WEAK MIXING SUSPENSION FLOWS OVER SHIFTS OF FINITE TYPE ARE UNIVERSAL

ANTHONY QUAS AND TERRY SOO

Abstract. Let S be an ergodic measure-preserving automorphism on a non-atomic probability space, and let T be the time-one map of a topologically weak mixing suspension flow over an irreducible subshift of finite type under a Hölder ceiling function. We show that if the measure-theoretic entropy of the S is strictly less than the topological entropy of T, then there exists an embedding from the measure-preserving automorphism into the suspension flow. As a corollary of this result and the symbolic dynamics for geodesic flows on compact surfaces of negative curvature developed by Bowen [4] and Ratner [20], we also obtain an embedding from the measure-preserving automorphism into a geodesic flow, whenever the measure-theoretic entropy of S is strictly less than the topological entropy of the time-one map of the geodesic flow.

1. Introduction

Theorem 1. *Let $(\Omega, \mathcal{D}, \nu)$ be a non-atomic probability space endowed with an ergodic measure-preserving automorphism S. Let $(G^t)_{t\in\mathbb{R}}$ be a geodesic flow on a compact surface of variable negative curvature $\mathcal{M}$, with unit tangent bundle $\mathrm{UT}(\mathcal{M})$. If the measure-theoretic entropy of S is strictly less than the topological entropy of G^1, then there exists a measurable mapping $\Psi : \Omega \to \mathrm{UT}(\mathcal{M})$ such that the restriction of Ψ to a set of full measure Ω' is an injection, and $\Psi(S(\omega)) = G^1(\Psi(\omega))$ for all $\omega \in \Omega'$.*

In Theorem 1, we say that the measurable mapping Ψ is an **embedding** since on a set of full measure Ψ is an injection and $\Psi \circ S = G^1 \circ \Psi$. Thus Theorem 1 says that geodesic flows on compact surfaces of negative curvature are universal. We will prove Theorem 1 by exploiting the symbolic representation of geodesic flows developed by Bowen [4] and Ratner [20], and proving our main result, that topologically weak

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mixing suspension flows over irreducible subshifts of finite type under Hölder continuous ceiling functions are also universal. In what follows, we review all the necessary terminology required for stating our main result.

Let V be a finite set of symbols of cardinality $\#V$; we will always assume that $\#V \geq 2$. Let $\theta : V^{\mathbb{Z}} \to V^{\mathbb{Z}}$ be the shift defined by $\theta(y)_i = y_{i+1}$ for all $y \in V^{\mathbb{Z}}$ and all $i \in \mathbb{Z}$. We endow $V^{\mathbb{Z}}$ with the standard product metric and Borel σ-algebra. Sometimes $V^{\mathbb{Z}}$ is called a full-shift. Let $\mathfrak{A}$ be a square matrix zero-one matrix of order $\#V$. Also assume that $\mathfrak{A}$ is irreducible; that is, for all $1 \leq i, j \leq \#V$, there exists $n \in \mathbb{Z}^+$ such that $\mathfrak{A}^n(i,j) > 0$. Define

$$Y := \left\{ y \in V^{\mathbb{Z}} : \mathfrak{A}(y_i y_{i+1}) = 1 \text{ for all } i \in \mathbb{Z} \right\}.$$

Thus Y is a set of all bi-infinite paths of the directed graph on V with adjacency matrix $\mathfrak{A}$; since $\mathfrak{A}$ is assumed to be irreducible, the graph is strongly connected. We say that Y is an **irreducible subshift of finite type**; see [17, Chapter 2] for background. More generally, we say that X is a **subshift**, if X is a closed shift-invariant subset of $V^{\mathbb{Z}}$ (of course X is endowed with the Borel σ-algebra). We will call a subshift *non-trivial* if it does not consist of a finite set of points. (When talking about subshifts, the map will always be the shift map, θ, and it will be left implicit).

Let Y be an irreducible subshift of finite type and let $f : Y \to (0, \infty)$ be Hölder continuous. Set

$$\operatorname{susp}(Y, f) := \{(y, s) \in Y \times [0, \infty) : y \in Y, 0 \leq s < f(y)\}.$$

We define the flow $(T^t)_{t \in \mathbb{R}}$ on $\operatorname{susp}(Y, f)$ by setting $T^t(y, s) = (y, s+t)$ and identifying the points $(\theta y, 0) = (y, f(y))$. More precisely, for all $y \in Y$, set $f^{(0)}(y) := 0$ and $f^{(n+1)}(y) := f^{(n)}(y) + f(\theta^n y)$ for all $n \in \mathbb{Z}$. For all $(y, s) \in \operatorname{susp}(Y, f)$, and $t \in \mathbb{R}$, there is a unique $n \in \mathbb{Z}$ such that $f^{(n)}(y) \leq s + t < f^{(n+1)}(y)$; we set

$$T^t(y, s) = \left(\theta^n y, s + t - f^{(n)}(y)\right).$$

Sometimes $(\operatorname{susp}(Y, f), (T^t)_{t \in \mathbb{R}})$ is called the suspension flow over Y under the ceiling function f. The Bowen-Walters distance [7] makes $\operatorname{susp}(Y, f)$ a compact metric space, where a neighbourhood about a point $(y, s) \in \operatorname{susp}(Y, f)$ contains all the points $T^t(w, s) \in \operatorname{susp}(Y, f)$, where $|t|$ is small and w is close to y. With respect to the topology generated, T^t is a homeomorphism on $\operatorname{susp}(Y, f)$ for all $t \in \mathbb{R}$.

Theorem 2. *Let* $(\Omega, \mathcal{D}, \nu)$ *be a non-atomic probability space endowed with an ergodic measure-preserving automorphism* S. *Let* $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ *be the suspension flow over an irreducible subshift of finite type* Y *under a Hölder continuous function* $f : Y \to (0, \infty)$. *If the measure-theoretic entropy of* S *is strictly less than the topological entropy of* T^1, *and if* $(T^t)_{t\in\mathbb{R}}$ *is topologically weak mixing, then there exists an embedding* $\Psi : \Omega \to \mathrm{susp}(Y, f)$, *so that* $\Psi \circ S = T^1 \circ \Psi$.

Recall that $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ is defined to be topologically weak mixing if there does not exist a non-constant continuous eigenfunction function F from the suspension space to the complex unit circle, so that for all $t \in \mathbb{R}$, we have $F \circ T^t = e^{2\pi \mathbf{i}\beta t}F$, for some eigenvalue $\beta > 0$. A simple, yet representative, example in which $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ is topologically weak mixing is the case $Y = \{a, b\}^{\mathbb{Z}}$, and $f(y) = y_0$ for all $y \in Y$, where a, b are positive real numbers whose ratio is irrational.

We will prove Theorem 2 by first proving it for the special case of a uniquely ergodic subshift; Theorem 3 below, in combination with the Jewett-Krieger theorem will imply Theorem 2. See [2, 14, 15, 11] for more on the Jewett-Krieger theorem.

Theorem 3. *Suppose that* X *is a non-trivial uniquely ergodic subshift with invariant measure* μ. *Let* $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ *be the suspension flow over an irreducible subshift of finite type* Y *under a Hölder continuous function* $f : Y \to (0, \infty)$. *If the topological entropy of* θ *is strictly less than the topological entropy of* T^1, *and if* $(T^t)_{t\in\mathbb{R}}$ *is topologically weak mixing, then there exists a set* $X' \subset X$ *of full measure and an embedding* $\Psi : X' \to \mathrm{susp}(Y, f)$, *so that* $\Psi \circ \theta(x) = T^1 \circ \Psi(x)$ *for all* $x \in X'$.

Proof of Theorem 2. By the Jewett-Krieger theorem [15], a non-atomic probability space endowed with an ergodic measure-preserving automorphism with finite measure-theoretic entropy is isomorphic to a uniquely ergodic subshift equipped with its invariant measure. Notice that by the variational principle [10], the topological entropy of the uniquely ergodic subshift is equal to the measure-theoretic entropy of the initial automorphism. Hence, composing this isomorphism with the embedding given by Theorem 3 produces the required embedding. □

Let us remark that the embedding that we define to prove Theorem 3 is not continuous.

Question 1. *Let* X *be a subshift equipped with an invariant measure* μ. *Let* $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ *be a topologically weak mixing suspension over an irreducible subshift of finite type* Y *under a Hölder continuous function* $f : Y \to (0, \infty)$.

If the measure-theoretic entropy of $\theta\colon X \to X$ is (strictly) less than the topological entropy of T^1, under which conditions must there exist a mapping $\Psi : X \to \mathrm{susp}(Y, f)$ where the continuity points have full measure?

Mappings satisfying the property in the question are said to be *finitary* (see [21] and [13] for more information).

It remains to prove Theorems 1 and 3, but next we discuss an application of Theorem 1 that was the original motivation of Theorem 2. We thank Jean-Paul Thouvenot [22] who suggested that Theorem 2 could be used to prove Theorem 1, from which one can obtain a negative answer to the following problem posed by Ledrappier [16]:

> Let $(G^t)_{t\in\mathbb{R}}$ be geodesic flow on a compact surface of variable negative curvature. If a measure λ is invariant and ergodic under the group $(G^n)_{n\in\mathbb{Z}}$, must it be invariant under $(G^t)_{t\in\mathbb{R}}$?

