega_0)}(x) = x_{11} + x_{21}$ so that the difference is $x_{12} - x_{21}$.

Rearranging we have $x_{22} = x_2 - x_{21}$ so that $-x_{21} = \mathrm{Pr}_{V(\omega_0)/\!\!/W(\omega)}(x_{22})$ so that $\|x_{21}\| < \epsilon\|x_{22}\|$. Similarly $\|x_{12}\| < \epsilon\|x_{11}\|$.

We have $\|x_{11} + x_{21}\| = \|\mathrm{Pr}_{V(\omega_0)/\!\!/W(\omega_0)}(x)\| \le C\|x\|$ so that $\|x_{11}\| \le C\|x\| + \|x_{21}\| < C\|x\| + \epsilon\|x_{22}\|$ and similarly $\|x_{22}\| < C\|x\| + \epsilon\|x_{11}\|$. Summing and rearranging we obtain $\|x_{11}\| + \|x_{22}\| < 2C/(1-\epsilon)\|x\|$. From the previous equations we obtain $\|R(\omega)(x) - R(\omega_0)(x)\| \le \|x_{12}\| + \|x_{21}\| < 2C\epsilon/(1-\epsilon)\|x\|$. Since ϵ may be chosen arbitrarily small, this establishes continuity of R at ω_0. □

Lemma 7. *Let $R\colon \Omega \to L(X, X)$ and $E\colon \Omega \to \mathcal{G}(X)$ be continuous. Then $\omega \mapsto \|R(\omega)|_{E(\omega)}\|$ is continuous.*

Proof. Let $\epsilon > 0$ and let $\omega_0 \in \Omega$. Choose a $\delta > 0$ such that $\delta + 2\delta\|R(\omega_0)\|/(1-\delta) < \epsilon$ and $2\delta\|R(\omega_0)\| + \delta < \epsilon$.

Fix an F_0 such that $E(\omega_0) \oplus F_0 = X$. Choose a neighbourhood N of ω_0 such that for $\omega \in N$, $E(\omega) \oplus F_0 = X$, $\|\mathrm{Pr}_{F_0/\!\!/E(\omega)}|_{E(\omega_0)}\| < \delta$ and $\|R(\omega) - R(\omega_0)\| < \delta$.

Let $x \in \overline{B}_{E(\omega)}$, the closed unit ball of $E(\omega)$. Since $E(\omega_0) \oplus F_0 = X$, x may be expressed uniquely as $a + b$ with $a \in E(\omega_0)$ and $b \in F_0$. We have $a = x - b$ so that $\mathrm{Pr}_{F_0/\!\!/E(\omega)}(a) = -b$ yielding $\|b\| < \delta\|a\|$. We have $\|a\| \le \|x\| + \|b\|$ so that $\|a\| < 1/(1-\delta)$.

We now have

$$\begin{aligned}
&\|R(\omega)x\| - \|R(\omega_0)((1-\delta)a)\| \\
&\le \|R(\omega)x - R(\omega_0)((1-\delta)a)\| \\
&\le \|R(\omega)x - R(\omega_0)x\| + \|R(\omega_0)(x - (1-\delta)a)\| \\
&\le \|R(\omega) - R(\omega_0)\| \cdot \|x\| + \|R(\omega_0)\| \cdot \|b + \delta a\| \\
&\le \delta + 2\delta\|R(\omega_0)\|/(1-\delta) \le \epsilon.
\end{aligned}$$

It follows that $\|R(\omega)|_{E(\omega)}\| < \|R(\omega_0)|_{E(\omega_0)}\| + \epsilon$.

Conversely let $x \in \overline{B}_{E(\omega_0)}$. We have $x = \mathrm{Pr}_{F_0/\!\!/E(\omega)}x + \mathrm{Pr}_{E(\omega)/\!\!/F_0}x$, which we write as $c + d$. By assumption $\|c\| < \delta$ so that $\|d\| < 1 + \delta$. We have

$$\begin{aligned}
&\|R(\omega_0)x\| - \|R(\omega)d/(1+\delta)\| \\
&\le \|R(\omega_0)x - R(\omega)d/(1+\delta)\| \\
&\le \|R(\omega_0)x - R(\omega_0)d/(1+\delta)\| + \|R(\omega_0)d/(1+\delta) - R(\omega)d/(1+\delta)\| \\
&\le \|R(\omega_0)\| \cdot \|x - d + \delta d/(1+\delta)\| + \|R(\omega_0) - R(\omega)\| \cdot \|d/(1+\delta)\| \\
&\le 2\delta\|R(\omega_0)\| + \delta < \epsilon.
\end{aligned}$$

It follows that $\|R(\omega_0)|_{E(\omega_0)}\| < \|R(\omega)|_{E(\omega)}\| + \epsilon$, which establishes the required continuity. □

Lemma 8. *Suppose that the map* $V\colon \Omega \to \mathcal{G}(X)$ *is continuous and that there are elements* E_0 *and* F_0 *of the Grassmannian such that* $V(\omega_0) \oplus E_0 \oplus F_0 = X$*. Then there is a neighbourhood* N *of* ω_0 *such that on* N*,* $\omega \mapsto V(\omega) \oplus F_0$ *is continuous.*

Proof. Since $E_0 \oplus F_0$ is a topological complementary subspace of $V(\omega_0)$, by continuity there is a neighbourhood N_1 of ω such that for $\omega \in N_1$, $V(\omega) \oplus E_0 \oplus F_0 = X$. In particular we see that E_0 is a topological complementary subspace to $V(\omega) \oplus F_0$ for $\omega \in N_1$. Let $\omega_1 \in N_1$ be fixed. We need to establish that for ω sufficiently close to ω_1, $\|\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}|_{V(\omega_1)\oplus F_0}\|$ is small. We demonstrate this by writing the operator as the composition of three parts: two of them bounded and the third one small.

Let $\epsilon > 0$. We note that $\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}|_{F_0}$ is zero so that we can rewrite $\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}|_{V(\omega_1)\oplus F_0}$ as $\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}|_{V(\omega_1)} \circ \mathrm{Pr}_{V(\omega_1) /\!\!/ F_0}$. We further decompose $\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}$ as $\mathrm{Pr}_{E_0 /\!\!/ F_0} \circ \mathrm{Pr}_{E_0\oplus F_0 /\!\!/ V(\omega)}$ so that

$$\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}|_{V(\omega_1)\oplus F_0} = \mathrm{Pr}_{E_0 /\!\!/ F_0} \circ \mathrm{Pr}_{E_0\oplus F_0 /\!\!/ V(\omega)}|_{V(\omega_1)} \circ \mathrm{Pr}_{V(\omega_1) /\!\!/ F_0}.$$

Since $V(\omega_1) \oplus F_0$ and $E_0 \oplus F_0$ are topological direct sums it follows that $C = \|\mathrm{Pr}_{V(\omega_1) /\!\!/ F_0}\|$ and $C' = \|\mathrm{Pr}_{E_0 /\!\!/ F_0}\|$ are finite. By continuity of $V(\omega)$ there is a neighbourhood N_2 of ω_1 on which $\|\mathrm{Pr}_{E_0\oplus F_0 /\!\!/ V(\omega)}|_{V(\omega_1)}\| < \epsilon/(CC')$. Multiplying the norms we see that for $\omega \in N_2$, $\|\mathrm{Pr}_{E_0 /\!\!/ V(\omega)\oplus F_0}\| < \epsilon$ as required. □

Lemma 9. *Let the maps* $V\colon \Omega \to \mathcal{G}_d(X)$ *and* $W : \Omega \to \mathcal{G}_{d'}(X)$ *be continuous and suppose that* $V(\omega) \cap W(\omega) = \{0\}$ *for each* $\omega \in \Omega$*. Then the map* $\omega \mapsto V(\omega) \oplus W(\omega)$ *is continuous.*

Proof. Fix $\omega_0 \in \Omega$. Let F_0 be a topological complement of $V(\omega_0) \oplus W(\omega_0)$ (this exists as all finite-dimensional subspaces have a topological complement). In order to demonstrate continuity we need to show that $\mathrm{Pr}_{F_0 /\!\!/ V(\omega)\oplus W(\omega)}|_{V(\omega_0)\oplus W(\omega_0)}$ has small norm. Since $V(\omega_0) \oplus W(\omega_0)$ is a topological direct sum it is sufficient to show that $\mathrm{Pr}_{F_0 /\!\!/ V(\omega)\oplus W(\omega)}|_{V(\omega_0)}$ is of small norm with a similar result for the restriction to $W(\omega_0)$. We write

$$\mathrm{Pr}_{F_0 /\!\!/ V(\omega)\oplus W(\omega)}|_{V(\omega_0)} = \mathrm{Pr}_{F_0 /\!\!/ W(\omega)} \circ \mathrm{Pr}_{F_0\oplus W(\omega) /\!\!/ V(\omega)}|_{V(\omega_0)}.$$

We have from Lemmas 6 and 8 that $\omega \mapsto \mathrm{Pr}_{F_0\oplus W(\omega) /\!\!/ V(\omega)}$ is continuous in a neighbourhood of ω_0 and from Lemma 7 that $\omega \mapsto \|\mathrm{Pr}_{F_0+W(\omega) /\!\!/ V(\omega)}|_{V(\omega_0)}\|$ is continuous on this neighbourhood. Since $\|\mathrm{Pr}_{F_0+W(\omega) /\!\!/ V(\omega)}|_{V(\omega_0)}\| = 0$ when $\omega = \omega_0$, it follows that this norm is arbitrarily small for ω in a neighbourhood of ω_0.

It remains to show that $\|\mathrm{Pr}_{F_0 /\!\!/ W(\omega)}\|$ remains bounded on a neighbourhood of ω_0. To see this we note from Lemma 6 that $\mathrm{Pr}_{F_0\oplus V(\omega_0) /\!\!/ W(\omega)}$ is continuous on a neighbourhood of ω_0 and $\mathrm{Pr}_{F_0 /\!\!/ V(\omega_0)}$ is a bounded operator since $F_0 \oplus V(\omega_0)$ is a topological direct sum. Composing these two operators gives the required result. □

Lemma 10. *Let* X *be a Banach space,* K *a compact metrizable space and let* $E : K \to \mathcal{G}_d(X)$ *be a continuous map. Let* $\mathbb{P}$ *be a finite measure on* K*. Then there exists an open and dense measurable subset* U *of* K *with full* $\mathbb{P}$*-measure and maps* $e_1, \ldots, e_d : K \to X$ *with* $e_i|U$ *continuous,* $i = 1, \ldots, d$ *such that for each* $\omega \in U$*,* $e_1(\omega), \ldots, e_d(\omega)$ *is a basis for* $E(\omega)$*.*

Furthermore, the basis can be chosen so that for each $\omega \in U$ and all $a \in \mathbb{R}^d$,

$$\|a\|_2 \leq \left\| \sum_{i=1}^{d} a_i e_i(\omega) \right\| \leq 4\sqrt{d}\|a\|_2.$$

Proof. Given $\omega_0 \in K$, there exists $F_{\omega_0} \in \mathcal{G}^d(X)$ such that $E(\omega_0) \oplus F_{\omega_0} = X$. By continuity of $E(\omega)$, there exists an open neighourhood U_{ω_0} of ω_0 such that $E(\omega) \oplus F_{\omega_0} = X$ for all $\omega \in U_{\omega_0}$. By a theorem of F. John (see [5, Chapter 4 Theorem 15] for example), there exists a basis $v_1, \dots, v_d$ for $E(\omega_0)$ satisfying

$$2\|a\|_2 \leq \left\| \sum_{i=1}^{d} a_i v_i \right\| \leq 2\sqrt{d}\|a\|_2, \tag{5}$$

for all $a \in \mathbb{R}^d$, where $\|a\|_2 = (\sum_{i=1}^d a_i^2)^{1/2}$ is the Euclidean norm on $\mathbb{R}^d$. Define $e_i^{\omega_0} : U_{\omega_0} \to X$ for each $i = 1, \dots, d$, by setting $e_i^{\omega_0}(\omega) = \mathrm{Pr}_{E(\omega) /\!\!/ F_{\omega_0}} v_i$. Notice that these vectors depend continuously on ω by Lemma 6. Replacing U_{ω_0} by a smaller neighbourhood of ω_0 if necessary, we may assume that for all $\omega \in U_{\omega_0}$ and $a \in \mathbb{R}^d$,

$$\|a\|_2 \leq \left\| \sum_{i=1}^{d} a_i e_i^{\omega_0}(\omega) \right\| \leq 4\sqrt{d}\|a\|_2.$$

It follows that the vectors are linearly independent and hence form a basis for $E(\omega)$.

