# BRATTELI DIAGRAMS WHERE RANDOM ORDERS ARE IMPERFECT

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Abstract. For the simple Bratteli diagrams $B$ where there is a single edge connecting any two vertices in consecutive levels, we show that a random order has uncountably many infinite paths if and only if the growth rate of the level-$n$ vertex sets is super-linear. This gives us the dichotomy: a random order on a quickly growing Bratteli diagram admits a homeomorphism, while a random order on a slowly growing Bratteli diagram does not. We also show that for a large family of infinite rank Bratteli diagrams $B$, a random order on $B$ does not admit a continuous Vershik map.

## 1. Introduction

Consider the following random process. For each natural number $n$, we have a collection of finitely many individuals. Each individual in the $n+1$-st collection randomly picks a parent from the $n$-th collection, and this is done for all $n$. If we know how many individuals there are at each stage, the question "How many infinite ancestral lines are there?" almost always has a common answer $j$: what is it? We can also make this game more general, by for each individual, changing the odds that he choose a certain parent, and ask the same question.

The information that we are given will come as a *Bratteli diagram* $B$ (Definition 2.1), where each "individual" at stage $n$ is represented by a vertex in the $n$-th vertex set $V_n$, and the chances that an individual $v \in V_{n+1}$ chooses $v' \in V_n$ as a parent is the ratio of the number of edges incoming to $v$ with source $v'$ to the total number of edges incoming to $v$. We consider the space $\mathcal{O}_B$ of *orders* on $B$ (Definition 2.4) as a measure space equipped with the uniform product measure $\mathbb{P}$. A result in [BKY14] (stated as Theorem 3.1 here) tells us that there is some $j$, either a positive integer or infinite, such that a $\mathbb{P}$-random order $\omega$ possesses $j$ maximal paths.

Bratteli diagrams, which were first studied in operator algebras, appeared explicitly in the measurable dynamical setting in [Ver81],[Ver85], where it was shown that any ergodic invertible transformation of a Lebesgue space can be represented as a measurable "successor" (or *Vershik*) map on the space of infinite paths $X_B$ in some Bratteli diagram $B$ (Definition 2.8). The successor map, which is defined using an order on $B$, is not defined on the set of maximal paths in $X_B$, but as this set is typically a null set, it poses no problem in the measurable framework. Similar results were discovered in the topological setting in [HPS92]: any minimal homeomorphism on a Cantor Space has a representation as a (continuous, invertible) Vershik map which is defined on all of $X_B$ for some Bratteli diagram $B$. To achieve this, the technique used in [HPS92] was to construct the order so that it had a

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unique minimal and maximal path, in which case the successor map extends uniquely to a homeomorphism of $X_B$. For such an order our quantity $j = 1$. We were curious to see whether such an order is typical, and whether a typical order defined a continuous Vershik map. Clearly this would depend on the Bratteli diagram being considered.

In this article we compute $j$ for a large family of *infinite rank* Bratteli diagrams (Definition 2.3). Namely, in Theorem 4.2, we show that $j$ is uncountable for the situation where any individual at stage $n$ is equally likely to be chosen as a parent by any individual at stage $n + 1$, whenever the generation growth rate is super-linear. If the generations grow at a slower rate than this, $j = 1$. We note that this latter situation has been studied in the context of gene survival in a variable size population, as in the Fisher-Wright model (e.g. [Sen74], [Don86]). We describe this connection in Section 4.

In Theorem 4.12 we generalise part of Theorem 4.2 to a large family of Bratteli diagrams. We can draw the following conclusions from these results. First we show in Corollary 4.5 that $j$ is not an invariant of $B$'s *dimension group* [Eff81]. Second, an order $\omega$ is called *perfect* if it admits a continuous Vershik map. For a large class of simple Bratteli diagrams (including the ones we identify in Theorems 4.1 and 4.12), if $j > 1$, then a $\mathbb{P}$-random order is almost surely not perfect (Theorem 3.3). This is in contrast to the case for finite rank diagrams, where almost any order put on "almost any" finite rank Bratteli diagram is perfect (Section 5, [BKY14]). Indeed, one wonders whether for a "reasonable" infinite rank diagram, it is always the case that $j = \infty$. Here the word reasonable needs to be defined in light of the results above.

## 2. Bratteli diagrams and Vershik maps

In this section, we collect the notation and basic definitions that are used throughout the paper.

### 2.1. **Bratteli diagrams.**

**Definition 2.1.** A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i\geq 0} V_i$ and the edge set $E = \bigcup_{i\geq 1} E_i$ are partitioned into disjoint subsets $V_i$ and $E_i$ where

(i) $V_0 = \{v_0\}$ is a single point;

(ii) $V_i$ and $E_i$ are finite sets;

(iii) there exists a range map $r$ and a source map $s$, both from $E$ to $V$, such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$.

Note that $E$ may contain multiple edges between a pair of vertices. The pair $(V_i, E_i)$ or just $V_i$ is called the *i-th level* of the diagram $B$. A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively.

For $m < n$, $v \in V_m$ and $w \in V_n$, let $E(v, w)$ denote the set of all paths $\overline{e} = (e_1, \ldots, e_p)$ with $s(e_1) = v$ and $r(e_p) = w$. For a Bratteli diagram $B$, let $X_B$ be the set of infinite paths starting at the top vertex $v_0$. We endow $X_B$ with the topology generated by cylinder sets $\{U(e_j, \ldots, e_n) : j,\ n \in \mathbb{N}, \text{ and } (e_j, \ldots, e_n) \in E(v, w), v \in V_{j-1}, w \in V_n\}$, where $U(e_j, \ldots, e_n) := \{x \in X_B : x_i = e_i,\ i = j, \ldots, n\}$. With this topology, $X_B$ is a 0-dimensional compact metric space.

**Definition 2.2.** Given a Bratteli diagram $B$, the $n$-th *incidence matrix* $F_n = (f_{v,w}^{(n)})$, $n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix whose entries $f_{v,w}^{(n)}$ are equal to the number of edges between the vertices $v \in V_{n+1}$ and $w \in V_n$, i.e.

$$f_{v,w}^{(n)} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|.$$

Next we define some families of Bratteli diagrams that we work with in this article.

**Definition 2.3.** Let $B$ be a Bratteli diagram.

(1) We say $B$ has *finite rank* if for some $k$, $|V_n| \leq k$ for all $n \geq 1$.
(2) We say that $B$ is *simple* if for any level $n$ there is $m > n$ such that $E(v, w) \neq \emptyset$ for all $v \in V_n$ and $w \in V_m$.
(3) We say that a Bratteli diagram is *completely connected* if all entries of its incidence matrices are positive.

