STATISTICAL PROPERTIES OF INVARIANT GRAPHS IN PIECEWISE AFFINE DISCONTINUOUS FORCED SYSTEMS

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ABSTRACT. When a contractive map is forced by a chaotic discontinuous system, the asymptotic response function that defines the attracting invariant set can be highly irregular. In this context, it is natural to ask whether the invariant distributions of the base and factor systems share the same characteristics and in particular, whether the factor distribution of an absolutely continuous measure in the base can be absolutely continuous. Here, we address this question in a basic example of linear real contractions forced by (generalized) baker's maps and we prove absolute continuity for almost every value of the factor contraction rate.

1. INTRODUCTION

Generalized synchronization is an essential feature of forced dissipative systems [10]. This term refers to the fact that the asymptotic dynamics of (discrete time) skew-product systems in Banach spaces with dissipative factor, *i.e.*

$$(w_{n+1}, z_{n+1}) = (F(w_n), G(w_n, z_n)) \quad \text{where} \quad \|G(w,z) - G(w,z')\| \leq \lambda \|z - z'\|$$

for some $\lambda < 1$ independent of w, is globally attracted by an invariant graph in phase space composed of points $(w, h(w))$. In other words, for arbitrary z_0, the iterated variable z_n approaches $h(w_n)$ uniformly in w_0 as $n \to \infty$. The corresponding synchronization function h, which conjugates the skew-product on the invariant graph to its base system F, naturally depends both on this forcing and on the factor G.

A standard problem in this context is to analyze the properties of the synchronization function. One aims to identify whose features of the forcing dynamics that carry over to the response system; see *e.g.* the introduction in [15] for a fairly complete overview. In the slightly different framework of the theory of inertial manifolds in dynamical systems, Hirsch, Pugh and Shub [6, 7] proved in early work that in the case where F is an homeomorphism, h turns out to be continuous and can be Hölder or Lipschitz continuous depending upon additional assumptions on the Lyapunov exponents. Proofs of these properties and improvements in the present context were given later on, see [1, 4, 15, 17]. In addition, Stark proved that h is smooth when F is a diffeomorphism [14, 15] and extended these results to non-uniform contractive responses (see also [8, 16] for recent developments).

Date: May 26, 2011.

BF is supported by the EU Marie Curie fellowship PIOF-GA-2009-235741.

AQ is supported by NSERC..

The authors thank Boris Solomyak for helpful conversations. This work grew out from discussions during the conference "Nonlinear Dynamics: New directions" in Guanajuato (MX) in May 2010. We are grateful to the organizers for inviting us there.

On the other hand, Afraimovich, Cordonnet and Chazottes [1] noticed that when the forcing is not continuous and the response is sensitive to discontinuities, h may fail to be continuous. Beside being of theoretical interest on their own, discontinuous forcing terms emerge in a variety of applications, especially in the modeling of low-pass filters in signal analysis [2, 3, 9]. In this case, the synchronization function could become singular in a measure-theoretic sense.

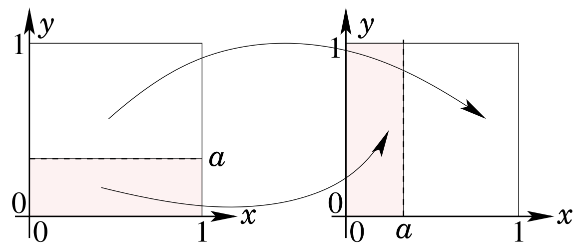

FIGURE 1. Sketch of the (generalized) baker's map F_a from the unit square into itself.

To the best of our knowledge, no study has been reported in the literature where properties of response statistical distributions have been described in the case of discontinuous synchronization functions. This paper investigates this issue in a basic example of discontinuous system. Assuming that the forcing is given by the iterations $(x_{n+1}, y_{n+1}) = F_a(x_n, y_n)$ ($n \in \mathbb{Z}$ and (x_n, y_n) plays the role of w_n here) of the generalized baker's transformation on the unit square $[0,1]^2$ (see Figure 1):

$$F_a(x,y) = \begin{cases} (ax, y/a) & \text{if } 0 \le y < a \\ (a + (1-a)x, (y-a)/(1-a)) & \text{if } a \le y < 1 \end{cases}$$

where $0 < a < 1$, we study the invariant distribution of the response variable in the case where

$$G(x,y,z) = x + \lambda z$$

depends on the forcing variable only via the first coordinate x.