We have that $\{U_\omega : \omega \in K\}$ is an open cover of K. Let ρ be a metric on K compatible with the topology and let $\delta > 0$ be the Lebesgue number of the cover: that is, for every $0 < r < \delta$ and $\omega \in K$, there exists $\omega' \in \Omega$ such that $B_r(\omega)$, the open ball of radius r centred at ω, is contained in $U_{\omega'}$. Fix $0 < r_0 < \delta$ and consider the open cover $\{B_{r_0}(\omega) : \omega \in K\}$ of K. By compactness, we have a finite subcover $\{B_{r_0}(\omega_i) : i = 1, \dots, k\}$. For each $i = 1, \dots, k$, the collection $\{\partial B_r(\omega_i) : r_0 < r < \delta\}$ is an uncountable family of pairwise disjoint sets (contained in the sphere of radius r about ω_i), and so there exists $r \in (r_0, \delta)$ such that $\mathbb{P}(\partial B_r(\omega_i)) = 0$ for each i.

We have that $\{B_i := B_r(\omega_i) : i = 1, \dots, k\}$ is a cover of K by open sets whose boundaries have zero $\mathbb{P}$-measure. These sets have the additional property that for each $i = 1, \dots, k$, there exists $\omega_i' \in K$ such that $B_i \subset U_{\omega_i'}$. Set $D_i = B_i \setminus \bigcup_{j<i} B_j$ and let $U = \bigcup_{i=1}^k \mathrm{int}(D_i)$. Since $K \setminus U \subset \bigcup_{i=1}^k \partial B_i$, U is an open dense set of full $\mathbb{P}$-measure and $\{D_i : i = 1, \dots, k\}$ is a partition of K. Setting $e_i(\omega) = e_i^{\omega_j'}(\omega)$ for $\omega \in D_j$, for each $i = 1, \dots, d$ and $j = 1, \dots, k$ gives maps with the required properties. □

3. Oseledets splitting

Thieullen [27] in his work on multiplicative ergodic theorems for operators introduced a framework on which this paper will be based. A key notion introduced in that paper is $\mathbb{P}$-continuity.

Definition 11. For a topological space Ω, equipped with a Borel probability $\mathbb{P}$, a mapping f from Ω to a topological space Y is said to be $\mathbb{P}$-*continuous* if Ω can be expressed as a countable union of Borel sets such that the restriction of f to each is continuous.

Remark 12. As noted in [27], if Ω is homeomorphic to a Borel subset of a separable complete metric space, then a function $f : \Omega \to Y$ is $\mathbb{P}$-continuous if and only if there exists a sequence $(K_n)_{n\geq 0}$ of pairwise disjoint compact subsets of X such that $\mu(\bigcup_{n\geq 0} K_n) = 1$ and the restriction $f|_{K_n}$ is continuous for each $n \geq 0$.

We shall call a Lyapunov filtration or Oseledets splitting $\mathbb{P}$-continuous if all of the exponents and all maps into the Grassmannian are $\mathbb{P}$-continuous (with respect to the topology defined in Section 2 in the case of maps into the Grassmannian).

Remark 13. If $\mathbb{P}$ is a *Radon* measure on Ω (that is, locally finite and tight) and Y is a metric space, then a map $f : \Omega \to Y$ is $\mathbb{P}$-continuous if and only if it is measurable (see [9]). In particular, this is the case in the 'Polish noise' setting (see, for example, [15, 1]), where Ω is a separable topological space with a complete metric, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbb{P}$ is any Borel probability.

Consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$. If σ is invertible with a measurable inverse, we say $\mathcal{R}$ has an *invertible base*. If Ω is a Borel subset of a complete separable metric space, $\mathcal{F}$ is the Borel sigma-algebra and σ is continuous (or a homeomorphism), we say $\mathcal{R}$ has a *continuous (or homeomorphic) base*.

Suppose $\mathcal{R}$ is a random dynamical system with a homeomorphic base. Provided $\omega \mapsto \mathcal{L}_\omega$ is $\mathbb{P}$-continuous we see that $\omega \mapsto \|\mathcal{L}_\omega^{(n)}\|$ is $\mathbb{P}$-continuous and hence $\mathcal{F}$-measurable. We shall assume throughout that $\int \log^+ \|\mathcal{L}_\omega\| \, \mathrm{d}\mathbb{P}(\omega) < \infty$. Since $\log \|\mathcal{L}_\omega^{(n)}\|$ is a subadditive sequence of functions it follows from the subadditive ergodic theorem that for almost every ω, $\frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|$ is convergent and hence the quantity $\lambda(\omega)$ defined in (3) may be re-expressed as

$$\lambda(\omega) = \lim_{n\to\infty} (1/n) \log \|\mathcal{L}_\omega^{(n)}\|.$$

The boundedness of $\int \log^+ \|\mathcal{L}_\omega\| \, \mathrm{d}\mathbb{P}(\omega)$ ensures that $\lambda(\omega)$ is finite $\mathbb{P}$-almost everywhere.

Proposition 14. *For each $\omega \in \Omega$, we have* $\sup \Lambda(\omega) = \lambda(\omega)$.

Proof. Clearly $\sup \Lambda(\omega) \leq \lambda(\omega)$, so we show that $\sup \Lambda(\omega) \geq \lambda(\omega)$. Fix $\omega \in \Omega$, let $r > \sup_{v\in X} \lambda(\omega, v)$. Set $A_N = \{v \in X : \|\mathcal{L}_\omega^{(n)} v\| \leq N e^{nr}, \forall n \in \mathbb{N}\}$. The set A_N is closed, and by the choice of r, we have $\bigcup_{N\in\mathbb{N}} A_N = X$ for each $\omega \in \Omega$. Thus by the Baire Category Theorem, there exists an A_N containing an interior point u. Let $\delta > 0$ be small enough that $B_\delta(u) \subset A_N$. For any $v \in B_\delta(0)$ and $n > 0$, we have $\|\mathcal{L}_\omega^{(n)} v\| \leq \|\mathcal{L}_\omega^{(n)}(v - u)\| + \|\mathcal{L}_\omega^{(n)} u\|$. So $\|\mathcal{L}_\omega^{(n)}\| \leq (2N/\delta)e^{nr}$, and hence $\lambda(\omega) \leq r$. Since r is an arbitrary quantity greater than $\Lambda(\omega)$, the result follows. □

We concentrate on the setting in which σ is ergodic. The function $\lambda(\omega)$ is then constant along orbits, and thus essentially constant. We denote by $\lambda^* \in \mathbb{R}$ the constant satisfying $\lambda(\omega) = \lambda^*$ for almost every $\omega \in \Omega$.

A second key concept introduced by Thieullen is that of the index of compactness of a random composition of operators. For a bounded operator A, $\|A\|_{\mathrm{ic}}$ is defined to be the infimal r such that $A(B_X)$ may be covered by a finite number of r-balls, where B_X is the unit ball in X. We have $\|AA'\|_{\mathrm{ic}} \leq \|A\|_{\mathrm{ic}} \|A'\|_{\mathrm{ic}}$ for any bounded linear operators on X. One can check that $|\|A\|_{\mathrm{ic}} - \|A'\|_{\mathrm{ic}}| \leq \|A - A'\|$ so that $\|A\|_{\mathrm{ic}}$ is a continuous function of the operator. In particular for each n, $\omega \mapsto \|\mathcal{L}_\omega^n\|_{\mathrm{ic}}$ is $\mathbb{P}$-continuous and hence $\mathcal{F}$-measurable. By sub-additivity we have $(1/n) \log \|\mathcal{L}_\omega^{(n)}\|_{\mathrm{ic}}$ is convergent.

Definition 15. The limit $\kappa(\omega) := \lim_{n\to\infty}(1/n)\log\|\mathcal{L}_\omega^{(n)}\|_{\mathrm{ic}}$ is called the *index of compactness* of the random composition of operators.

Since $\kappa(\omega)$ is σ-invariant it is equal almost everywhere to a constant which we call κ^*.

Theorem 16 (Thieullen [27])**.** *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ be a random dynamical system with an ergodic continuous base, and suppose $\omega \mapsto \mathcal{L}_\omega$ is $\mathbb{P}$-continuous, and that $\int \log^+ \|\mathcal{L}_\omega\| \,\mathrm{d}\mathbb{P} < +\infty$. If $\kappa^* < \lambda^*$, then $\mathcal{R}$ is quasi-compact, with $\alpha(\omega) = \kappa^*$ a. e., and admits a $\mathbb{P}$-continuous Lyapunov filtration.*

Moreover, if the base is invertible and $\mathcal{L}_\omega$ is injective a. e., then $\mathcal{R}$ admits a $\mathbb{P}$-continuous Oseledets splitting.

Our main result in this article is the extension of Thieullen's theorem to show that one obtains an Oseledets splitting in Thieullen's setting without making the assumption of invertibility of the $\mathcal{L}_\omega$.

Theorem 17. *Let Ω be a Borel subset of a separable complete metric space, $\mathcal{F}$ the Borel sigma-algebra and $\mathbb{P}$ a Borel probabilty. Let X be a Banach space and consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ with base transformation $\sigma : \Omega \to \Omega$ an ergodic homeomorphism, and suppose that the generator $\mathcal{L} : \Omega \to L(X, X)$ is $\mathbb{P}$-continuous and satisfies*

$$\int \log^+ \|\mathcal{L}_\omega\| \,\mathrm{d}\mathbb{P} < +\infty.$$

If $\kappa^* < \lambda^*$ for almost every ω, then $\mathcal{R}$ is quasi-compact and admits a unique $\mathbb{P}$-continuous Oseledets splitting.

The proof of this theorem (which makes extensive use of Theorem 16) is given in the next two subsections, in which existence and uniqueness of the Oseledets splitting, respectively, are proved.

3.1. **Existence of an Oseledets splitting.** Consider a random dynamical system $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$ with an ergodic homeomorphic base. Suppose $\mathcal{L}$ is $\mathbb{P}$-continuous and $\int \log^+ \|\mathcal{L}_\omega\| \,\mathrm{d}\mathbb{P} < +\infty$. Let $\mathrm{EX}(\mathcal{R}) = \{(\lambda_i, d_i)\}_{i=1}^p$ be the exceptional Lyapunov spectrum of $\mathcal{R}$, and $(V_i : \Omega \to \mathcal{G}^\infty(X))_{i=1}^p$ the Lyapunov filtration.

Following Thieullen, we construct an extension Banach space $\tilde{X}$, and a new generator $\tilde{\mathcal{L}} : \Omega \to L(\tilde{X}, \tilde{X})$ whose cocycle retains all the dynamical information of the original system but has the advantage that $\tilde{\mathcal{L}}_\omega$ is injective.

