Explicit bounds for separation between Oseledets subspaces

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Abstract

We consider a two-sided sequence of bounded operators in a Banach space which are not necessarily injective and satisfy two properties (SVG) and (FI). The *singular value gap* (SVG) property says that two successive singular values of the cocycle at some index d admit a uniform exponential gap; the *fast invertibility* (FI) property says that the cocycle is uniformly invertible on the fastest d-dimensional direction. We prove the existence of a uniform equivariant splitting of the Banach space into a fast space of dimension d and a slow space of codimension d. We compute an explicit constant lower bound on the angle between these two spaces using solely the constants defining the properties (SVG) and (FI). We extend the results obtained by Bochi and Gourmelon in the finite-dimensional case for bijective operators and the results obtained by Blumenthal and Morris in the infinite dimensional case for injective norm-continuous cocycles, in the direction that the operators are not required to be globally injective, that no dynamical system is involved and no compactness of the underlying system or smoothness of the cocycle is required. Moreover we give quantitative estimates of the angle between the fast and slow spaces that are new even in the case of finite-dimensional bijective operators in Hilbert spaces.

1 Introduction

Let X be a real Banach space and $(A_k)_{k\in\mathbb{Z}}$ be a bi-infinite sequence of bounded operators of X which are not required to be injective. The *cocycle* associated to

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$(A_k)_{k\in\mathbb{Z}}$ is the sequence of iterated operators

$$A(k,n) := A_{k+n-1}\cdots A_{k+1}A_k, \quad \forall k\in\mathbb{Z} \text{ and } \forall n\geqslant 0,$$

with the convention $A(k,0) := \mathrm{Id}$. Our main objective is to find simple conditions on the sequence $(A_k)_{k\in\mathbb{Z}}$ which guarantee the existence of constants $d\geqslant 1$, $K_d\geqslant 1$ and $\tau>0$, and a uniform equivariant splitting $X = E_k\oplus F_k$ of fast/slow subspaces satisfying the following condition:

- $\forall k\in\mathbb{Z}, \quad \dim(E_k) = d, \ (A_k|E_k)$ is injective,
- $\forall k\in\mathbb{Z}, \quad A_kE_k = E_{k+1}$ and $A_kF_k\subset F_{k+1}$, (the equivariance property),
- $\inf_{k\in\mathbb{Z}}\gamma(E_k,F_k)>0$, (the uniform minimal gap property),
- $\forall k\in\mathbb{Z},\ \forall n\geqslant 1,\ \dfrac{\|A(k,n)|F_k\|}{\|(A(k,n)|E_k)^{-1}\|^{-1}}\leqslant K_de^{-n\tau}$, (the slow/fast ratio property)

where $\gamma(E_k,F_k)$ denotes the minimal gap between E_k and F_k (a notion of minimal angle between two complementary spaces, see definition A.19),

$$\gamma(E_k,F_k) := \inf\{\mathrm{dist}(u,F_k) : u\in E_k,\ \|u\|=1\},$$

and $\|(A(k,n)|E_k)^{-1}\|^{-1}$ and $\|A(k,n)|F_k\|$ denote respectively the lowest and largest expansion of the cocycle restricted to E_k and F_k,

$$\|A(k,n)|F_k\| := \sup\{\|A(k,n)v\| : v\in F_k,\ \|v\|=1\},$$
$$\|(A(k,n)|E_k)^{-1}\|^{-1} := \inf\{\|A(k,n)u\| : u\in E_k,\ \|u\|=1\}.$$

(The notation $\|(A|E)^{-1}\|^{-1}$ will be used only when $\dim(E)<+\infty$ and $A:E\to X$ is injective). In order to distinguish the two equivariant subspaces in this exponential dichotomy, we will use the terminology *fast space* for E_k and *slow space* for F_k although both operators $A(k,n):E_k\to E_{k+n}$ and $A(k,n):F_k\to F_{k+n}$ may be expanding or contracting. The index k denotes the position of the cocycle and n represents the order of iteration. We interpret $A(k,n)$ as an operator acting from a space above k to a space above $k+n$; in particular the dual operator $A(k,n)^*$ acts on the dual space as an operator from a space above $k+n$ to a space above k.

Our main assumption is related to the existence of a uniform gap in the singular value decomposition at index d. The notion of singular values for an operator in a general Banach space is not well defined. We define the *singular value* of index $d\geqslant 1$ of an operator A, to be the number

$$\sigma_d(A) := \sup_{\dim(E)=d}\ \inf_{u\in E\setminus\{0\}}\frac{\|Au\|}{\|u\|}.$$

Equivalent definitions $\sigma'_d(A), \sigma''_d(A)$ are given in A.29 and A.31. In the Hilbert case, all these definitions are equal. To simplify the notations, we use

$$\sigma_d(k,n) := \sigma_d(A(k,n)).$$

The top singular value is $\sigma_1(k,n) = \|A(k,n)\|$ and, in the particular case $\dim X = d$ and $A(k,n)$ is invertible, the bottom singular value is $\sigma_d(k,n) = \|A(k,n)^{-1}\|^{-1}$.

Main hypothesis 1.1. Let X be a real Banach space and $(A_k)_{k\in\mathbb{Z}}$ be a sequence of bounded operators (not necessarily injective nor surjective). We assume there exist an integer $d \geqslant 1$ and constants $D_{\text{SVG}}, D_{\text{FI}} \geqslant 1$, $\tau, \mu > 0$ such that

- the sequence admits a uniform *singular value gap* at index d

$$\text{(SVG)} \qquad \forall k \in \mathbb{Z}, \forall n \geqslant 0, \quad \begin{cases} \dfrac{\sigma_{d+1}(k,n)\|A_{k+n}\|}{\sigma_d(k,n+1)} \leqslant D_{\text{SVG}}\, e^{-n\tau} \\[2ex] \dfrac{\|A_k\|\sigma_{d+1}(k+1,n)}{\sigma_d(k,n+1)} \leqslant D_{\text{SVG}}\, e^{-n\tau} \end{cases}$$

  (We implicitly assume that $\sigma_d(k,n) > 0$ for every $k \in \mathbb{Z}$ and $n \geqslant 0$),

- the sequence is *d-dimensionally fast invertible*

$$\text{(FI)} \qquad \forall m \geqslant 0, \quad \inf_{k\in\mathbb{Z},\ n\geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-m,m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} \geqslant D_{\text{FI}}^{-1} e^{-m\mu}.$$

Property (FI) is a new property that can be used as a substitute for uniform invertibility along d-dimensional spaces. It is an asymmetric property with respect to forward and backward iterations related to the fact that the fast space (respectively the slow space) has dimension d (respectively codimension d). We will show, thanks to the super-multiplicative property of a similar quotient, that (FI) is equivalent to a seemingly weaker property with $m = 1$,

$$\text{(FI)} \iff \text{(FI)}_{\text{weak}} \qquad e^{-\nu} := \inf_{k\in\mathbb{Z},\ n\geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-1,1+n)}{\sigma_i(k-1,1)\sigma_i(k,n)} > 0.$$

We have chosen the other form to quantify precisely the minimal gap between the fast and slow spaces in our main theorem 1.2 in the Banach spaces case. In the Hilbert spaces case we may choose $D_{\text{FI}} = 1$ and $\nu = \mu$.

Property (FI) is used as a sufficient and necessary hypothesis in a bootstrap argument. Our main result actually shows that the cocycle must satisfy a stronger property $\text{(FI)}_{\text{strong}}$ with a uniform lower bound independent of m,

$$\text{(FI)}_{\text{strong}} \qquad \inf_{m\geqslant 0} \inf_{k\in\mathbb{Z},\, n\geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-m, m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} > 0.$$

We will show

$$\text{(SVG) and (FI)} \implies \text{(FI)}_{\text{strong}}.$$

Notice that we do not assume that the norm of the operators A_k is uniformly bounded from above. Notice also that A_k may not be invertible.

If the cocycle is *uniformly invertible* (UI) in the sense

$$\text{(UI)} \qquad \sup_{k\in\mathbb{Z}} \|A_k\| \leqslant M^* \ \text{ and } \ \inf_{k\in\mathbb{Z}} \|A_k^{-1}\|^{-1} \geqslant M_*$$

for some constants $M^*, M_* > 0$, property (FI) is automatically true with $D_{\text{FI}} = 1$ and $\mu := d\log(M^*/M_*)$. In that case our main result implies

$$\text{(UI)} \implies \text{(FI)}, \qquad \text{(SVG) and (UI)} \implies \text{(FI)}_{\text{strong}}.$$

The singular value gap property (SVG) admits a weaker form. This weaker form is actually equivalent to the strong one for uniformly invertible cocycles and was introduced by Bochi and Gourmelon in [3] for the first time,

$$\text{(SVG)}_{\text{weak}} \qquad \forall k \in \mathbb{Z}, \forall n \geqslant 0, \quad \frac{\sigma_{d+1}(k,n)}{\sigma_d(k,n)} \leqslant D_{\text{SVG}}\, e^{-n\tau}.$$

The strong form (SVG) was introduced by Blumenthal and Morris in [2] in order to extend the results of Bochi and Gourmelon to the infinite-dimensional case. They nevertheless assume the cocycle to be norm-continuous over a compact dynamical system and each operator A_k to be injective. Our property (FI) is used instead of the injectiveness assumption. Moreover we do not assume that the cocycle is defined over a dynamical system, nor do we require regularity conditions as in [3, 2]. Our main objective is to obtain an effective splitting of the Banach space into a fast and a slow space, equivariant under the cocycle, for which the angle between the two spaces can be explicitly bounded from below using only the constants $(D_{\text{SVG}}, D_{\text{FI}}, \tau, \mu)$ while avoiding the use of compactness of the underlying dynamical system and regularity assumptions on the cocycle.

Our estimates depend on a constant K_d which is only a function of the dimension d and the Banach space. For a Hilbert space $K_d = 1$, for a general Banach space, K_d is explicitly computed given a volume distortion $\Delta_d(X)$ (see definition A.4) which measures the distortion of the unit Banach ball to the best fitted Euclidean ball. We have that $\Delta_d(X) \leqslant \sqrt{d}$ for Banach spaces and $\Delta_d(X) = 1$ for Hilbert spaces. We give an estimate of $\Delta_d(X)$ in proposition A.5 when $X = \ell_d^p$ is the space of dimension d equipped the p-norm. We do not intend to undertake a systematic study of $\Delta_d(X)$. We have chosen to give a unified proof for both Banach and Hilbert spaces in such a way the constants appearing in the estimates become optimal in the Hilbert case.

Our main result is the following.