Since F_a is invertible and the anterior value of x of the preimage of (x,y) is determined by x alone, the synchronization function h only depends on x and its explicit expression is given by [1]

$$h(x) = \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x), \; x \in [0,1] \tag{1}$$

where

$$T_a(x) = \begin{cases} \frac{x}{a} & \text{if } \; 0 \le x < a \\ \frac{x-a}{1-a} & \text{if } \; a \le x \le 1 \end{cases}$$

This function h appears to be extremely choppy (see examples in Figure 2). Indeed, there is no interval where it is monotonic and the function is discontinuous with negative jumps $h(x) < h(x-0)$ for every x in the (dense) set of pre-images of a under iterations of T_a. Yet, the image $h([0,1])$ coincides with the interval $[h(0), h(1)]$.

In [5], the first author analyzes properties of the function h. In particular, it is shown that the total variation is finite iff $\lambda < 1/2$. Moreover, when h is of bounded variation, a decomposition into the difference $h_c - h_d$ of increasing functions, where

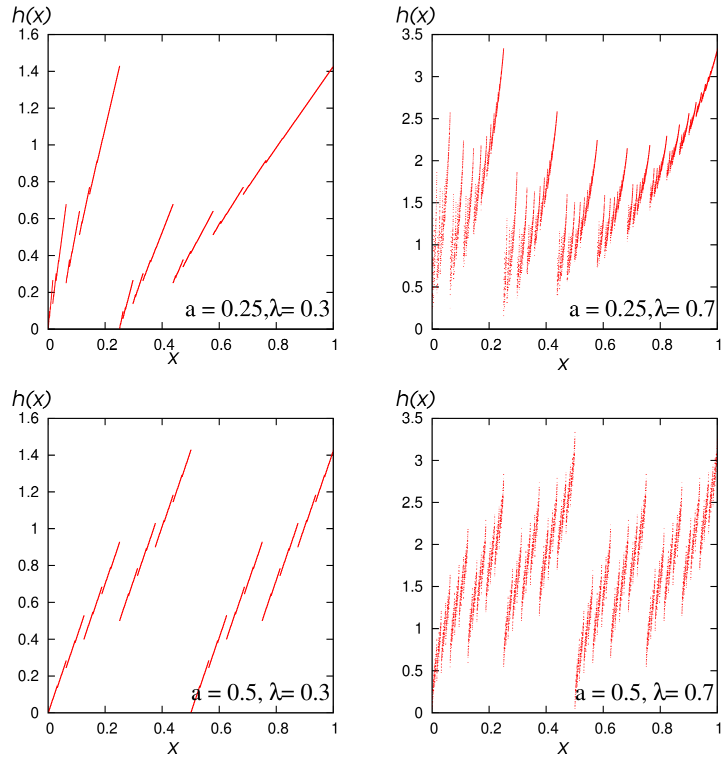

FIGURE 2. Examples of graph of h for $a = 0.25$ (top pictures) and $a = 0.5$ (bottom pictures). In the left pictures ($\lambda = 0.3 < 1/2$), h is of bounded variation; in the right ones ($\lambda = 0.7 > 1/2$), it has infinite variation. Notice the symmetry $h(x) = 1 - h(1 - x - 0)$ for $a = 0.5$.

in addition h_c is continuous and h_d is a step function, is provided. (This is the only result of [5] that is being used here, see Lemma 3 below.)

To complete these results, we would like to know here whether the distribution of the (z_n) inherits the absolute continuity of the system to which it is synchronized. It is well known that two-dimensional Lebesgue measure is an (absolutely continuous) invariant measure for T_a. Accordingly we ask whether the measure Leb $\circ\, h^{-1}$ is absolutely continuous where Leb denotes the Lebesgue measure on $[0, 1]$.

Notice that while h can be regarded as the uniform limit of non-singular functions - take the approximation h_n obtained by truncating the series in (1) to the first $n+1$ terms - this property does not suffice to guarantee that h is non singular. For instance a sequence of linear functions on a bounded interval with slopes tending to zero provides a counter-example.

To address absolute continuity of Leb$\circ h^{-1}$, we need to separate the cases $\lambda < 1/2$ and $\lambda \geq 1/2$. In the first case, we use that h is of bounded variation to show that it has finitely many pre-images almost everywhere and to obtain lower bounds on the measure of specific neighborhoods. In that way we obtain the following statement.