The extended random dynamical system $\tilde{\mathcal{R}} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \tilde{X}, \tilde{\mathcal{L}})$ is defined as follows:

$$\tilde{X} = \left\{(v_n)_{n=0}^\infty : \forall n,\ v_n \in X,\ \sup_n \|v_n\| < \infty\right\},$$
$$\tilde{\mathcal{L}}_\omega(v_0, v_1, v_2, \ldots) = (\mathcal{L}_\omega v_0, \alpha_0 v_0, \alpha_1 v_1, \alpha_2 v_2, \ldots),$$

for a positive sequence $(\alpha_n)_{n=0}^\infty$ decaying to zero. We endow $\tilde{X}$ with the norm $\|\tilde{v}\|_{\tilde{X}} = \sup_n \|v_n\|_X$ where $\tilde{v} = (v_n)_{n=0}^\infty$. Every $\tilde{\mathcal{L}}_\omega$ is injective on $\tilde{X}$. In Thieullen's article sufficient conditions on the speed of decay of the sequence (α_n) are given to ensure that the indices of compactness of $\tilde{\mathcal{R}}$ and $\mathcal{R}$ are equal ($\tilde{\kappa}^* = \kappa^*$) and that $\tilde{\lambda}^* = \lambda^*$. In fact we check in Subsection 3.4 that this holds for any sequence (α_n) of positive numbers tending to 0.

Provided $\kappa^* < \lambda^*$, we may apply the invertible form of Thieullen's Theorem to $\tilde{\mathcal{R}}$ to obtain the σ-invariant subset $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, $\mathbb{P}$-continuous Lyapunov filtration $(\tilde{V}_i : \Omega \to \mathcal{G}^\infty(\tilde{X}))_{i=1}^p$ and Oseledets subspaces $(\tilde{E}_i : \Omega \to \mathcal{G}_\infty(\tilde{X}))_{i=1}^p$. We denote by $\tilde{\lambda}(\omega, v) := \lim_{n\to\infty} \log \|\tilde{\mathcal{L}}_\omega^{(n)} \tilde{v}\|$ the Lyapunov exponents for $\tilde{\mathcal{R}}$.

Let $\pi : \tilde{X} \to X$ denote the (continuous) mapping onto the zeroth coordinate. We have $\mathcal{L} \circ \pi = \pi \circ \tilde{\mathcal{L}}$. Thieullen proves that for all $\omega \in \Omega'$ and $\tilde{v} \in \tilde{X}$, $\lambda(\omega, \pi(\tilde{v})) = \tilde{\lambda}(\omega, \tilde{v})$, and that $\mathcal{R}$ and $\tilde{\mathcal{R}}$ have the same expectional exponents. He then defines $V_i(\omega) = \pi(\tilde{V}_i(\omega))$ and proves that the $(V_i : \Omega \to \mathcal{G}^\infty(X))_{i=1}^p$ form a Lyapunov filtration for the one-sided system.

For each $1 \le i \le p$, we define $E_i(\omega) = \pi\tilde{E}_i(\omega)$. As the linear image of a finite dimensional space, $E_i(\omega)$ is a closed subspace. We now demonstrate that $(E_i : \Omega \to \mathcal{G}(X))_{i=1}^p$ is the splitting we seek.

Claim 18. *The maps $(E_i : \Omega \to \mathcal{G}(X))_{i=1}^p$ form a $\mathbb{P}$-continuous Oseledets splitting for $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$.*

Proof. Let $v \in V_{i+1}(\omega) \cap E_i(\omega)$. Then $v = \pi(\tilde{v})$ with $\tilde{v} \in \tilde{V}_{i+1}(\omega)$ so that $\lambda(\omega, v) = \tilde{\lambda}(\omega, \tilde{v}) \le \lambda_{i+1}$. On the other hand if $v \neq 0$, then $v = \pi(\tilde{v}')$ for some $\tilde{v}' \in \tilde{E}_i(\omega)\backslash\{0\}$. In this case we obtain $\lambda(\omega, v) = \tilde{\lambda}(\omega, \tilde{v}') = \lambda_i$. Since this contradicts the fact that $\lambda(\omega, v) \le \lambda_{i+1}$ it follows that $V_{i+1}(\omega) \cap E_i(\omega) = \{0\}$. Since $\tilde{V}_i(\omega) = \tilde{V}_{i+1}(\omega) + \tilde{E}_i(\omega)$, any $v \in V_i(\omega)$ can be written as $u + w$ with $u \in V_{i+1}(\omega)$ and $w \in E_i(\omega)$. By the triviality of the intersection this decomposition is unique. Since both spaces are closed ($V_{i+1}(\omega)$ by Thieullen's theorem and $E_i(\omega)$ by finiteness of dimension) we obtain $V_i(\omega) = V_{i+1}(\omega) \oplus E_i(\omega)$.

We now show that $\dim E_i(\omega) = \dim \tilde{E}_i(\omega)$. If $\tilde{v} \in \tilde{X}$ satisfies $\pi(\tilde{v}) = 0$, then $\tilde{\lambda}(\omega, \tilde{v}) = -\infty$, and so $\ker(\pi) \cap \tilde{E}_i(\omega) = \{0\}$. Since $E_i(\omega) = \pi(\tilde{E}_i(\omega))$, we see that $\dim E_i(\omega) \le \dim \tilde{E}_i(\omega)$. Suppose the subspace $\tilde{E}_i(\omega)$ is k-dimensional and can be written as $\tilde{E}_i(\omega) = \langle \tilde{v}^1, \dots, \tilde{v}^k \rangle$. Then $\langle v_0^i := \pi(\tilde{v}^i) : i = 1, \dots, k \rangle = E_i(\omega)$. If we had $c_1 v_0^1 + \cdots + c_k v_0^k = 0$ for some $c \in \mathbb{R}^k \backslash \{0\}$, then we would have $c_1 \tilde{v}^1 + \cdots c_k \tilde{v}^k \in \ker(\pi) \cap \tilde{E}_i(\omega) = \{0\}$, contradicting linear independence of the $\tilde{v}^1, \dots, \tilde{v}^k$. Hence, $v_0^1, \dots, v_0^k$, are also linearly independent, and so $d_i = \dim E_i(\omega) = \dim \tilde{E}_i(\omega)$.

We end by proving the $\mathbb{P}$-continuity of the subspaces E_i. Since the function $\tilde{E}_i : \Omega \to \mathcal{G}_{d_i}(\tilde{X})$ is $\mathbb{P}$-continuous, there exists a sequence $(K_n)_{n\ge0}$ of pairwise disjoint compact subsets of Ω such that $\mathbb{P}(\bigcup_{n\ge0} K_n) = 1$ and $\tilde{E}_i|_{K_n}$ is continuous for each $n \ge 0$. By Lemma 10, for each $n \ge 0$ there exists an open and dense subset $U_n \subset K_n$, $\mathbb{P}(U_n) = \mathbb{P}(K_n)$ and continuous functions $\tilde{e}_j^{i,n} : U_n \to \tilde{X}$, $j = 1, \dots, d_i$, with $\tilde{e}_1^{i,n}(\omega), \dots, \tilde{e}_{d_i}^{i,n}(\omega)$ forming a basis for $\tilde{E}_i(\omega)$ for each $\omega \in U_n$. Since $\pi : \tilde{X} \to X$ is continuous, the functions $e_j^{i,n} := \pi \circ \tilde{e}_j^{i,n} : U_n \to X$ are continuous, and $e_1^{i,n}(\omega), \dots, e_{d_i}^{i,n}(\omega)$ forms a basis for $E_i(\omega)$ as shown above. Take functions $e_j^i : \Omega \to X$, $j = 1, \dots, d_i$, satisfying $e_j^i(\omega) = e_1^{i,n}(\omega)$ for $\omega \in U_n$, $n \ge 0$. Applying Lemma 9 inductively we see that $\omega \mapsto \langle e_1^i(\omega), \dots, e_{d_i}^i(\omega) \rangle \in \mathcal{G}_{d_i}(X)$ is continuous on U_n, for each $n \ge 0$ and so $\mathbb{P}$-continuous on Ω, which shows that $E_i : \Omega \to \mathcal{G}_{d_i}(X)$ is $\mathbb{P}$-continuous. □

3.2. **Uniqueness of the Oseledets splitting.** Consider $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, X, \mathcal{L})$, a quasi-compact random dynamical system, and assume that σ is ergodic and $\int \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P}(\omega) < \infty$. Let $\mathrm{EX}(\mathcal{R}) = \{(\lambda_i, d_i)\}_{i=1}^p$ be the exceptional Lyapunov spectrum, $(V_i : \Omega \to \mathcal{G}^{c_i}(X))_{i=1}^p$ the Lyapunov filtration and $(E_i : \Omega \to \mathcal{G}_{d_i}(X))_{i=1}^p$ the Oseledets subspaces constructed above.

The following lemma gives us exponential uniformity in a finite-dimensional subspace all of whose Lyapunov exponents are equal. A result of this type first appeared in the Euclidean case in a paper of Barreira and Silva [4] (see also [10] for an independent proof). The proof here follows by choosing a suitable basis.

Lemma 19. *Let $B\colon \Omega \to L(X,X)$ be a $\mathbb{P}$-continuous family of operators and let $E\colon \Omega \to \mathcal{G}_d(X)$ be $\mathbb{P}$-continuous. Suppose that $B(\omega)$ maps $E(\omega)$ bijectively to $E(\sigma\omega)$. If for almost every ω, $\lim_{n\to\infty}(1/n)\log\|B_\omega^{(n)}v\| \to \lambda$ for all $v \in E(\omega)\backslash\{0\}$ (i.e. if all Lyapunov exponents of B are equal to λ) then*

$$\lim_{n\to\infty}\frac{1}{n}\log\inf_{x\in S_{E(\omega)}}\|B_\omega^{(n)}x\| = \lim_{n\to\infty}\frac{1}{n}\log\sup_{x\in S_{E(\omega)}}\|B_\omega^{(n)}x\| = \lambda,$$

where $S_{E(\omega)}$ denotes the unit sphere $\{x \in E(\omega) : \|x\| = 1\}$.

Proof. By Lemma 10, since $E : \Omega \to \mathcal{G}_d(X)$ is $\mathbb{P}$-continuous, we have $\mathbb{P}$-continuous functions $f_i : \Omega \to X$ satisfying

$$\|a\|_2 \le \left\|\sum_{i=1}^d a_i f_i(\omega)\right\| \le 4\sqrt{d}\|a\|_2, \tag{6}$$

where $\|\cdot\|_2$ represents the Euclidean norm on $\mathbb{R}^d$. Let $A(\omega) : \mathbb{R}^d \to E(\omega)$ be the map given by $A(\omega)a = \sum_{i=1}^d a_i f_i(\omega)$. By (6) we have $\|a\|_2 \le \|A(\omega)a\| \le 4\sqrt{d}\|a\|_2$. The linear map A is invertible a. e. and satisfies $1/(4\sqrt{d})\|v\| \le \|A(\omega)^{-1}v\|_2 \le \|v\|$ for $v \in E(\omega)$. We have a cocycle τ on $\mathbb{R}^d$ given by $\tau^{(n)}(\omega) := A(\sigma^n\omega)^{-1}B_\omega^{(n)}A(\omega)$. If $a \in \mathbb{R}^d$, then

$$\begin{aligned}\|\tau^{(n)}(\omega)a\|_2 &\le \|A(\sigma^n\omega)^{-1}\| \cdot \|B_\omega^{(n)}A(\omega)a\| \\ &\le \|B_\omega^{(n)}(A(\omega)a)\| \text{ and} \\ \|\tau^{(n)}(\omega)a\|_2 &\ge \|B_\omega^{(n)}(A(\omega)a)\|/\|A(\sigma^n\omega)\| \\ &\ge \frac{1}{4\sqrt{d}}\|B_\omega^{(n)}(A(\omega)a)\|.\end{aligned}$$

Since $A(\omega)$ is a bijection, it follows that $\lim_{n\to\infty}(1/n)\log\|\tau^{(n)}(\omega)a\|_2 = \lambda_i$ for each $a \in \mathbb{R}^d\backslash\{0\}$. Applying the theorem of Barreira and Silva [4, Theorem 2] (or see [10, Proof of Theorem 4.1]), we have that

$$\begin{aligned}&\lim_{n\to\infty}(1/n)\log\inf\{\|\tau^{(n)}(\omega)a\|_2 : a \in \mathbb{R}^d, \|a\|_2 = 1\} \\ =&\lim_{n\to\infty}(1/n)\log\sup\{\|\tau^{(n)}(\omega)a\|_2 : a \in \mathbb{R}^d, \|a\|_2 = 1\} = \lambda.\end{aligned}$$

Reusing the above inequalities the proof of the Lemma is complete. □

A sequence $(v_n)_{n\in\mathbb{Z}}$ is called a *full orbit* at $\omega \in \Omega$ if $\mathcal{L}(\sigma^n\omega)v_n = v_{n+1}$ for all $n \in \mathbb{Z}$. For full orbits, we may consider growth rates as $n \to -\infty$.