Corollary 4. *Let $(G^t)_{t\in\mathbb{R}}$ be geodesic flow on a compact surface of variable negative curvature. There is a measure λ that is invariant and ergodic under the group $(G^n)_{n\in\mathbb{Z}}$, but not invariant under $(G^t)_{t\in\mathbb{R}}$.*

Corollary 4 is immediate from Theorem 1, and the simple fact that there exists an ergodic measure-preserving automorphism that has arbitrarily small entropy and does not admit a square root. Let us remark that by appealing to a result of Ornstein [18], which says that there exists a K-automorphism that does not admit a square root, one can also require that $(UT(\mathcal{M}), \lambda, T^1)$ is a K-automorphism.

In fact, the question above was posed as an approach to the following question due to Ledrappier and Federico and Jana Rodriguez-Hertz.

> Let $(G^t)_{t\in\mathbb{R}}$ be geodesic flow on a compact surface of variable negative curvature. If a set is minimal under the action of $(G^t)_{t\in\mathbb{R}}$, must it be minimal under $(G^n)_{n\in\mathbb{Z}}$?

In the appendix, we give a negative answer to this question (we note that this implies a negative answer to the earlier question).

We prove Theorem 1 and Corollary 4 in Section 5. In the next section, we give an outline of the proof of Theorem 3 that will also give criteria for topological weak mixing for suspensions flows of subshifts of finite type under Hölder continuous ceiling functions. In Section 3, we will assemble some lemmas that will help us define the embedding of Theorem 3 in Section 4.

2. Background and proof sketch

In this section, we first introduce some basic terminology. Second, we will discuss topological weak mixing for suspension flows and thirdly, we will discuss the basic approach and highlight the main ideas of the proof.

2.1. **Basic terminology.** Let X be a non-trivial subshift. Let $x \in X$ and $n \in \mathbb{Z}^+$. We say that the finite string $x_0 \cdots x_{n-1} = x_0^{n-1}$ is an **block of size n**. Let $B^n(X) := \left\{x_0^{n-1} : x \in X\right\}$ denote the set of all blocks of size n in X. We refer to all members of $B(X) = \bigcup_{n\in\mathbb{Z}^+} B^n(X)$, as **X-blocks**. Given two X-blocks x_0^{n-1} and z_0^{m-1} we let their concatenation be given by

$$x_0 \ldots x_{n-1} z_0 \ldots z_{m-1} = x_0^{n-1} z_0^{m-1} = \boxed{x_0^{n-1}}\boxed{z_0^{m-1}}, \tag{1}$$

whenever the concatenation is also a X-block. Sometimes, as in (1) we will enclose blocks to make their concatenations easier to parse. We say that a block x_0^{n-1} **appears** in z_0^{m-1} or $z \in X$ if there exists $k \in \mathbb{Z}$ such that $z_k^{k+n-1} = x_0^{n-1}$. In order to distinguish between $x \in X$ and $\theta x \in X$, sometimes we will use the symbol "**.**" to indicate the position of the origin, so that if $x = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots$ then $\theta x = \cdots x_{-2}x_{-1}x_0.x_1x_2\cdots$.

Let Y be an irreducible subshift of finite type, and let $f : Y \to (0,\infty)$. Given $(y,s) \in \mathrm{susp}(Y,f)$ for each $m \in \mathbb{Z}$, there exists a $t \in \mathbb{R}$ such that $T^t(y,s) = (\theta^m y, 0)$; for any $n \geq m$, we say that the Y-block y_m^{n-1} **begins at t (in (y,s))**. For $y \in Y$, we define the **length in the suspension** of the Y-block y_m^{n-1} in y by

$$\mathrm{Len}(y,f;m,n-1) = f^{(n-m)}(\theta^m y) = \sum_{i=m}^{n-1} f(\theta^i y) \tag{2}$$

Note the difference between the size of a block (number of symbols) and length of a block in the suspension. In general, the length in the suspension of a Y-block y_m^n may depend on the whole bi-infinite sequence y. We define the **maximum length in the suspension** of a block y_0^{m-1} by

$$\mathrm{MLen}(y_0^{m-1}, f) := \max\left\{f^{(m)}(z) : z \in Y, z_0^{m-1} = y_0^{m-1}\right\}.$$

Let $t > 0$, for $Y' \subseteq Y$ define

$$B_t(Y', f) := \{B \in B(Y') : \mathrm{MLen}(B,f) \leq t\}. \tag{3}$$

We say that $(\mathrm{susp}(Y,f), (T^t)_{t\in\mathbb{R}})$ satisfies **specification** if for all $\varepsilon > 0$, there exists L_ε such that given any two blocks $A_0, A_1 \in B(Y)$, and a real number $L > L_\varepsilon + \mathrm{MLen}(A_0, f)$, there exists $(y, 0) \in \mathrm{susp}(Y, f)$ such that A_0 begins at 0 and A_1 begins within ε of L. Note that specification is a much stronger condition than requiring that the subshift of finite type Y has a irreducible adjacency matrix. Consider again the special case where $Y = \{a, b\}^{\mathbb{Z}}$, and $f(y) = y_0$ for all $y \in Y$, where a, b are positive real numbers. It is easy to verify that $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ satisfies specification if and only if a/b is irrational.

2.2. **Topological weak mixing.** We discuss in this sub-section some criteria equivalent to topological weak mixing of suspension flows.

Let Y be a subshift of finite type, and let $f : Y \to (0, \infty)$. If $z \in Y$ is periodic, we let $\mathrm{per}(z)$ be the least positive integer such that $\theta^{\mathrm{per}(z)} z = z$. The **period in the suspension** is

$$\mathrm{ALen}(z, f) := \mathrm{Len}(z, f; 1, \mathrm{per}(z)).$$

For any non-empty $A \subset \mathbb{R} \setminus \{0\}$, let $\mathrm{span}(A)$ denote the closure of the finite integer combinations of elements of A. Since this is a closed subgroup of $\mathbb{R}$, it is either $\mathbb{R}$ or a subgroup of the form $c\mathbb{Z}$ for some $c > 0$. In the latter case, we define $\gcd(A) = c$ and if $\mathrm{span}(A) = \mathbb{R}$, we define $\gcd(A) = 0$.

We say that $f : Y \to (0, \infty)$ is **filling** if it satisfies the following properties.

(i) The function f is Hölder continuous.
(ii) The set of all periods in the suspension has greatest common divisor equal to zero.

We say that two functions f and g are **Hölder cohomologous** if there exists a Hölder continuous function h such that $f = g + h - h \circ \theta$.

Proposition 5 (Equivalent notions of topological weak mixing)**.** *Let $(\mathrm{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ be the suspension flow over an irreducible subshift of finite type Y under a Hölder continuous ceiling function $f : Y \to (0, \infty)$. The following conditions are equivalent.*

(a) The suspension flow is topologically weak mixing.
(b) The ceiling function is filling.
(c) The suspension flow satisfies specification.
(d) The ceiling function f is not Hölder cohomologous to a function h taking values in $\beta\mathbb{Z}$ for some $\beta > 0$.

This proposition is essentially standard. We remark that our proofs of Theorems 3 and Theorem 1 only require Proposition 5 ((a) $\Rightarrow$ (b)); we give brief details of this implication. A slightly generalized version of (b)$\Rightarrow$ (c) appears in our proof of Lemma 7(F).

To see (a) implies (b), we argue by the contrapositive. If f fails to be filling, the periods in the suspension must all lie in a subgroup $\beta\mathbb{Z}$ of the reals. In this case a version of the Livschitz theorem appearing in the book of Parry and Pollicott [19, Proposition 5.2] implies that f is cohomologous to a function taking values in $\beta\mathbb{Z}$ (so that (d) fails). This can then be used to construct a continuous eigenfunction

$$F(y,s) = e^{2\pi \mathbf{i}(s-h(y))/\beta}$$

which satisfies $F(T_t(y,s)) = e^{2\pi \mathbf{i}t/\beta} F(y,s)$, contradicting (a).

Proposition 5 explains the role of the assumption that the suspension flow is topologically weak mixing in Theorem 3. If the suspension flow is not assumed to be topologically weak-mixing, the conclusion of Theorem 3 may fail – in particular if the r is not of the form $1/n$, then any measure-preserving transformation that can be embedded into $\mathrm{susp}(Y,f)$ must have a (measurable) eigenfunction and hence cannot be weak-mixing (in the ergodic sense). On the other hand if r is of the form $1/n$, then more straightforward methods establish the existence of an embedding.

2.3. **The basic approach.** Suppose that X is a non-trivial uniquely ergodic subshift with invariant measure μ. Let $(\mathrm{susp}(Y,f),(T^t)_{t\in\mathbb{R}})$ be a topologically weak mixing suspension over an irreducible subshift of finite type Y under a Hölder continuous function $f : Y \to (0,\infty)$. Assume that the topological entropy of $\theta\colon X \to X$ is strictly less than topological entropy of T^1.