Theorem 1.2. *Let X be a Banach space, $d \geqslant 1$, and $(A_k)_{k\in\mathbb{Z}}$ be a sequence of bounded operators satisfying the two assumptions* (SVG) *and* (FI) *at the index d, for some constants $D_{SVG}, D_{FI} \geqslant 1$ and $\tau, \mu > 0$. Then there exist a constant K_d depending only on the dimension d and the Banach norm such that,*

1. *there exists an equivariant splitting $X = E_k \oplus F_k$ satisfying for every $k \in \mathbb{Z}$,*
   - $\dim(E_k) = d,\ A_k(E_k) = E_{k+1},\ A_k(F_k) \subset F_{k+1},$
   - $\gamma(E_k, F_k) \geqslant \dfrac{1}{5K_dD_{FI}}\Big[\dfrac{(3d+7)^{-2}}{2K_dD_{FI}}\dfrac{1-e^{-\tau}}{D_{SVG}e^{\tau}}\Big]^{\frac{\mu(\mu+4\tau)}{2\tau^2}},$
2. (FI) $\Leftrightarrow$ $(\text{FI})_{\text{strong}}$. *More precisely for every $k \in \mathbb{Z}$, $m, n \geqslant 1$,*

$$\prod_{i=1}^{d}\frac{\sigma_i(k-m-n,m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} \geqslant \frac{3}{25K_dD_{FI}^3}\Big[\frac{(3d+7)^{-2}}{2K_dD_{FI}}\frac{1-e^{-\tau}}{D_{SVG}e^{\tau}}\Big]^{\mu(\mu^2+5\mu\tau+8\tau^2)/2\tau^3},$$

3. *The spaces E_k and F_k are called the* fast *and* slow *spaces respectively and satisfy: for every $k \in \mathbb{Z}$ and n such that,*

$$n \geqslant \Big(1+\frac{\mu(\mu+4\tau)}{2\tau^2}\Big)\frac{1}{\tau}\log\Big(\frac{D_{SVG}e^{\tau}}{1-e^{-\tau}}2(3d+7)^2K_d\Big),$$

   - $\|(A(k,n)|E_k)^{-1}\|^{-1} \geqslant \frac{3}{5}K_d^{-1}\gamma(E_k,F_k)\sigma_d(k,n),$
   - $\|A(k,n)|F_k\| \leqslant 3K_d\gamma(F_{k+n},E_{k+n})^{-1}\sigma_{d+1}(k,n).$

Using the definition of $\bar{\Delta}_d(X)$ in equation (A.3)*, and the constants $C_{0,d}$ and $\widehat{C}_{0,d}$ in theorems A.35 and A.43, with $\epsilon = 0$, we obtain*

$$K_d := \widehat{C}_{0,d}^{7}C_{0,d}^{8d+5}\bar{\Delta}_2(X)^{4d}\bar{\Delta}_d(X)^{8d} \leqslant (2d)^{2000d^3}.$$

If X is a real Hilbert space then $K_d = 1$ and D_{FI} may be chosen equal to 1 in (FI).

Our main result extends the results of Bochi and Gourmelon [3] in the case $X = \mathbb{R}^d$ in three ways: we do not assume the cocycle to be invertible, we do not introduce a dynamical system, we do not assume either C^0 regularity or compactness. The proof used in [3] requires all these assumptions and actually needs the ergodic Oseledets theorem for invariant probability measures. We have chosen to work in two directions: a direction which gives explicit estimates, especially for the lower bound of the angle, with respect to the initial data, and a direction which gives an unified proof for Banach and Hilbert spaces. In order not to introduce artificial constants in the Banach setting, we found it necessary to develop in appendix A a theory of volume distortion $\bar{\Delta}_d(X)$ which enables us to quantify on each d-dimensional space the distortion of the Banach norm with respect to the best fitted Euclidean norm. The volume distortion $\bar{\Delta}_d(X)$ is 1 in the Hilbert case. We express all estimates in terms of a constant K_d that is only a function of $\bar{\Delta}_d(X)$ and satisfies $K_d = 1$ in the Hilbert case.

In item 1 we obtain an explicit lower bound of the angle between the fast and slow spaces depending only on $D_{\text{SVG}}, D_{\text{FI}}, \tau, \mu$ and the dimension d. We have chosen to give a uniform estimate for every $k \in \mathbb{Z}$ instead of an asymptotic estimate as $k \to \pm\infty$. This choice has led to additional computation.

In item 2 we prove the strong form $(\text{FI})_{\text{strong}}$. This is actually a simple consequence of lemma A.44 and the uniform bound $\inf_{k\in\mathbb{Z}} \gamma(E_k, F_k) > 0$. We nevertheless give a precise estimate valid for all iterates m, n and not just for $m, n \to +\infty$. In the Hilbert case, the estimate is simpler with $K_d = 1$ and $D_{\text{FI}} = 1$ in (FI).

In item 3 we show that the two equivariant splittings correspond indeed to the fast and slow spaces; we again made the decision to give explicit but not optimal estimates. The singular value of index d of the cocycle restricted to the fast space is comparable up to a factor given by the minimal gap $\gamma(E_k, F_k)$ to the original d-dimensional singular value. A similar result is obtained for the slow space. For large n and in the Hilbert case, the two constants $\frac{3}{5}K_d^{-1}$ and $3K_d$ may be replaced by 1.

The proof of our main result is divided into 3 parts. In section 2, we show how property (SVG) implies the existence of two fast and slow spaces that may not be complementary. This mechanism is standard since Raghunathan [13] in finite dimension, Ruelle [14] in Hilbert spaces, Blumenthal-Morris [2] in Banach spaces, and González-Tokman-Quas [8] for a shorter proof. Our proof quantifies precisely the speed of convergence of the approximate spaces. In section 3, we show how property (FI) implies that the two fast and slow spaces give a splitting of the ambient space. This part is the heart of the proof and is new. In section 4, we show that (FI) is a necessary and sufficient condition and actually equivalent to a stronger condition $(\text{FI})_{\text{strong}}$. In appendix A, we recall basic definitions of the geometric theory of Banach spaces. We recall different notions of distance between subspaces, several notions of singular values, some facts about the projective norm on the exterior product. The main purpose of this appendix is to recall without proofs the standard approximate singular value decomposition theorem A.35.

2 Construction of the fast and slow spaces

The proof of our main result is based on a version of the singular value decomposition (SVD) theorem for a single bounded operator in the Banach setting. The (SVD) theorem is well known for compact operators in a Hilbert space (see [12]). We did not find a version of the (SVD) theorem adapted to our needs in the literature. Appendix A fills in this missing piece. The main interest of Appendix A is theorem A.35 which shows the existence of approximate singular spaces at every index d. The singular spaces may not be exact because of the non compactness of the operators and are thus non canonical. They depend for instance on an arbitrarily small constant $\epsilon > 0$ coming from the fact that, in the case of infinite Banach or Hilbert spaces, the norm of an operator may not be attained by a vector of the unit sphere. Notice that we shall not use the (FI) condition in this section.

The following theorem is a special version of theorem A.35 applied to each operator $A(k,n) = A_{k+n-1} \cdots A_{k+1}A_k$. We fix $\epsilon > 0$ and the index $d \geqslant 1$. We show there exist a pair of complementary spaces $X = U(k,n) \oplus V(k,n)$ of the source space and a pair of complementary spaces $X = \tilde{U}(k+n,n) \oplus \tilde{V}(k+n,n)$ of the target space that are related by $A(k,n)$ and $A(k,n)^*$. We replace the usual notion of orthogonality by a weaker notion using C-Auerbach families (see definition A.12 for more details). We show that the two splittings are $C_{\epsilon,d}$-orthogonal in the sense of the following definition.

Definition 2.1. Let X be a Banach space, $d \geqslant 1$, $C \geqslant 1$.

- We say that a family of vectors $(u_1, \ldots, u_d)$ is *C-Auerbach* if

$$\forall j = 1, \ldots, d, \quad C^{-1} \leqslant \operatorname{dist}(u_j, \operatorname{span}(u_i : i \neq j)) \leqslant \|u_j\| \leqslant C.$$

- We say a splitting $X = U \oplus V$ with $\dim(U) = d$ is *C-orthogonal* if there exist a C-Auerbach basis $(e_1, \ldots, e_d)$ spanning U and a C-Auerbach basis $(\phi_1, \ldots, \phi_d)$ spanning $V^\perp$ in the dual space X^* which are dual to each other, that is $\langle \phi_i | e_j \rangle = \delta_{i,j}, \ \forall i, j = 1, \ldots, d$.

If $V \subset X$ is a subspace of X, the *annihilator* of U is the subspace in the dual space, $U^\perp := \{\phi \in X^* : \langle \phi | u \rangle = 0, \ \forall u \in U\}$. If $H \subset X^*$, the *pre-annihilator* of H is the subspace in X, $H^{\perp\!\!\perp} := \{v \in X : \langle \eta | v \rangle = 0, \ \forall \eta \in H\}$.

Theorem 2.2 (Approximate singular value decomposition). *Let X be a Banach space, $d \geqslant 1$, $\epsilon > 0$, and $(A_k)_{k \in \mathbb{Z}}$ be a sequence of bounded operators. Then there exists a constant $K_d \geqslant 1$ depending only on the Banach norm and d, such that for every $k \in \mathbb{Z}$, $n \geqslant 1$, and $C_{\epsilon,d} := (1+\epsilon)K_d$,*

1. *there exist two $C_{\epsilon,d}$-orthogonal splittings:*
   - $X = U(k,n) \oplus V(k,n), \quad X = \tilde U(k,n) \oplus \tilde V(k,n),$
   - $\dim(U(k,n)) = \dim(\tilde U(k,n)) = d,$
   - $A(k,n)U(k,n) = \tilde U(k+n,n), \quad A(k,n)V(k,n) \subset \tilde V(k+n,n),$
   - $A(k,n)^*\tilde U(k+n,n)^\perp \subset U(k,n)^\perp, \quad A(k,n)^*\tilde V(k+n,n)^\perp = V(k,n)^\perp,$
2. *the singular values of $A(k,n)$ and $A(k,n)^*$ restricted to this splitting are comparable to those of $A(k,n)$ on X: for every $1 \leqslant i \leqslant d$,*
   - $\sigma_i(k,n) \geqslant \sigma_i(A(k,n)|U(k,n)) \geqslant \sigma_i(k,n)/C_{\epsilon,d},$
   - $\sigma_i(k,n) \geqslant \sigma_i(A(k,n)^*|\tilde V(k+n,n)^\perp) \geqslant \sigma_i(k,n)/C_{\epsilon,d},$
   - $\sigma_{d+1}(k,n) \leqslant \|A(k,n)|V(k,n)\| \leqslant \sigma_{d+1}(k,n)C_{\epsilon,d},$
   - $\sigma_{d+1}(k,n) \leqslant \|A(k,n)^*|\tilde U(k+n,n)^\perp\| \leqslant \sigma_{d+1}(k,n)C_{\epsilon,d},$
3. *the minimal gap of the two splittings is uniformly bounded from below,*
$$\gamma(U(k,n),V(k,n)) \geqslant 1/C_{\epsilon,d}, \quad \gamma(V(k,n),U(k,n)) \geqslant 1/C_{\epsilon,d},$$
$$\gamma(\tilde U(k,n),\tilde V(k,n)) \geqslant 1/C_{\epsilon,d}, \quad \gamma(\tilde V(k,n),\tilde U(k,n)) \geqslant 1/C_{\epsilon,d},$$
4. *there exits a pair of $C_{\epsilon,d}$-Auerbach families of (the source space) X, X^*,*
$$(e_1(k,n),\ldots,e_d(k,n)), \quad (\phi_1(k,n),\ldots,\phi_d(k,n))$$
   and a pair of $C_{\epsilon,d}$-Auerbach families of (the target space) X, X^*,
$$(\tilde e_1(k+n,n),\ldots,\tilde e_d(k+n,n)), \quad (\tilde\phi_1(k+n,n),\ldots,\tilde\phi_d(k+n,n))$$
   satisfying
   - $\langle \phi_i(k,n)|e_j(k,n)\rangle = \delta_{i,j}, \quad \langle \tilde\phi(k,n)|\tilde e_j(k,n)\rangle = \delta_{i,j},$
   - $A(k,n)e_i(k,n) = \sigma_i(k,n)\tilde e_i(k+n,n),$
   - $A(k,n)^*\tilde\phi_i(k+n,n) = \sigma_i(k,n)\phi_i(k,n),$
   - $U(k,n) = \mathrm{span}(e_1(k,n),\ldots,e_d(k,n)),$
   - $V(k,n) = \mathrm{span}(\phi_1(k,n),\ldots,\phi_d(k,n))^{\perp\!\!\!\perp},$
   - $\tilde U(k+n,n) = \mathrm{span}(\tilde e_1(k+n,n),\ldots,\tilde e_d(k+n,n)),$
   - $\tilde V(k+n,n) = \mathrm{span}(\tilde\phi_1(k+n,n),\ldots,\tilde\phi_d(k+n,n))^{\perp\!\!\!\perp}.$
5. *Moreover $K_d = 1$ if X is a Hilbert space and ϵ may be chosen to be zero if X is finite-dimensional.*