Lemma 20. *Let $(v_n)_{n\in\mathbb{Z}} \subset X$ be a full orbit for $\omega \in \Omega'$ and suppose $v_n \in V_i(\sigma^n\omega)$ for all $n \in \mathbb{Z}$. Then*

$$\liminf_{n\to\infty}(1/n)\log\|v_{-n}\| \ge -\lambda_i.$$

If we have $v_n \in E_i(\sigma^n\omega)$ for all $n \in \mathbb{Z}$, then we have the stronger statement

$$\lim_{n\to\infty}(1/n)\log\|v_{-n}\| = -\lambda_i.$$

Proof. We have $\lim_{n\to\infty}(1/n)\log\|\mathcal{L}_\omega^{(n)}|V_i(\omega)\| = \lambda_i$, and by [10, Lemma 8.2], it follows that $\lim_{n\to\infty}(1/n)\log\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)|V_i(\sigma^{-n}\omega)\| = \lambda_i$. Thus for any full orbit $\{v_n\}_{n\in\mathbb{Z}}$ satisfying $0 \neq v_n \in V_i(\sigma^n\omega)$ for all $n \in \mathbb{Z}$, we have

$$\limsup_{n\to\infty}(1/n)\log(\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)v_{-n}\|/\|v_{-n}\|) \leq \lambda_i. \tag{7}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log\|v_{-n}\| &= -\limsup_{n\to\infty}\frac{1}{n}\log\frac{\|v_0\|}{\|v_{-n}\|}\\ &= -\limsup_{n\to\infty}\frac{1}{n}\log\frac{\|\mathcal{L}^{(n)}(\sigma^{-n}\omega)v_{-n}\|}{\|v_{-n}\|} \geq -\lambda_i.\end{aligned}$$

For the second statement we shall assume that $v_n \in E_i(\sigma^n\omega)$ for all $n \in \mathbb{Z}$. The mapping $\mathcal{L}_\omega|E_i(\omega)$ is a bijection, so we denote by $S(\omega) : E_i(\sigma\omega) \to E_i(\omega)$ the inverse map. We let $S^{(n)}(\omega) := S(\sigma^{-n}\omega)\cdots S(\sigma^{-1}\omega) = [\mathcal{L}^{(n)}_{\sigma^{-n}\omega}|_{E_i(\sigma^{-n}\omega)}]^{-1}$ denote the cocycle for the map σ^{-1} generated by S. As $\log\|S^{(n)}(\omega)\|$ is a subadditive sequence of functions over σ^{-1}, using [10, Lemma 8.2] again we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\|S^{(n)}(\omega)\| &= \lim_{n\to\infty}\frac{1}{n}\log\|S^{(n)}(\sigma^n\omega)\|\\ &= -\lim_{n\to\infty}\frac{1}{n}\log\inf_{0\neq v\in E_i(\omega)}\frac{\|\mathcal{L}_\omega^{(n)}v\|}{\|v\|}\\ &= -\lambda_i,\end{aligned}$$

where the last equality follows from Lemma 19. Suppose now that $0 \neq v_n \in E_i(\sigma^n\omega)$ for all $n \in \mathbb{Z}$. Then we have

$$\limsup_{n\to\infty}\frac{1}{n}\log\|v_{-n}\| = \limsup_{n\to\infty}\frac{1}{n}\log\frac{\|S^{(n)}(\omega)v_0\|}{\|v_0\|} \leq -\lambda_i.$$

□

Claim 21. *The $\mathbb{P}$-continuous Oseledets splitting is unique on a full measure subset of Ω.*

Proof. Fix $1 \leq i \leq p$. Consider a $\mathbb{P}$-continuous map $E_i' : \Omega \to \mathcal{G}_{d_i}(X)$ satisfying $\mathcal{L}_\omega E_i'(\omega) = E_i'(\sigma\omega)$ and $E_i'(\omega) \oplus V_{i+1}(\omega) = V_i(\omega)$ for almost every $\omega \in \Omega$. Assume for a contradiction that there is a measurable subset $J \subset \Omega$, $\mathbb{P}(J) > 0$, such that $E_i(\omega) \neq E_i'(\omega)$ for all $\omega \in J$.

Let $F_i(\omega) = \bigoplus_{j<i} E_j(\omega)$. We have $V_{i+1}(\omega) \oplus E_i(\omega) \oplus F_i(\omega) = X$ for all $\omega \in \Omega'$. Let $(U_n)_{n\geq 0}$ be a sequence of measurable subsets of Ω, $\mathbb{P}(\bigcup_{n\geq 0} U_n) = 1$, such that the maps $V_{i+1}|_{U_n}$, $E_i|_{U_n}$ and $F_i|_{U_n}$ are continuous. By Lemma 9, the map $E_i \oplus F_i$ is continuous on U_n for each $n \geq 0$. By Lemma 6, the map $R(\omega) := \mathrm{Pr}_{V_{i+1}(\omega)/\!\!/E_i(\omega)\oplus F_i(\omega)}$ is continuous on U_n for each $n \geq 0$. Thus, by Lemma 7, the mapping $g(\omega) = \|R(\omega)|_{E_i'(\omega)}\|$ is $\mathbb{P}$-continuous, and in particular, is $\mathcal{F}$-measurable.

We first prove that $\lim_{n\to\infty} g(\sigma^n\omega) = 0$ for almost all ω. Let $\omega \in \Omega'$ be given. For any fixed $u \in E_i'(\omega) \setminus \{0\}$, we have $R(\omega)u \in V_{i+1}(\omega)$ so that for any $\epsilon > 0$ there exists a $C < \infty$ with $\|\mathcal{L}_\omega^{(n)} R(\omega)u\| \leq Ce^{n(\lambda_{i+1}+\epsilon)}$ for all $n > 0$.

On the other hand since $u \in V_i(\omega) \setminus V_{i+1}(\omega)$, there is a $C' > 0$ such that $\|\mathcal{L}_\omega^{(n)}u\| \geq C'e^{n(\lambda_i-\epsilon)}$ for all n. Fix $\epsilon < \frac{1}{4}(\lambda_i - \lambda_{i-1})$. We have for each fixed u

there is a constant C_u such that

$$\frac{\|\mathcal{L}_\omega^{(n)}R(\omega)u\|}{\|\mathcal{L}_\omega^{(n)}u\|} \le C_u e^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)} \text{ for all } n>0.$$

We now use a Baire category argument. Define D_N by

$$D_N = \{u \in E_i'(\omega)\colon \|\mathcal{L}_\omega^{(n)}R(\omega)u\| \le Ne^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)}\|\mathcal{L}_\omega^{(n)}u\| \,\forall n>0\}.$$

Since these sets are closed and their union is all of $E_i'(\omega)$, one of them must contain a ball $\overline{B_\delta(u)} \cap E_i'(\omega)$. By scale-invariance it contains a ball $\overline{B_1(u/\delta)} \cap E_i'(\omega)$. Set $u_0 = u/\delta$ and let $x \in E_i'(\omega)$ satisfy $\|x\| = 1$. Then we have for each n

$$\|\mathcal{L}_\omega^{(n)}R(\omega)(u_0+x)\| \le Ne^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)}\|\mathcal{L}_\omega^{(n)}(u_0+x)\|$$
$$\|\mathcal{L}_\omega^{(n)}R(\omega)u_0\| \le Ne^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)}\|\mathcal{L}_\omega^{(n)}u_0\|.$$

Using $\mathcal{L}_\omega^{(n)}R(\omega) = R(\sigma^n\omega)\mathcal{L}_\omega^{(n)}$, subtracting the above two inequalities and using the triangle inequality we obtain

$$\|R(\sigma^n\omega)\mathcal{L}_\omega^{(n)}x\| \le Ne^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)}(\|\mathcal{L}_\omega^{(n)}(u_0+x)\| + \|\mathcal{L}_\omega^{(n)}u_0\|).$$

Since $\mathcal{L}_\omega^{(n)}x/\|\mathcal{L}_\omega^{(n)}x\|$ is a general point of the intersection of the unit sphere with $E_i'(\sigma^n\omega)$ we obtain

$$g(\sigma^n\omega) \le Ne^{-n(\lambda_i-\lambda_{i+1}-2\epsilon)} \frac{\sup_{x\in S_{E_i'(\omega)}} \|\mathcal{L}_\omega^{(n)}(u_0+x)\| + \|\mathcal{L}_\omega^{(n)}u_0\|}{\inf_{x\in S_{E_i'(\omega)}} \|\mathcal{L}_\omega^{(n)}x\|},$$

The numerator is bounded above by an expression of the form $Ce^{n(\lambda_i+\epsilon)}$. Similarly, by Lemma 19 the denominator is bounded below by an expression of the form $C'e^{n(\lambda_i-\epsilon)}$. It follows that $g(\sigma^n\omega) \le (NC/C')e^{-n(\lambda_i-\lambda_{i+1}-4\epsilon)}$. By our choice of ϵ we see that $g(\sigma^n\omega) \to 0$ as claimed.

Now let $\omega \in J$ and let (v_n) be a full orbit over ω with $v_0 \in E_i'(\omega)\setminus E_i(\omega)$. Such an orbit exists since $\mathcal{L}_\omega$ maps $E_i'(\omega)$ bijectively to $E_i'(\sigma(\omega))$. Let $u_n = v_n - R(\sigma^n\omega)v_n$ and $w_n = R(\sigma^n\omega)v_n$. Since $E_i'(\omega) \subset E_i(\omega) \oplus V_{i+1}(\omega)$ we see that $u_n \in E_i(\sigma^n\omega)$ (i.e. u_n has no component in $F_i(\sigma^n\omega)$). We also have $w_n \in V_{i+1}(\sigma^n\omega)$. Since $w_0 \neq 0$ we have $w_n \neq 0$ for all $n<0$.

We now have $R(\sigma^{-n}\omega)(w_{-n}+u_{-n}) = w_{-n}$. Lemma 20 tells us that $\|u_{-n}\| \le Ce^{-n(\lambda_i-\epsilon)}$ and that $\|w_{-n}\| \ge C'e^{-n(\lambda_{i+1}+\epsilon)}$. We deduce that

$$\begin{aligned}\liminf_{n\to\infty} g(\sigma^{-n}\omega) &= \liminf_{n\to\infty} \|R(\sigma^{-n}\omega)|_{E_i'(\sigma^{-n}\omega)}\| \\ &\ge \liminf_{n\to\infty} \|w_{-n}\|/(\|w_{-n}\| + \|u_{-n}\|) = 1.\end{aligned}$$

If we consider the set $A = \{\omega \in \Omega : g(\omega) < 1/2\}$ we have for almost every ω, $\sigma^n\omega \in A$ for all large positive n whereas $\sigma^n\omega \notin A$ for all large negative n. This contradicts the Poincaré recurrence theorem, and hence the promised uniqueness is established. □

3.3. **Necessity of invertibility of the base.** The Main Theorem provides an invariant splitting in the absence of invertibility of the operators as long as the base is invertible. It is natural to ask whether one can obtain an invariant splitting in the absence of invertibility of the base. The following example establishes that in general this is not possible.