In this article we work only with completely connected Bratteli diagrams.

### 2.2. **Orderings on a Bratteli diagram.**

**Definition 2.4.** A Bratteli diagram $B = (V, E)$ is called *ordered* if a linear order '$>$' is defined on every set $r^{-1}(v)$, $v \in \bigcup_{n\geq 1} V_n$. We use $\omega$ to denote the corresponding partial order on $E$ and write $(B, \omega)$ when we consider $B$ with the ordering $\omega$. Denote by $\mathcal{O}_B$ the set of all orderings on $B$.

Every $\omega \in \mathcal{O}_B$ defines a *lexicographic* partial ordering on the set of finite paths between vertices of levels $V_k$ and $V_l$: $(e_{k+1}, ..., e_l) > (f_{k+1}, ..., f_l)$ if and only if there is $i$ with $k + 1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i > f_i$. It follows that, given $\omega \in \mathcal{O}_B$, any two paths from $E(v_0, v)$ are comparable with respect to the lexicographic ordering generated by $\omega$. If two infinite paths are *tail equivalent*, i.e. agree from some vertex $v$ onwards, then we can compare them by comparing their initial segments in $E(v_0, v)$. Thus $\omega$ defines a partial order on $X_B$, where two infinite paths are comparable if and only if they are tail equivalent.

**Definition 2.5.** We call a finite or infinite path $e = (e_i)$ *maximal (minimal)* if every $e_i$ is maximal (minimal) amongst the edges from $r^{-1}(r(e_i))$.

Notice that, for $v \in V_i$, $i \geq 1$, the minimal and maximal (finite) paths in $E(v_0, v)$ are unique. Denote by $X_{\max}(\omega)$ and $X_{\min}(\omega)$ the sets of all maximal and minimal infinite paths in $X_B$, respectively. It is not hard to show that $X_{\max}(\omega)$ and $X_{\min}(\omega)$ are *non-empty closed subsets* of $X_B$. If $B$ is completely connected, then $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have no interior points.

Given a Bratteli diagram $B$, we can describe the set of all orderings $\mathcal{O}_B$ in the following way. Given a vertex $v \in V\backslash V_0$, let $P_v$ denote the set of all orders on $r^{-1}(v)$; an element in $P_v$ is denoted by $\omega_v$. Then $\mathcal{O}_B$ can be represented as

$$\mathcal{O}_B = \prod_{v\in V\backslash V_0} P_v. \tag{2.1}$$

The set of all orderings $\mathcal{O}_B$ on a Bratteli diagram $B$ can be considered also as a *measure space* whose Borel structure is generated by cylinder sets. On the set $\mathcal{O}_B$ we take the uniform product measure $\mathbb{P} = \prod_{v\in V\backslash V_0} \mathbb{P}_v$ where $\mathbb{P}_v$ is the uniformly distributed measure on $P_v$: $\mathbb{P}_v(\{i\}) = (|r^{-1}(v)|!)^{-1}$ for every $i \in P_v$ and $v \in V\backslash V_0$. We will make use of

conditional probability arguments and the finite $\sigma$-algebras $\mathcal{F}_N$ generated by the cylinder sets $\prod_{v\in\bigcup_{i=1}^{N} V_i} P_v$.

The uniform measure $\mathbb{P}$ is the only measure we consider in this article, so we will often use the term "almost every" without explicit reference to $\mathbb{P}$.

**Definition 2.6.** Let $B$ be a Bratteli diagram, and $n_0 = 0 < n_1 < n_2 < \dots$ be a strictly increasing sequence of integers. The *telescoping of $B$ to $(n_k)$* is the Bratteli diagram $B'$, whose $k$-level vertex set $V'_k = V_{n_k}$ and whose incidence matrices $(F'_k)$ are defined by

$$F'_k = F_{n_{k+1}-1} \circ \dots \circ F_{n_k},$$

where $(F_n)$ are the incidence matrices for $B$.

If $B'$ is a telescoping of $B$, then there is a natural injection $L : \mathcal{O}_B \to \mathcal{O}'_B$. Note that unless $|V_n| = 1$ for all but finitely many $n$, $L(\mathcal{O}_B)$ is a set of zero measure in in $\mathcal{O}_{B'}$.

### 2.3. **Vershik maps.**

**Definition 2.7.** Let $(B,\omega)$ be an ordered Bratteli diagram. We say that $\varphi = \varphi_\omega : X_B \to X_B$ is a *continuous Vershik map* if it satisfies the following conditions:

(i) $\varphi$ is a homeomorphism of the Cantor set $X_B$;

(ii) $\varphi(X_{\max}(\omega)) = X_{\min}(\omega)$;

(iii) if an infinite path $x = (x_1, x_2, \dots)$ is not in $X_{\max}(\omega)$, then $\varphi(x_1, x_2, \dots) = (x^0_1, \dots, x^0_{k-1}, \overline{x_k}, x_{k+1}, x_{k+2}, \dots)$, where $k = \min\{n \geq 1 : x_n \text{ is not maximal}\}$, $\overline{x_k}$ is the successor of $x_k$ in $r^{-1}(r(x_k))$, and $(x^0_1, \dots, x^0_{k-1})$ is the minimal path in $E(v_0, s(\overline{x_k}))$.

If $\omega$ is an ordering on $B$, then one can always define the map $\varphi_0$ that maps $X_B \setminus X_{\max}(\omega)$ onto $X_B \setminus X_{\min}(\omega)$ according to (iii) of Definition 2.7. If there exists a measure $\mu$ on $X_B$ such that $\mu(X_{\max}(\omega)) = \mu(X_{\min}(\omega)) = 0$, and such that $\mu$ is invariant under $\phi_0$, then we may extend $\phi_0$ to a measure-preserving transformation $\phi$ of $X_B$. In this case, we call $\phi$ a measurable Vershik map of $(X_B, \mu)$.

The question about the existence of a continuous Vershik map is equivalent to that of an extension of $\varphi_0 : X_B \setminus X_{\max}(\omega) \to X_B \setminus X_{\min}(\omega)$ to a homeomorphism of the entire set $X_B$. Note that if $X_{\max}(\omega)$ and $X_{\min}(\omega)$ have empty interiors, and there is a continuous extension of the measurable Vershik map to the whole space, then this extension is unique.

**Definition 2.8.** Let $B$ be a Bratteli diagram $B$. We say that an ordering $\omega \in \mathcal{O}_B$ is *perfect* if $\omega$ admits a continuous Vershik map $\varphi_\omega$ on $X_B$. If $\omega$ is not perfect, we call it *imperfect*.