Theorem 1. *For every $a \in (0,1)$ and every $\lambda < 1/2$, the push-forward distribution Leb $\circ\, h^{-1}$ is absolutely continuous with density given* a.e. *by $x \mapsto \sum_{y\in h^{-1}(x)} \frac{1}{h'(y)}$.*

In the case $\lambda \geq 1/2$, we rely on the symbolic dynamics of the map T_a to rewrite h as a random series and to regard the distribution Leb $\circ\, h^{-1}$ as a Bernoulli convolution. Then we adapt the proof in [11] to show absolute continuity for *a.e.* λ. Despite that Leb $\circ\, h^{-1}$ can be expressed as an infinite convolution for arbitrary value of a, we were only able to obtain the required estimates in the case where $a = 1/2$. As a result, the conclusion in this case writes as follows.

Theorem 2. *Assume that $a = 1/2$. Then Leb $\circ\, h$ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$.*

2. The Case $\lambda < 1/2$

In this section, we deal with functions of bounded variation and take advantage of their multiple properties. Recall in particular that any discontinuity is necessarily of jump type, *i.e.* the left and right limits exist at every point.

In addition, we only consider functions that are right continuous, so that they are *càdlàg*. The jump of a càdlàg function f that occurs at x is denoted by $J_f(x) = f(x) - f(x-0)$. A càdlàg function f on an interval $[a,b]$ is said to be a *step function* if

$$f(x) = f(a) + \sum_{a<t\leq x} J_f(t), \quad \forall x \in [a,b]$$

viz. if all changes in the value of f as the argument changes may be accounted for by the jumps.

Any càdlàg function f of bounded variation has a unique decomposition into three parts: $f = f_d + f_c + f_s$, a step function f_d, an absolutely continuous part f_c and a singular continuous part f_s. In our case, when $\lambda < 1/2$ the synchronization function h defined by (1) is of bounded variation and has no singular continuous part in its decomposition:

Lemma 3. *When $\lambda < 1/2$, we have $h(x) = h_c(x) - h_d(x)$ for all $x \in [0,1]$ where h_c is strictly increasing and absolutely continuous and h_d is a strictly increasing step function with a dense set of discontinuities.*

Proof. We already know from [5] that h_d is a strictly increasing step function with a dense set of discontinuities and h_c is strictly increasing and continuous. There remains only to prove that h_c is absolutely continuous.

Consider the approximation h_n obtained by truncating the series in (1) to the first $n+1$ terms. The function h_n has a similar decomposition $h_{n,c} - h_{n,d}$ where the continuous part $h_{n,c}$ consists of finitely many affine pieces; hence it is absolutely continuous. By the fundamental theorem of calculus, we have

$$h_{n,c}(x) - h_{n,c}(0) = \int_0^x h'_{n,c}(y)dy, \quad \forall x \in [0,1] \tag{2}$$

where the derivative is given by

$$h'_{n,c}(x) = \sum_{t=0}^{n} \lambda^t (T_a^{t+1}(x))', \quad \text{for a.e. } x.$$

By monotony, each sequence $\{h'_{n,c}(x)\}$ converges as $n \to \infty$ to

$$h'_c(x) = \sum_{t=0}^{\infty} \lambda^t (T_a^{t+1}(x))'.$$

Moreover, each $h_{n,c}(x)$ converges to $h_c(x)$ (and $h_{n,c}(0) = h_c(0) = 0$). Thus relation (2) implies that each sequence $\{\int_0^x h'_{n,c}(y)dy\}$ is bounded. By Lebesgue's monotone convergence theorem, we conclude that h'_c is integrable over $[0,1]$ and we have

$$h_c(x) = \int_0^x h'_c(y)dy, \quad \forall x \in [0,1]$$

which implies that h_c is absolutely continuous. □

2.1. **Proof of Theorem 1.** The crucial point is to obtain a lower bound for the Lebesgue measure of the pre-image $h^{-1}U$ of small neighborhoods U of points in an appropriate subset of $h([0,1])$ with full measure. In order to select this subset, we need a series of preliminary results.

1st step: Selecting a suitable subset in $h([0,1])$.
Image measure of sets with zero measure. Since h is of bounded variation, it has a finite derivative a.e. Since h_d is a step function, its derivative vanishes almost everywhere and thus $h' = h'_c$ a.e. Let E be the exceptional set of points where either the derivative of h fails to exist or we have $h' \neq h'_c$. The following statement guarantees that $h(E)$ has zero Lebesgue measure.

Lemma 4. *Let f and g be real functions defined on $[0,1]$ where f is absolutely continuous and increasing and g is an increasing step function. Then defining $i(x) = f(x) - g(x)$, we have $Leb(i(A)) = 0$ whenever $Leb(A) = 0$.*

Proof. Let $\epsilon > 0$. By definition of absolute continuity, there exists $\delta > 0$ such that if $(a_1, b_1), (a_2, b_2), \ldots$ are disjoint intervals whose lengths sum to at most δ, then $\sum_i |f(b_i) - f(a_i)| < \epsilon/2$.