Our approach is to partition an element $x \in X$ into X-blocks $\big(x_{n_i}^{n_{i+1}-1}\big)_{i\in\mathbb{Z}}$, and then encode each of these blocks into corresponding Y-blocks $\big(\phi\big(x_{n_i}^{n_{i+1}-1}\big)\big)_{i\in\mathbb{Z}}$, which we will piece together to produce $\Psi(x) = (y(x), s(x)) \in \mathrm{susp}(Y,f)$ in such a way as to guarantee that $\Psi(\theta(x)) = T^1(\Psi(x))$. Clearly, some care is needed to ensure that we can recover x from $\Psi(x)$.

Alpern's multiple Rokhlin tower theorem [1, 8] implies that as a function on X, we can choose an equivariant subset of $\mathbb{Z}$; that is, for $x \in X$, if $M(x)$ is the subset of $\mathbb{Z}$ assigned to x, then $M(\theta x)$ is obtained by subtracting 1 from each elements of $M(x)$. Furthermore, for any $n \in \mathbb{Z}^+$ we may specify that the distance between two successive points of $M(x)$ is either n or $n+1$. Let $(n_i)_{i\in\mathbb{Z}} = (n_i(x))_{i\in\mathbb{Z}}$ be an enumeration of $M(x)$ such that $\ldots < n_{-2}(x) < n_{-1}(x) < n_0(x) \le 0 < n_1(x) < n_2(x) < \ldots$. The point x can then be partitioned into X-blocks as

$$x = \cdots \boxed{x_{n_{-1}}^{n_0-1}}\boxed{x_{n_0}^{n_1-1}}\boxed{x_{n_1}^{n_2-1}}\cdots .$$

As a consequence of the fact that the topological entropy of θ is strictly less than that topological entropy of T^1, there will exist $\alpha \in (0,1)$ such that for all n sufficiently large, we have

$$\#(B^n(X) \cup B^{n+1}(X)) < \#B_{(1-\alpha)n}(Y, f).$$

Thus there exists an injection from $\phi_1 : B^n(X) \cup B^{n+1}(X) \to B_{(1-\alpha)n}(Y, f)$, which encodes n-blocks and $(n+1)$-blocks of X into (shorter) Y-blocks of length in the suspension at most $(1-\alpha)n$.

For each $i \in \mathbb{Z}$, we will set $A_i = x_{n_i}^{n_{i+1}-1}$ and $\psi_1(x) = \cdots.\phi_1(A_0)\cdots$, where $\phi_1(A_0)$ is suitably extended to become an element of Y. Let $\Psi_1(x) = T^{-n_0(x)}(\psi_1(x), 0)$. At this stage Ψ_1 encodes only the block A_0, but we will define subsequent Ψ_j that will encode successively more of the blocks A_i that appear in x. Observe that since the $n_i(x)$ were chosen in an equivariant way, we have that $\Psi_1(\theta^k x) = T^k\Psi_1(x)$ for all $k \in [n_0, n_1)$. In this way, Ψ_1 maps x in such a way that the block A_0 is encoded and starts *exactly* at time $n_0(x)$ in the image. The above will ensure that the encoded block has length in the suspension of approximately $(1-\alpha)n$, so that in the suspension, the encoded version of the block is approximately αn shorter than the original block.

By an additional application of Alpern's multiple Rokhlin tower theorem, we may choose a much sparser (two-sided) subsequence $(n_i')_{i\in\mathbb{Z}}$ of (n_i) in an equivariant way, where again $\ldots n_{-1}' < n_0' \leq 0 < n_1' \ldots$.

Define level two blocks by $A_i' = x_{n_i'}^{n_{i+1}'-1}$, so that A_0' consists of the concatenation of A_a, $A_{a+1}, \ldots, A_{b-1}$ for some $a \leq 0 < b$. As long as the n above is taken sufficiently large (depending on α and how long it takes for specification to 'kick in'), it is possible to place filler blocks in the gaps of length in the suspension approximately $n\alpha$ between the encodings of the $(A_i)_{a\leq i<b}$ to make a second order block in which each encoded block starts within a fixed precision, 2^{-1} say, of the desired starting location $n_i(x)$. Again this finite block is extended to a point $\psi_2(x)$ in Y. The second approximation to the embedding is then given by $\Psi_2(x) = T^{-n_0'(x)}(\psi_2(x), 0)$. Now the encoded A_0' block starts exactly at $n_0'(x)$, while the $(A_i)_{a\leq i<b}$ start approximately at $n_i(x)$.

In order to iterate this construction, one minor issue is that to achieve the packing at the next stage, it is necessary to ensure again that the encoded level two blocks are shorter (by an amount that depends on the placement accuracy that we want to obtain in the third level) than the level two blocks in the source. We achieve this by using shorter filler between some of the last level one blocks (the *crumple zone*) forming the level two block.

The inductive step is illustrated in figure 1.

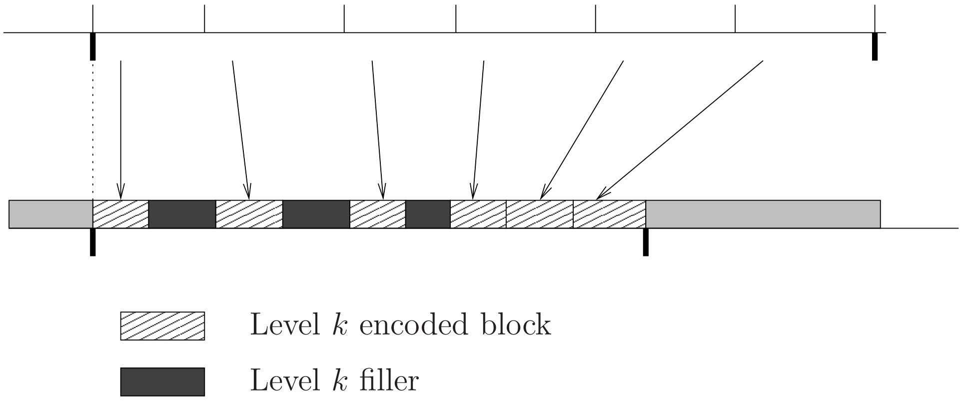

FIGURE 1. Inductive step building the $(k+1)$st level map from the kth level map. Notice the crumple zone on the right is compressed to ensure the image block is shorter than the source, allowing subsequent levels to accurately position blocks

The Borel-Cantelli lemma will be used to ensure that almost every point only is in finitely many crumple zones. For a point that it is in finitely many crumple zones, we will check that $\Psi_n(x)$ forms a Cauchy sequence, and therefore converges to a limit. This will allow us to define the embedding.

More formally, we will obtain a sequence of mappings from X to $\operatorname{susp}(Y, f)$ with the following properties.

Proposition 6. *Suppose that X is a non-trivial uniquely ergodic subshift with invariant measure μ. Let $(\operatorname{susp}(Y, f), (T^t)_{t\in\mathbb{R}})$ be the suspension flow over an irreducible subshift of finite type Y under a Hölder continuous function $f : Y \to (0, \infty)$. Let W be an arbitrary Y-block. If the topological entropy of θ is strictly less than the topological entropy of T^1, and if $(T^t)_{t\in\mathbb{R}}$ is topologically weak mixing, then there exists a sequence of measurable maps $\Psi_i : X \to \operatorname{susp}(Y, f)$ and a subset $X' \subset X$ of full measure with the following properties.*

(a) For all $x \in X'$ and for all sufficiently large $i \in \mathbb{Z}^+$ we have

$$\Psi_i(\theta x) = T^1\Psi_i(x).$$

(b) For all $x \in X'$, we have $\lim_{i\to\infty} \Psi_i(x) := \Psi(x) = (\mathbf{y}(x), \mathbf{s}(x)) \in \operatorname{susp}(Y, f)$ exists.

(c) For all $x \in X'$, we have that $\mathbf{y}(x)$ is aperiodic; that is, there does not exist $m \in \mathbb{Z} \setminus \{0\}$ such that $\theta^m\mathbf{y}(x) = \mathbf{y}(x)$.

(d) For all $x \in X'$, we have that $\mathbf{y}^{-1}(\mathbf{y}(x)) \subseteq \{\theta^n x : n \in \mathbb{Z}\}$.

(e) For all $x \in X'$, the block W appears in $\mathbf{y}(x)$ infinitely often in the positive coordinates and in the negative coordinates.

The proof of Theorem 3 follows easily from Proposition 6. Condition (e) is not required for the proof of Theorem 3, but is necessary for our proof of Theorem 1.

Proof of Theorem 3. We apply Proposition 6 with an arbitrary Y-block W. That $\Psi(\theta(x)) = T^1(\Psi(x))$ follows from (a) and (b). To see that Ψ is an injection, let $x \neq x' \in X'$ and assume $\Psi(x) = \Psi(x')$. By Property (d), $x' = \theta^n x$ for some $n \in \mathbb{Z}$. This implies $\Psi(x) = \Psi(x') = T^n(\Psi(x))$ which gives a contradiction using (c).