Let $\mathcal{P}_B \subset \mathcal{O}_B$ denote the set of perfect orders on $B$.

## 3. The size of certain sets in $\mathcal{O}_B$.

The following result was shown for finite rank Bratteli diagrams in [BKY14]; the proof for non-finite rank diagrams is very similar.

**Theorem 3.1.** *Let $B$ be a simple Bratteli diagram. Then there exists $j \in \mathbb{N} \cup \{\aleph_0, 2^{\aleph_0}\}$ such that for $\mathbb{P}$-almost all orderings, $|X_{\max}(\omega| = |X_{\min}(\omega)| = j$.*

**Example 3.2.** It is not difficult, though contrived, to find a simple finite rank Bratteli diagram $B$ where almost all orderings are not perfect. Let $V_n = V = \{v_1, v_2\}$ for $n \geq 1$, and

define $m_{v,w}^{(n)} := \frac{f_{v,w}^{(n)}}{\sum_w f_{v,w}^{(n)}}$: i.e. $m_{v,w}^{(n)}$ is the proportion of edges with range $v \in V_{n+1}$ that have source $w \in V_n$. Suppose that $\sum_{n=1}^{\infty} m_{v_i,v_j}^{(n)} < \infty$ for $i \neq j$. Then for almost all orderings, there is some $K$ such that for $k > K$, the sources of the two maximal/minimal edges at level $n$ are distinct, i.e. $j = 2$. The assertion follows from [BKY14, Theorem 5.4].

The following result is proved for finite rank diagrams in Theorem 5.4 of [BKY14].

**Theorem 3.3.** *Suppose that $B$ is a completely connected Bratteli diagram of infinite rank such that $\mathbb{P}$-almost all orderings have $j$ maximal and minimal elements, with $j > 1$. Then $\mathbb{P}$-almost all orderings are imperfect.*

Before proving the theorem, we need a combinatorial lemma.

**Lemma 3.4.** *Let $S$ be a finite set of size $n$ and let $F$ and $G$ be maps from $S$ into a set $R$ with $G$ non-constant. Let the set of $n!$ total orderings on $S$ be equipped with the uniform probability measure, and let $\sigma : \{1, 2, \ldots, n\} \to S$ be such an order. Then*

$$\mathbb{P}\big(F(\sigma(i)) = G(\sigma(i+1)) \text{ for all non-maximal } i \in S\big) \leq \frac{1}{n-1}.$$

*Proof.* We can represent each total ordering of $S$ as a permutation $\sigma : \{1, 2, \ldots, n\} \to S$, where $\sigma(1)$ is the minimal element in the ordering, and for $1 \leq i < n$, $\sigma(i+1)$ is the successor of $\sigma(i)$. Consider a permutation $\sigma$ to be *good* if

$$F(\sigma(i)) = G(\sigma(i+1)) \text{ for all } 1 \leq i < n. \tag{3.1}$$

We shall find an upper bound for the size set of good permutations.

Let $V$ be the union of the range of $F$ and the range of $G$. Form a directed multigraph $\mathcal{G} = (V, E)$ as follows. For $1 \leq i \leq n$, define the ordered pair $e_i = (G(i), F(i))$. Let $E = \{e_1, e_2, \ldots, e_n\}$. Now let $\sigma$ be a good permutation. Then for $1 \leq i < n$, the range of $e_{\sigma(i)}$ equals the source of $e_{\sigma(i+1)}$. Therefore, $e_{\sigma(1)} e_{\sigma(2)} \ldots e_{\sigma(n)}$ is an Eulerian trail in $\mathcal{G}$.

It is straightforward to check that the map from good permutations to Eulerian trails is bijective, and thus we need to bound the number of Eulerian trails in $\mathcal{G}$. To do this, note that each Eulerian trail induces an ordering on the out-edges of each vertex. Let $V = \{v_1, \ldots, v_k\}$, and let $n_i$ be the number of out-edges of $v_i$. Since $g$ is non-constant, there are at least two directed edges with different sources, and thus $n_i \leq n - 1$ for $1 \leq i \leq k$. The number of orderings of out-edges equals $n_1! n_2! \ldots n_k!$.

We distinguish two cases. If all vertices have out-degree equal to in-degree, then each Eulerian trail is in fact an Eulerian circuit. An Eulerian circuit corresponds to $n$ different Eulerian trails, distinguished by their starting edge. To count the number of circuits, we may fix a starting edge $e^*$, and then note that each circuit induces exactly one out-edge ordering if we start following the circuit at this edge. Note that in each such ordering, the edge $e^*$ must be the first in the ordering of the out-edges of its source. We may choose $e^*$ such that its source, say $v_1$, has maximum out-degree. Thus the number of compatible out-edge orderings is at most $(n_1 - 1)! n_2! \ldots n_k!$ This expression is maximized, subject to the conditions $n_1 + n_2 + \cdots + n_k \leq n$ and $n_i \leq n_1 \leq n - 1$ for $1 \leq i \leq k$, when $k = 2$ and $n_1 = n - 1$, $n_2 = 1$. Therefore, there are at most $(n-2)!$ Eulerian circuits and at most $n(n-2)!$ Eulerian trails and good permutations.

If not all vertices have out-degree equal to in-degree, then either no Eulerian trail exists and the lemma trivially holds, or exactly one vertex, say $v_1$, has out-degree greater than

in-degree, and this vertex must be the starting vertex of every trail. In this case, an ordering of out-edges precisely determines the trail. The number of out-edge orderings (and good permutations) in this case is bounded above by $(n-1)!$.

Therefore, there are at most $n(n-2)!$ out of $n!$ total orderings satisfying (3.1), and the lemma follows.

□

*Proof of Theorem 3.3.* Note that if $|V_n| = 1$ for infinitely many $n$, then any order on $B$ has exactly one maximal and one minimal path. So we shall have that $|V_n| \geq 2$ for all large $n$.

We first define some terminology. Recall that $s(e)$ and $r(e)$ denote the source and range of the edge $e$ respectively. Given an order $\omega \in \mathcal{O}_B$, we let $e_{\alpha,\omega}(v)$ be the edge labelled $\alpha$ and whose range is $v$. If $v \in V_{N'}$ for some $N' > n$, we let $t_{n,\omega}(v)$ be the element of $V_n$ that the maximal incoming path to $v$ goes through. We call $t_{n,\omega}(v)$ the *$n$-tribe* of $v$. Similarly the *$n$-clan* of $v$, $c_{n,\omega}(v)$ is the element of $V_n$ through which the minimal incoming path to $v$ passes. If $n$ is such that for any $N > n$, the elements of $V_N$ belong to at least two $n$-clans (or $n$-tribes), we shall say that $\omega$ has at least two infinite $n$-clans (or $n$-tribes.)