On the other hand, since g is increasing and bounded, it has at most countably many jump discontinuities. Let $\{x\colon g(x) - g(x-0) > 0\}$ be enumerated as $\{x_1, x_2, \ldots\}$. Let n be chosen such that $\sum_{i>n} g(x_i) - g(x_i - 0) < \epsilon/2$.

Since $\mathrm{Leb}(A) = 0$, the set A is contained in a countable union of open intervals whose lengths sum to at most δ. Removing $\{x_1, \ldots, x_n\}$, A is contained in the union of a finite set and a countable collection of open intervals whose lengths sum to at most δ.

For any $x < y$ we have $|i(y) - i(x)| \leq f(y) - f(x) + \sum_{x<z\leq y}(g(z) - g(z-0))$. Applying this inequality to the intervals covering A we obtain that $i(A)$ is contained in the union of a collection of intervals whose lengths sum to at most ϵ with a finite set of points. It follows that $i(A)$ has outer measure 0. □

Determining pre-images. It is convenient to use notions from the symbolic dynamics associated with T_a. Given a finite word $\theta_{0,N} := \theta_0\theta_1 \cdots \theta_N \in \{0,1\}^{N+1}$, let $I_{\theta_{0,N}}$ be the corresponding cylinder, that is

$$I_{\theta_{0,N}} = \bigcap_{k=0}^{N} T_a^{-k}(I_{\theta_k})$$

where $I_0 = [0, a)$ and $I_1 = [a, 1)$. Let also $I_{0,N} := [x^-_{\theta_{0,N}}, x^+_{\theta_{0,N}})$ implicitly introduce notations for the cylinder boundaries. Consider the quantity

$$h^{-1}_{\theta_{0,N}}(y) = \inf\{t \in I_{\theta_{0,N}} \ : \ h(t) \geq y\}.$$

Right continuity of h implies $h \circ h^{-1}_{\theta_{0,N}}(y) \geq y$. Let $\{\theta^i_{0,N}\}_{i=1}^{k(N)}$ be an enumeration of words for which we indeed have equality, *i.e.* $h \circ h^{-1}_{\theta_{0,N}}(y) = y$. The next statement claims that this collection remains bounded as $N \to \infty$.

Lemma 5. *There are finitely many pre-images $h^{-1}(y)$ for a.e. $y \in h([0, 1])$.*

Proof. Using symbol concatenation we have

$$I_{\theta_{0,N}} = I_{\theta_{0,N}0} \cup I_{\theta_{0,N}1} \quad \text{and} \quad x^+_{\theta_{0,N}0} = x^-_{\theta_{0,N}1}.$$

It easily follows from the definition (1) of h that the discontinuities are negative jumps that lie at the $x^+_{\theta_{0,N}}$. Write $D_{\theta_{0,N}} = [h(x^+_{\theta_{0,N}}), h(x^+_{\theta_{0,N}} - 0))$ for the corresponding gaps.

Lemma 6. *There can be pre-images $h^{-1}(y)$ on both sides of $x^+_{\theta_{0,N}0}$ in $I_{\theta_{0,N}}$ only if $y \in D_{\theta_{0,N}0}$.*

Proof. Let $x \in I_{\theta_{0,N}1}$ be such that $x^-_{\theta_{0,N}1} < x$. Both points belong to the same cylinder $I_{\theta_{0,N}1}$. Hence we have

$$T^k_a(x^-_{\theta_{0,N}1}) < T^k_a(x), \text{ for } k = 0, \cdots, N + 1.$$

Moreover, the point $x^-_{\theta_{0,N}1}$ is a pre-image of 0 by definition, *i.e.*

$$0 = T^k_a(x^-_{\theta_{0,N}1}) \leq T^k_a(x), \ \forall k \geq N + 2$$

hence $h(x^-_{\theta_{0,N}1}) < h(x)$ for all $x \in I_{\theta_{0,N}1}$. Therefore, if $y < h(x^-_{\theta_{0,N}1})$ there cannot be any pre-image $h^{-1}(y)$ in $[x^-_{\theta_{0,N}1}, x^+_{\theta_{0,N}})$.