□

3. Key Ingredients

In this section, we will define, using Alpern's multiple Rokhlin tower theorem [1, 8], equivariant subsets of $\mathbb{Z}$, as a function on a non-trivial uniquely ergodic subshift; following Keane and Smorodinsky [12, 13], we think of elements of these subsets as *markers*. These two ingredients will allow us to define the embedding of Theorem 3 in Section 4.

Before we state Lemma 7, we need some additional notation. Let Y be an irreducible subshift of finite type. Given any $A, B \in B(Y)$ there is a $C \in B(Y)$ such that $ACB \in B(Y)$. In the case, where $AB \notin B(Y)$, we define the **connecting block** of A and B to be the block C such that ACB is legal with the least maximum length in the suspension, where we break ties by assigning a lexicographic order on $B(Y)$; in the case where $AB \in B(Y)$, we let $A\boxed{C(A,B)}B = AB$, and say that $C(A, B)$ is an empty connecting block, and assign it zero maximum length in the suspension. Clearly, the maximum lengths in the suspension of all connecting blocks are bounded above.

Given a block $A \in B(Y)$, we let

$$Y \setminus A := \{y \in Y : \ A \text{ does not appear in } y\}.$$

Note that $Y \setminus A$ is a subshift of finite type.

For any $B \in B(Y)$ such that $BB \in B(Y)$, let $B^k := \overbrace{B \cdots B}^{k \text{ times}}$ for any integer $k \geq 1$, and let $\overline{B} \in Y$ be the periodic point with

$$\overline{B} = \cdots BB.BB \cdots .$$

Lemma 7. *Let $(\mathrm{susp}(Y, f), (T^t)_{t \in \mathbb{R}})$ be a topologically weak mixing suspension flow over an irreducible subshift of finite type Y under a Hölder continuous ceiling function f. Let X be a non-trivial uniquely ergodic* *subshift. Suppose that the topological entropy of θ is strictly less than the topological entropy of T^1*

Given any $W \in B(Y)$, there exist Y-blocks $\langle, \rangle, \boldsymbol{L}, \boldsymbol{R}_1, \boldsymbol{R}_2 \ldots$, non-decreasing integers $(K_n, k_n, L_n)_{n\in\mathbb{Z}^+}$ with the following properties.

(A) *The concatenations $\boldsymbol{L}\,\boldsymbol{L}$ and $\boldsymbol{R}_i\,\boldsymbol{R}_i$ for all $i \in \mathbb{Z}^+$, are also Y-blocks.*

(B) *We have* $\gcd\big(\mathrm{ALen}(\overline{\mathbf{L}}, f), \mathrm{ALen}(\overline{\mathbf{R_i}}, f)\big) \leq 2^{-i}$ *for every $i \geq 1$.*

(C) *The block W appears in the blocks $\langle$ and $\rangle$.*

(D) *The concatenations $\langle\,\langle, \rangle\,\rangle$ are also Y-blocks. The block $\langle$ does not appear in the periodic point $\overline{\rangle}$, and the block $\rangle$ does not appear in the periodic point $\overline{\langle}$.*

(E) *For all nonnegative integers ℓ and r, the Y-block*

$$\boxed{C(\langle, \boldsymbol{L})}\,\boldsymbol{L}^{\ell}\,\boxed{C(\boldsymbol{L}, \boldsymbol{R}_i)}\,\boldsymbol{R}_i^{r}\,\boxed{C(\boldsymbol{R}_i, \rangle)}$$

does not contain $\langle^{k_i}$ or $\rangle^{k_i}$.

(F) *For any two blocks $A_0, A_1 \in B(Y)$ and for all $L > L_i+\mathrm{MLen}(A_0, f)$, there exists $(z, 0) \in \mathrm{susp}(Y, f)$ with the following properties.*

(i)

$$z = \cdots \rangle\,\rangle\,\boxed{C(\rangle, A_0)}\,\boldsymbol{.}A_0 D_0 A_1\,\boxed{C(A_1, \langle)}\,\langle\,\langle \cdots,$$

where D_0 is given by

$$\boxed{C(A_0, \langle)}\,\langle^{K_i}\,\boxed{C(\langle, \boldsymbol{L})}\,\boldsymbol{L}^{\ell}\,\boxed{C(\boldsymbol{L}, \boldsymbol{R}_i)}\,\boldsymbol{R}_i^{r}\,\boxed{C(\boldsymbol{R}_i, \rangle)}\,\rangle^{K_i}\,\boxed{C(\rangle, A_1)}, \quad (4)$$

for some positive integers ℓ and r.

(ii) *The block A_1 begins within $2 \cdot 2^{-i}$ of L.*

(iii) *For all $w \in Y$, and all positive integers ℓ' and r', we have that the block A_1 begins within $3 \cdot 2^{-i}$ of L in $(w', 0)$, where*

$$w' = \cdots\cdots \rangle\,\rangle\, C(\rangle, A_0)\boldsymbol{.}\,\boxed{A_0 D_0}\,\boxed{A_1 D}\,w_0 w_1 \cdots,$$

and D is the block

$$\boxed{C(A_1, \langle)}\,\langle^{K_i}\,\boxed{C(\langle, \boldsymbol{L})}\,\boldsymbol{L}^{\ell'}\,\boxed{C(\boldsymbol{L}, \boldsymbol{R}_i)}\,\boldsymbol{R}_i^{r'}\,\boxed{C(\boldsymbol{R}_i, \rangle)}\,\rangle^{K_i}\,\boxed{C(\rangle, w_0)}$$

(G) *The sequence (K_n) satisfies*

$$K_n > 2K_{n-1} + 2k_{n-1} + n. \quad (5)$$

(H) *There exists $\alpha \in (0, 1)$ such that for all i sufficiently large, there exists $\mathbf{F} \in B(Y)$ and an injection $\phi : B^i(X) \cup B^{i+1}(X) \to B_{(1-\alpha)i}(Y \setminus \mathbf{F}, f)$ such that the following hold.*

(i) *The block $\mathbf{F}$ appears in both $\langle$ and $\rangle$.*

(ii) *For all $B \in B^i(X) \cup B^{i+1}(X)$, we have $\rangle\,\boxed{\phi(B)}\,\langle \in B(Y)$.*

(iii) For all $B \in B^i(X) \cup B^{i+1}(X)$*, the only appearance of* $\rangle$ *in* $\rangle\, \phi(B)\, \langle$ *is in the leftmost position; and the only appearance of* $\langle$ *is in the rightmost position.*

The map from Lemma 7 (H) allows us to code X-blocks into Y-blocks, which we think of as blocks of information. The resulting blocks of information are extended to become elements of Y using the blocks $\langle$ and $\rangle$, and then joined together using Lemma 7 (F), which allows us to prescribe where they begin within a certain accuracy, at the cost of placing a filler block in between two blocks of information. Thus we obtain an alternating sequence of information and filler blocks.

We need to be able to distinguish the information from the filler, and this is done by bracketing the filler with concatenations of the blocks $\langle$ and $\rangle$. One technicality is that although we can demand that the blocks $\langle$ and $\rangle$ do not appear in the information, it turns out that it is too much to ask that they do not appear in the filler; see Example 1 in Section 6. However, we can demand that the number of $\langle$ and $\rangle$ that appear in the filler is smaller than the number of $\langle$ and $\rangle$ blocks used to bracket the filler. Although, the blocks $\langle$ and $\rangle$ do not appear in the information, we need condition (Hiii) of Lemma 7, to assure us that we are able to distinguish the beginning or end of a block of information with the end of beginning of a $\rangle$ or $\langle$, respectively.

Proof of Lemma 7 (A) *and* (B). We first select the blocks $\mathbf{L}$ and $(\mathbf{R}_i)$. We consider two cases. If there exist $\mathbf{L}$ and $\mathbf{R}$ for which $\mathbf{L}\,\mathbf{L}$ and $\mathbf{R}\,\mathbf{R}$ are both legal and such that $\gcd(\mathrm{ALen}(\overline{\mathbf{L}}, f), \mathrm{ALen}(\overline{\mathbf{R}}, f)) = 0$, then let $\mathbf{R}_i = \mathbf{R}$ for each i.

Otherwise, let $z^1 \in Y$ be a periodic point. Let $\mathbf{L}$ be the periodic block forming z^1. Fix an integer $i \geq 1$. There exists a positive integer n so that $\mathrm{ALen}(z^1, f)/n < 2^{-i}$. By Proposition 5 (a) $\to$ (b), we have that f is filling; thus there exists a periodic point $z^2 \in Y$ such that $\mathrm{ALen}(z^2, f) \notin (\mathrm{ALen}(z^1, f)/n!)\mathbb{Z}$. Let $\mathbf{R}_i$ be the periodic block forming z^2. Let $\gcd\big(\mathrm{ALen}(z^1, f), \mathrm{ALen}(z^2, f)\big) = r$. Since $r > 0$, but is not a multiple of $\mathrm{ALen}(z^1, f)/n!$, we must have $\mathrm{ALen}(z^1, f) = rm$ for some integer $m > n$, from which we conclude that $r < 2^{-i}$.