Let $C_{n,N}$ be the set of orders $\omega$ such that if the non-maximal paths $x$ and $y$ agree to level $N$, then $\varphi_\omega(x)$ and $\varphi_\omega(y)$ agree to level $n$.

Fix $n$ and $N$ with $N > n$, and take any $N' > N$. Any order $\omega \in C_{n,N}$ must satisfy the following constraints: if $\alpha$ and $\beta$ are two non-maximal edges whose sources in $V_{N'}$ belong to the same $N$-tribe, then their successors must belong to the same $n$-clan. In particular, if $v$ is any vertex in $V_{N'}$ such that the sources of $e_{\alpha,\omega}(v)$ and $e_{\beta,\omega}(v)$ belong to the same $N$-tribe, where $\alpha$ and $\beta$ are both non-maximal, then the sources of $e_{\alpha+1,\omega}(v)$ and $e_{\beta+1,\omega}(v)$ must belong to the same $n$-clan. That is there is a map $f\colon V_N \to V_n$ such that for any $v \in V_{N'}$ and any non-maximal $\alpha$, $f(t_{N,\omega}(s(e_{\alpha,\omega}(v)))) = c_{n,\omega}(s(e_{\alpha+1,\omega}(v)))$. We think of this $f$ as mapping $N$-tribes to $n$-clans.

Motivated by the preceding remark, if $N' > N > n$, we define two subsets of $\mathcal{O}_B$. We let $D_{n,N'}$ be the set of orders such that $V_{N'}$ contains members of at least two $n$-clans; and $E_{n,N,N'}$ to be the subset of orders in $\mathcal{D}_{n,N'-1}$ which additionally satisfy the condition (*):

> There is a function $f\colon V_N \to V_n$ such that for all $v \in V_{N'}$, if $\alpha$ is a non-maximal edge entering $v$ then $f(t_{N,\omega}(s(e_{\alpha,\omega}(v)))) = c_{n,\omega}(s(e_{\alpha+1,\omega}(v)))$.

We observe that $D_{n,N'}$ and $E_{n,N,N'}$ are $\mathcal{F}_{N'}$-measurable. We compute $\mathbb{P}(E_{n,N,N'}|\mathcal{F}_{N'-1})$. Since $D_{n,N'-1}$ is $\mathcal{F}_{N'-1}$ measurable, we have $\mathbb{P}(E_{n,N,N'}|\mathcal{F}_{N'-1})(\omega)$ is 0 for $\omega \notin D_{n,N'-1}$. For a fixed map $f\colon V_N \to V_n$, and a fixed vertex $v \in V_{N'}$, and $\omega \in D_{n,N'-1}$, the conditional probability given $\mathcal{F}_{N'-1}$ that (*) with the specific function $f$ is satisfied at $v$ is at most $1/(|V_{N'-1}| - 1)$. To see this, notice that for $\omega \in D_{n,N'-1}$, the $n$-clan is a non-constant function of $V_{N'-1}$, so that the hypothesis of Lemma 3.4 is satisfied, with $F = f \circ t_{N,\omega} \circ s$ and $G = c_{n,\omega} \circ s$, both applied to the set of incoming edges to $v$. Also, since $B$ is completely connected, there are at least $|V_{N'-1}|$ edges coming into $v$.

Since these are independent events conditioned on $\mathcal{F}_{N'-1}$, the conditional probability that (*) is satisfied for the fixed function $f$ over all $v \in V_{N'}$ is at most $1/(|V_{N'-1}| - 1)^{|V'_N|}$. There are $|V_n|^{|V_N|}$ possible functions $f$ that might satisfy (*). Hence we obtain

$$\mathbb{P}(E_{n,N,N'}) \leq \frac{|V_n|^{|V_N|}\mathbb{P}(D_{n,N'-1})}{(|V_{N'-1}| - 1)^{|V_{N'}|}},$$

so that for fixed $n$ and $N$ with $n < N$, one has $\liminf_{N' \to \infty} \mathbb{P}(E_{n,N,N'}) = 0$. By the hypothesis, for any $\epsilon > 0$, there exists $m(\epsilon)$ such that $\mathbb{P}(R_n) > 1 - \epsilon$ for all $n > m(\epsilon)$, where $R_n = \{\omega \in \mathcal{O}_B \colon \omega \text{ has at least 2 infinite } n\text{-clans}\}$.

Since $C_{n,N} \cap R_n \subset E_{n,N,N'}$ for all $N' > N > n$, we conclude that $\mathbb{P}(C_{n,N} \cap R_n) = 0$ for $N > n$, so that $\mathbb{P}(C_{n,N}) \leq \epsilon$ for $N > n > m(\epsilon)$. Now since $\mathcal{P}_B = \bigcap_{n=1}^{\infty} \bigcup_{N=n}^{\infty} C_{n,N}$ and $C_{n,N} \subset C_{n,N+1}$ for each $N \geq n$, we conclude that $\mathbb{P}(\mathcal{P}_B) = 0$.

□

## 4. Diagrams whose orders are almost always imperfect

4.1. **Bratteli diagrams and the Wright-Fisher model.** Let $J$ denote a matrix (size determined by the context) all of whose entries are 1. If $V_n$ is the $n$-th vertex set in B, define $M_n = |V_n|$. In this section, all Bratteli diagrams that we consider have incidence matrices $F_n = J$ for each $n$, where the size of $J$ can vary with $n$.

We wish to give conditions on $(M_n)$ so that a $\mathbb{P}$-random order has infinitely many maximal paths. We first comment on the relation between our question and the Wright-Fisher model in population genetics. Given a subset $A_k \subset V_k$, and an ordering $\omega \in \mathcal{O}_B$, we let $A_n$ for $n > k$ be the collection of vertices $v$ in $V_n$ such that the unique upward maximal path through $v$ passes through $A_k$. If informally, we can consider the tree formed by all maximal edges, whose levels are the sets $V_n$, then $A_n$ is the set of vertices in $V_n$ that have "ancestors" in $A_k$.