We call $U(k,n)$ and $V(k,n)$, the *approximate fast and slow forward spaces* above k. Similarly we will call $\tilde{U}(k,n)$ and $\tilde{V}(k,n)$, defined using $A(k-n,n)$, the *approximate fast and slow backward spaces* above k. Since the approximate forward spaces are built using the sequence of operators $(A_k, A_{k+1}, \ldots, A_{k+n-1})$ and the approximate backward spaces are built using $(A_{k-n}, A_{k-n+1}, \ldots, A_{k-1})$, the two splittings above k, $X = U(k,n) \oplus V(k,n)$ and $X = \tilde{U}(k,n) \oplus \tilde{V}(k,n)$, need not be closely related.

We first consider the construction of the slow spaces $(F_k)_{k\in\mathbb{Z}}$ using the forward cocycle $(A_n)_{n=k}^{+\infty}$ and their approximate slow forward spaces $V(k,n)$.

The following lemma shows an exponential contraction between the two approximate slow forward spaces. The maximal gap $\delta(V,W)$ between V and W is a standard notion of distance between two subspaces (see definition A.17 and equivalent formulations – note the asymmetry in the definition).

$$\delta(V,W) = \sup\{\mathrm{dist}(v,W) : v \in V,\ \|v\| = 1\}.$$

Lemma 2.3 (Raghunathan estimate I). *Suppose that the sequence of operators (A_k) satisfies (SVG). Then for every $k \in \mathbb{Z}$ and $n \geqslant 1$,*

$$\begin{aligned} &\delta(V(k,n), V(k,n+1)) \leqslant C_{\epsilon,d}^2 D_{\text{SVG}}\, e^{-n\tau}, \\ &\delta(V(k,n+1), V(k,n)) \leqslant C_{\epsilon,d}^2 D_{\text{SVG}}\, e^{-n\tau}/(1 - C_{\epsilon,d}^2 D_{\text{SVG}}\, e^{-n\tau}). \end{aligned} \tag{1}$$

Proof. Let $v \in V(k,n)$ and $\phi \in V(k,n+1)^\perp$ be of norm 1. Choose $\tilde{\phi} \in \tilde{V}(k+n+1, n+1)^\perp$ such that $\phi = A(k,n+1)^*\tilde{\phi}$. Using item 2 of theorem 2.2 one obtains on the one hand

$$\|\phi\| = \|A(k,n+1)^*\tilde{\phi}\| \geqslant \frac{\sigma_d(k,n+1)}{C_{\epsilon,d}} \|\tilde{\phi}\|,$$

and on the other hand

$$\begin{aligned} \langle \phi | v \rangle &= \langle \tilde{\phi} | A(k,n+1) v \rangle \\ &\leqslant \|\tilde{\phi}\|\, \|A(k,n+1)v\| \leqslant \|\tilde{\phi}\| \|A_{k+n}\| \|A(k,n)v\| \\ &\leqslant C_{\epsilon,d} \|A_{k+n}\| \sigma_{d+1}(k,n) \|\tilde{\phi}\| \|v\| \\ &\leqslant C_{\epsilon,d}^2 \frac{\|A_{k+n}\| \sigma_{d+1}(k,n)}{\sigma_d(k,n+1)} \|\phi\| \|v\| \\ &\leqslant C_{\epsilon,d}^2 D_{\text{SVG}}\, e^{-n\tau}, \end{aligned}$$

where the last line follows from (SVG). The first estimate in (1) then follows from (A.7). The second estimate is obtained using equation (A.11),

$$\delta(V(k,n+1), V(k,n)) \leqslant \frac{\delta(V(k,n), V(k,n+1))}{1 - \delta(V(k,n), V(k,n+1))}. \qquad \square$$

The previous lemma shows that the gap between two successive $V(k,n)$ is exponentially small. This implies in particular that $(V(k,n))_{n\geqslant 1}$ is a Cauchy sequence and that $V(k,n) \to F_k$ uniformly in k to a subspace F_k of codimension d that we will call the slow space. We will need a more precise statement where F_k is understood as a graph over a fixed splitting uniformly in k (see definition A.22). The reference splitting will be given by $X = U(k,N_*) \oplus V(k,N_*)$ for some N_* chosen sufficiently large. An initial choice of N_* is made in the following lemma and will be subsequently tightened in lemma 3.3, 3.8, and finally in Assumption 3.11. It will be convenient to choose at each step of the proof N_* depending on a parameter $\theta_* \in (0,1)$ as in (2), (5) and (7).

Lemma 2.4 (Existence of the slow space). *Let $\theta_* \in (0,1)$ and N_* satisfy*

$$D_{SVG}\, e^{-N_*\tau} \leqslant \theta_*(1-\theta_*)^6 \frac{1-e^{-\tau}}{C_{\epsilon,d}^4}. \tag{2}$$

Then for every $k \in \mathbb{Z}$, for every $n \geqslant N_*$, the following 5 items are satisfied.

1. $V(k,n) = \mathrm{Graph}(\Theta(k,n))$ *for some* $\Theta(k,n) \in \mathcal{B}(V(k,N_*), U(k,N_*))$

$$\delta(V(k,N_*), V(k,n)) \leqslant \|\Theta(k,n)\| \leqslant \theta_*, \quad \delta(V(k,n), V(k,N_*)) \leqslant \theta_*.$$

2. $(\Theta(k,n))_{n\geqslant N_*}$ *is a Cauchy sequence, for every* $n \geqslant 1$

$$\|\Theta(k,n+1) - \Theta(k,n)\| \leqslant \theta_* e^{-(n-N_*)\tau}(1-e^{-\tau}).$$

3. *Let* $\Theta_k(N_*) := \lim_{n\to+\infty} \Theta(k,n)$ *and* $F_k := \mathrm{Graph}(\Theta_k(N_*))$. *Then*

$$\delta(V(k,N_*), F_k) \leqslant \|\Theta_k(N_*)\| \leqslant \theta_*, \quad \delta(F_k, V(k,N_*)) \leqslant \theta_*.$$

   F_k is called the slow space of index d; F_k is independent of the choice of N_*.

4. $V(k,n)^\perp = \mathrm{Graph}(\Theta^\perp(k,n))$ *for the bounded operator*

$$\Theta^\perp(k,n) = -\pi(k,N_*)^*\Theta(k,n)^*\rho(k,N_*)^* \in \mathcal{B}(V(k,N_*)^\perp, U(k,N_*)^\perp),$$

   where $\pi(k,n)$ is the projection onto $V(k,n)$ parallel to $U(k,n)$ and $\rho(k,n)$ is the inclusion operator $U(k,n) \hookrightarrow X$. Moreover

$$\Theta_k^\perp(N_*) := \lim_{n\to+\infty} \Theta^\perp(k,n) \quad \textit{exists},$$
$$F_k^\perp = \mathrm{Graph}(\Theta_k^\perp(N_*)), \quad \|\Theta^\perp(k,n)\| \leqslant \theta_*, \quad \|\Theta_k^\perp(N_*)\| \leqslant \theta_*.$$

5. $\|(A(k,n)|U(k,N_*)^{-1}\|^{-1}/\sigma_d(k,n)$ *is uniformly bounded from below,*

- $X = U(k,N_*) \oplus F_k$,
- $\forall u \in U(k,N_*), \quad \|A(k,n)u\| \geqslant C_{\epsilon,d}^{-2}(1-\theta_*)^2\sigma_d(k,n)\|u\|$,
- $\gamma(U(k,N_*),F_k) \geqslant C_{\epsilon,d}^{-1}(1-\theta_*)^2$.

Proof. In order to simplify the notations, fix k and denote

$$V_n := V(k,n),\ V_* := V(k,N_*),\ \ U_* := U(k,N_*).$$

We want to apply lemma A.25 for the initial splitting $X = U_* \oplus V_*$ where V_* plays the role of U_0. An additional complication comes from the fact that the minimal angle is not symmetric. We shall show by induction for every $n \geqslant N_*$

- $\|\Theta_n - \Theta_{n-1}\| \leqslant \theta_{n-1}(1-\theta_*), \quad (\Theta_{N_*-1} = 0$ by convention$)$,
- $V_n = \mathrm{Graph}(\Theta_n)$ for some $\Theta_n \in \mathcal{B}(V_*,U_*)$ with $\|\Theta_n\| \leqslant \theta_*(1-\theta_*)\gamma(U_*,V_*)$,
- $\delta(V_n,V_*) \leqslant \theta_*\gamma(U_*,V_*)$,

where $\theta_n := \theta_* e^{-(n-N_*)\tau}(1-e^{-\tau})\gamma(U_*,V_*) \leqslant \theta_*$.