Example 22. Let $\Sigma = \{0,1\}^{\mathbb{Z}}$ be equipped with the shift-transformation σ and the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure and let A_0 and A_1 be two non-commuting invertible 2×2 matrices which we consider as operators on $\mathbb{R}^2$. Let $\mathcal{L}_\omega \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\mathcal{L}_\omega = A_{\omega_0}$. We assume further that the two Lyapunov exponents of the random dynamical system differ. As is standard we define for $n > 0$, $\mathcal{L}_\omega^{(-n)} = A_{\omega_{-n}}^{-1} \circ \cdots \circ A_{\omega_{-1}}^{-1}$. We call this random dynamical system $\mathcal{R}$.

Oseledets' theorem then guarantees that there is a decomposition $\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega)$ such that for $v \in E_i(\omega) \setminus \{0\}$, $(1/n) \log \|\mathcal{L}_\omega^{(n)} v\| \to \lambda_i$ both as $n \to \infty$ and as $n \to -\infty$; and $\mathcal{L}_\omega(E_i(\omega)) = E_i(\sigma(\omega))$ and $\mathcal{L}_{\sigma^{-1}\omega}^{-1}(E_i(\omega)) = E_i(\sigma^{-1}\omega)$ for almost every ω. By uniqueness (Theorem 17), this splitting is unique.

We define an inverse system $\overline{\mathcal{R}}$ as follows: $\overline{\Sigma} = \Sigma$ where the base map is $\overline{\sigma} = \sigma^{-1}$. We define the operators on this inverse system by $\overline{\mathcal{L}}_\omega = A_{\omega_{-1}}^{-1}$. Oseledets' theorem guarantees that the splitting $E_1(\omega) \oplus E_2(\omega)$ also works for $\overline{\mathcal{R}}$.

We now define non-invertible systems $\mathcal{R}^+$ and $\mathcal{R}^-$ obtained by truncating the shifts in $\mathcal{R}$ and $\overline{\mathcal{R}}$ to one-sided shifts. Σ^+ is defined to be $\{0,1\}^{\mathbb{Z}^+}$ (where $\mathbb{Z}^+ = \{0,1,2,\ldots\}$) and Σ^- is defined to be $\{0,1\}^{\mathbb{Z}^-}$ (where $\mathbb{Z}^- = \{-1,-2,\ldots\}$). Let Σ^+ and Σ^- be equipped with their Borel σ-algebras $\mathcal{B}^+$ and $\mathcal{B}^-$. The maps σ and $\overline{\sigma}$ factor naturally onto maps σ^+ on Σ^+ and σ^- on Σ^- through $\pi^+(\omega) = (\omega_n)_{n \in \mathbb{Z}^+}$ and $\pi^-(\omega) = (\omega_n)_{n \in \mathbb{Z}^-}$. Define $\mathcal{L}_\eta^+$ for $\eta \in \Sigma^+ = A_{\eta_0}$ as before and similarly $\mathcal{L}_\xi^- = A_{\xi_{-1}}^{-1}$ for $\xi \in \Sigma^-$ as before. Let $\mathcal{R}^+$ and $\mathcal{R}^-$ be the two one-sided dynamical systems.

Suppose now for a contradiction that there are Oseledets splittings for $\mathcal{R}^+$ and $\mathcal{R}^-$: $E_1^+(\eta) \oplus E_2^+(\eta)$ for $\eta \in \Sigma^+$ and $E_1^-(\xi) \oplus E_2^-(\xi)$ for $\xi \in \Sigma^-$ respectively.

Then one can check that $E_1^+(\pi^+(\omega)) \oplus E_2^+(\pi^+(\omega))$ is an invariant splitting for $\mathcal{R}$ which gives the correct rates of expansion as $n \to \infty$. Theorem 17 guarantees that there is only one such splitting and hence we see that

$$E_i(\omega) = E_i^+(\pi^+(\omega)) \text{ for almost every } \omega. \tag{8}$$

Similarly $E_1^-(\pi^-(\omega)) \oplus E_2^-(\pi^-(\omega))$ is an invariant splitting for $\overline{\mathcal{R}}$ which gives the correct rates of expansion as the power n of the inverse random dynamical system approaches ∞. Since the splitting for $\overline{\mathcal{R}}$ was the same as that for $\mathcal{R}$ we deduce that

$$E_i(\omega) = E_i^-(\pi^-(\omega)) \text{ for almost every } \omega. \tag{9}$$

From (8) we deduce that E_i is $\mathcal{F}^+$-measurable where $\mathcal{F}^+ = {\pi^+}^{-1}\mathcal{B}^+$ whereas from (9) we deduce that E_i is $\mathcal{F}^-$-measurable where $\mathcal{F}^- = {\pi^-}^{-1}\mathcal{B}^-$. It follows that the E_i are $\mathcal{F}^- \cap \mathcal{F}^+$-measurable. Since the intersection $\mathcal{F}^- \cap \mathcal{F}^+$ is the trivial sigma-algebra it follows that E_i is constant almost everywhere, equal to E_i^* say. From this it follows that $A_0(E_i^*) = A_1(E_i^*) = E_i^*$ so that A_0 and A_1 have common eigenspaces and hence are simultaneously diagonalizable. Since they do not commute by assumption this is a contradiction.

3.4. **Reduction to the invertible case in Thieullen's Theorem.** As mentioned above Thieullen deduces the non-invertible version of his theorem from the invertible case by constructing an invertible extension of the given system. More specifically if the original system has maps $\mathcal{L}_\omega$ acting on a Banach space X the new system has maps $\tilde{\mathcal{L}_\omega}$ acting on a Banach space $\tilde{X}$ where

$$\tilde{X} = \{(x_0, x_1, \ldots) \colon x_i \in X, \sup \|x_i\| < \infty\}; \text{ and}$$
$$\tilde{\mathcal{L}_\omega}(x_0, x_1, \ldots) = (\mathcal{L}_\omega(x_0), \alpha_0 x_0, \alpha_1 x_1, \ldots).$$

Thieullen then defines $\gamma_n = \sum_{k \le n} \log \alpha_k$ and states conditions on the (α_n) and (γ_n) which suffice to ensure that the exceptional spectrum of the extension agrees with the exceptional spectrum of the original system.

His conditions are as follows:

(1) (α_n) is a strictly decreasing sequence converging to 0;
(2) $\lim_{n\to\infty} \gamma_n/n = -\infty$;
(3) $\forall \mu < 0, \sup\{p \ge 0 \colon \gamma_p \ge (n+p)\mu\} = o(n)$.

We claim that Condition (1) implies the other two conditions. That (1) implies (2) is immediate. We now indicate a brief proof that (2) implies (3).

Let $\mu < 0$ and $\epsilon > 0$ be given. Set $M = -\mu/\epsilon - \mu$. By (2) there is a p_0 such that for $p \ge p_0$ we have $\gamma_p/p < -M$. At this point, choose an n. If $\gamma_p \ge (n+p)\mu$ then either (i) $p < p_0$ or (ii) $p \ge p_0$ and therefore $(n+p)\mu \le \gamma_p < -pM$. In the latter case $-p\mu/\epsilon = p(M+\mu) \le -n\mu$. We then see that $p \le \min(p_0, \epsilon n)$. Since ϵ is arbitrary we see that $\sup\{p \ge 0 \colon \gamma_p \ge (n+p)\mu\} = o(n)$ as required and Condition (3) is established.

We remark that in Thieullen's proofs it is sufficient to take a sequence (α_k) for which (2) is satisfied. Clearly the most natural way to do this is to take any sequence satisfying (1).

4. Applications

The motivation for the development of Theorem 4 is the desire to extend transfer operator approaches for the global analysis of dynamical systems from deterministic autonomous dynamical systems to random or non-autonomous dynamical systems.

A common setting for deterministic systems is: $M \subset \mathbb{R}^m$ is a smooth manifold and $T : M \to M$ a C^1 map with some additional regularity properties. The (deterministic) dynamical system $T : M \to M$ has an associated *Perron–Frobenius operator* $\mathcal{L}_T : X \to X$ defined by $\mathcal{L}_T f(x) = \sum_{y \in T^{-1}x} f(y)/|\det DT(y)|$, where X is a Banach space of complex-valued functions on M. The Perron–Frobenius operator evolves density functions on M forward in time, just as the map T evolves single points $x \in M$ forward in time.

More generally, the "weight" $1/|\det DT(y)|$ may be replaced with a sufficiently regular generalised weight $g(y)$ to form a *transfer operator.* Perron–Frobenius operators and transfer operators have proven to be indispensable tools for studying the long term behaviour of dynamical systems. An ergodic absolutely continuous invariant probability measure (ACIP) describes the long term distribution of forward trajectories $\{T^k x\}_{k=0}^\infty$ in M for Lebesgue almost-all initial points in $x \in M$. An early use of Perron–Frobenius operators was to prove the existence of ACIMs for piecewise C^2 expanding maps [16]. A study of the peripheral spectrum of $\mathcal{L}_T$ yielded information on the number of ergodic ACIPs [13, 25]. The particular weight function $1/|\det DT(y)|$ is attuned to ACIPs. Other "equilibrium states" can be read off from the leading eigenfunction of the transfer operator by varying the weight function g (in statistical mechanics terms, g describes the local energy of states in M).

The spectrum of the Perron–Frobenius operator provides information on the exponential rate at which observables become temporally decorrelated. The essential spectral radius of Perron–Frobenius operators [14] establishes a threshold beyond which spectral values are necessarily isolated. Furthermore, this radius is typically connected with the average rate at which nearby trajectories separate. Thus, these isolated spectral values are of particular interest in applications because they predict decorrelation rates slower than one expects to be produced by local separation of trajectories. The eigenfunctions associated with these isolated eigenvalues have been used to detect slowly mixing structures in a variety of physical systems, see, for example, [26, 8, 12, 7].

From a physical applications point of view, it is natural to study random or time-dependent (non-autonomous) dynamical systems using a transfer operator methodology. Theorem 4 considered this question in the setting of a finite number of piecewise linear, expanding interval maps, sharing a joint Markov partition, where the Perron–Frobenius operators acted on the space of functions of bounded variation. In the present work, in our first application, we remove the assumptions of finiteness, piecewise linearity and Markovness, and allow random compositions that are expanding-on-average. Our second application is to subshifts of finite type with random continuously-parametrized weight functions.

4.1. **Application I: Interval maps.** We now show that Theorem 17 can be applied in the context of random compositions of expanding-on-average mappings acting through their Perron–Frobenius operators on the space BV of functions of bounded variation. In this context a major drawback of the Thieullen approach becomes clear: if T_1 and T_2 are *any* two distinct expanding mappings then their Perron–Frobenius operators $\mathcal{L}_{T_1}$ and $\mathcal{L}_{T_2}$ are far apart in the operator norm on BV. In fact the set of Perron–Frobenius operators acting on BV is discrete. As a consequence, in order for $\omega \mapsto \mathcal{L}_{T_\omega}$ to be a continuous map on a compact space, the maps range of $\omega \mapsto T_\omega$ is forced to be finite. If we want $\omega \mapsto \mathcal{L}_{T_\omega}$ to be $\mathbb{P}$-continuous then it can have at most countable range.

Let $I = [0,1] \subset \mathbb{R}$ denote the closed unit interval, $\mathcal{B}$ denote the Borel σ-algebra and m denote Lebesgue measure.

Definition 23. We say a map $T : I \to I$ is *Rychlik* if

(1) T is differentiable on a dense open subset $U_T \subset I$ of full measure;
(2) for each connected component B of U_T, $T|_B$ extends to a homeomorphism from $\overline{B}$ to a subinterval of I;
(3) the function $g_T : I \to \mathbb{R}$ has bounded variation, where

$$g_T(x) = \begin{cases} \frac{1}{|DT(x)|} & x \in U_T \\ 0 & \text{otherwise.} \end{cases}$$

The class of Rychlik maps is closed under composition. Recall that the *variation* of a function $f : I \to \mathbb{R}$ is the quantity

$$\text{var}(f) := \sup_{0=p_1<p_2<\ldots<p_k=1} \sum_{i=1}^{k} |f(p_i) - f(p_{i-1})|. \tag{10}$$

A function on the interval is said to be of bounded variation if $\text{var}(f) < \infty$.