Similarly, one proves that $h(x) < h(x^+_{\theta_{0,N}0} - 0)$ for all $x \in I_{\theta_{0,N}0}$ and then there cannot be any pre-image $h^{-1}(y)$ in $[x^-_{\theta_{0,N}}, x^+_{\theta_{0,N}0})$ if $y \geq h(x^+_{\theta_{0,N}0} - 0)$. The lemma easily follows. □

Since $\sum_N \sum_{\theta_{0,N}} \text{Leb}(D_{\theta_{0,N}}) < \infty$ the first Borel-Cantelli lemma implies that Lebesgue a.e. y belongs to finitely many gaps $D_{\theta_{0,N}}$, *i.e.* for almost every y, there exists $N_y \in \mathbb{Z}^+$ such that no gap $D_{\theta_{0,N}}$ with $N \geq N_y$ contains y.

It follows from Lemma 6 that any pre-image $h^{-1}(y) \in I_{\theta_{0,N}}$ with $N \geq N_y$, must belong to either $I_{\theta_{0,N}0}$ or to $I_{\theta_{0,N}1}$. By induction, for each word θ_{0,N_y} there exists a unique sequence $\{I_{\theta_{0,N_k}}\}_{k \geq N_y}$ of nested cylinders whose diameters asymptotically vanish such that any pre-image in $I_{\theta_{0,N_y}}$ belongs to every $I_{\theta_{0,N_k}}$. By the Nested Balls theorem, it follows that there is at most one pre-image $h^{-1}(y)$ in each interval $\text{clos}(I_{\theta_{0,N_y}})$ and thus there are finitely many pre-images for a.e. $y \in h([0, 1])$. □

We will make use of the following approximate version of h in which small-scale discontinuities are removed. Given $x \in [0, 1)$ and $N \in \mathbb{Z}^+$, let $h_N(x)$ be the function defined by

$$h_N(x) = \sup\{h(t) \colon t \leq x, \ t \in I_{\theta_{0,N}}(x)\},$$

where $I_{\theta_{0,N}}(x)$ denotes the cylinder of length $N+1$ containing x. This function is increasing on each $I_{\theta_{0,N}}$. It follows that there can be at most countably many y such that $h_N^{-1}(y)$ contains an interval for some N. Let

$$F_1 = \{y \in h([0,1])\colon \text{for each } N, h_N^{-1}(y) \text{ contains no interval}\}$$

and let F_0 be the subset of $h([0,1])$ composed of points with finitely many pre-images under h. Each $y \in F_0 \cap F_1$ has only finitely many pre-images under h and each h_N.

Let $G = (h([0,1]) \setminus h(E)) \cap F_0 \cap F_1$. Since we have

$$h_c'(x) \geq \sum_{t=0}^{\infty} \lambda^t (a \wedge (1-a))^{t+1} \geq a \wedge (1-a)$$

for all $x \notin E$, the derivative h' exists and is bounded below at each pre-image of every $y \in G$ under h and h_N. Accordingly the following functions are well-defined in this set

$$\psi_N(y) = \sum_{\{x\colon h_N(x)=y\}} \frac{1}{h'(x)} \quad \text{and} \quad \psi(y) = \sum_{\{x\colon h(x)=y\}} \frac{1}{h'(x)}.$$

Notice that if $h_N(x) = h(x)$ then $h_{N+1}(x) = h(x)$ also so that for $y \in G$, $\{x\colon h_N(x) = y\}$ is a subset of $\{x\colon h_{N+1}(x) = y\}$. It follows that the ψ_N are pointwise increasing. Further since the preimages of y all lie in distinct cylinders for $N \geq N_y$ it follows that for $N \geq N_y$, $\psi_N(y) = \psi(y)$.

2nd step: Lower bound on the measure of neighborhood pre-images and absolute continuity.

Let $x \in h^{-1}(y)$ for $y \in G$ and $\epsilon > 0$. Since by assumption h is differentiable at x there exists $\mu > 0$ such that

$$|h(x+\eta) - y - h'(x)\eta| < \epsilon h'(x)|\eta| \text{ for all } 0 < |\eta| < \mu.$$

Since y has finitely many pre-images, there exists a $\bar{\mu}$ for which the above equation holds simultaneously for all $x \in h^{-1}(y)$. As a consequence, for every $0 < \delta < \bar{\mu}(1+\epsilon)\min_{x\notin E} h'(x)$ and each pre-image $x \in h^{-1}(y)$, the inequality $|t-x| < \delta/(h'(x)(1+\epsilon))$ implies $|h(t) - y| < \delta$. It follows that

$$\mathrm{Leb}(h^{-1}(y-\delta, y+\delta)) \geq 2\delta\psi(y)/(1+\epsilon) \text{ for all } \delta < \bar{\mu}(1+\epsilon)\min_{x\notin E} h'(x).$$