Notice that the periods in the suspension of the $\overline{\mathbf{R}_i}$ must approach infinity as $i \to \infty$ as otherwise infinitely many $\overline{\mathbf{R}_i}$ would be identical by the pigeonhole principle and could be expressed as $\overline{\mathbf{R}}$. This would fall into the case considered above. $\square$

Proof of Lemma 7 (C) *and* (D). Let $V = W\boxed{C(W, W)}$ be the shortest word containing W that can be periodically concatenated. Fix a $p \geq 2$ such that $h_{\mathrm{top}}(X, \theta) < h_{\mathrm{top}}(\mathrm{susp}(Y \setminus V^p, f), T^1)$ (such a p exists as the right side converges to $h_{\mathrm{top}}(\mathrm{susp}(Y,f),T^1)$ as $p\to\infty$). Set $\mathbf{F}=V^p$. We may additionally assume that p was chosen so that the $\mathrm{ALen}(\overline{\mathbf{F}},f)>\mathrm{ALen}(\overline{\mathbf{L}},f)$ and also $\mathrm{ALen}(\overline{\mathbf{F}},f)>\mathrm{ALen}(\overline{\mathbf{R}},f)$ in the case that there was a single right block.

Let $\langle$ and $\rangle$ be two blocks of the form $\mathbf{F}^{10}U\mathbf{F}^{10}$, where the U's are chosen to ensure that $\langle$ does not appear in $\overline{\rangle}$, $\rangle$ does not appear in $\overline{\langle}$ and neither of $\langle$ or $\rangle$ can be written as a power of a shorter block. □

Proof of Lemma 7 (E). Observe that (E) is satisfied for some (k_i) if and only if $\overline{\mathbf{L}}$ and $\overline{\mathbf{R}_i}$ are not in the orbit of $\overline{\langle}$ or $\overline{\rangle}$ for all $i\in\mathbb{Z}^+$. For this it suffices to check that the periods in the suspension of $\overline{\mathbf{L}}$ and $\overline{\mathbf{R}_i}$ differ from those of $\overline{\langle}$ and $\overline{\rangle}$.

Since $\overline{\langle}$ and $\overline{\rangle}$ have longer period in the suspension than $\overline{\mathbf{L}}$ by construction, the conclusion is straightforward for $\overline{\mathbf{L}}$. Similarly for the $\overline{\mathbf{R_i}}$ if all of the $\mathbf{R}_i$ are equal to $\mathbf{R}$.

Otherwise, by the remark made in the proof of parts (A) and (B), for sufficiently large i, the period in the suspension of $\overline{\mathbf{R}_i}$ would exceed the periods in the suspension of $\overline{\langle}$ and $\overline{\rangle}$. Discarding those $\mathbf{R}_i$ for which this fails and re-numbering (this can be done without affecting any previous stage of the construction), we obtain the desired conclusion. □

Proof of Lemma 7 (F) *and* (G). It will become obvious that it does not matter how we define K_i for the purposes of conditions (Fi) and (Fii); thus satisfying (5) is trivial. We will need to make K_i large for condition (Fiii).

A straightforward calculation shows that for all $v,v'\in Y$, and $-\iota(v,v')\le i\le j\le\iota(v,v')$, we have that

$$|\,\mathrm{Len}(v,f;i,j)-\mathrm{Len}(v',f;i,j)|\le M2^{c\left(-\iota(v,v')+\min(|i|,|j|)\right)}, \tag{6}$$

where $\iota(v,v')=\min\left\{|j|\in\mathbb{Z}:v_j\ne v'_j\right\}$, $M\ge\|f\|$ is some constant independent of v,v' and $c\in(0,1)$ is the Hölder exponent of f.

Let $G(m,n)$ be the Y-block given by the concatenation

$$\boxed{C(A_0,\langle)}\,\langle^{K_i}\,\boxed{C(\langle,\boldsymbol{L})}\,\boldsymbol{L}^{2N+m}\,\boxed{C(\boldsymbol{L},\boldsymbol{R}_i)}\,\boldsymbol{R}_i^{2N+n}\,\boxed{C(\boldsymbol{R}_i,\rangle)}\,\rangle^{K_i}\,\boxed{C(\rangle,A_1)},$$

where N,m,n are nonnegative integers. We first choose a suitable value for N. Consider

$$z(m,n)=\cdots\rangle\,\rangle\,C(\rangle,A_0).\boxed{A_0}G(m,n)\boxed{A_1}C(A_1,\langle)\,\langle\cdots$$

We think of the blocks $\boldsymbol{L}^{2N+m}$ as concatenations $\boldsymbol{L}^N\ \boldsymbol{L}^m\ \boldsymbol{L}^N$ and similarly with the $\boldsymbol{R}_i$ blocks. Writing it this way makes it clear, using (6), that the length in the suspension of the first $\boldsymbol{L}^N$ block is exponentially close (in N) to the corresponding block in $z(0,0)$ uniformly in m (and n). Similarly with the second $\boldsymbol{L}^N$ block. Additionally, the length in the suspension of the $\boldsymbol{L}^m$ block is exponentially close in N to $m \cdot \mathrm{ALen}(\overline{L}, f)$, again uniformly in m and n. Similar statements hold for the $\boldsymbol{R}_i$ blocks.

Similarly, the differences of the lengths in the suspension of the other corresponding sections appearing in the concatenations forming $z(m,n)$ and $z(0,0)$ approach 0 exponentially in N, uniformly in m and n.

Hence if the length in the suspension of the block $A_0\boxed{G(0,0)}$ in $(z(0,0),0)$ is $b(N)$ (the N dependence appears in $G(0,0)$), the arguments above show that N can be chosen so that the length in the suspension of the block $A_0\boxed{G(m,n)}$ in $z(m,n)$ is within 2^{-i} of $b(N) + m\,\mathrm{ALen}(\overline{\boldsymbol{L}}, f) + n\,\mathrm{ALen}(\overline{\boldsymbol{R}_i}, f)$ uniformly in m and n. Let

$$\begin{aligned} b' \;=\; & \mathrm{MLen}(A_0, f) + \mathrm{MLen}\left(\boxed{C(A_0, \langle)}\,\langle^{K_i}\,\boxed{C(\langle, \boldsymbol{L})}\,\boldsymbol{L}^N, f\right) + \\ & \mathrm{MLen}\left(\boxed{C(\boldsymbol{L}, \boldsymbol{R}_i)}\,\boldsymbol{R}_i^N, f\right) + \mathrm{MLen}\left(\boxed{C(\boldsymbol{R}_i, \rangle)}\,\rangle^{K_i}\,\boxed{C(\rangle, A_1)}, f\right). \end{aligned}$$

Note that $b \le b'$. By part (B) $\gcd\big(\mathrm{ALen}(\overline{\boldsymbol{L}}, f), \mathrm{ALen}(\overline{\boldsymbol{R}_i}, f)\big) \le 2^{-i}$, thus conditions (Fi) and (Fii) follow by choosing $L_i \ge b' + M_i - \mathrm{MLen}(A_0, f)$, where M_i is such that for all $M > M_i$, there is a non-negative integer combination of $\mathrm{ALen}(\overline{\boldsymbol{L}}, f)$ and $\mathrm{ALen}(\overline{\boldsymbol{R}_i}, f)$ that approximates M to within an error of $2 \cdot 2^{-i}$.

To see that condition (Fiii) holds compare z with w'. Observe that by (6) and the definition of D, the difference between the length in the suspension of $A_0 D_0$ in z and w' is bounded by a function of i that goes to 0 as $K_i \to \infty$. Thus we may choose K_i sufficiently large so that A_1 begins within $3 \cdot 2^{-i}$ of L in $(w', 0)$ for all $w \in Y$. □

Proof of Lemma 7 (H). Let P be the set of all periodic points of $Y \setminus \mathbf{F}$. By [3, Theorem 4.1], we have that

$$\kappa_1 \frac{e^{h'u}}{u} \le \#\{y \in P : \mathrm{ALen}(y, f) \le u\} \le \kappa_2 \frac{e^{h'u}}{u}, \tag{7}$$

for some constants $\kappa_1, \kappa_2 > 0$.

Note that by (6), for some constant $\kappa > 0$, the difference between the length in the suspension of a block and its maximum length in the suspension is at most κ. By (7), by taking p large, we have that there exists $\alpha \in (0,1)$, such that for all i sufficiently large

$$\#\big(B^i(X) \cup B^{i+1}(X)\big) \le \#B_{(1-\alpha)i}(Y \setminus \mathbf{F}, f).$$

Condition (Hii) is easily satisfied by taking i sufficiently large, since the maximum lengths in the suspension of all connecting blocks are bounded above. Conditions (Hi) and (Hiii) are easily verified with the definitions of $\mathbf{F}$, $\langle$, and $\rangle$. $\square$

For a set A, let $\mathrm{pow}(A)$ denote its power set.