The *Perron–Frobenius operator* for a Rychlik map T is defined, for a function $f \in L^1(I)$ by

$$\mathcal{L}_T f(x) = \sum_{y \in T^{-1}(x)} g_T(y) f(y). \tag{11}$$

The Perron–Frobenius operator is a *Markov operator*: that is, if $f \in L^1(I)$, then $\int \mathcal{L}_T f \, \mathrm{d}m = \int f \, \mathrm{d}m$, and if $f \geq 0$, then $\mathcal{L}_T f \geq 0$.

We consider the action of $\mathcal{L}_T$ on the Banach space

$$\mathrm{BV} := \left\{ f \in L^\infty(I) : f \text{ has a version } \tilde{f} \text{ with } \operatorname{var} \tilde{f} < \infty \right\}$$

with norm $\|f\| := \max(\|f\|_1, \inf\{\operatorname{var}(\tilde{f}) : \tilde{f} \text{ is a version of } f\})$. A version $\tilde{f}$ of $f \in \mathrm{BV}$ has minimal variation if and only if $\tilde{f}(x) \in [\lim_{y \to x^-} f(y), \lim_{y \to x^+} f(y)]$ for all x. We shall assume versions are chosen so as to satisfy this condition, unless stated otherwise.

We shall need a lemma that is a combination of Lemmas 4, 5 and 6 from Rychlik [25].

Lemma 24 (Rychlik [25])**.** *Let T be a Rychlik map of the unit interval and let $\mathcal{L}_T$ be its Perron-Frobenius operator. Suppose* $\operatorname{ess\,inf}_x |T'(x)| > 1$. *Let $a = 3/\operatorname{ess\,inf} |T'|$. Then there is a partition $\mathcal{P}$ of the unit interval into finitely many subintervals and a constant D such that for all $f \in BV$*

$$\operatorname{var} \mathcal{L}_T f \leq a \operatorname{var} f + D \sum_{J \in \mathcal{P}} \left| \int_J f \right|.$$

We define a random composition of Rychlik maps as follows. Let $\{T_i\}_{i \in I}$, be a finite or countably infinite set of Rychlik maps. Let $\overline{I}$ denote the one-point compactification of I (with the discrete topology) and let $S = \overline{I}^{\mathbb{Z}}$. Let $\sigma \colon S \to S$ be the shift map and let $\mathbb{P}$ be an ergodic shift-invariant probability measure supported on $\Omega = I^{\mathbb{Z}}$. For $\omega \in \Omega$ let $\mathcal{L}_\omega = \mathcal{L}_{T_{\omega_0}}$ be the Perron–Frobenius operator of the map T_{ω_0} acting on the space BV. We make the further assumption that $\int \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P}(\omega) < \infty$ (or equivalently $\sum_{i \in I} \mathbb{P}(\{i\}) \log \|\mathcal{L}_{T_i}\| < \infty$). If these conditions are satisfied we refer to the 6-tuple $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathrm{BV}, \mathcal{L})$ as a *Rychlik random dynamical system*.

One can then verify that the system $\mathcal{R}$ satisfies the assumptions of Theorem 17.

We denote the n-fold composition $T_{\sigma^{n-1}\omega} \circ \cdots T_{\sigma\omega} \circ T_\omega$ by $T_\omega^{(n)}$. It is well known that the composition, $\mathcal{L}_\omega^{(n)}$, of the Perron–Frobenius operators of $T_\omega, T_{\sigma\omega}, \ldots, T_{\sigma^{n-1}\omega}$ is equal to the Perron–Frobenius operator of $T_\omega^{(n)}$. A random composition may also be considered as a single transformation on the space $\Omega \times I$ which we endow with the sigma-algebra $\mathcal{F} \otimes \mathcal{B}$: the *skew product* $\Theta : \Omega \times I \to \Omega \times I$ is given by $\Theta(\omega, x) = (\sigma\omega, T_\omega x)$.

We shall need a well-known inequality relating the index of compactness to the essential spectral radius. For a version of the converse inequality the reader is referred to work of Morris [21]. Let $A \colon X \to X$ be a linear operator on a Banach space. We write $\|A\|_{\mathrm{fr}}$ for $\inf\{\|A - F\| \colon F \text{ has finite rank}\}$. Recall from earlier $\|A\|_{\mathrm{ic}}$ is defined to be $\inf\{r \colon A(B_X) \text{ may be covered by a finite number of } r\text{-balls}\}$.

Lemma 25. *For a linear operator A between Banach spaces $\|A\|_{ic} \leq \|A\|_{fr}$.*

Proof. Let $A = F + R$ where F has finite rank and $\|R\| = r$. Let $\epsilon > 0$. Since $F(B_X)$ is compact it may be covered by a finite number of ϵ-balls for any $\epsilon > 0$, $\bigcup_{n=1}^N B_\epsilon(x_n)$. Hence $A(B_X) \subset F(B_X) + R(B_X) \subset \bigcup_{n=1}^N B_\epsilon(x_n) + B_r(0) = \bigcup_{n=1}^N B_{r+\epsilon}(x_n)$ so that for each $\epsilon > 0$, $\|A\|_{\mathrm{ic}} \le r + \epsilon$. Since it is possible to find decompositions with r arbitrarily close to $\|A\|_{\mathrm{fr}}$ the lemma follows. □

Keller [14] used Lemma 24 together with a supplementary argument to identify the essential spectral radius of the Perron–Frobenius operator of an expanding Rychlik map acting on the space of functions of bounded variation. We show that Keller's argument applies equally in our context of random dynamical systems.

Theorem 26. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, BV, \mathcal{L})$ be a Rychlik random dynamical system. Then there exists a χ such that for $\mathbb{P}$-almost every ω,*

$$\left(1/\operatorname{ess\,inf}\left|T_\omega^{(n)\prime}(x)\right|\right)^{1/n} \to \chi. \tag{12}$$

Further if $\chi < 1$ then $\|\mathcal{L}_\omega^{(n)}\|_{ic}^{1/n} \to \chi$.

Definition 27. We say that the Rychlik random dynamical system appearing in the theorem is *expanding-on-average* if $\chi < 1$.

Proof. We note that both $\|\mathcal{L}_\omega^{(n)}\|_{\mathrm{ic}}$ and $a_n(\omega) = 1/\operatorname{ess\,inf}_x \left|T_\omega^{(n)\prime}(x)\right|$ are submultiplicative. It follows from the subadditive ergodic theorem that both of the limits appearing in the statement of the theorem exist for $\mathbb{P}$-almost every ω. In the case where $\chi < 1$ we claim the following inequalities:

$$a_n(\omega) \le \left\|\mathcal{L}_\omega^{(n)}\right\|_{\mathrm{ic}} \le \left\|\mathcal{L}_\omega^{(n)}\right\|_{\mathrm{fr}} \le 3a_n(\omega) \text{ provided } a_n(\omega) < 1. \tag{13}$$

The middle inequality is Lemma 25. To see the upper bound, let $\mathcal{P}$ be the partition of the interval into subintervals guaranteed by Lemma 24. Let $E_{\mathcal{P}}$ be the conditional expectation operator defined by

$$E_{\mathcal{P}} f(t) = \frac{1}{|J|}\int_J f \text{ for } t \in J.$$

We then have $\mathcal{L}_\omega^{(n)} = \mathcal{L}_\omega^{(n)} \circ (1 - E_{\mathcal{P}}) + \mathcal{L}_\omega^{(n)} \circ E_{\mathcal{P}}$. The second term has finite rank and Lemma 24 guarantees that $\operatorname{var}(\mathcal{L}_\omega^{(n)} \circ (1 - E_{\mathcal{P}})f) \le 3a_n(\omega) \operatorname{var} f$. Since $\mathcal{L}_\omega^{(n)}$ preserves integrals and $(1 - E_{\mathcal{P}})f$ has integral 0, it follows that $\mathcal{L}_\omega^{(n)} \circ (1 - E_{\mathcal{P}})f$ has integral 0 and therefore that the L^1 norm is bounded above by the variation. This yields $\|\mathcal{L}_\omega^{(n)} \circ (1 - E_{\mathcal{P}})\| \le 3a_n(\omega)$ so that $\|\mathcal{L}_\omega^{(n)}\|_{\mathrm{fr}} \le 3a_n(\omega)$.

For the lower bound fix an $\epsilon > 0$ and suppose that $1/|T_\omega^{(n)\prime}(x)| > (1 - \epsilon)a_n(\omega)$ for x in an interval J. Suppose further that J lies in a single branch of $T_\omega^{(n)}$. Let I and I' be two subintervals of J, with no endpoints in common and let $f_I = \frac{1}{2}\mathbf{1}_I$ and $f_{I'} = \frac{1}{2}\mathbf{1}_{I'}$. Then we have $\|\mathcal{L}_\omega^{(n)} f_I - \mathcal{L}_\omega^{(n)} f_{I'}\| > 2(1 - \epsilon)a_n(\omega)$. It follows that no $(1 - \epsilon)a_n(\omega)$ ball contains more than two $\mathcal{L}_\omega^{(n)} f_I$'s with distinct endpoints and so in particular $\mathcal{L}_\omega^{(n)} B_{\mathrm{BV}}$ does not have a finite cover by $(1 - \epsilon)a_n(\omega)$ balls. We see that $\left\|\mathcal{L}_\omega^{(n)}\right\|_{\mathrm{ic}} \ge (1 - \epsilon)a_n(\omega)$. Since ϵ is arbitrary, we see that (13) follows.

Taking nth roots and taking the limit, the theorem follows. □

We now demonstrate that $\lambda^* = 0$. As a Perron–Frobenius operator $\mathcal{L}_\omega^{(n)}$ is a stochastic operator for each $\omega \in \Omega$, for any density $0 \not\equiv f \in \mathrm{BV}$ we have $\|\mathcal{L}_\omega^{(n)} f\| \geq \|\mathcal{L}_\omega^{(n)} f\|_1 = \|f\|_1$, which shows that $\lambda(\omega) \geq 0$. To show $\lambda^* \leq 0$, since $\|\mathcal{L}_\omega\|_1 \leq 1$, it suffices to consider the growth of the variation of $\mathcal{L}_\omega^{(n)} f$. As $\chi < 1$, for almost every $\omega \in \Omega$ there exists $n \in \mathbb{N}$, $0 < \alpha < 1$ and $\beta \geq 0$ such that $\operatorname{var} \mathcal{L}_\omega^{(n)} f \leq \alpha \operatorname{var} f + \beta \|f\|_1$ by Lemma 24. Iterating this inequality gives a bound for the sequence $(\operatorname{var} \mathcal{L}_\omega^{(kn)})_{k\in\mathbb{N}}$, and so $\liminf_{k\to\infty} (1/(nk)) \log \|\mathcal{L}_\omega^{(nk)}\| \leq 0$. As $\lim_{n\to\infty} (1/n) \log \|\mathcal{L}_\omega^{(n)}\|$ exists for $\mathbb{P}$-almost every ω, we have $\lambda^* \leq 0$.

Corollary 28. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, BV, \mathcal{L})$ be a Rychlik random dynamical system. Assume that $\mathcal{R}$ is expanding-on-average. Then $\mathcal{R}$ is quasi-compact, with*

$$\kappa^* = \lim_{n\to\infty} \frac{1}{n} \log \left(1/ \operatorname*{ess\,inf}_x \left| T_\omega^{(n)\prime}(x) \right| \right) < 0 = \lambda^* \text{ for } \mathbb{P}\text{-almost every } \omega.$$

The random dynamical system therefore admits a $\mathbb{P}$-continuous Oseledets splitting.

The Oseledets splitting provides information on the invariant measures and rates of mixing of the random system. A natural generalisation of the notion of 'invariant measure' to the random setting is the concept of 'sample measure'. A family $\{\mu_\omega\}_{\omega\in\Omega}$ of *sample measures* (see [2]), is a family of probability measures μ_ω on I satisfying

(1) for all $U \in \mathcal{F}$, the map $\omega \mapsto \mu_\omega(U)$ is $\mathcal{F}$-measurable.
(2) $T_\omega \mu_\omega = \mu_{\sigma\omega}$ for a. e. $\omega \in \Omega$.