Suppose that the above conditions are satisfied for the index n. We first claim that the choice of N_* implies

$$\delta(V_{n+1},V_n) \leqslant \theta_n(1-\theta_n)(1-\theta_*)^2\gamma(U_*,V_n) \leqslant \theta_n.$$

To see this, on the one hand, from equation (A.16), we have

$$\begin{aligned}\gamma(U_*,V_n) &\geqslant \frac{\gamma(U_*,V_*) - \delta(V_n,V_*)}{1+\delta(V_n,V_*)} \\ &\geqslant \frac{(1-\theta_*)\gamma(U_*,V_*)}{1+\theta_*\gamma(U_*V_*)} \geqslant (1-\theta_*)^2\gamma(U_*,V_*).\end{aligned}$$

On the other hand, from the definition of N_* we have

$$\begin{aligned}C_{\epsilon,d}^2 D_{\mathrm{SVG}}\, e^{-n\tau} &\leqslant \theta_*(1-\theta_*)^6 e^{-(n-N_*)\tau}(1-e^{-\tau})\gamma(U_*,V_*)^2, \\ &\leqslant \theta_n(1-\theta_*)^6\gamma(U_*,V_*).\end{aligned}$$

Combining both estimates, lemma 2.3 and equation (A.11), one obtains

$$\delta(V_n,V_{n+1}) \leqslant C_{\epsilon,d}^2 D_{\mathrm{SVG}}\, e^{-n\tau} \leqslant \theta_n(1-\theta_*)^4\gamma(U_*,V_n) \leqslant \theta_n \leqslant \theta_*,$$

$$\delta(V_{n+1},V_n) \leqslant \frac{\theta_n(1-\theta_*)^4\gamma(U_*,V_n)}{1-\theta_*} \leqslant \theta_n(1-\theta_n)(1-\theta_*)^2\gamma(U_*,V_n).$$

The claim is proved. We now show the three conditions for the index $n + 1$. From item 2 of lemma A.25, $V_{n+1} = \mathrm{Graph}(\Theta_{n+1})$ for some $\Theta_{n+1} \in \mathcal{B}(V_*, U_*)$ and

$$\|\Theta_{n+1} - \Theta_n\| \leqslant \frac{\delta(V_{n+1}, V_n)}{\gamma(U_*, V_n) - \delta(V_{n+1}, V_n)} \frac{\gamma(U_*, V_*)}{\gamma(U_*, V_*) - \delta(V_n, V_*)} \leqslant \theta_n(1 - \theta_*),$$

$$\delta(V_*, V_{n+1}) \leqslant \|\Theta_{n+1}\| \leqslant \sum_{k=N_*}^{n} \theta_k(1 - \theta_*) \leqslant \theta_*(1 - \theta_*)\gamma(U_*, V_*),$$

$$\delta(V_{n+1}, V_*) \leqslant \frac{\delta(V_*, V_{n+1})}{1 - \delta(V_*, V_{n+1})} \leqslant \theta_*\gamma(U_*, V_*).$$

The induction is complete and the three first items are proved.

The fact that F_k is independent of the initial choice N_* is proved in the following way. Let $w \in F_k$, $w = v + \Theta_k(N_*)v$ for some $v \in V(k, N_*)$. Then

$$w - [v + \Theta(k, n)v] = [\Theta_k(N_*)v - \Theta(k, n)v],$$

$$\mathrm{dist}(w, V(k, n)) \leqslant \|\Theta_k(N_*) - \Theta(k, n)\| \, \|v\| \leqslant \frac{\|\Theta_k(N_*) - \Theta(k, n)\|}{\gamma(V_*, U_*)} \|w\|,$$

$$\delta(F_k, V(k, n)) \leqslant \frac{\|\Theta_k(N_*) - \Theta(k, n)\|}{\gamma(V_*, U_*)} \leqslant \theta_* e^{-(n-N_*)\tau} \frac{\gamma(U_*, V_*)}{\gamma(V_*, U_*)}.$$

Let F'_k as in item 3 with another choice of θ'_* and N'_*. Using the weak triangle inequality

$$\delta(F_k, F'_k) \leqslant 2\delta(F_k, V(k, n)) + 2\delta(V(k, n), F'_k)$$

and letting $n \to +\infty$, one obtains $\delta(F_k, F'_k) = 0$ and $F_k = F'_k$.

Item 4 is a consequence of lemma A.23. Item 5 is a consequence of item 2 of theorem 2.2 and equation (A.16),

$$\gamma(U_*, V_n) \geqslant \frac{\gamma(U_*, V_*) - \delta(V_n, V_*)}{1 + \delta(V_n, V_*)} \geqslant \gamma(U_*, V_*) \frac{1 - \theta_*}{1 + \theta_*} \geqslant \gamma(U_*, V_*)(1 - \theta_*)^2,$$

$$\gamma(U_*, F_k) \geqslant \gamma(U_*, V_*)(1 - \theta_*)^2, \quad \text{(by taking the limit } n \to +\infty).$$

Moreover for every $u \in U_*$ such that $\|u\| = 1$,

$$\begin{aligned}
\|A(k, n)u\| &\geqslant \sup\{\langle \tilde{\phi} | A(k, n)u \rangle : \tilde{\phi} \in \tilde{V}(k + n, n)^{\perp}, \ \|\tilde{\phi}\| = 1\} \\
&\geqslant \sup\{\langle \phi | u \rangle : \phi \in V_n^{\perp}, \ \|\phi\| = 1\} \inf \left\{ \frac{\|A(k, n)^* \tilde{\phi}\|}{\|\tilde{\phi}\|} : \tilde{\phi} \in V(k + n, n)^{\perp} \right\} \\
&\geqslant \mathrm{dist}(u, V_n) \frac{\sigma_d(k, n)}{C_{\epsilon,d}} \geqslant \gamma(U_*, V_n) \frac{\sigma_d(k, n)}{C_{\epsilon,d}} \\
&\geqslant \gamma(U_*, V_*)(1 - \theta_*)^2 \frac{\sigma_d(k, n)}{C_{\epsilon,d}}.
\end{aligned}$$

□

Lemma 2.5 (**Equivariance of the slow space**). *For every $k \in \mathbb{Z}$,*

$$A_k F_k \subset F_{k+1}.$$

Proof. Let $v \in V(k, n+1)$, and $\phi \in V(k+1, n)^\perp$. Then there exists $\tilde{\phi} \in \tilde{V}(k+n+1, n)^\perp$ such that $\phi = A(k+1, n)^* \tilde{\phi}$. On the one hand, item 2 of theorem 2.2 implies

$$\|\phi\| \geqslant \sigma_d(A(k+1,n)^* | \tilde{V}(k+n+1,n)^\perp) \|\tilde{\phi}\| \geqslant \frac{\sigma_d(k+1,n)}{C_{\epsilon,d}} \|\tilde{\phi}\|.$$

On the other hand, item 2 also shows

$$\begin{aligned}
\langle \phi | A_k v \rangle = \langle \tilde{\phi} | A(k, n+1) v \rangle &\leqslant \|\tilde{\phi}\| \|A(k,n+1) | V(k,n+1)\| \, \|v\|, \\
&\leqslant C_{\epsilon,d}^2 \frac{\sigma_{d+1}(k,n+1)}{\sigma_d(k+1,n)} \|\phi\| \, \|v\| \\
&\leqslant C_{\epsilon,d}^2 \|A_k\| \frac{\|A_{k+1}\| \sigma_{d+1}(k+2,n-1)}{\sigma_d(k+1,n)} \|\phi\| \, \|v\| \\
&\leqslant C_{\epsilon,d}^2 \|A_k\| D_{\text{SVG}} e^{-(n-1)\tau} \|\phi\| \, \|v\|.
\end{aligned}$$

We have thus obtained for every $v \in V(k, n+1)$,

$$\begin{aligned}
\operatorname{dist}(A_k v, V(k+1,n)) &= \sup\{\langle \phi | A_k v \rangle : \phi \in V(k+1,n)^\perp, \ \|\phi\| = 1\}, \\
&\leqslant C_{\epsilon,d}^2 \|A_k\| D_{\text{SVG}} e^{-(n-1)\tau} \|v\|.
\end{aligned}$$

Let θ_* and N_* satisfy equation (2). Assume $n \geqslant N_*$. Let $v_* \in V(k, N_*)$ and $w_n := \Theta(k, n+1) v_* + v_*$. Then there exists $v_n' \in V(k+1, N_*)$ such that

$$w_n' := \Theta(k+1,n) v_n' + v_n' \ \text{ satisfies } \ \|A_k w_n - w_n'\| \to 0.$$

Since $w_n \to w := \Theta_k(N_*) v_* + v_*$, the sequences $(A_k w_n)_n$, $(w_n')_n$ and $(v_n')_n$ are Cauchy sequences. We obtain therefore the convergence of $v_n' \to v' \in V(k+1, N_*)$ and $A_k(\Theta_k(N_*) v_* + v_*) = \Theta_{k+1}(N_*) v' + v'$. □

We now consider the construction of the fast spaces $(E_k)_{k \in \mathbb{Z}}$ using the backward cocycle $(A_n)_{-\infty}^{n=k-1}$ and their approximate fast backward spaces $\tilde{U}(k, n)$. The following lemma is analogous to lemma 2.3.

Lemma 2.6 (**Raghunathan estimate II**). *For every $n \geqslant 1$, $k \in \mathbb{Z}$,*

$$\begin{aligned}
&\delta(\tilde{U}(k,n+1), \tilde{U}(k,n)) \leqslant C_{\epsilon,d}^2 D_{\mathit{SVG}} e^{-n\tau}, \\
&\delta(\tilde{U}(k,n), \tilde{U}(k,n+1)) \leqslant C_{\epsilon,d}^2 D_{\mathit{SVG}} e^{-n\tau} / (1 - C_{\epsilon,d}^2 D_{\mathit{SVG}} e^{-n\tau}).
\end{aligned} \tag{3}$$

Proof. Let $\tilde{u} \in \tilde{U}(k, n+1)$ and $\tilde{\phi} \in \tilde{U}(k,n)^\perp$ of norm 1. On the one hand $\tilde{u} = A(k-n-1, n+1)u$ for some $u \in U(k-n-1, n+1)$ and item 2 of theorem 2.2 implies

$$\|\tilde{u}\| \geqslant \sigma_d(k-n-1, n+1)\|u\|/C_{\epsilon,d}.$$

On the other hand, item 2 also implies

$$\begin{aligned}
\langle\tilde{\phi}|\tilde{u}\rangle = \langle\tilde{\phi}|A(k-n-1,n+1)u\rangle &= \langle A(k-n-1,n+1)^*\tilde{\phi}|u\rangle \\
&\leqslant \|A_{k-n-1}\|\|A(k-n,n)^*\tilde{\phi}\|\|u\| \\
&\leqslant \|A_{k-n-1}\|\sigma_{d+1}(k-n,n)C_{\epsilon,d}\|\tilde{\phi}\|\|u\| \\
&\leqslant C_{\epsilon,d}^2 \frac{\|A_{k-n-1}\|\sigma_{d+1}(k-n,n)}{\sigma_d(k-n-1,n+1)}\|\tilde{\phi}\|\|\tilde{u}\|.
\end{aligned}$$

The second inequality is a consequence of equation (A.11). □

The following lemma is analogous to lemma 2.4. We show that the sequence of subspaces $(\tilde{U}(k,n))_{n\geqslant 1}$ is a Cauchy sequence converging uniformly in k to a subspace E_k of dimension d. We see E_k as a graph over $\tilde{U}(k, N_*)$ in the splitting $X = \tilde{U}(k,N_*) \oplus \tilde{V}(k,N_*)$ for some large N_* defined in (2).