Lemma 8 (Markers upstairs)**.** *Let X be a non-trivial subshift with an invariant measure μ. Let $(\ell_i)_{i=1}^{\infty}$ be a sequence of positive integers such that $\ell_{i+1}/\ell_i \geq 4$. There exists a measurable function $M : \mathbb{Z}^+ \times X \to \mathrm{pow}(\mathbb{Z})$ and a set $X' \subset X$ of full measure with the following properties.*

(a) For all $i \in \mathbb{Z}^+$ and for all $x \in X'$, we have $M(i, \theta(x)) = M(i, x) - 1$.
(b) For all $0 < i < j$ and all $x \in X'$, we have $M(j, x) \subset M(i, x)$.
(c) For all $x \in X'$, the distance between two successive elements of $M(1, x)$ is ℓ_1 or $\ell_1 + 1$.
(d) For all $i \geq 2$ and $x \in X'$, if $u < v$ are two consecutive elements of $M(i, x)$ then $\ell_i \leq v - u \leq 2\ell_i$. The number of elements of $M(i-1, x)$ in $[u, v]$ is between $\ell_{i+1}/(2\ell_i)$ and $2\ell_{i+1}/\ell_i$.

Proof. From Alpern's multi-tower theorem, [1], it follows that there exist two measurable sets $Q_1, Q_2 \in \mathcal{F}$, such that $\{\theta^j(Q_1)\}_{j=0}^{\ell_1-1} \cup \{\theta^j(Q_2)\}_{j=0}^{\ell_1}$ give a partition of X (modulo a null set). For each $x \in X$, we let

$$M(1, x) = \{u \in \mathbb{Z} : \theta^u(x) \in Q_1 \cup Q_2\}.$$

Clearly, conditions (a) and (c) are satisfied.

Assume that $M(i, x)$ has been defined so that conditions (a), (b), and (d) hold for all $j \leq i$. It remains to define $M(i+1, x)$. Again, by Alpern's theorem, there exist two sets $P_1, P_2 \in \mathcal{F}$, such that $\{\theta^j(P_1)\}_{j=0}^{\ell_{i+1}+2\ell_i-1} \cup \{\theta^j(P_2)\}_{j=0}^{\ell_{i+1}+2\ell_i}$ give a partition of X. Consider the set

$$M(i+1, x)' = \{u \in \mathbb{Z} : \theta^u(x) \in P_1 \cup P_2\}.$$

For each $u' \in M(i+1, x)'$, let $\kappa(u') = \inf\{u \in M(i, x) : u' < u\}$. Set $M(i+1, x) = \{\kappa(u') : u' \in M(i+1, x)'\}$. $\square$

The following definitions will be important in the defining the sequence of maps in Proposition 6. Let M be the function from Lemma 8. For each $i \in \mathbb{Z}^+$ and each $x \in X$, let

$$N_i^-(x) := \sup\{z \in M(i, x) : z \leq 0\} \tag{8}$$
$$\text{and } N_i^+(x) := \inf\{z \in M(i, x) : z > 0\}. \tag{9}$$

Corollary 9. *Let $(a_i)_{i=1}^{\infty}$ be a sequence of positive integers. If*

$$\ell_{i+1} \geq 2^{i+2} a_i \ell_i, \tag{10}$$

then the function M from Lemma 8 has the additional property that

$$\mu(\{x \in X : N^+_{i+1}(x) - N^+_i(x) \le 2a_i\ell_i\}) \le 2^{-i}; \text{ and}$$
$$\mu(\{x \in X : N^-_i(x) - N^-_{i+1}(x) \le 2a_i\ell_i\}) \le 2^{-i}.$$

Hence for μ-almost every $x \in X'$, $N^+_i(x) \to \infty$ and $N^-_i(x) \to -\infty$ as $i \to \infty$.

Proof. We estimate $\mu(\{x \in X\colon N^+_{i+1}(x) - N^+_i(x) \le 2a_i\ell_i\})$, the other estimate being identical. From Lemma 8, $N^+_i(x)$ is less than $2\ell_i$ for all x. Hence $N^+_{i+1}(x)-N^+_i(x) \le 2a_i\ell_i$ implies $N^+_{i+1}(x) < 2(a_i+1)\ell_i \le 4a_i\ell_i$.

Next, notice since consecutive $(i+1)$-markers are separated by at least ℓ_{i+1}, one has for each $0 \le k < \ell_{i+1}$, $N^+_{i+1}(x) = \ell_{i+1}$ if and only if $N^+_{i+1}(\theta^k(x)) = \ell_{i+1} - k$. Hence the sets $(N^+_{i+1})^{-1}(k)$ have equal measure for k in the range $1 \le k \le \ell_{i+1}$ so that $\mu(\{x\colon N^+_{i+1}(x) = j\}) \le 1/\ell_{i+1}$ for all $j \le \ell_{i+1}$. We then see that $\mu(\{x\colon N^+_{i+1}(x) \le 4a_i\ell_i\}) \le 4a_i\ell_i/\ell_{i+1} < 2^{-i}$ as required. $\square$

4. Definition of Injection

Proof of Proposition 6. Suppose that X is a uniquely ergodic subshift and $\mathrm{susp}(Y, f)$ is a topologically mixing suspension flow whose topological entropy exceeds the entropy of X as in the statement of the proposition. Let W be a Y-block. Let $\langle, \rangle, \boldsymbol{L}, \boldsymbol{R}_1, \boldsymbol{R}_2 \ldots, (K_n, k_n, L_n)_{n\in\mathbb{Z}^+}$, and $\alpha \in (0, 1)$ be given by Lemma 7 (K_n being the number of $\langle$ and $\rangle$ used to delimit the filler blocks between level n blocks, L_n being a length in the suspension such that any gap of size exceeding L_n can be filled up to within 2^{-n} accuracy by level n filler blocks, and α being the compression that can be achieved in encoding the first level information blocks). The block W appears in $\langle$ and $\rangle$.

Note that by (6), for some constant $\kappa > 0$, the difference between the length in the suspension of a block and its maximum length in the suspensions is at most κ. Let $(a_i)_{i\in\mathbb{Z}^+}$ be a sequence of positive integers such that

$$a_i(L_i - \kappa) > 2L_{i+1}. \tag{11}$$

The a_i represent the number of level i blocks forming the crumple zone when building the level $i+1$ blocks. Choose $\ell_1 > 0$ large enough so that by Lemma 7 (H), there exists an injection $\phi : B^{\ell_1+1}(X) \cup B^{\ell_1}(X) \to B_{(1-\alpha)\ell_1}(Y \setminus \mathbf{F}, f)$ such that $\rangle\, \phi(B) \,\langle \in B(Y)$ for all $B \in B^{\ell_1+1}(X) \cup B^{\ell_1}(X)$. We can additionally require that ℓ_1 is large enough that

$$\alpha\ell_1 > 2L_1. \tag{12}$$

We define the subsequent ℓ_i by setting

$$\ell_{i+1} = 2^{i+2} a_i \ell_i. \tag{13}$$

Let M and $X' \subset X$ be given be Lemma 8, and N_i^+ and N_i^- be given by (8) and (9), respectively. By Corollary 9, $N_i^+(x) \to \infty$ and $N_i^-(x) \to -\infty$ for μ-almost all $x \in X'$.

We will recursively define sequences of maps $\phi_i : X \to B(Y)$. Each ϕ_i will be extended to a map $\psi_i \colon X \to Y$ by the relation

$$\psi_i(x) = \cdots \rangle \, \rangle . \phi_i(x) \, \langle \, \langle \cdots . \tag{14}$$

We set

$$\Psi_i(x) := T^{-N_i^-(x)}(\psi_i(x), 0) \tag{15}$$

for all $x \in X'$. Given $(y, s) \in \mathrm{susp}(Y, f)$, let $\pi_1(y, s) = y$ and $\pi_2(y, s) = s$. We shall inductively construct maps ϕ_i, ψ_i, Ψ_i satisfying the following properties (previously illustrated in Figure 1).

(I) For all $x \in X'$; and for all $i \geq 1$, we have $\rangle \boxed{\phi_i(x)} \langle \in B(Y)$.

(II) For all $x \in X'$ and for all $i \geq 1$, we have

$$N_i^+(x) - N_i^-(x) > 2L_i + \mathrm{MLen}(\phi_i(x), f).$$

(III) For all $x \in X'$, $i \geq 1$, and all $k \in [N_i^-(x), N_i^+(x))$, we have

$$\Psi_i(\theta^k x) = T^k \Psi_i(x).$$

(IV) For all $x \in X'$ and for all $i \geq 2$, if $|N_i^+(x) - N_{i-1}^+(x)| > 2a_i\ell_i$, then there exists $|\varepsilon| \leq 3 \cdot 2^{-i}$ such that

$$\pi_1\big(T^{N_{i-1}^-(x)+\varepsilon}\Psi_i(x)\big)_0^{n-1} = \phi_{i-1}(x) = \pi_1\big(T^{N_{i-1}^-(x)}\Psi_{i-1}(x)\big)_0^{n-1}; \text{ and}$$

$$\pi_2\big(T^{N_{i-1}^-(x)+\varepsilon}\Psi_i(x)\big) = 0 = \pi_2\big(T^{N_{i-1}^-(x)}\Psi_{i-1}(x)\big),$$

where n is the size of $\phi_{i-1}(x)$.