Given a family $\{\mu_\omega\}_{\omega\in\Omega}$ of sample measures, the measure μ on $\Omega \times I$ given by $\mathrm{d}\mu(\omega, x) := \mathrm{d}\mu_\omega(x)\mathrm{d}\mathbb{P}(\omega)$ is an invariant probability for the associated skew product $\Theta(\omega, x) = (\sigma\omega, T_\omega x)$. Conversely, any Θ-invariant probability measure μ with marginal $\mathbb{P}$ on Ω may be disintegrated to give a family of sample measures for the original system.

Sample measures for random compositions of expanding interval maps have previously been studied by Pelikan [23], Morita [20], and in a more general setting by Buzzi [6]. He considers random compositions of Lasota–Yorke maps that have neither too many branches nor too large distortion, and proves that the associated skew product transformation possesses a finite number of mutually singular ergodic ACIPs μ, each giving a family $\{\mu_\omega\}_{\omega\in\Omega}$ of sample measures with densities of bounded variation. Returning to the present setting of a random composition of Rychlik maps, any such family $\{f_\omega\}_{\omega\in\Omega}$ of sample measures with densities of bounded variation satisfies $\mathrm{d}\mu_\omega/\mathrm{d}m \in E_1(\omega)$ for $\mathbb{P}$-almost every ω. It follows that the number of such mutually singular ergodic ACIPs (whose sample measure densities are necessarily linearly independent for $\mathbb{P}$-a. e. ω) is bounded by d_1, the dimension of the Oseledets subspace $E_1(\omega)$.

Furthermore, the exceptional Lyapunov spectral values strictly less than 0, and their corresponding Oseledets subspaces, provide information on exponential decay rates that are slower than the decay produced by local separation of trajectories. The authors discuss and provide examples of such spectral values and Oseledets subspaces in [10]. Corollary 28 provides conditions under which Oseledets subspaces exist in much greater generality than in [10], removing the assumptions of piecewise linearity and Markovness, and allowing the system to be expanding on average. In non-rigorous numerical experiments, Oseledets subspaces have been shown to effectively capture so-called "coherent sets" in aperiodic fluid flow [11]. The present work represents a first step toward making such calculations rigorous by extending the study of Perron–Frobenius operator cocycles to Banach spaces that are more representative of fluid flow.

4.2. **Application II: Transfer Operators with Random Weights.** Let Σ be a one-sided 1-step shift of finite type on N symbols. We assume that for each symbol j in the alphabet there is at least one i for which ij is a legal transition (if not we restrict our attention to the subset of Σ obtained by deleting all symbols that have no preimage). For $x, y \in \Sigma$ we let $\Delta(x, y)$ be $\min\{n\colon x_n \neq y_n\}$ (or ∞ if $x = y$). The θ-metric on Σ is $d_\theta(x, y) = \theta^{\Delta(x,y)}$ (so that the standard metric is $d_{1/2}$).

We will write S for the usual left shift map on Σ. If $x \in \Sigma$ and v is a word of some length $k \geq 1$ in the alphabet such that $v_{k-1}x_0$ is a legal transition then we will write vx for the point in $S^{-k}x$ obtained by concatenating v and x.

Let $\mathcal{C}_\theta$ denote the set of θ-Lipschitz functions: those functions f for which there is a C such that $|f(x) - f(y)| \leq Cd_\theta(x, y)$ for all x and y. We define $|f|_\theta$ to be the smallest C for which such an inequality holds. As usual we endow $\mathcal{C}_\theta$ with the topology generated by the norm $\|f\|_\theta = \max(|f|_\theta, \|f\|_\infty)$. Let $\mathcal{W}_\theta$ be the collection of those functions g in $\mathcal{C}_\theta$ such that $\min_x g(x) > 0$.

Denote $P_g f(x) = \sum_{y\in S^{-1}x} f(y)g(y)$ and consider P_g as an operator on $(\mathcal{C}_\theta, \|\cdot\|_\theta)$. For the purposes of the following lemma we consider arbitrary $g \in \mathcal{C}_\theta$ but we shall later restrict to $g \in \mathcal{W}_\theta$.

Lemma 29. *The map $P\colon \mathcal{C}_\theta \to L(\mathcal{C}_\theta, \mathcal{C}_\theta)$ is continuous with respect to the operator norm on $L(\mathcal{C}_\theta, \mathcal{C}_\theta)$.*

Proof. P is clearly linear. Let $g \in \mathcal{C}$. We want to bound $\|P_g f\|_\theta$. We first estimate $\|P_g f\|_\infty$. Let $x \in \Sigma$. Then $|P_g f(x)| \leq \sum_{y\in\sigma^{-1}x} |f(y)| \cdot |g(y)| \leq N\|f\|_\infty\|g\|_\infty \leq N\|f\|_\theta\|g\|_\theta$. This yields

$$\|P_g f\|_\infty \leq N\|f\|_\theta\|g\|_\theta. \tag{14}$$

We now bound $|P_g f|_\theta$. Let $x \neq y \in \Sigma$. We need to estimate $|P_g f(x) - P_g f(y)|/d_\theta(x, y)$. If $x_0 \neq y_0$ then the denominator is 1 and the numerator is at most $2N\|g\|_\infty\|f\|_\infty \leq 2N\|g\|_\theta\|f\|_\theta$. If $x_0 = y_0$ then

$$\begin{aligned} |P_g f(x) - P_g f(y)| &= \sum_{\{i\colon ix_0 \text{ legal}\}} \big(g(ix)f(ix) - g(iy)f(iy)\big) \\ &\leq N(\|g\|_\infty|f|_\theta + \|f\|_\infty|g|_\theta)d_\theta(ix, iy) \\ &\leq 2N\|g\|_\theta\|f\|_\theta\theta d_\theta(x, y). \end{aligned}$$

Combined with the estimate in the case $x_0 \neq y_0$, this shows $|P_g f|_\theta \leq 2N\|g\|_\theta \cdot \|f\|_\theta$ and so $\|P_g\| \leq 2N\|g\|_\theta$. ☐

Baladi's book [3] contains a number of detailed calculations of the spectral radii and essential spectral radii of Perron–Frobenius operators acting on the Lipschitz spaces. We now develop some of these arguments in the case of random compositions.

Suppose that $G\colon \Omega \mapsto \mathcal{W}_\theta;\ \omega \mapsto g_\omega$ is a continuous mapping. Since Ω will be assumed to be compact there will be a constant γ such that $g_\omega(x) \geq \gamma$ for all $x \in \Sigma$ and $\omega \in \Omega$. Similarly there will be a constant such that $\|g_\omega\|_\theta \leq C$ for all $\omega \in \Omega$. We assume as usual that $\sigma\colon \Omega \to \Omega$ is ergodic. We write $P^{(n)}_\omega$ for the composition of Perron-Frobenius operators $P_{g_{\sigma^{n-1}\omega}} \circ \cdots \circ P_{g_\omega}$.

A linear map on $\mathcal{C}_\theta$ is said to be *positive* if it maps non-negative functions to non-negative functions. In particular if $g \in \mathcal{W}_\theta$ then P_g is positive.

Lemma 30. *Let $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{C}_\theta, P)$ be a continuous ergodic random dynamical system of Perron-Frobenius operators with random weights on a shift of finite type Σ. Suppose that Ω is compact and $P\colon \Omega \to \mathcal{W}_\theta$ is continuous. Let $R_n(\omega) = \|P_\omega^{(n)}\mathbf{1}\|_\infty$. Then $R_n(\omega)^{1/n}$ converges $\mathbb{P}$-almost everywhere to a constant R^*.*

Proof. Since the operators P_g are positive (g being positive), we have $P_\omega^{(n+m)}\mathbf{1} = P_{\sigma^m\omega}^{(n)}(P_\omega^{(m)}\mathbf{1}) \leq P_{\sigma^m\omega}^{(n)} R_m(\omega)\mathbf{1} \leq R_n(\sigma^m\omega)R_m(\omega)\mathbf{1}$. It follows that $\log R_n(\omega)$ is a subadditive sequence of functions so that by the subadditive ergodic theorem, for $\mathbb{P}$-almost all ω, $R_n(\omega)^{1/n}$ converges to a quantity $R(\omega)$. Since this quantity is σ-invariant, there is a constant R^* such that $R(\omega) = R^*$ for $\mathbb{P}$-almost every $\omega \in \Omega$. □

Lemma 31 (Bounded Distortion)**.** *Let $\mathcal{R}$ be as in the previous lemma. Let $g_\omega^{(n)}(x)$ denote the product $g_\omega(x)g_{\sigma(\omega)}(Sx)\ldots g_{\sigma^{n-1}\omega}(S^{n-1}x)$. There exists a $D > 0$ such that for all $\omega \in \Omega$, if $x_0 = y_0$ and v is a word of an arbitrary length k such that $v_{k-1}x_0$ is a legal transition then*

$$\left|1 - \frac{g_\omega^{(k)}(vy)}{g_\omega^{(k)}(vx)}\right| \leq Dd_\theta(x, y).$$

Proof. As mentioned above there is a $\gamma > 0$ such that $g_\omega(x) \geq \gamma$ for all $\omega \in \Omega$ and all $x \in \Sigma$. Similarly there is a Γ such that $g_\omega(x) \leq \Gamma$ for all ω and x and also a C such that $\|g_\omega\|_\theta \leq C$ for all $\omega \in \Omega$. We make use of the fact that there exists a constant K such that if $\gamma < a, b < \Gamma$ then $|\log(b/a)| \leq K|b - a|$.

We then have

$$\begin{aligned}
\left|\log \frac{g_\omega^{(k)}(vy)}{g_\omega^{(k)}(vx)}\right| &\leq \sum_{j=0}^{k-1} \left|\log \frac{g_{\sigma^j\omega}(S^j(vy))}{g_{\sigma^j\omega}(S^j(vx))}\right| \\
&\leq K \sum_{j=0}^{k-1} C d_\theta(S^j(vx), S^j(vy)) \\
&= CK \sum_{j=0}^{k-1} \theta^{k-j} d_\theta(x, y) \\
&\leq (CK/(1-\theta))d_\theta(x, y),
\end{aligned}$$

Exponentiating we see $g_\omega^{(k)}(vy)/g_\omega^{(k)}(vx)$ lies between the values $\exp(-rd_\theta(x, y))$ and $\exp(rd_\theta(x, y))$ where $r = CK/(1-\theta)$. Since exp is Lipschitz on $[-r, r]$ there exists a D such that $|\exp(t) - 1| \leq (D/r)|t|$ on $[-r, r]$. It follows that

$$\left|\frac{g_\omega^{(k)}(vy)}{g_\omega^{(k)}(vx)} - 1\right| \leq Dd_\theta(x, y)$$

as required. □

The next lemma appears as an exercise in the deterministic case in Baladi's book [3].

Lemma 32. *Let $\mathcal{R}$ be as above. Then there exists a constant K such that for $f \in \mathcal{C}_\theta$*

$$|P_\omega^{(n)} f|_\theta \leq R_n(\omega)(\theta^n |f|_\theta + K\|f\|_\infty).$$

Proof. We need to estimate $\sup_{x\neq y} |P_\omega^{(n)} f(x) - P_\omega^{(n)} f(y)|/d_\theta(x,y)$. If x and y differ in the zeroth coordinate, the denominator is 1 and we bound the numerator above by $R_n(\omega)\|f\|_\infty$ giving a bound of the given form (with K=1).