Lemma 2.7 (Existence of the fast space). *Let $\theta_* \in (0,1)$ and N_* satisfy equation (2). Then for every $k \in \mathbb{Z}$, for every $n \geqslant N_*$, the following 4 items are satisfied.*

1. $\tilde{U}(k,n) = \mathrm{Graph}(\tilde{\Theta}(k,n))$ *for some* $\tilde{\Theta}(k,n) \in \mathcal{B}(\tilde{U}(k,N_*), \tilde{V}(k,N_*))$,

   $$\delta(\tilde{U}(k,N_*), \tilde{U}(k,n)) \leqslant \|\tilde{\Theta}(k,n)\| \leqslant \theta_*, \quad \delta(\tilde{U}(k,n), \tilde{U}(k,N_*)) \leqslant \theta_*.$$

2. $(\tilde{\Theta}(k,n))_{n\geqslant N_*}$ *is a Cauchy sequence, for every* $n \geqslant 1$

   $$\|\tilde{\Theta}(k,n+1) - \tilde{\Theta}(k,n)\| \leqslant \theta_* e^{-(n-N_*)\tau}(1-e^{-\tau}),$$

3. *Let* $\tilde{\Theta}_k(N_*) := \lim_{n\to+\infty} \tilde{\Theta}(k,n)$ *and* $E_k := \mathrm{Graph}(\tilde{\Theta}_k(N_*))$. *Then*

   $$\delta(\tilde{U}(k,N_*), E_k) \leqslant \|\tilde{\Theta}_k(N_*)\| \leqslant \theta_*, \quad \delta(E_k, \tilde{U}(k,N_*)) \leqslant \theta_*.$$

   E_k is called the fast space of index d; E_k is independent of the choice of N_*.

4. $\|(A(k-n,n)^*|\tilde{V}(k,N_*)^\perp)^{-1}\|^{-1}/\sigma_d(k-n,n)$ *is bounded from below,*

   - $X = E_k \oplus \tilde{V}(k,N_*)$,
   - $\forall\tilde{\phi} \in \tilde{V}(k,N_*)^\perp, \quad \|A(k-n,n)^*\tilde{\phi}\| \geqslant C_{\epsilon,d}^{-2}(1-\theta_*)^2\sigma_d(k-n,n)\|\tilde{\phi}\|$,
   - $\gamma(\tilde{U}(k,n), \tilde{V}(k,N_*)) \geqslant (1-\theta_*)^2 C_{\epsilon,d}^{-1}$.

Proof. The proof of items 1 – 3 is similar to the one in lemma 2.4 by permuting the role of U and V. For instance we also obtain by induction

$$\delta(\tilde{U}(k,N_*),\tilde{U}(k,n)) \leqslant \theta_* \gamma(\tilde{U}(k,N_*),\tilde{V}(k,N_*)).$$

For the last item, we choose $\tilde{\phi} \in \tilde{V}(k,N_*)^\perp$, $\|\tilde{\phi}\| = 1$, then using (A.6),

$$\begin{aligned}
&\|A(k-n,n)^*\tilde{\phi}\| \\
&\quad \geqslant \sup\{\langle\tilde{\phi}|A(k-n,n)u\rangle : u \in U(k-n,n),\ \|u\| = 1\} \\
&\quad \geqslant \sup\{\langle\tilde{\phi}|\tilde{u}\rangle : \tilde{u} \in \tilde{U}(k,n),\ \|\tilde{u}\| = 1\} \inf\left\{\frac{\|A(k-n,n)u\|}{\|u\|} : u \in U(k-n,n)\right\} \\
&\quad \geqslant \operatorname{dist}(\tilde{\phi},\tilde{U}(k,n)^\perp)\frac{\sigma_d(k-n,n)}{C_{\epsilon,d}} \geqslant \gamma(\tilde{V}(k,N_*)^\perp,\tilde{U}(k,n)^\perp)\frac{\sigma_d(k-n,n)}{C_{\epsilon,d}},
\end{aligned}$$

and by using equations (A.16) and (A.17) one concludes

$$\begin{aligned}
\gamma(\tilde{V}(k,N_*)^\perp,\tilde{U}(k,n)^\perp) &= \gamma(\tilde{U}(k,n),\tilde{V}(k,N_*)) \\
&\geqslant \frac{\gamma(\tilde{U}(k,N_*),\tilde{V}(k,N_*)) - \delta(\tilde{U}(k,N_*),\tilde{U}(k,n))}{1+\delta(\tilde{U}(k,N_*),\tilde{U}(k,n))} \\
&\geqslant \frac{1-\theta_*}{1+\theta_*}\gamma(\tilde{U}(k,N_*),\tilde{V}(k,N_*)) \geqslant (1-\theta_*)^2 C_{\epsilon,d}^{-1}.
\end{aligned}$$

□

Lemma 2.8 (Equivariance of the fast space). *For every $k \in \mathbb{Z}$,*

$$A_k E_k = E_{k+1}.$$

Proof. Let $\tilde{u} \in \tilde{U}(k,n)$ and $\tilde{\phi} \in \tilde{U}(k+1,n+1)^\perp$. Then there exists $u \in U(k-n,n)$ such that $\tilde{u} = A(k-n,n)u$. On the one hand

$$\|\tilde{u}\| \geqslant \sigma_d(k-n,n)\|u\|/C_{\epsilon,d}.$$

On the other hand

$$\begin{aligned}
\langle\tilde{\phi}|A_k\tilde{u}\rangle &= \langle A(k-n,n+1)^*\tilde{\phi}|u\rangle \leqslant \|A(k-n,n+1)^*\tilde{\phi}\|\|u\| \\
&\leqslant C_{\epsilon,d}\sigma_{d+1}(k-n,n+1)\|\tilde{\phi}\|\|u\| \leqslant C_{\epsilon,d}^2\frac{\sigma_{d+1}(k-n,n+1)}{\sigma_d(k-n,n)}\|\tilde{\phi}\|\,\|\tilde{u}\| \\
&\leqslant C_{\epsilon,d}^2\|A_k\|\frac{\sigma_{d+1}(k-n,n-1)\|A_{k-1}\|}{\sigma_d(k-n,n)}\|\tilde{\phi}\|\,\|\tilde{u}\| \\
&\leqslant C_{\epsilon,d}^2\|A_k\|D_{\mathrm{SVG}}e^{-(n-1)\tau}\|\tilde{\phi}\|\,\|\tilde{u}\|.
\end{aligned}$$

We just have proved for every $\tilde{u} \in \tilde{U}(k,n)$,

$$\operatorname{dist}(A_k\tilde{u},\tilde{U}(k+1,n+1) \leqslant C_{\epsilon,d}^2\|A_k\|D_{\mathrm{SVG}}e^{-(n-1)\tau}\|\tilde{u}\|.$$

Let θ_*, N_* as in equation (2). Let $\tilde{u}_* \in \tilde{U}(k, N_*)$ and $w_n := \tilde{u}_* + \tilde{\Theta}(k,n)\tilde{u}_*$. Then there exists $\tilde{u}'_n \in \tilde{U}(k+1, N_*)$ such that

$$w'_n := \tilde{u}'_n + \tilde{\Theta}(k+1, n+1)\tilde{u}'_n \ \text{ satisfies } \ \|A_k w_n - w'_n\| \to 0.$$

Since $w_n \to \tilde{u}_* + \tilde{\Theta}_k(N_*)\tilde{u}_*$, $\tilde{u}'_n \to \tilde{u}'$, $w'_n \to w' = \tilde{u}' + \tilde{\Theta}_{k+1}(N_*)\tilde{u}'$. We have proved $A_k(\tilde{u}_* + \tilde{\Theta}_k(N_*)\tilde{u}_*) = \tilde{u}' + \tilde{\Theta}_{k+1}(N_*)\tilde{u}'$ and the equivariance of the fast space. □

3 Proof of item 1 of theorem 1.2

We present the proof of the bound from below (item 1 of theorem 1.2) of the angle between E_k and F_k uniformly in $k \in \mathbb{Z}$. We use for the first time the property (FI). Although there should exist a direct proof for any dimension d, we reduce our analysis to the case $d = 1$ by introducing the exterior product $\bigwedge^d X$. The cocycle $A(k,n)$ admits a canonical extension to the exterior product that we denote

$$\widehat{A}(k,n) := \bigwedge{}^d A(k,n).$$

The approximate singular value decomposition obtained in theorem 2.2 for the cocycle $A(k,n)$ can be extended to the cocycle $\widehat{A}(x,n)$ by applying theorem A.43 to each $A(k,n)$. We use definition A.39 for the notation $\widehat{U}$ and $\check{V}$, for every subspace U of dimension d and V of codimension d, respectively. We obtain the following theorem.

Theorem 3.1. *Let X be a Banach space, $d \geqslant 1$, $\epsilon > 0$, and $(A_k)_{k\in\mathbb{Z}}$ be a sequence of bounded operators. Let $X = U(k,n) \oplus V(k,n) = \tilde{U}(k,n) \oplus \tilde{V}(k,n)$ be the approximate singular value decomposition given in theorem 2.2 spanned respectively by the bases $(e_1, \ldots, e_d)$, $(\phi_1, \ldots, \phi_d)$, $(\tilde{e}_1, \ldots, \tilde{e}_d)$, $(\tilde{\phi}_1, \ldots, \tilde{\phi}_d)$. Then there exists a constant $\widehat{K}_d$ depending only on the Banach norm and d, such that, for every $k \in \mathbb{Z}$, $n \geqslant 1$, $\widehat{C}_{\epsilon,d} := (1+\epsilon)\widehat{K}_d$,*

1. $\bigwedge^d X = \widehat{U}(k,n) \oplus \check{V}(k,n)$, $\quad \bigwedge^d X = \widehat{\tilde{U}}(k,n) \oplus \check{\tilde{V}}(k,n)$,
2. $\widehat{U}(k,n) = \operatorname{span}(\bigwedge_{i=1}^d e_i(k,n))$, $\quad \check{V}(k,n) = \operatorname{span}(\bigwedge_{i=1}^d \phi_i(k,n))^{\perp\!\!\perp}$,
3. $\widehat{\tilde{U}}(k,n) = \operatorname{span}(\bigwedge_{i=1}^d \tilde{e}_i(k,n))$, $\quad \check{\tilde{V}}(k,n) = \operatorname{span}(\bigwedge_{i=1}^d \tilde{\phi}_i(k,n))^{\perp\!\!\perp}$,
4. $\dim(\widehat{U}(k,n)) = \dim(\widehat{\tilde{U}}(k,n)) = 1$,
5. $\widehat{A}(k,n)\widehat{U}(k,n) = \widehat{\tilde{U}}(k+n,n)$, $\quad \widehat{A}(k,n)\check{V}(k,n) \subset \check{\tilde{V}}(k+n,n)$,

6. $\widehat{C}_{\epsilon,d}^{-1} \prod_{i=1}^d \sigma_i(k,n) \leqslant \|\widehat{A}(k,n)|\widehat{U}(k,n)\| \leqslant \widehat{C}_{\epsilon,d} \prod_{i=1}^d \sigma_i(k,n)$,

7. $\widehat{C}_{\epsilon,d}^{-1} \prod_{i=1}^d \sigma_i(k,n) \leqslant \|\widehat{A}(k,n)^*|\check{\tilde{V}}(k+n,n)^\perp\| \leqslant \widehat{C}_{\epsilon,d} \prod_{i=1}^d \sigma_i(k,n)$,

8. $\|\widehat{A}(k,n)|\check{V}(k,n)\| \leqslant \widehat{C}_{\epsilon,d}\, \sigma_1(k,n) \cdots \sigma_{d-1}(k,n)\sigma_{d+1}(k,n)$,

9. $\gamma(\widehat{U}(k,n), \check{V}(k,n)) \geqslant \widehat{C}_{\epsilon,d}^{-1}, \quad \gamma(\check{V}(k,n), \widehat{U}(k,n)) \geqslant \widehat{C}_{\epsilon,d}^{-1}$.