Let $Y_n = |A_n|/M_n$. We observe that conditional on $Y_n$, $Y_{n+1}$ is distributed as the average of $M_{n+1}$ independent Bernoulli random variables with parameter $Y_n$ (i.e. $M_{n+1}Y_{n+1}$ is a binomial random variable with parameters $M_{n+1}$ and $Y_n$). In particular, $(Y_n)$ is a martingale with respect to the natural filtration $(\mathcal{F}_n)$, where $\mathcal{F}_n$ is the $\sigma$-algebra generated by the first $n$ levels of $B$. Since $(Y_n)$ is a bounded martingale, it follows from the martingale convergence theorem that $(Y_n)$ almost surely converges to some limit $Y_\infty$ where $0 \leq Y_\infty \leq 1$.

It turns out that this is exactly the same as the Wright-Fisher model in population genetics. Here one studies populations where there are disjoint generations; each population member inherits an allele (gene type) from a uniformly randomly chosen member of the previous generation. Analogous to $Y_n$, one studies the proportion of the population that have various alleles. If one declares the vertices in $A_k \subset V_k$ to have allele type **A** and the other vertices in that level to have allele type **a**, then there is a maximal path through $A_k$ if and only if in the Wright-Fisher model, the allele **A** persists - that is there exist individuals in all levels beyond the $n$th with type **A** alleles.

In a realization of the Wright-Fisher model, an allele type is said to *fixate* if the proportion $Y_n$ of individuals with that allele type in the $n$th level converges to 0 or 1 as $n \to \infty$. An allele type is said to become *extinct* if $Y_n = 0$ for some finite level, or to *dominate* if $Y_n = M_n$ for some finite level.

**Theorem 4.1.** [Don86, Theorem 3.2] *Consider a Wright-Fisher model with population structure $(M_n)_{n \geq 0}$. Then domination of one of the alleles occurs almost surely if and only if $\sum_{n \geq 0} 1/M_n = \infty$.*

Theorem 4.1 is also true if in the Wright-Fisher model, individuals can inherit one of $k$ alleles with $k \geq 2$. We indicate a proof of the simpler fact that if $\sum_{n \geq 0} 1/M_n = \infty$ then

each allele type fixates, that is that its density converges to 0 or 1 ([Don86, Theorem 3.2 ]). To see this, let $Q_n = Y_n(1-Y_n)$. Now we have

$$\mathbb{E}(Q_n|\mathcal{F}_{n-1}) = Y_{n-1} - Y_{n-1}^2 - (\mathbb{E}(Y_n^2|Y_{n-1}) - \mathbb{E}(Y_n|Y_{n-1})^2) = Q_{n-1} - \mathrm{Var}(Y_n|Y_{n-1}).$$

Since $M_nY_n$ is binomial with parameters $M_n$ and $Y_{n-1}$,

$$\mathrm{Var}(Y_n|Y_{n-1}) = (1/M_n^2)(M_nY_{n-1}(1-Y_{n-1})) = Q_{n-1}/M_n.$$

This gives $\mathbb{E}(Q_n|\mathcal{F}_{n-1}) = (1-1/M_n)Q_{n-1}$. Now using the tower property of conditional expectations, we have $\mathbb{E}Q_n = \mathbb{E}(Q_n|\mathcal{F}_0) = \prod_{j=1}^n(1-1/M_j)\mathbb{E}Q_0$, which converges to 0. As noted above, the sequence $(Y_n(\omega))$ is convergent for almost all $\omega$ to $Y_\infty(\omega)$ say. It follows that $Q_n(\omega)$ converges pointwise to $Y_\infty(1-Y_\infty)$. By the bounded convergence theorem, we deduce that $\mathbb{E}Y_\infty(1-Y_\infty) = 0$, so that $Y_\infty$ is equal to 0 or 1 almost everywhere.

We shall use Theorem 4.1 to prove the first part of the following theorem.

**Theorem 4.2.** *Consider a Bratteli diagram with $M_n \geq 1$ vertices in the nth level and whose incidence matrices are all of the form $J$. We have the following dichotomy:*

*If $\sum_n 1/M_n = \infty$, then there is $\mathbb{P}$-almost surely a unique maximal path.*

*If $\sum_n 1/M_n < \infty$, then there are $\mathbb{P}$-almost surely uncountably many maximal paths.*

To prove the second part of this result we will need the following tools.

**Lemma 4.3.** *Let $(Z_n)_{n\geq 0}$ be a bounded sub-martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $\tau_1$ and $\tau_2$ be two stopping times such that $\tau_1 \leq \tau_2$ almost surely. Then $\mathbb{E}Z_{\tau_1} \leq \mathbb{E}Z_{\tau_2}$.*

*Proof.* Assume initially that $\tau_1$ and $\tau_2$ are bounded. Then

$$Z_{\tau_2} - Z_{\tau_1} = \sum_n \mathbf{1}_{\{\tau_1 \leq n < \tau_2\}}(Z_{n+1} - Z_n).$$

Notice that the event $\{\tau_1 \leq n < \tau_2\} = \{\tau_1 \leq n\} \setminus \{\tau_2 \leq n\}$, so that it is $\mathcal{F}_n$-measurable. Hence

$$\begin{aligned}\mathbb{E}\big(\mathbf{1}_{\{\tau_1\leq n<\tau_2\}}(Z_{n+1}-Z_n)\big) &= \mathbb{E}\big(\mathbb{E}(\mathbf{1}_{\{\tau_1\leq n<\tau_2\}}(Z_{n+1}-Z_n)|\mathcal{F}_n)\big)\\ &= \mathbb{E}\big(\mathbf{1}_{\{\tau_1\leq n<\tau_2\}}\mathbb{E}(Z_{n+1}-Z_n|\mathcal{F}_n)\big).\end{aligned}$$

By the submartingale property, this quantity is non-negative, so that $\mathbb{E}Z_{\tau_2} \geq \mathbb{E}Z_{\tau_1}$. In the case where $\tau_1$ and $\tau_2$ are unbounded, we use the stopping times $\min(\tau_1, N)$ and $\min(\tau_2, N)$ and take the limit (using the bounded convergence theorem). □

**Proposition 4.4.** *Consider a Wright-Fisher model with population structure $(M_n)_{n\geq 0}$. If $\sum_{n\geq 0} 1/M_n < \infty$, then for each $\epsilon > 0$ and $\eta > 0$, there exists an $l$ such that then with probability $1-\epsilon$, one has $|Y_n - Y_l| < \eta$ for all $n > l$.*

*Proof.* Let $l$ be chosen so that $\sum_{n=l+1}^\infty 1/M_n < 4\epsilon\eta^2$. Consider the (possibly infinite) stopping time $\tau = \min\{n\colon Y_n \notin [Y_l - \eta, Y_l + \eta]\}$. Set $Z_n = (Y_n - Y_l)^2$ and notice that $(Z_n)_{n\geq l}$ is a bounded sub-martingale by the conditional expectation version of Jensen's inequality. Since $Y_n \to Y_\infty$, it follows by continuity that $Z_n \to Z_\infty := (Y_\infty - Y_l)^2$.