If x and y agree in the zeroth coordinate then we estimate as follows. We let W_n be the set of words v of length n such that $v_{n-1}x_0$ is legal.

$$\begin{aligned}
&|P_\omega^{(n)} f(x) - P_\omega^{(n)} f(y)| \\
&= \left| \sum_{v \in W_n} \left(g_\omega^{(n)}(vx) f(vx) - g_\omega^{(n)}(vy) f(vy) \right) \right| \\
&\leq \sum_{v\in W_n} g_\omega^{(n)}(vx) \cdot |f(vx) - f(vy)| + \sum_{v\in W_n} |f(vy)| \cdot |g_\omega^{(n)}(vx) - g_\omega^{(n)}(vy)| \\
&\leq \sum_{v\in W_n} g_\omega^{(n)}(vx) \left(|f|_\theta d_\theta(vx,vy) + \|f\|_\infty \left| 1 - \frac{g_\omega^{(n)}(vy)}{g_\omega^{(n)}(vx)} \right| \right) \\
&\leq R_n(\omega) \left(|f|_\theta \theta^n d_\theta(x,y) + \|f\|_\infty D d_\theta(x,y) \right).
\end{aligned}$$

We therefore see that $|P_\omega^{(n)} f|_\theta \leq R_n(\omega) \left(\theta^n |f|_\theta + D\|f\|_\infty\right)$ as required. □

Let $n > 0$ and let $[w_1], \ldots, [w_k]$ be an enumeration of the n-cylinders. For each $1 \leq j \leq k$, let x_j be a point of $[w_j]$. Given these choices, define a finite rank operator $\Pi_n : \mathcal{C}_\theta \to \mathcal{C}_\theta$ by

$$(\Pi_n f)(x) = f(x_j) \text{ for } x \in [w_j].$$

Lemma 33. *For $f \in \mathcal{C}_\theta$ and Π_n as above we have*

$$\begin{aligned}
&\|(I - \Pi_n)f\|_\infty \leq \theta^n |f|_\theta \\
&|(I - \Pi_n)f|_\theta \leq \max(2\theta, 1)|f|_\theta.
\end{aligned}$$

Proof. Let $Q: \mathcal{C}_\theta \to \mathcal{C}_\theta$ denote $I - \Pi_n$. Let $x \in [w_j]$. Then $Qf(x) = f(x) - f(x_j)$. Since $\Delta(x, x_j) \geq n$, we have $|Qf(x)| \leq |f|_\theta \theta^n$.

Now let $x, y \in \Sigma$. If they lie in the same n-cylinder set then $|Qf(x) - Qf(y)| = |f(x) - f(y)| \leq |f|_\theta d_\theta(x,y)$. On the other hand if x and y lie in distinct n-cylinders, $[w_i]$ and $[w_j]$ respectively then we have

$$\begin{aligned}
|Qf(x) - Qf(y)| &= |(f(x) - f(x_i)) - (f(y) - f(x_j))| \\
&\leq |f(x) - f(x_i)| + |f(y) - f(x_j)|.
\end{aligned}$$

Since $\Delta(x, x_i)$ and $\Delta(y, x_j)$ are each at least n the right side is bounded above by $2|f|_\theta \theta^n \leq (2\theta)|f|_\theta d_\theta(x,y)$. □

Theorem 34. *Let $\mathcal{R}$ be as above. Then there are $C_1 > 0$ and $C_2 > 0$ such that*

$$R_n(\omega) \leq \|P_\omega^{(n)}\| \leq C_1 R_n(\omega)$$
$$(1/4)\theta^n R_n(\omega) \leq \|P_\omega^{(n)}\|_{ic} \leq C_2 \theta^n R_n(\omega).$$

In particular $\lambda(\omega) = R^*$ and $\kappa(\omega) = \theta R^*$ for $\mathbb{P}$-almost every ω so that the random dynamical system is quasi-compact.

Proof. Let K be as in Lemma 32. Let $f \in \mathcal{C}_\theta$. We have $|P_\omega^{(n)} f|_\theta \le (K+1)R_n(\omega)\|f\|_\theta$. Also $P_\omega^{(n)} f \le P_\omega^{(n)}(\|f\|_\infty \mathbf{1}) \le R_n(\omega)\|f\|_\infty$. Combining these we see that $\|P_\omega^{(n)} f\|_\theta \le (K+1)R_n(\omega)\|f\|_\theta$.

On the other hand we have $\|P_\omega^{(n)}\mathbf{1}\|_\theta \ge R_n(\omega)$ while $\|\mathbf{1}\|_\theta = 1$ so the bounds on $\|P_\omega^{(n)}\|$ are established.

For the upper bound on $\|P_\omega^{(n)}\|_{\mathrm{ic}}$ we use Lemma 25 to compare with $\|P_\omega^{(n)}\|_{\mathrm{fr}}$ and we let Π_n be as above and give bounds on $\|P_\omega^{(n)} \circ (I - \Pi_n)\|$. Let $f \in \mathcal{C}_\theta$. We have $\|P_\omega^{(n)} \circ (I-\Pi_n) f\|_\infty \le R_n(\omega)\|(I-\Pi_n)f\|_\infty \le \theta^n R_n(\omega)|f|_\theta$ where the last inequality made use of Lemma 33. Using Lemmas 32 and 33 we see $|P_\omega^{(n)}((I-\Pi_n)f)|_\theta \le R_n(\omega)(\theta^n \max(2\theta, 1)|f|_\theta + K\theta^n\|f\|_\infty)$. Combining these two inequalities leads to an upper bound of the desired form for $\|P_\omega^{(n)}\|_{\mathrm{ic}}$.

For the lower bound on $\|P_\omega^{(n)}\|_{\mathrm{ic}}$ there exists (by continuity) an open set U on which $P_\omega^{(n)}\mathbf{1}(x) > R_n(\omega)/2$. We show that the index of compactness is large by exhibiting an infinite collection of points in the unit sphere of $\mathcal{C}_\theta$ whose images under $P_\omega^{(n)}$ are uniformly separated.

Let $u \in U$ and let C_k be the k-cylinder about u. Since U is open there exists a k_0 such that $C_{k_0} \subset U$. Since Σ is an irreducible shift of finite type there exists an infinite sequence $k_0 < k_1 < k_2 < \ldots$ such that C_{k_i} is a proper subset of $C_{k_i - 1}$ for all $i \ge 1$. We let $f_i = \theta^{k_i+n-1}\mathbf{1}_{C_{k_i}} \circ S^n$. To check that $\|f_i\|_\theta = 1$ we note that if x and y agree on at least the first $k_i + n$ symbols then $f_i(x) = f_i(y)$. Since the numerator of $|f_i(x) - f_i(y)|/\theta^{\Delta(x,y)}$ takes only the values 0 and θ^{k_i+n-1} the maximum in this expression is obtained by taking x and y that agree for as many symbols as possible, but for which $f_i(x) \ne f_i(y)$. By the assumption on Σ and choice of k_i there are points agreeing for $k_i + n - 1$ symbols but disagreeing on the $k_i + n - 1$st symbol for which $f_i(x) \ne f_i(y)$ so that $\|f_i\|_\theta = 1$ as required.

We then calculate

$$\begin{aligned} P_\omega^{(n)} f_i(x) &= \sum_{\{w_0^{n-1}\colon\, w_{n-1}x_0 \text{ is legal}\}} g_\omega^{(n)}(wx)\theta^{k_i+n-1}\mathbf{1}_{C_{k_i}} \circ S^n(wx) \\ &= \theta^{k_i+n-1}\mathbf{1}_{C_{k_i}}(x) P_\omega^{(n)}\mathbf{1}(x). \end{aligned}$$

Letting $h = P_\omega^{(n)} f_i - P_\omega^{(n)} f_j$, we have $h = (\theta^{k_i+n-1}\mathbf{1}_{C_{k_i}} - \theta^{k_j+n-1}\mathbf{1}_{C_{k_j}})P_\omega^{(n)}\mathbf{1}$. Let $i < j$ and pick $x \in C_{k_j-1}\setminus C_{k_j}$ and $y \in C_{k_i-1}\setminus C_{k_i}$. Then we have $\Delta(x,y) = k_i - 1$, $h(x) = \theta^{k_i+n-1}P_\omega^{(n)}\mathbf{1}(x) \ge (1/2)\theta^{k_i+n-1}R_n(\omega)$ whereas $h(y) = 0$ giving $\|h\|_\theta \ge (1/2)\theta^n R_n(\omega)$. It follows that no ball of radius less than $(1/4)\theta^n R_n(\omega)$ can contain two $P_\omega^{(n)} f_i$'s and so $\|P_\omega^{(n)}\|_{\mathrm{ic}} \ge (1/4)\theta^n R_n(\omega)$.

□

Example 35. Let $\sigma\colon \Omega \to \Omega$ be any homeomorphic dynamical system defined on a compact space Ω preserving an ergodic probability measure $\mathbb{P}$. Let $\Sigma = \{0,1\}^{\mathbb{Z}^+}$. Fix $0 < \theta < 1$ and let $\{h_\omega : \omega \in \Omega\}$ be a continuously-parameterized family of antisymmetric monotonic elements of $\mathcal{C}_\theta(\Sigma)$, where a function is *antisymmetic* if it satisfies $h(\bar{x}) = -h(x)$ for $x \in \Sigma$, where $\bar{x}_i = 1 - x_i$. A function will be called *monotonic* if it satisfies $h(x) \le h(y)$ whenever $x \preceq y$, where $x \preceq y$ means $x_i \le y_i$ for each i.

We will assume that $\|h_\omega\|_\infty < a < 1/2$ for all $\omega \in \Omega$. We then define a continuously parameterized family of elements g_ω of $\mathcal{W}_\theta$ by

$$g_\omega(1x) = \tfrac{1}{2} + h_\omega(x)$$
$$g_\omega(0x) = \tfrac{1}{2} - h_\omega(x).$$

From the choice of g_ω, we see that $P_\omega \mathbf{1} = \mathbf{1}$ for all ω so that $R^* = 1$ and $\lambda(\omega) = 1$ for a.e. ω and hence $\kappa(\omega) = \theta$ for a.e. ω by Theorem 34.

One can verify that the P_ω map antisymmetric functions to antisymmetric functions and monotonic functions to monotonic functions. Following Liverani [18] we define a cone $\mathcal{K}_a = \{f\colon f(x) > 0,\ \forall x; f(x)/f(y) \le e^{ad_\theta(x,y)},\ \forall x, y\}$. For suitably large a, there is $a' < a$ such that $P_\omega(\mathcal{K}_a) \subset \mathcal{K}_{a'}$. Since $\mathbf{1}$ is a fixed point the theory of cones guarantees that if f is a positive function in $\mathcal{C}_\theta$, then $P_\omega^{(n)} f$ converges at an exponential rate to a constant uniformly in ω. In particular an antisymmetric function f can be written as the difference of two positive functions: $f_1 - f_2$. Since $P_\omega^{(n)} f_1$ converges exponentially fast to a constant $C_1(\omega)$ and $P_\omega^{(n)} f_2$ converges exponentially to a constant $C_2(\omega)$, the fact that $P_\omega^{(n)} f$ remains antisymmetric implies that $C_1(\omega) = C_2(\omega)$. It follows that $P_\omega^{(n)} f$ converges at an exponential rate to 0 uniformly over $\omega \in \Omega$.

Choosing $f(x) = \mathbf{1}_{[1]} - \mathbf{1}_{[0]}$, f is both monotone and antisymmetric. It follows that $P_\omega^{(n)} f$ decays exponentially. We are able to give a lower bound on the decay rate that guarantees the presence of non-trivial exceptional spectrum. Specifically, using the fact that $g_\omega(0x) + g_\omega(1x) = 1$, we have

$$\begin{aligned} P_\omega f(1111\ldots) &= g_\omega(1111\ldots)f(1111\ldots) + g_\omega(0111\ldots)f(0111\ldots) \\ &= g_\omega(1111\ldots)f(1111\ldots) + (1 - g_\omega(1111\ldots))f(0000\ldots) \\ &= (2g_\omega(1111\ldots) - 1)f(1111\ldots). \end{aligned}$$

If the h_ω are chosen in such a way that $g_\omega(1111\ldots)$ is uniformly close to 1 as ω varies then we will ensure that there is non-trivial exceptional spectrum.

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