This theorem is a direct consequence of theorem A.43. We now recall some notations introduced in item 3 and 4 of lemma 2.4. We consider E_k and F_k as graphs over a fixed splitting $X = \tilde{U}(k,N_*) \oplus \tilde{V}(k,N_*)$ and $X = U(k,N_*) \oplus V(k,N_*)$ respectively.

Notations 3.2. Let $\theta_* \in (0,1)$ and N_* satisfy equation (2). Then

- $E_k = \mathrm{Graph}(\tilde{\Theta}_k(N_*))$ for some $\tilde{\Theta}_k(N_*) : \tilde{U}(k,N_*) \to \tilde{V}(k,N_*)$,
- $F_k = \mathrm{Graph}(\Theta_k^\perp(N_*))^{\perp\!\!\perp}$ for some $\Theta_k^\perp(N_*) : V(k,N_*)^\perp \to U(k,N_*)^\perp$,
- $\widehat{E}_k = \mathrm{span}\big(\bigwedge_{i=1}^d (\mathrm{Id} \oplus \tilde{\Theta}_k(N_*))\tilde{e}_i(k,N_*)\big)$,
- $\check{F}_k := \mathrm{span}\big(\bigwedge_{i=1}^d (\mathrm{Id} \oplus \Theta_k^\perp(N_*)\phi_i(k,N_*)\big)^{\perp\!\!\perp}$,
- $\check{F}_k = \mathrm{Graph}(\widehat{\Theta}_k(N_*))$ for some $\widehat{\Theta}_k(N_*) : \check{V}(k,N_*) \to \widehat{U}(k,N_*)$,
- $\|\tilde{\Theta}_k(N_*)\| \leqslant \theta_*, \quad \|\Theta_k^\perp(N_*)\| \leqslant \theta_*, \quad \|\widehat{\Theta}_k(N_*)\| \leqslant C_{\epsilon,d}^{2d} K_d \theta_* (1+\theta_*)^{d-1}$, (using lemma A.42 for some constant $K_d = \bar{\Delta}_d(X)^d$ given by (A.3)).

The strategy of the proof is based on two steps. In the first step we show that, for some N_* large enough,

$$\forall k \in \mathbb{Z}, \quad \gamma(\widehat{A}(k-N_*,N_*)\widehat{U}(k-N_*,N_*), \check{F}_k) \geqslant c(N_*),$$

with a constant that depends on N_* (and goes to zero as $N_* \to +\infty$). This estimate may be considered as a bootstrap argument; this is the only place where property (FI) is used.

In the second part, we analyze the special backward cocycle associated to the sequence of operators $(\widehat{A}(k-nN_*,N_*))_{n=1}^{+\infty}$. We improve the previous estimate and show that actually

$$\forall n \geqslant 1,\ \forall k \in \mathbb{Z}, \quad \gamma(\widehat{A}(k-nN_*,nN_*)\widehat{U}(k-nN_*,nN_*), \check{F}_k) \geqslant \text{constant}.$$

The proof is complicated by the fact that we are in a Banach space and look for an explicit lower bound. The proof is also new in the finite dimensional setting. We conclude the proof by observing

$$\widehat{A}(k-nN_*, nN_*)\widehat{U}(k-nN_*, nN_*) = \widehat{\widehat{U}}(k, nN_*) \to \widehat{E}_k.$$

We obtain a uniform bound from below of $\gamma(\widehat{E}_k, \widehat{F}_k)$ and therefore a uniform bound from below of $\gamma(E_k, F_k)$ by using lemma A.40.

We show in the following lemma that the smallest expansion of $\widehat{A}(k,n)$ on $\widehat{U}(k,m)$ is bounded from below by $\prod_{i=1}^d \sigma_i(k,n)$ uniformly in m,n large enough,

$$\forall k \in \mathbb{Z},\ \forall m,n \geqslant N_*, \quad \|\widehat{A}(k,n)|\widehat{U}(k,m)\| \geqslant \text{constant}\Big[\prod_{i=1}^d \sigma_i(k,n)\Big]. \tag{4}$$

We now choose N_* satisfying a more restrictive condition than the one in (2).

Lemma 3.3. *Let $\theta_* \in (0,1)$ and N_* satisfy*

$$D_{SVG}\, e^{-N_*\tau} \leqslant \theta_*(1-\theta_*)^7 \frac{1-e^{-\tau}}{C_{\epsilon,d}^5}. \tag{5}$$

Then for every $n,m \geqslant N_*$ and $k \in \mathbb{Z}$,

$$\forall u \in \hat{U}(k,m), \quad \|\hat{A}(k,n)u\| \geqslant C_{\epsilon,d}^{-4d} K_d^{-1}(1-\theta_*)^d \Big(\prod_{i=1}^d \sigma_i(k,n)\Big)\|u\|,$$

where $K_d := \bar{\Delta}_d(X)^{3d}$.

Proof. Part 1. We prove in both cases, $n \geqslant m$ and $m \geqslant n$, that there exists an operator $\Theta^\perp : V(k,m)^\perp \to U(k,m)^\perp$ such that $V(k,n)^\perp = \mathrm{Graph}(\Theta^\perp)$ and $\|\Theta^\perp\| \leqslant \theta_*$.

For $n \geqslant m$ the existence of $\Theta^\perp$ is a consequence of item 4 of lemma 2.4 taking $N_* = m$.

For $m \geqslant n$, let $\theta' := \theta_*(1-\theta_*)/C_{\epsilon,d}$, then

$$D_{\text{SVG}}\, e^{-n\tau} \leqslant D_{\text{SVG}}\, e^{-N_*\tau} \leqslant \theta'(1-\theta')^6 \frac{1-e^{-\tau}}{C_{\epsilon,d}^4},$$
$$\delta(V(k,m), V(k,n)) \leqslant \theta' \leqslant \theta_*(1-\theta_*)\gamma(V(k,m), U(k,m)).$$

In particular, from item 1 of lemma A.25,

$$\delta(V(k,m), V(k,n)) < \gamma(V(k,m), U(k,m)),$$
$$\delta(V(k,n)^\perp, V(k,m)^\perp) < \gamma(U(k,m)^\perp, V(k,m)^\perp),$$
$$V(k,n)^\perp = \mathrm{Graph}(\Theta^\perp), \quad \text{for some} \quad \Theta^\perp : V(k,m)^\perp \to U(k,m)^\perp,$$
$$\|\Theta^\perp\| \leqslant \frac{\delta(V(k,n)^\perp, V(k,m)^\perp)}{\gamma(U(k,m)^\perp, V(k,m)^\perp) - \delta(V(k,n)^\perp, V(k,m)^\perp)} \leqslant \theta_*.$$

Part 2. We now prove the relative rate of expansion of $\widehat{A}(k,n)$. From lemma A.26, one obtains with $K'_d = \bar{\Delta}_d(X)^{2d}$,

$$\det\left(\left[\langle \phi_i(k,n) | e_j(k,m) \rangle\right]_{ij}\right) \geqslant (K'_d)^{-1} C_{\epsilon,d}^{-2d} (1-\theta_*)^d.$$

As $A^*(k,n)\tilde{\phi}_i(k+n,n) = \sigma_i(k,n)\phi_i(k,n)$, using equations (A.21) and (A.22), one obtains

$$\begin{aligned} \det\left(\left[\langle \phi_i(k,n) | e_j(k,m) \rangle\right]_{ij}\right) &= \frac{\det\left(\left[\langle \tilde{\phi}_i(k+n,n) | A(k,n) e_j(k,m) \rangle\right]_{ij}\right)}{\prod_{i=1}^d \sigma_i(k,n)}, \\ &\leqslant \Sigma_d(X) \frac{\| \bigwedge_{i=1}^d \tilde{\phi}_i(k+n,n) \| \, \| \hat{A}(k,n) \bigwedge_{i=1}^d e_i(k,m) \|}{\prod_{i=1}^d \sigma_i(k,n)}. \end{aligned}$$

From proposition A.34, we have $\Sigma_d(X) \leqslant \bar{\Delta}_d(X)^d$. From the definition of the projective norm (A.20), we have

$$\| \textstyle\bigwedge_{i=1}^d \tilde{\phi}(k+n,n) \| \leqslant C_{\epsilon,d}^d \ \text{ and } \ \| \textstyle\bigwedge_{i=1}^d e_j(k,m) \| \leqslant C_{\epsilon,d}^d. \qquad \square$$

The next lemma gives a lower bound of the angle between the approximate fast space $\widehat{W}_k := \widehat{A}(k-N_*, N_*)\widehat{U}(k-N_*, m)$ and the slow space $\check{F}_k$ for $m \geqslant N_*$. This estimate is non trivial as $\widehat{W}_k$ is defined using the operators $(A_{k-n})_{n\geqslant 1}$ and $\check{F}_k$ is defined using the operators $(A_{k+n})_{n\geqslant 0}$. Property (FI) forces the two spaces to be complementary. It is the only place where (FI) is used.

Lemma 3.4 (First crucial step). *Let $\theta_* \in (0,1)$, N_* satisfy equation* (5)*, $k \in \mathbb{Z}$, and $m \geqslant N_*$. Denote $\widehat{W}_k := \widehat{A}(k-N_*, N_*)\widehat{U}(k-N_*, m)$. Then*

$$\gamma(\widehat{W}_k, \check{F}_k) \geqslant \widehat{C}_{\epsilon,d}^{-3} C_{\epsilon,d}^{-4d} K_d^{-1} (1-\theta_*)^d D_{FI}^{-1} e^{-N_*\mu},$$

where $K_d := \bar{\Delta}_d(X)^{3d}$.