We will use the following weaker version valid for all $N \geq 0$:

$$\mathrm{Leb}(h^{-1}(y-\delta, y+\delta)) \geq 2\delta\psi_N(y)/(1+\epsilon) \text{ for all } \delta < \bar{\mu}(1+\epsilon)\min_{x\notin E} h'(x).$$

Being finite sums of uniformly bounded functions, the ψ_N are integrable over $h([0,1])$. Then the Lebesgue differentiation theorem implies that for a.e. $y \in h([0,1])$, we have

$$2\delta\psi_N(y) \geq \int_{y-\delta}^{y+\delta} (\psi_N(t) - \epsilon)\, dt \text{ for sufficiently small } \delta.$$

By Lemma 5, the set G has full Lebesgue measure. Let the subset of $H \subset G$ with full measure for which the previous inequality holds. Combining the previous two inequalities we conclude that for $y \in H$, there exists a $\bar{\delta} > 0$ such that

$$\mathrm{Leb}(h^{-1}(y-\delta, y+\delta)) \geq \frac{1}{1+\epsilon}\int_{y-\delta}^{y+\delta} (\psi_N(t) - \epsilon)\, dt \text{ when } 0 < \delta < \bar{\delta}. \tag{3}$$

For any subinterval J of $h([0,1])$, the collection of intervals $U_{\delta,y} \subset J$ with $y \in J \cap H$ and $\delta < \Delta$ is a Vitali cover. By the Vitali Covering theorem, there exists a countable collection of such intervals that disjointly cover all of J up to a set of measure 0. By countable additivity of the two sides of (3) we deduce that

$$|h^{-1}J| \geq \frac{1}{1+\epsilon} \int_J (\psi_N(t) - \epsilon)\, dt.$$

Since this inequality holds for every $\epsilon > 0$ we get

$$|h^{-1}J| \geq \int_J \psi_N(t)\, dt \text{ for each subinterval } J \text{ of } h([0,1]). \tag{4}$$

Since this holds for all N, we deduce that ψ (the increasing pointwise limit) is integrable and

$$|h^{-1}J| \geq \int_J \psi(t)\, dt \text{ for each subinterval } J \text{ of } h([0,1]). \tag{5}$$

To continue the proof, we now make the following claim

$$\int_{h([0,1])} \psi(t)\, dt = 1 \tag{6}$$

and we proceed as follows. For any interval $J = [a,b] \subset h([0,1])$, let $J_- = [0,a)$ and $J_+ = (b, h(1)]$. Then by (4) and (6), we have

$$\begin{aligned} 1 = |h^{-1}(h([0,1]))| &= |h^{-1}J_-| + |h^{-1}J| + |h^{-1}J_+| \\ &\geq \int_{J_-} \psi(t)\, dt + \int_J \psi(t)\, dt + \int_{J_+} \psi(t)\, dt \\ &= \int_{h([0,1])} \psi(t)\, dt = 1. \end{aligned}$$

Since the left and right sides are equal, all of the inequalities in the middle are equalities; so we see that

$$|h^{-1}J| = \int_J \psi(t)\, dt$$

for any interval J. Since the left and right sides define two measures that agree on all intervals, they must be the same measure and hence we see that

$$|h^{-1}B| = \int_B \psi(t)\, dt$$

for any measurable set B, as desired.

It remains to prove (6). Since the functions ψ_N pointwise increase to ψ it is enough to show that $\lim_{N\to\infty} \int \psi_N = 1$. Let $S_N = \{x \in [0,1) \colon h_N(x) > h(x)\}$. We show that $\int \psi_N = 1 - \mathrm{Leb}(S_N)$ and that $\mathrm{Leb}(S_N) \to 0$. Firstly, since $h'(x) = h_N'(x)$ for every $x \notin E$, the push-forward under h_N of the Lebesgue measure restricted to $[0,1] \setminus S_N$ is absolutely continuous with density exactly ψ_N. It follows that $\int \psi_N = 1 - \mathrm{Leb}(S_N)$ as claimed. Secondly, notice that S_N consists of a union of countably many intervals. Indeed if $h_N(x) > h(x)$, then by right continuity there exists $\delta > 0$ such that

$$h_N(t) = h_N(x) > h(t) \ \forall t \in [x, x+\delta).$$

Let J be a maximal interval in S_N and let $\bar{J} = [a, b]$. Since we must have $h(b-0) = h(a)$ it follows that $h_c(b) - h_c(a) = h_d(b-0) - h_d(a)$. Since the left side exceeds $m(b-a)$ where $m = \text{ess inf } h' > 0$, we deduce that

$$b - a \leq \frac{1}{m} \sum_{x \in J} h(x-0) - h(x).$$

Moreover, the function h is discontinuous at $x \in I_{\theta_{0,N}}$ iff $T_a^n(x) = a$ for some $n > N$ and we have $h(x-0) - h(x) = \frac{\lambda^n}{1-\lambda}$ [5]. Summing the previous inequality over all intervals included in the cylinders of length $N+1$, we deduce that $\text{Leb}(S_N) \leq \frac{1}{m(1-\lambda)} \sum_{n=N+1}^{\infty} 2^n \lambda^n$. Since this converges to 0 as $N \to \infty$ the proof is complete.