We have

$$\begin{aligned}\mathbb{E}Z_n &= \mathbb{E}\big(\mathbb{E}((Y_n - Y_l)^2|\mathcal{F}_l)\big)\\ &= \mathbb{E}\big(\mathbb{E}(Y_n^2 - Y_l^2|\mathcal{F}_l)\big)\\ &= \mathbb{E}(Y_n^2 - Y_l^2)\\ &= \sum_{j=l}^{n-1} \mathbb{E}(Y_{j+1}^2 - Y_j^2).\end{aligned}$$

A calculation shows that

$$\begin{aligned}\mathbb{E}(Y_{j+1}^2 - Y_j^2|\mathcal{F}_j) &= \mathbb{E}(Y_{j+1}^2|\mathcal{F}_j) - \mathbb{E}(Y_{j+1}|\mathcal{F}_j)^2\\ &= \mathrm{Var}(Y_{j+1}|\mathcal{F}_j)\\ &= \frac{Y_j(1-Y_j)}{M_{j+1}}\end{aligned}$$

so that $\mathbb{E}(Y_{j+1}^2 - Y_j^2) \le 1/(4M_{j+1})$ and we obtain $\mathbb{E}Z_n \le \sum_{j=l+1}^{n} 1/(4M_j)$. In particular we have for all $n > l$, $\mathbb{E}Z_n \le \epsilon\eta^2$.

Applying Lemma 4.3, we have for all $l < n < m$,

$$\mathbb{E}Z_{\min(n,\tau)} \le \mathbb{E}Z_m \le \epsilon\eta^2. \tag{4.1}$$

Hence we have $\mathbb{E}Z_{\min(\tau,n)} \le \epsilon\eta^2$, for each $n > l$. Now

$$\lim_{n\to\infty} Z_{\min(\tau,n)} = Y := \begin{cases} Z_\tau & \text{if } \tau < \infty; \text{ or}\\ Z_\infty & \text{otherwise.}\end{cases}$$

The bounded convergence theorem implies $\mathbb{E}Y \le \epsilon\eta^2$, but $Y \ge \eta^2$ if $\tau < \infty$, so that $\mathbb{P}(\tau < \infty) \le \epsilon$. This establishes the claim in the proposition.

□

*Proof of Theorem 4.2.* Suppose first that $\sum_n 1/M_n = \infty$. We show for all $k$, with probability 1, there exists $n > k$ such that all maximal paths from each level $n$ vertex to the root vertex pass through a single vertex at level $k$.

To do this, we consider the $M_k$ vertices at level $k$ to each have a distinct allele type. By Theorem 4.1, there is for almost every $\omega$, a level $n$ such that by level $n$ one of the $M_k$ allele types has dominated all the others. This is, of course, a direct translation of the statement that we need.

Now we consider the case $\sum_n 1/M_n < \infty$. In this case we consider the vertices to have one of two possible allele types. The event that there are uncountably many maximal paths has $\mathbb{P}$-measure 0 or 1. Hence to show that it has measure 1, it suffices to show that the measure is positive.

Using Proposition 4.4, choose an increasing sequence of levels $(n_k)_{k\ge1}$ with the properties that $n_k > 4^k$ and that if an allele at level $n_k$ has density $Y_k$, then with probability at least $1 - 8^{-k}$, it has density in the range $[Y_k - 4^{-(k+1)}, Y_k + 4^{-(k+1)}]$ in all subsequent levels.

In particular, if an allele has density $Y_k$ at the $n_k$th level, then with probability at least $1 - 8^{-k}$, it has density in the range $[Y_k - 4^{-(k+1)}, Y_k + 4^{-(k+1)}]$ at the $n_{k+1}$st level. Given

this, we can establish the positivity of the measure of the set of orderings with uncountably many maximal paths as follows.

We will show that for a set of orders of positive mass, we can realize a Cantor set as a subset of $X_{\max}(\omega)$. We do this by defining, for each finite binary string $x$ of length $j$, disjoint sets $A_x = A_x(\omega) \subset V_{n_j}$, such that if $x$ is a prefix of $y$, then vertices in $A_y$ have level $n_j$ maximal ancestors in $A_x$. We then show that for a set of orders of positive mass, $A_x \neq \emptyset$ for any word $x$, so that given any increasing sequence of words $(x_j)$, there is a maximal path which runs through vertices in $A_{x_j}$ for each $j$.

The symbol $\lambda$ will denote the empty string. For a finite string $x \in \{0,1\}^j$, let $x0$ and $x1$ be the extensions of the finite word $x$ by 0 and 1 respectively. Let $A_\lambda$ be the set consisting of the vertex at the top level. Given a subset $A$ of $V_{n_j}$, we let $\phi_j(A)$ consist of those vertices in $V_{n_{j+1}}$ whose (unique) maximal upward path passes through $A$. In other words, in the tree formed by the maximal edges, $\phi_j(A)$ is the set of vertices in level $n_{j+1}$ that have ancestors in $A$.

The inductive hypothesis is that at each stage $j < k$, one has disjoint subsets $A_x \subset V_{n_j}$ for each $x \in \{0,1\}^j$ satisfying the following:

(1) $|A_x|/M_{n_j} \geq 4^{-j}$ for each $x \in \{0,1\}^j$.
(2) $\phi_j(A_x) = A_{x0} \cup A_{x1}$ for each $x \in \{0,1\}^j$;

We will show that assuming the stages up to the $j$th of the induction are satisfied, then the $(j+1)$st stage can be satisfied with probability at least $1 - 4^{-j}$. The initial stage of the induction is the set $A_\epsilon$.

Suppose that all stages up to the $j$th are satisfied. Then the sets $A_x$ for $x \in \{0,1\}^j$ form a partition of $V_{n_j}$, with each one consisting of at least $M_{n_j}/4^j$ elements. Then for each $x \in \{0,1\}^j$, let $\rho_x = |A_x|/M_{n_j}$. With probability at least $1 - 8^{-j}$, $|\phi_j(A_x)|/M_{n_{j+1}}$ is in the range $[\rho_x - 4^{-(j+1)}, \rho_x + 4^{-(j+1)}]$. Let $A_{x0}$ and $A_{x1}$ be the almost equal division of $\phi_j(A_x)$ obtained by putting the first $\lceil |\phi_j(A_x)|/2 \rceil$ into $A_{x0}$ and the rest into $A_{x1}$. The densities of these are at least $\frac{1}{2}|\phi_j(A_x)|/M_{n_{j+1}} - 1/(2M_{n_{j+1}})$. Provided that the density of $\phi_j(A_x)$ exceeds $\rho_x - 4^{-(j+1)}$, the densities of $A_{x0}$ and $A_{x1}$ exceed $\frac{1}{2}\rho_x - \frac{1}{2}4^{-(j+1)} - 1/(2M_{n_{j+1}}) \geq \frac{1}{2}\rho_x - 4^{-(j+1)} \geq 4^{-(j+1)}$.