Proof. As $\check{V}(k,n) \to \check{F}_k$ in the co-Grassmannian topology, it is enough to bound from below $\gamma(\widehat{W}_k, \check{V}(k,n))$ for large $n \geqslant m$. We first show that $\widehat{W}_k$ is the graph of some operator $\widehat{\Gamma}(k,n) : \widehat{U}(k,n) \to \check{V}(k,n)$. We then give an upper bound for $\|\mathrm{Id} \oplus \widehat{\Gamma}(k,n)\|$; or equivalently a lower bound for the angle $\gamma(\widehat{W}_k, \check{V}(k,n))$. Let

$$w \in \widehat{W}_k, \ \ w = w' + w'', \ \ w' \in \widehat{U}(k,n) \ \text{ and } \ w'' \in \check{V}(k,n).$$

On the one hand $w = \widehat{A}(k-N_*, N_*)u$ for some $u \in \widehat{U}(k-N_*, m)$. Then using lemma 3.3 with $K_d = \bar{\Delta}_d(X)^{3d}$ and item 6 of theorem 3.1, one gets

$$\begin{aligned} \|\widehat{A}(k,n)w\| &= \|\widehat{A}(k-N_*, N_*+n)u\| \\ &\geqslant C_{\epsilon,d}^{-4d} K_d^{-1} (1-\theta_*)^d \textstyle\prod_{i=1}^d \sigma_i(k-N_*, N_*+n) \, \|u\|, \\ \|w\| &\leqslant \widehat{C}_{\epsilon,d} \textstyle\prod_{i=1}^d \sigma_i(k-N_*, N_*) \|u\|. \end{aligned}$$

Thus

$$\|\widehat{A}(k,n)w\| \geqslant \widehat{C}_{\epsilon,d}^{-1} C_{\epsilon,d}^{-4d} K_d^{-1} (1-\theta_*)^d \frac{\prod_{i=1}^d \sigma_i(k-N_*, N_*+n)}{\prod_{i=1}^d \sigma_i(k-N_*, N_*)} \|w\|.$$

On the other hand using items 6 and 8 of theorem 3.1,

$$\begin{aligned} \|\widehat{A}(k,n)w'\| &\leqslant \widehat{C}_{\epsilon,d} \big[\textstyle\prod_{i=1}^d \sigma_i(k,n)\big] \|w'\|, \\ \|\widehat{A}(k,n)w''\| &\leqslant \widehat{C}_{\epsilon,d} \big[\textstyle\prod_{i=1}^{d-1} \sigma_i(k,n)\big] \sigma_{d+1}(k,n) \|w''\|, \\ \|\widehat{A}(k,n)w\| &\leqslant \widehat{C}_{\epsilon,d} \big[\textstyle\prod_{i=1}^d \sigma_i(k,n)\big] \Big[\|w'\| + \tfrac{\sigma_{d+1}(k,n)}{\sigma_d(k,n)} \|w''\|\Big]. \end{aligned}$$

Property (FI) implies

$$\frac{\prod_{i=1}^d \sigma_i(k-N_*, N_*+n)}{\prod_{i=1}^d \sigma_i(k-N_*, N_*) \prod_{i=1}^d \sigma_i(k,n)} \geqslant D_{\text{FI}}^{-1} e^{-N_*\mu}.$$

Combining the two estimates of $\|\widehat{A}(k,n)w\|$ and using property (SVG), one obtains,

$$\begin{aligned} &\|(\mathrm{Id} \oplus \widehat{\Gamma}(k,n))w'\| = \|w\| \leqslant \\ &\qquad \widehat{C}_{\epsilon,d}^2 C_{\epsilon,d}^{4d} K_d (1-\theta_*)^{-d} D_{\text{FI}} e^{N_*\mu} \big[1 + D_{\text{SVG}} e^{-n\tau} \|\widehat{\Gamma}(k,n)\|\big] \|w'\|. \end{aligned}$$

In particular $\|\widehat{\Gamma}(k,n)\|$ is uniformly bounded from above. Using lemma A.24 and item 9 of theorem 3.1

$$\begin{aligned} \gamma(\widehat{W}, \check{V}(k,n)) &\geqslant \frac{\gamma(\widehat{U}(k,n), \check{V}(k,n))}{\|\mathrm{Id} \oplus \widehat{\Gamma}(k,n)\|} \geqslant \frac{\widehat{C}_{\epsilon,d}^{-1}}{\|\mathrm{Id} \oplus \widehat{\Gamma}(k,n)\|} \\ &\geqslant \widehat{C}_{\epsilon,d}^{-3} C_{\epsilon,d}^{-4d} K_d^{-1} (1-\theta_*)^d D_{\text{FI}}^{-1} e^{-N_*\mu} \big[1 + D_{\text{SVG}} e^{-n\tau} \|\widehat{\Gamma}(k,n)\|\big]^{-1}. \end{aligned}$$

We conclude by letting $n \to +\infty$. □

Similarly to lemma 3.3, we show that the largest expansion of $\widehat{A}(k,n)$ restricted to $\check{F}_k$ is bounded from above by $[\prod_{i=1}^d \sigma_i(k,n)] e^{-n\tau}$ uniformly for n large enough,

$$\forall k \in \mathbb{Z},\ \forall n \geqslant N_*, \quad \|\widehat{A}(k,n) | \check{F}_k\| \leqslant \text{constant} \Big(\prod_{i=1}^d \sigma_i(k,n)\Big) e^{-n\tau}. \tag{6}$$

Equation (6) together with equation (4) show that the cocycle $\widehat{A}(k,n)$ satisfies property (SVG) at index 1. Estimate (6) is the main reason to introduce the exterior product. The simplest proof based on the original cocycle seems to require a comparison between the two ratios $\sigma_d(k,n)/\sigma_1(k,n)$ and $\sigma_{d+1}(k,n)/\sigma_d(k,n)$.

Lemma 3.5. *Let $\theta_* \in (0,1)$ and N_* satisfy equation* (5)*. Then for every $n \geqslant N_*$ and $k \in \mathbb{Z}$,*

$$\|\widehat{A}(k,n)|\check{F}_k\| \leqslant 2\widehat{C}^2_{\epsilon,d} C^{2d}_{\epsilon,d} K_d \theta_* (1+\theta_*)^{d-1} \Big(\prod_{i=1}^d \sigma_i(k,n)\Big) e^{-(n-N_*)\tau},$$

where $K_d = \bar{\Delta}_d(X)^d$.

Proof. Let $F_k = \mathrm{Graph}(\Theta_k^\perp(n))^{\perp\!\!\!\perp}$ and $\check{F}_k = \mathrm{Graph}(\widehat{\Theta}_k(n))$ as in notations 3.2. We first notice

$$D_{\text{SVG}} e^{-n\tau} \leqslant e^{-(n-N_*)\tau} D_{\text{SVG}} e^{-N_*\tau} \leqslant \theta'(1-\theta')^6 \frac{1-e^{-\tau}}{C^4_{\epsilon,d}}$$

with $\theta' := \theta_* e^{-(n-N_*)\tau}$. Substituting θ' for θ_* and n for N_* in item 4 of lemma 2.4, one obtains $\|\Theta_k^\perp(n)\| \leqslant \theta'$. Then lemma A.42 and proposition A.34 imply

$$\|\widehat{\Theta}_k(n)\| \leqslant C^{2d}_{\epsilon,d} K_d \theta' (1+\theta_*)^{d-1}.$$

Let $w \in \check{F}_k$, $w = w' + w''$, $w'' \in \check{V}(k,n)$ and $w' = \widehat{\Theta}_k(n) w'' \in \widehat{U}(k,n)$. Then

$$\begin{aligned}
\|w''\| &\leqslant \|\pi_{\check{V}(k,n)|\widehat{U}(k,n)}\| \|w\| \leqslant \widehat{C}_{\epsilon,d}\|w\|, \\
\|\widehat{A}(k,n)w'\| &\leqslant \widehat{C}_{\epsilon,d}\big[\textstyle\prod_{i=1}^d \sigma_i(k,n)\big] \|\widehat{\Theta}_k(n)\| \|w''\|, \\
\|\widehat{A}(k,n)w''\| &\leqslant \widehat{C}_{\epsilon,d}\, \sigma_1(k,n) \ \cdots\ \sigma_{d-1}(k,n)\sigma_{d+1}(k,n)\|w''\|, \\
\|\widehat{A}(k,n)w\| &\leqslant \widehat{C}^2_{\epsilon,d}\big[\textstyle\prod_{i=1}^d \sigma_i(k,n)\big]\Big[\|\widehat{\Theta}_k(n)\| + \tfrac{\sigma_{d+1}(k,n)}{\sigma_d(k,n)}\Big]\|w\|.
\end{aligned}$$

We conclude using property (SVG),

$$\frac{\sigma_{d+1}(k,n)}{\sigma_d(k,n)} \leqslant D_{\text{SVG}} e^{-n\tau} \leqslant \theta' \leqslant C^{2d}_{\epsilon,d} K_d \theta' (1+\theta_*)^{d-1}. \qquad \square$$

We now change notation and rewrite the cocycle $(\widehat{A}(k-nN_*, N_*))_{n=1}^{+\infty}$ as block matrices along the following splitting. Notice the small circumflex for the new notation. Define

- $\hat{A}_{-n} := \widehat{A}(k-nN_*, N_*),\ \forall n \geqslant 1,$
- $\hat{U}_{-n} := \widehat{U}(k-nN_*, nN_*),\ \hat{V}_{-n} := \check{V}(k-nN_*, nN_*),\ \forall n \geqslant 1,$
- $\hat{U}_0 := \widehat{\widehat{U}}(k, N_*),\ \hat{V}_0 := \check{\check{V}}(k, N_*),$
- $\hat{E}_{-n} := \widehat{E}_{k-nN_*},\ \hat{F}_{-n} := \widehat{F}_{k-nN_*},\ \forall n \geqslant 0,$

- $\bigwedge^d X = \hat U_{-n} \oplus \hat F_{-n}, \forall n \geqslant 0$.

Notice that the first crucial step, lemma 3.4, implies that $\hat U_0 = \hat A_{-1} \hat U_{-1}$ and $\hat F_0$ are indeed two complementary spaces. We consider the following block splitting

- $\hat p_{-n}$ the projector onto $\hat U_{-n}$ parallel to $\hat F_{-n},\ \forall n \geqslant 0$,
- $\hat q_{-n}$ the projector onto $\hat F_{-n}$ parallel to $\hat U_{-n},\ \forall n \geqslant 0$,
- $\hat A_{-n} := \begin{bmatrix} \hat a_{-n} & 0 \\ \hat c_{-n} & \hat d_{-n} \end{bmatrix},\ \forall n \geqslant 1$
- $\hat a_{-n} = p_{-(n-1)} \circ (\hat A_{-n} | \hat U_{-n}) : \hat U_{-n} \to \hat U_{-(n-1)}$,
- $\hat c_{-n} = q_{-(n-1)} \circ (\hat A_{-n} | \hat U_{-n}) : \hat U_{-n} \to \hat F_{-(n-1)}$,
- $\hat d_{-n} = (\hat A_{-n} | \hat F_{-n}) : \hat F_{-n} \to \hat F_{-(n-1)}$.