3. The Case $\lambda \geq 1/2$

When $\lambda \geq 1/2$, the variation of h becomes infinite [5] and a decomposition into the difference of bounded functions $h = h_c - h_d$ as in Lemma 3 no longer exists. In order to circumvent this difficulty, we use the symbolic dynamics associated with the map T_a to regard h as a random series.

The map T_a is semi-conjugate to the left shift σ on $\{0,1\}^{\mathbb{Z}^+}$, *i.e.* we have $T_a \circ \pi = \pi \circ \sigma$ where $\pi : \{0,1\}^{\mathbb{Z}^+} \to [0,1]$ is given explicitly by

$$\pi(\theta) = a \sum_{k=0}^{\infty} \theta_k \prod_{j=0}^{k-1} \alpha(\theta_j), \quad \forall \theta = \{\theta_k\}_{k=0}^{\infty} \in \{0,1\}^{\mathbb{Z}^+}$$

where $\alpha(0) = a$ and $\alpha(1) = 1 - a$ (and by convention $\prod_{j=0}^{-1} \alpha(\theta_j) = 1$).

The coding map $\pi^{-1}(x)$ is unique for a.e. x and the push forward of Leb (which is T_a-invariant) under π^{-1} is the (σ-invariant) Bernoulli measure μ_a where the symbol 0 appears with probability a and 1 appears with probability $1-a$, *viz.* $\mu_a = \text{Leb} \circ \pi$. Therefore, the measure $\text{Leb} \circ h$ can be regarded as $\mu_a \circ g^{-1}$ where the symbolic function $g = h \circ \pi$ is obtained explicitly by inserting the expression of π into (1)

$$g(\theta) = a \sum_{n=1}^{\infty} \theta_n \lambda^n \sum_{k=1}^{n} \lambda^{-k} \prod_{j=n-k+1}^{n-1} \alpha(\theta_j), \quad \forall \theta \in \{0,1\}^{\mathbb{Z}^+}.$$

Using instead of θ, the sequence $\omega \in \{-1,+1\}^{\mathbb{Z}^+}$ of independent identically distributed random variables taking the values $+1$ and -1 with probability $1-a$ and a respectively, the question of absolute continuity of the measure $\mu_a \circ g^{-1}$ reduces to the following: what can be said about the set of λ for which the random variable

$$Z = \sum_{n=1}^{\infty} \omega_n \lambda^n \sum_{k=1}^{n} \lambda^{-k} \prod_{j=n-k+1}^{n-1} \alpha((\omega_j+1)/2)$$

has an absolutely continuous distribution?

In [13] Solomyak showed that for $a = 1/2$ and Lebesgue a.e. $\lambda \in [1/2, 1)$, the random variable $\sum_{n=1}^{\infty} \omega_n \lambda^n$ has an absolutely continuous distribution. His proof made use of Fourier transform arguments and a transversality condition. Shortly afterwards Peres and Solomyak [11] gave a new proof based on correlation dimension. This approach allowed them to extend later their results to arbitrary $a \in [1/3, 2/3]$ [12].

Here we are able to establish the transversality condition of the function g in the case where $a = 1/2$. By adapting the argument of [11], we obtain the following result.

Theorem 2. *Assume that $a = 1/2$. Then Leb $\circ\, h$ is absolutely continuous for a.e. $\lambda \in (1/2, 1)$.*

Proof. When $a = 1/2$, the random variable Z takes the following simpler form

$$\frac{2}{\lambda - 2^{-1}} \sum_{n=1}^{\infty} \omega_n (\lambda^n - 2^{-n}).$$

The first thing to note is that it is sufficient to demonstrate the absolute continuity of a random variable of the form

$$\tilde{Z}_\lambda = \sum_{n=M}^{\infty} \omega_n (\lambda^n - 2^{-n})$$

– that is we can omit any subset of terms – as the convolution of an absolutely continuous random variable with another random variable is always absolutely continuous.