Let $x \in X'$. Set $\phi = \phi_1$. By Lemma 7 (H) (Hii), we may set

$$\psi_1(x) := \cdots \rangle \, \rangle \, . \phi_1(x_{N_1^-(x)} \cdots x_{N_1^+(x)-1}) \, \langle \, \langle \cdots .$$

Note that by Lemma 8 (c), we have that $N_1^+(x) - N_1^-(x) \in \{\ell, \ell+1\}$. Clearly, by (12) and the definition of ϕ conditions (I), (II) and (III) are satisfied.

Suppose ϕ_i and ψ_i have been defined and satisfy conditions (I), (II), (III), and (IV). Consider the set

$$\left\{ v \in M(i, x) : v \in [N_{i+1}^-(x), N_{i+1}^+(x)) \right\}.$$

Let $N_{i+1}^-(x) = n^1 < n^2 < \cdots < n^{j-1} < n^j < N_{i+1}^+(x)$ be an enumeration of the set. Let $j' = j - a_i$ (this is positive because the number of level i blocks in a level $i+1$ block is at least $\ell_{i+1}/(2\ell_i) > 2^{i+1}a_i$ by Lemma 8 (d) and by (13). The point $\psi_{i+1}(x)$ will be of the form

$$\cdots\rangle\,\rangle\,\mathbf{.}B_1F_1B_2F_2\cdots F_{j-1}B_j\,\langle\,\langle$$

where $B_k = \phi_i(\theta^{n^k}x)$ for each $1 \le k \le j$ and the F_k are given by $F(\ell_k, r_k)$, where

$$F(\ell,r) := \langle^{K_i}\boxed{C(\langle,\boldsymbol{L})}\,\boldsymbol{L}^\ell\,\boxed{C(\boldsymbol{L},\boldsymbol{R}_i)}\,\boldsymbol{R}_i^r\,\boxed{C(\boldsymbol{R}_i,\rangle)}\,\rangle^{K_i}. \tag{16}$$

We call such an F a **level i filler block**.

By repeated applications of Lemma 7 we can choose the ℓ_k and r_k such that for $1 \le k \le j'$, B_k begins within $3\cdot 2^{-i}$ of n^k in $\Psi_{i+1}(x) = T^{-n^1}\psi_{i+1}(x)$. For $k \in [j', j)$ we simply set $F_k = F(1,1)$ to create the crumple zone mentioned earlier.

We set $\phi_{i+1}(x) = B_1F_1B_2\cdots F_{j-1}B_j$, and note that property (I) is trivially satisfied. We call the $\phi_{i+1}(x)$ **level $\mathbf{i+1}$ blocks**.

Notice that the crumple zone consists of a_i level i blocks. Each of these consists of at most $2\ell_i$ symbols by Lemma 8 (d), so that $|N_i^+(x) - N_{i-1}^+(x)| > 2a_i\ell_i$ guarantees that x lies in a level i block outside the crumple zone. The construction above then ensures that (IV) holds for $i+1$.

By construction, we have that $\mathrm{MLen}(F_k, f) \le L_i + \kappa$ for $k \in [j', j)$, so using (II) at level i for these k's we have

$$n^{k+1} - n^k - \mathrm{MLen}(F_k, f) \ge L_i - \kappa.$$

Since there are a_i such k's, we obtain

$$N_{i+1}^+(x) - N_{i+1}^-(x) - \mathrm{MLen}(\phi_{i+1}(x), f) \ge a_i(L_i - \kappa)$$

Hence by (11), condition (II) is satisfied at level $i+1$. Note that property (III) follows immediately from the definition of Ψ_{i+1}. Hence the inductive step is complete.

Conclusion (a) (of Proposition 6) follows immediately from property (III) and Corollary 9. Conclusion (b) follows from property (IV), Corollary 9, and the Borel-Cantelli Lemma.

It remains to show that properties (c), (d), and (e) (of Proposition 6) hold. Suppose that

$$\lim_{i\to\infty}\Psi_i(x) := \Psi(x) = (\mathbf{y}(x), \mathbf{s}(x)) \in \mathrm{susp}(Y, f),$$

for all $x \in X'' \subset X'$, where X'' is also a set of full measure. Let $x \in X''$. Let $(n_i)_{i\in\mathbb{Z}}$ be an enumeration of the set $M(1,x)$, where $n_0 = N_1^-(x)$. For each $i \in \mathbb{Z}$, let

$$B_i := \varphi(x_{n_i}\cdots x_{n_{i+1}-1}).$$

Recall that in the construction, level 1 blocks (images of ϕ) are combined into longer level 2 blocks by placing level 1 filler between the pairs of blocks while the leftmost level 1 block in a level 2 block does not (yet) have filler next to it and similarly for the rightmost level 1 block in a level 2 block. At the next stage level 2 blocks are interspersed with level 2 filler, again leaving the end blocks bare. Hence, as long as $N_i^+(x) \to \infty$ and $N_i^-(x) \to \infty$, then the level of the filler on the left of $\phi(x)$ is $\min\{k\colon N_{k+1}^-(x) \neq N_1^-(x)\}$ with a similar expression for the level of filler on the right. Hence for each $x \in X''$, $\mathbf{y}(x)$, the first coordinate of $\Psi(x)$, is of the form

$$\cdots B_{-1}F_{-1}B_0F_0B_1F_1\cdots, \tag{17}$$

where the B_i are of the form $\phi(C)$ for $C \in B^{\ell_i}(X) \cup B^{\ell_i+1}(X)$ and the F_i are Y-blocks of the form $\langle^{K_n} C \rangle^{K_n}$ for some n and some Y-block C; furthermore, by Lemma 7 (D) and (G), we have that the block C does not contain a $\langle^{K_j}$ or $\rangle^{K_j}$, with $j \geq n$.

Note that the blocks $\rangle^n$ and $\langle^n$ appear in $\mathbf{y}(x)$ for all $n \in \mathbb{Z}^+$. Hence by Lemma 7 (D), property (c) holds. Since W appears in $\langle$, property (e) holds.

Let $y \in Y$ be of the form (17). If y_0 belongs to one of the F blocks, then one of the following happens:

(1) there exist $m \leq 0 < n$ such that y_m^{n-1} is $\langle$ or $\rangle$; Or
(2) there exists an $i \geq 1$ and an $n > 0$ such that $\rangle^{K_i}$ occurs in y_0^{n-1} but $\langle^{K_i}$ does not.

On the other hand, if y_0 belongs to one of the B blocks, then (2) is ruled out by the properties of the filler blocks. (1) is ruled out by properties of the level one blocks in Lemma 7 (Hi) and (Hiii).

For each symbol in y, one can therefore decide whether it belongs to one of the B blocks or one of the F blocks. Hence given the first coordinate of $\Psi(x)$, (the element of Y) one can recover the sequence of B_i blocks and hence applying ϕ_1^{-1} to each of them, one can recover x up to translation, completing the proof of property (d).

□

5. Proof of Theorem 1

Bowen and Ratner constructed symbolic dynamics for geodesic flows, proving the following theorem.

Theorem 10 (Bowen, R. [4] and Ratner, M. [20])**.** *Let $(W^t)_{t\in\mathbb{R}}$ be geodesic flow on a compact surface of variable negative curvature $\mathcal{M}$, with unit tangent bundle* $\mathrm{UT}(\mathcal{M})$*. There exists a suspension flow*

$(\mathrm{susp}(Y,f),(T^t)_{t\in\mathbb{R}})$ *over a subshift of finite type* Y *under a Hölder continuous function* $f : Y \to (0,\infty)$*, and a finite-to-one continuous surjection from* $\pi : \mathrm{susp}(Y,f) \to \mathrm{UT}(\mathcal{M})$ *such that* $\pi \circ W^t = T^t \circ \pi$ *for all* $t \in \mathbb{R}$*.*

Furthermore, the subshift of finite type Y can be chosen to be irreducible [6, Lemma 2.1], [5, Section 3], and π is one-to-one on a set of full measure for any measure that is ergodic and fully supported [19, Theorem III.8].

Note that if map π in Theorem 10 were one-to-one for all ergodic measures, then Theorem 1 would follow immediately from Theorem 2. We will not be able to use Theorem 10 directly because the map we define for Theorem 3 has an image which is in general not fully supported. We will use the following corollary, which is a consequence of the Proof of Theorem 10 given by Bowen [4].