In order for this induction step to fail, for one of the $2^j$ sets $A_x$ at the $j$th level we must have a drop in density from $A_x$ to $\phi(A_x)$ of more than $4^{-(j+1)}$. Using the union bound, the probability that this happens is at worst $2^j \times 8^{-j}$. Since these probabilities sum to less than 1, we see that with positive probability the induction steps can all be completed.

**Corollary 4.5.** *The number of maximal paths that a random order on $B$ possesses is not invariant under telescoping of $B$.*

*Proof.* Consider the Bratteli diagram $B$ where $M_{2n+1} = 1$ and $M_{2n} = n^2$ for each $n$, and where the incidence matrices of $B$ are all $F_n = J$. Any order on $B$ has one maximal path. Let $B'$ be the diagram with $M_n = n^2$ for each $n$, and let the incidence matrices of $B'$ all be $F_n = J$. By Theorem 4.2, a random order on $B'$ has infinitely many maximal paths. On the other hand, $B$ can be telescoped to $B'$. □

4.2. **Other Bratteli diagrams whose orders support many maximal paths.** Next we partially extend the results in Section 4.1 to a larger family of Bratteli diagrams.

**Definition 4.6.** Let $B$ be a Bratteli diagram.

- We say that $B$ is *superquadratic* if there exists $\delta > 0$ so that $M_n \geq n^{2+\delta}$ for all large $n$.
- Let $B$ be superquadratic with constant $\delta$. We say that $B$ is *exponentially bounded* if $\sum_{n=1}^{\infty} |V_{n+1}| \exp(-|V_n|/n^{2+2\delta/3})$ converges.

We remark that the condition that $B$ is exponentially bounded is very mild.

In Theorem 4.12 below we show that Bratteli diagrams satisfying these conditions have infinitely many maximal paths. Given $v \in V_{n+1}$, define

$$V_n^{v,i} := \{w \in V_n : f_{v,w}^{(n)} = i\},$$

so that if the incidence matrix entries for $B$ are all positive and bounded above by $r$, then $V_n = \bigcup_{i=1}^{r} V_n^{v,i}$ for each $v \in V_{n+1}$.

**Definition 4.7.** Let $B$ be a Bratteli diagram with positive incidence matrices. We say that $B$ is *impartial* if there exists an integer $r$ so that all of $B$'s incidence matrix entries are bounded above by $r$, and if there exists some $\alpha \in (0,1)$ such that for any $n$, any $i \in \{1, \ldots, r\}$ and any $v \in V_{n+1}$, $|V_n^{v,i}| \geq \alpha |V_n|$.

In other words, $B$ is impartial if for any row of any incidence matrix, no entry occurs disproportionately rarely or often with respect to the others. Note that our diagrams in Theorem 4.2 are impartial. However the vertex sets can grow as fast as we want, so the diagrams are not necessarily exponentially bounded. We remark also that if a Bratteli diagram is impartial, then it is completely connected, which means that we can apply Theorem 3.3 if $j > 1$.

**Definition 4.8.** Suppose that $B$ is a Bratteli diagram each of whose incidence matrices has entries with a maximum value of $r$. We say that $A \subset V_n$ is $(\beta, \epsilon)$-*equitable* for $B$ if for each $v \in V_{n+1}$ and for each $i = 1, \ldots, r$,

$$\left| \frac{|V_n^{v,i} \cap A|}{|V_n^{v,i}|} - \beta \right| \leq \epsilon.$$

In the case $\beta = \frac{1}{2}$, we shall speak simply of $\epsilon$-equitability.

Given $v \in V \backslash V_0$ and an order $\omega \in \mathcal{O}_B$, recall that we use $\widetilde{e}_v = \widetilde{e}_v(\omega)$ to denote the maximal edge with range $v$.

**Lemma 4.9.** *Suppose that $B$ is impartial. Let $A \subset V_n$ be $(\beta, \epsilon)$-equitable, and $v \in V_{n+1}$. Let the random variable $X_v$ be defined as*

$$X_v(\omega) = \begin{cases} 1 & \text{if } s(\widetilde{e}_v) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

*Then $\beta - \epsilon \leq \mathbb{E}(X_v) \leq \beta + \epsilon$.*

*Proof.* We have

$$\begin{aligned} \mathbb{E}(X_v) &= \frac{\sum_{j=1}^{r} j |A \cap V_n^{v,j}|}{\sum_{j=1}^{r} j |V_n^{v,j}|} \\ &\leq \frac{\sum_{j=1}^{r} j |V_n^{v,j}| (\beta + \epsilon)}{\sum_{j=1}^{r} j |V_n^{v,j}|} = \beta + \epsilon, \end{aligned}$$

the last inequality following since $A$ is $\epsilon$-equitable. Similarly, $\mathbb{E}(X_v) \geq \beta - \epsilon$. □

**Lemma 4.10.** *Let $B$ be an impartial Bratteli diagram with impartiality constant $\alpha$ and the property that each entry of each incidence matrix is between 1 and $r$. Let $\beta$, $\delta$ and $\epsilon$ be positive, let $(p_v)_{v\in V_N}$ satisfy $|p_v - \beta| < \delta$ for each $v \in V_N$ and let $A \subset V_N$ be a randomly chosen subset, where each $v$ is included with probability $p_v$ independently of the inclusion of all other vertices. Then the probability that $A$ fails to be $(\beta, \delta+\epsilon)$-equitable is at most $2r|V_{N+1}|e^{-\alpha|V_N|\epsilon^2}$.*

*Proof.* Let $(Z_v)_{v\in V_N}$ be $\mathbf{1}_{v\in A}$, so that these are independent Bernoulli random variables, where $Z_v$ takes the value 1 with probability $p_v$

For $u \in V_{N+1}$ and $1 \le i \le r$, define

$$Y_{u,i} := \frac{1}{|V_N^{u,i}|}\sum_{v\in V_N^{u,i}} Z_v = \frac{|\{v \in V_N^{u,i} : v \in A\}|}{|V_N^{u,i}|} = \frac{|A \cap V_N^{u,i}|}{|V_N^{u,i}|}. \tag{4.2}$$