For ω, τ in $\{\pm 1\}^{\mathbb{Z}^+}$ following Peres and Solomyak we define

$$\tilde{\phi}_{\omega,\tau}(\lambda) = \sum_{n=M}^{\infty} (\tau_n - \omega_n)(\lambda^n - 2^{-n}).$$

For comparison with [11] we also define

$$\phi_{\omega,\tau}(\lambda) = \sum_{n=M}^{\infty} (\tau_n - \omega_n)\lambda^n.$$

Notice that $\tilde{\phi}_{\omega,\tau}(\lambda) = \phi_{\omega,\tau}(\lambda) - C_{\omega,\tau}$ where $C_{\omega,\tau} = \sum_{n=M}^{\infty} (\tau_n - \omega_n) 2^{-n}$.

The absolute continuity of the random series $\sum_{n=1}^{\infty} \omega_n \lambda^n$ was shown in [11] to follow from an estimate of the form

$$S := \liminf_{r \to 0} \int_\Omega \int_\Omega \text{Leb}\,\{\lambda\colon \phi_{\omega,\tau}(\lambda) \le r\}\, d\mu(\tau)\, d\mu(\omega) < \infty.$$

We want to show a similar estimate for the quantity $\tilde{S}$ obtained in a similar way by using $\tilde{\phi}_{\omega,\tau}$ instead of $\phi_{\omega,\tau}$.

For now fix ω and τ and let $k(\omega,\tau) = \min\{n \ge M\colon \omega_n \ne \tau_n\}$ (which we will denote by k when ω and τ are fixed). We assume without loss of generality that $\tau_k = 1$ and $\omega_k = -1$ and we define

$$\tilde{g}(\lambda) := \tilde{\phi}_{\omega,\tau}(\lambda)/(2\lambda^k).$$

In particular, we have $\tilde{g}(\lambda) = g(\lambda) - C_{\tau,\omega}/(2\lambda^k)$ where the function g is as in [11]

$$g(\lambda) = 1 + \sum_{n=1}^{\infty} b_n \lambda^n \text{ with } b_n \in \{-1, 0, 1\}.$$

Notice that $0 \le C_{\omega,\tau}/(2\lambda^k) \le 2(2\lambda)^{-M}$ since $0 \le C_{\omega,\tau} \le 2^{2-k}$.

Peres and Solomyak then showed for an explicit $\delta > 0$ that

$$\text{Leb}\{\lambda \in [2^{-1}, 2^{-2/3}]\colon |g(\lambda)| \le \rho\} \le 2\delta^{-1}\rho \quad \forall \rho > 0.$$

The proof works by showing $|g'(\lambda)|$ exceeds δ when $g(\lambda)$ is δ-close to 0.

Fix $\lambda_0 \in (1/2, 2^{-2/3})$ and choose an M such that $2(2\lambda_0)^{-M} < \delta/2$. For $\lambda \in [\lambda_0, 2^{-2/3}]$, $\tilde{g}$ differs from g by a constant that is at most $2(2\lambda)^{-M} < \delta/2$ so that $|\tilde{g}'(\lambda)|$ exceeds δ when $|\tilde{g}(\lambda)| < \delta/2$. This is sufficient to allow us to conclude in the same way as [11] that $S < \infty$ and hence $\tilde{Z}_\lambda$ is absolutely continuous for almost every λ in $[\lambda_0, 2^{-2/3}]$. Since this holds for any $\lambda_0 > 1/2$ this establishes absolute continuity of $\tilde{Z}_\lambda$ for almost every λ in $[2^{-1}, 2^{-2/3}]$.

For the range $[2^{-2/3}, 2^{-1/2}]$ the same modifications as in [11] give the result on $[2^{-1}, 2^{-1/2}]$.

For λ in the range $J_k = [2^{-1/2^k}, 2^{-1/2^{k+1}}]$ we need to make a small modification to the Peres-Solomyak argument (in their case it was argued that this case followed immediately from the previously understood cases). For us if $\lambda \in J_k$ we take only the terms of the sequence whose indices are multiples of k. Showing absolute continuity of the partial series arising in this way is sufficient to prove absolute continuity of the entire series $\tilde{Z}_\lambda$. Setting $\rho = \lambda^k$, the partial series that we are now summing is

$$\sum_{n=M}^{\infty} \omega_{kn}(\rho^n - 2^{-kn}).$$

The previous proof works verbatim to show that this series is absolutely continuous. In fact, as pointed out in [11], applying this technique to each of the k parts separately shows that the original random variable is the sum of k parts that are each absolutely continuous. This implies that the resulting random variable is absolutely continuous with a continuous density function. □

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