Corollary 11 (Corollary of Bowen's proof of Theorem 10)**.** *There exists* $W \in B^m(Y)$ *for some* m *such that* π *is one-to-one on the set of all* $(y,s) \in \mathrm{susp}(Y,f)$ *such that for infinitely many positive* $n \geq 0$ *and infinitely many* $n < 0$*, we have* $(\theta^n y)_0^{m-1} = W$*.*

Proof of Theorem 1. As in the proof of Theorem 2, by the Jewett-Krieger theorem [15], we may make the simplifying assumption that the measure-preserving system that is to be embedded is a uniquely ergodic subshift. Let $(\mathrm{susp}(Y,f),(T^t)_{t\in\mathbb{R}})$, π, and W be given by Theorem 10 and Corollary 11. Now repeating verbatim the proof of Theorem 3, but using the specific word W gives and embedding of X into Y, where W appears in the image points infinitely many times in the future and the past. Corollary 11 then implies that π is one-to-one on the image of Ψ. The composition yields the result. $\square$

Proof of Corollary 4. Let $(\Omega',\mathcal{D}',\nu')$ be a non-atomic probability space endowed with an ergodic measure-preserving automorphism S'. Consider the automorphism S defined on $\Omega := \Omega' \times \{0,1\}$ given by $S(x,0) := (x,1)$ and $S(x,1) := (S'(x),0)$ for all $x \in \Omega'$, and the measure p on $\{0,1\}$ such that $p(0) = p(1) = 1/2$. It is easy to verify that S is ergodic and preserves the measure $\nu := \nu' \times p$. Furthermore, by [9, Lemma 8.7], S does not have a square-root; that is, there does not exist a subset of $\Omega \times \{0,1\}$ of full measure for which there is measure-preserving automorphism U such that $U \circ U = S$ on the subset. Hence the result follows immediately from Theorem 1 and choosing S' and thus S with measure-theoretic entropy sufficiently small. $\square$

Proof of Corollary 11. Let $\mathcal{P}$ be the Markov partition of $UT(\mathcal{M})$. It is known that the elements of $\mathcal{P}$ are the closure of their interiors, and that their interiors are disjoint. Let $B \in \mathcal{P}$. By expansiveness of geodesic flow, there exists a cylinder set W in Y and an interval J such that $\pi(W \times J) \subset \operatorname{int} B$. The map π is one-to-one on the preimage of the set of points whose orbits never intersect the stable or unstable boundaries of the Markov partition. Further, the union of the stable boundaries of the elements of the partition is forwards-invariant, while the union of the unstable boundaries of the elements of the partition is backwards-invariant.

If W appears infinitely often in the future of y, then $\pi(y, s)$ cannot belong to the stable boundary. If W appears infinitely often in the past of y, then $\pi(y, s)$ cannot belong to the unstable boundary. By the above observations about invariance, the orbit of $\pi(y, s)$ never hits the stable or unstable boundaries of the partition and so $\pi^{-1}(\pi(y, s)) = \{(y, s)\}$ as required. □

6. Appendix

The arguments given above also allow us to resolve a second question about geodesic flow due to Ledrappier and Federico and Jana Rodriguez-Hertz, who asked if a minimal subset for the $\mathbb{R}$-action of geodesic flow is necessarily minimal for the $\mathbb{Z}$-action of the time one map of the geodesic flow. We give a counterexample to this using Corollary 11 above.

Proposition 12. *Let $(G^t)_{t\in\mathbb{R}}$ be the geodesic flow on a compact surface of variable negative curvature $\mathcal{M}$, with unit tangent bundle $\mathrm{UT}(\mathcal{M})$. Then there exists a minimal subset C for the $\mathbb{R}$-action on $\mathrm{UT}(\mathcal{M})$ that is not minimal for the $\mathbb{Z}$-action.*

Proof. As before, let $\operatorname{susp}(Y, f)$ be a suspension over an irreducible shift of finite type and π be a continuous factor map from the suspension flow to the geodesic flow. By Corollary 11, let W be a Y-block such that on the set of points for which W appears infinitely often in the past and the future, π is one-to-one.

If Y contains a periodic point $\overline{A}$ with rational period in the suspension, then let $x = \pi(\overline{A}, 0)$. Clearly the $\mathbb{R}$-orbit of x is closed and the $\mathbb{Z}$-orbit is a discrete subset.

Suppose then that $\operatorname{susp}(Y, f)$ has no periodic points with rational period in the suspension. Let B_0 and B_1 be two Y-blocks in which W appears, such that $B_i B_j$ is a Y-block for all pairs $i, j \in \{0, 1\}$. Let them agree on sufficiently many symbols at each end that switching any number of B_0's for B_1's or vice versa in a point $y \in Y$ does not change the length in the suspension of any contiguous block (disjoint from the blocks being switched) by more than $\frac{1}{10}$ (see (6)). We may also assume that $\overline{B_0}$ does not contain any B_1's and $\overline{B_1}$ does not contain any B_0's. Finally we will assume that the B_i are not powers of smaller words. See Lemma 7 (D) for a similar construction.

We build a point y in Y in the following way: Let

$$y^{(n)} = \cdots B_0\, B_0 . \boxed{B_{i_1} \cdots B_{i_n}}\, B_0\, B_0 \cdots$$

where the (i_n) are defined recursively by

$$i_n = \begin{cases} 1 & \text{if the block } B_{i_{n-1}} \text{ ends within } \frac{1}{10} \text{ of } \mathbb{Z} \text{ in } y^{(n-1)}; \\ 0 & \text{otherwise.} \end{cases}$$

The limit point is then $y = \cdots B_0\, B_0 . B_{i_1}\, B_{i_2} \cdots$. Notice that in y the B_0 and B_1 both appear with bounded gaps (because in any sufficiently long block of B_0's, say, their lengths in the suspension become arbitrarily close to $\mathrm{ALen}(\overline{B_0}, f)$, the fractional parts of whose multiples are dense in $[0, 1)$). Also all B_1's start at points of $\mathbb{Z} + [-\frac{1}{5}, \frac{1}{5}]$ (when subsequent blocks are altered the starting points may move by up to $\frac{1}{10}$). Let Ω denote the ω-limit set of y under the $\mathbb{R}$-action. By the above observation, all points of Ω have B_0's and B_1's appearing with bounded gaps. Further, for each $(z, s) \in \Omega$, there is a $\beta \in [0, 1)$ such that all B_1's begin at points of $\mathbb{Z} + \beta + [-\frac{1}{5}, \frac{1}{5}]$. Let C be a minimal subset of Ω (under the $\mathbb{R}$-action) and let $(z, s) \in C$. Suppose that B_1's start at points of $\mathbb{Z} + \beta + [-\frac{1}{5}, \frac{1}{5}]$. Then the same is true of the $\mathbb{Z}$-orbit closure of (z, s), $\Omega' \subseteq C$). So Ω' is a closed $\mathbb{Z}$-invariant subset of C. Since $T^{\frac{1}{2}}(z, s) \notin \Omega'$, we see that Ω' is a proper closed $\mathbb{Z}$-invariant subset of C, so that C is not minimal for the $\mathbb{Z}$-action.

By Corollary 11, since C consists of points with infinitely many W's in the past and the future, the restriction of π to C is one-to-one. Since C is compact, π is a conjugacy between the real action on C and the real action on $\pi(C)$. Hence (lack of) minimality is preserved and $\pi(C)$ is minimal for the $\mathbb{R}$-action on $\mathrm{UT}(\mathcal{M})$ but not for the $\mathbb{Z}$ action. $\square$

In the following example, we construct a filling function over the full shift taking only rational values, but for which the greatest common divisor of any finite number of orbits is positive. Further it has the property that $\mathrm{susp}(X \setminus F, f)$ fails to be topologically weak mixing for any word F.

Example 1. Let X be the full two shift on $\{0, 1\}$. Let $N_k = 3^k$, $E_k = \{x : x_0^{N_k - 1} \text{ contains all } k\text{-blocks}\}$ and $a_k = 1 + 2^{-N_{k+1}}$. Let $K(x) = \min\{k : x \notin E_k\}$ and define $f(x) = a_{K(x)}$.

This is a Lipschitz function taking only rational values. Now fix any word w, of size k, say. All points of $X \setminus w$ take f values that are a multiple of $2^{-N_{k+1}}$ so that the sums of f over periodic points that don't contain any w's are multiples of $2^{-N_{k+1}}$.

However the greatest common divisor of the full set of periods is 0. To see this, let w_1 and w_2 be two distinct words of size k. Assume further that $0^k w_1$ does not contain a w_2 and $w_2 0^k$ does not contain a w_1. Take a block of size N_k containing w_1 as its first k symbols and w_2 as its last k symbols with no w_1's or w_2's between. Further ensure that the block contains all k words. Extend this to a block of size $2N_k$ by adding N_k 0's and let x be the periodic orbit obtained by concatenating this word. Summing f over the period, one obtains exactly one value of $1 + 2^{-N_{k+2}}$ while all the other values are multiples of $2^{-N_{k+1}}$ so that the period in the suspension is an odd multiple of $2^{-N_{k+2}}$. Since the point $\overline{0}$ has a period in the suspension of $1 + 2^{-9}$, the greatest common divisor of these two periods is a factor of $513/2^{N_{k+2}}$. Since this holds for all k, the example is complete.

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(A. Quas) Department of Mathematics and Statistics, University of Victoria, PO BOX 3060 STN CSC, Victoria, BC V8W 3R4, Canada

E-mail address: `aquas at uvic.ca`

URL: `http://www.math.uvic.ca/∼aquas/`

(T. Soo) Department of Mathematics and Statistics, University of Victoria, PO BOX 3060 STN CSC, Victoria, BC V8W 3R4, Canada

E-mail address: `tsoo at uvic.ca`

URL: `http://www.math.uvic.ca/∼tsoo/`