Using Hoeffding's inequality [Hoe63], since $\beta - \delta \le \mathbb{E}(Y_{u,i}) \le \beta + \delta$ we have that

$$\begin{aligned}\mathbb{P}(\{|Y_{u,i} - \beta| \ge (\delta+\epsilon)\}) &\le \mathbb{P}(\{|Y_{u,i} - \mathbb{E}(Y_{u,i})| \ge \epsilon\}) \\ &\le 2e^{-2|V_N^{u,i}|\epsilon^2} \le 2e^{-2\alpha|V_N|\epsilon^2}.\end{aligned}$$

This implies that

$$\mathbb{P}\left(\bigcup_{i=1}^{r}\bigcup_{u\in V_{N+1}}\{|Y_{u,i} - \beta| \ge \delta+\epsilon\}\right) \le 2r|V_{N+1}|e^{-2|V_N|\alpha\epsilon^2}, \tag{4.3}$$

□

**Lemma 4.11.** *Suppose that $B$ is impartial, superquadratic and exponentially bounded. Then for any $\epsilon$ small there exist $n$ and $A \subset V_n$ such that $A$ is $(\frac{1}{2}, \epsilon)$-equitable.*

*Proof.* We apply the probabilistic method. Let $r$ and $\alpha$ be as in the statement of Lemma 4.10 and apply that lemma with $p_v = \frac{1}{2}$ for each $v \in V_n$. By the superquadratic and exponentially bounded properties, one has $2r|V_{n+1}|e^{-2\alpha|V_n|\epsilon^2} < 1$ for large $n$. Since the probability that a randomly chosen set is $(\frac{1}{2}, \epsilon)$-equitable is positive, the existence of such a set is guaranteed. □

**Theorem 4.12.** *Suppose that $B$ is a Bratteli diagram that is impartial, superquadratic and exponentially bounded. Then $\mathbb{P}$-almost all orders on $B$ have infinitely many maximal paths.*

We note that in the special case where $B$ is defined as in Section 4.1, the following proof can be simplified and does not require the condition that $B$ is exponentially bounded. Instead of beginning our procedure with an equitable set, which is what we do below, we can start with any set $A_N \subset V_N$ whose size relative to $V_N$ is around 1/2.

*Proof.* Since $B$ is superquadratic, we find a sequence $(\epsilon_j)$ such that

$$\sum_{j=1}^{\infty}\epsilon_j < \infty \text{ and} \tag{4.4}$$

$$M_j\epsilon_j^2 \ge j^\gamma \text{ for some } \gamma > 0 \text{ and large enough } j. \tag{4.5}$$

Fix $N$ so that (4.5) holds for all $j \ge N$, and let $N$ be large enough so that $\sum_{j=N}^{\infty}\epsilon_j < \frac{1}{2}$. Moreover, we can also choose our sequence $(\epsilon_j)$ and our $N$ large enough so that there exists

a set $A_N \subset V_N$ which is $\epsilon_N$-equitable: by Lemma 4.11, this can be done. For all $k \geq 0$, define also

$$\delta_{N+k} = \sum_{i=0}^{k} \epsilon_{N+i}.$$

Finally, let $r$ be so that all entries of all $F_n$ are bounded above by $r$.

Define recursively, for all integers $k > 0$ and all $v \in V_{N+k}$, the Bernoulli random variables $\{X_v : \mathcal{O}_B \to \{0,1\} : v \in V_{N+k}\}$, and the random sets $\{A_{N+k} : \mathcal{O}_B \to 2^{V_{N+k}} : k \geq 1\}$, where $X_v(\omega) = 1$ if $s(\widetilde{e}_v) \in A_{N+k-1}$, and 0 otherwise, and $A_{N+k} = \{v \in V_{N+k} \,:\, X_v = 1\}$.

We shall show that for a large set of $\omega$, each set $A_{N+k}$ is $\delta_{N+k}$-equitable. This implies that the size of $A_{N+k}$ is not far from $\frac{1}{2}|V_{N+k}|$. For, if $k \geq 1$, define the event

$$D_{N+k} := \{\omega : A_{N+k} \text{ is } \delta_{N+k} - \text{equitable}\}.$$

We claim that

$$\mathbb{P}(D_{N+k+1}|D_{N+k}) \geq 1 - 2r|V_{N+k+2}|e^{-2\alpha|V_{N+k+1}|\epsilon^2_{N+k+1}}.$$

To see this, notice that if $\omega \in D_{N+k}$, then by Lemma 4.9, given $\mathcal{F}_{N+k}$, each vertex in $V_{N+k+1}$ is independently present in $A_{N+k+1}$ with probability in the range $[\beta - \delta_{N+k}, \beta + \delta_{N+k}]$. Hence by Lemma 4.10, $A_{N+k+1}$ is $\delta_{N+k+1}$-equitable with probability at least $1 - 2r|V_{N+k+2}|e^{-\alpha|V_{N+k+1}|\epsilon^2_{N+k+2}}$.

Next we show that our work implies that a random order has at least two maximal paths. Let $\gamma = \frac{1}{2} - \sum_{j=N}^{\infty} \epsilon_j$. Notice that if $A_n \neq V_n$ for all $n > N$, then there are at least two maximal paths. By our choice of $N$ and $\gamma > 0$ we have that

$$\begin{aligned}
\mathbb{P}(\{\omega : |X_{\max}(\omega)| \geq 2\}) &\geq \mathbb{P}\left(\bigcap_{k=1}^{\infty} \left\{\omega : \gamma \leq \frac{|A_{N+k}|}{|V_{N+k}|} \leq 1 - \gamma \right\}\right) \\
&\geq \mathbb{P}\left(\bigcap_{k=1}^{\infty} D_{N+k}\right) \\
&= \lim_{n\to\infty} \mathbb{P}(D_{N+1}) \prod_{k=1}^{n} \mathbb{P}(D_{N+k+1}|D_{N+k}) \\
&\geq \lim_{n\to\infty} \mathbb{P}(D_{N+1}) \prod_{k=1}^{n} (1 - 2r|V_{N+k+2}|e^{-2|V_{N+k+1}|\alpha\epsilon^2_{N+k+1}}),
\end{aligned}$$

and the condition that $B$ is superquadratic and exponentially bounded ensures that this last term converges to a non-zero value.

We can repeat this argument to show that for any natural $k$, a random order has at least $k$ maximal paths. We remark also that the techniques of Section 4.1 could be generalized to show that a random order would have uncountably many maximal paths.

We now apply Theorem 3.3.

□

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