By the equivariance of the slow space $\hat A_{-n} \hat F_{-n} \subset \hat F_{-(n-1)}$, we obtain

- $\hat A^n_{-n} := \hat A_{-1} \hat A_{-2} \ \cdots \ \hat A_{-n} = \widehat A(k - nN_*, nN_*)$,
- $\hat a^n_{-n} := \hat a_{-1} \hat a_{-2} \cdots \ \hat a_{-n} = \hat p_0 \circ (\widehat A(k-nN_*, nN_*) | \widehat U(k - nN_*, nN_*))$,
- $\hat d^n_{-n} := \hat d_{-1} \hat d_{-2} \cdots \ \hat d_{-n} = (\widehat A(k-nN_*, nN_*) | \widehat F_{k-nN_*})$.

Lemma 3.4 implies that $\hat A_{-n} \hat U_{-n}$ and $\hat F_{-(n-1)}$ are complementary. In particular $\hat a_{-n} : \hat U_{-n} \to \hat U_{-(n-1)}$ is bijective. Define for $n \geqslant 1$,

- $\hat A_{-n-1} \hat U_{-n-1} = \mathrm{Graph}(\hat\Gamma_{-n})$ for some operator $\hat \Gamma_{-n} : \hat U_{-n} \to \hat F_{-n}$, by convention, $\hat\Gamma_0 := 0$,
- $\hat A^n_{-n} \hat U_{-n} = \mathrm{Graph}(\hat{\bar\Xi}^n_0)$ for some operator $\hat{\bar\Xi}^n_0 : \hat U_0 \to \hat F_0$. Notice that the choice of $\hat U_0$ implies $\hat{\bar\Xi}^1_0 = 0$.

Lemma 3.6. *Let $\theta_* \in (0,1)$ and N_* satisfy equation* (5). *Then*

$$\forall n \geqslant 1, \quad \|\hat q_{-n}\| \leqslant \widehat C_{\epsilon,d} C^{2d}_{\epsilon,d} K_d (1+\theta_*)^d,$$

where $K_d = \bar\Delta_d(X)^d$.

Proof. From notations 3.2 one obtains $\hat F_{-n} = \mathrm{Graph}(\hat\Theta_{-n})$ for some operator $\hat\Theta_{-n} := \widehat\Theta_{k-nN_*}(nN_*) : \hat V_{-n} \to \hat U_{-n}$. Moreover

$$\begin{aligned}
\hat q_{-n} &= (\mathrm{Id} \oplus \hat\Theta_{-n}) \circ \pi_{\hat V_{-n} | \hat U_{-n}}, \\
\|\hat\Theta_{-n}\| &\leqslant C^{2d}_{\epsilon,d} K_d \theta_* (1+\theta_*)^{d-1}, \\
\|\hat q_{-n}\| &\leqslant \widehat C_{\epsilon,d} (1 + \|\hat\Theta_{-n}\|) \leqslant \widehat C_{\epsilon,d} C^{2d}_{\epsilon,d} K_d (1+\theta_*)^d.
\end{aligned}$$

□

Lemma 3.7. *Let $\theta_* \in (0,1)$ and N_* satisfy equation* (5)*. Then*

$$\forall n \geqslant 1, \quad \|\hat{\Gamma}_{-n}\| \leqslant \widehat{C}_{\epsilon,d}^4 C_{\epsilon,d}^{6d} K_d (1-\theta_*)^{-2d} D_{FI} e^{N_*\mu},$$

where $K_d := \bar{\Delta}_d(X)^{4d}$.

Proof. Since $\hat{\Gamma}_{-n} = \hat{q}_{-n}(\mathrm{Id} \oplus \hat{\Gamma}_{-n})$, we obtain using lemmas A.24, 3.6 and 3.4

$$\|\hat{\Gamma}_{-n}\| \leqslant \frac{\|\hat{q}_{-n}\|}{\gamma(\hat{A}_{-n-1}\hat{U}_{-n-1}, \hat{F}_{-n})} \leqslant \|\hat{q}_{-n}\| \widehat{C}_{\epsilon,d}^3 C_{\epsilon,d}^{4d} K_d' (1-\theta_*)^{-d} D_{\mathrm{FI}} e^{N_*\mu},$$

with $K_d' = \bar{\Delta}_d(X)^{3d}$. □

We now show that the minimal gap between $\hat{A}_{-n}^n \hat{U}_{-n}$ and $\hat{F}_0$ is bounded from below uniformly in n. Since $\hat{A}_{-n}^n \hat{U}_{-n} = \mathrm{Graph}(\hat{\Xi}_0^n)$ for some $\hat{\Xi}_0^n : \hat{U}_0 \to \hat{F}_0$, it is enough to bound from above $\|\mathrm{Id} \oplus \hat{\Xi}_0^n\|$. We show how to estimate $\|\mathrm{Id} \oplus \hat{\Xi}_0^{n+1}\|$ in terms of $\|\mathrm{Id} \oplus \hat{\Xi}_0^n\|$. Since $\hat{A}_{-n}^n \hat{U}_{-n} = \widehat{U}(k, nN^*) \to \hat{E}_0$, we obtain a bound from below of $\gamma(\hat{E}_0, \hat{F}_0)$.

Lemma 3.8 (Second crucial step). *Let $\theta_* \in (0,1)$ and N_* satisfy*

$$D_{SVG} e^{-N_*\tau} \leqslant \frac{\theta_*(1-\theta_*)^{3d-1}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d} K_d D_{FI}} (1-\theta_*)^7 \frac{1-e^{-\tau}}{C_{\epsilon,d}^5}, \tag{7}$$

with $K_d := \bar{\Delta}_d(X)^{5d}$. Then for every $n \geqslant 1$,

$$\gamma(\hat{A}_{-n}^n \hat{U}_{-n}, \hat{F}_0) \geqslant \frac{(1-\theta_*)^d D_{FI}^{-1}}{\widehat{C}_{\epsilon,d}^3 C_{\epsilon,d}^{4d} K_d} e^{-N_*\mu} \prod_{k=0}^{n-2} \Big[1 + e^{N_*\mu} e^{-kN_*\tau}\Big]^{-1}.$$

Proof. Define

$$\theta' := \frac{\theta_*(1-\theta_*)^{3d-1}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d} K_d D_{\mathrm{FI}}}.$$

Notice that N_* satisfies equation (5) with θ' instead of θ_*

$$D_{\mathrm{SVG}} e^{-N_*\tau} \leqslant \theta'(1-\theta')^7 \frac{1-e^{-\tau}}{C_{\epsilon,d}^5},$$

Part 1. We estimate the norms $\|(\hat{a}_{-n}^n)^{-1}\|$ and $\|\hat{d}_{-n}^n\|$. On the one hand, using item 6 of theorem 3.1, one gets

$$(\hat{a}_{-n}^n)^{-1} = (\hat{A}_{-n}^n | \hat{U}_{-n})^{-1} \circ (\mathrm{Id} \oplus \hat{\Xi}_0^n),$$

$$\|(\hat{a}_{-n}^n)^{-1}\| \leqslant \widehat{C}_{\epsilon,d} \Big[\prod_{i=1}^d \sigma_i(k - nN_*, nN_*)\Big]^{-1} \|\mathrm{Id} \oplus \hat{\Xi}_0^n\|.$$

On the other hand, using lemma 3.5, one gets

$$\|\hat{d}^n_{-n}\| \leqslant 2\widehat{C}^2_{\epsilon,d} C^{2d}_{\epsilon,d} K'_d \theta'(1+\theta')^{d-1} \Big[\prod_{i=1}^{d} \sigma_i(k - nN_*, nN_*)\Big] e^{-(n-1)N_*\tau},$$

with $K'_d = \bar{\Delta}_d(X)^d$.

Part 2. We bound from above $\|\mathrm{Id} \oplus \hat{\Xi}^{n+1}_0\|$ in terms of $\|\mathrm{Id} \oplus \hat{\Xi}^n_0\|$. Notice first that $\hat{\Gamma}_{-n} = \hat{c}_{-n-1}(\hat{a}_{-n-1})^{-1}$. Moreover

$$\hat{A}^{n+1}_{-n-1} = \begin{bmatrix} \hat{a}^{n+1}_{-n-1} & 0 \\ \hat{c}^{n+1}_{-n-1} & \hat{d}^{n+1}_{-n-1} \end{bmatrix} = \begin{bmatrix} \hat{a}^n_{-n} & 0 \\ \hat{c}^n_{-n} & \hat{d}^n_{-n} \end{bmatrix} \begin{bmatrix} \hat{a}_{-n-1} & 0 \\ \hat{c}_{-n-1} & \hat{d}_{-n-1} \end{bmatrix},$$

$$\hat{c}^{n+1}_{-n-1} = \hat{c}^n_{-n}\hat{a}_{-n-1} + \hat{d}^n_{-n}\hat{c}_{-n-1},$$

$$\hat{c}^{n+1}_{-n-1}(\hat{a}^{n+1}_{-n-1})^{-1} = \hat{c}^n_{-n}(\hat{a}^n_{-n})^{-1} + \hat{d}^n_{-n}\hat{c}_{-n-1}(\hat{a}_{-n-1})^{-1}(\hat{a}^n_{-n})^{-1}.$$

Since $\hat{\Xi}^n_0 = \hat{c}^n_{-n}(\hat{a}^n_{-n})^{-1}$, we obtain $(\mathrm{Id} \oplus \hat{\Xi}^{n+1}_0) = (\mathrm{Id} \oplus \hat{\Xi}^n_0) + \hat{d}^n_{-n}\hat{\Gamma}_{-n}(\hat{a}^n_{-n})^{-1}$,

$$\|\mathrm{Id} \oplus \hat{\Xi}^{n+1}_0\| \leqslant \|\mathrm{Id} \oplus \hat{\Xi}^n_0\| \Big(1 + \frac{\|\hat{d}^n_{-n}\| \|\hat{\Gamma}_{-n}\| \|(\hat{a}^n_{-n})^{-1}\|}{\|\mathrm{Id} \oplus \hat{\Xi}^n_0\|}\Big).$$

Using the estimates of part 1 and θ' instead of θ_* in lemma 3.7, we obtain

$$\begin{aligned} \frac{\|\hat{d}^n_{-n}\| \|\hat{\Gamma}_{-n}\| \|(\hat{a}^n_{-n})^{-1}\|}{\|\mathrm{Id} \oplus \hat{\Xi}^n_0\|} &\leqslant 2\widehat{C}^7_{\epsilon,d} C^{8d}_{\epsilon,d} K_d \theta'(1-\theta')^{-3d+1} D_{\mathrm{FI}}\, e^{N_*\mu} e^{-(n-1)N_*\tau} \\ &\leqslant e^{N_*\mu} e^{-(n-1)N_*\tau}. \end{aligned}$$

Using $\|\mathrm{Id} \oplus \hat{\Xi}^1_0\| = 1$, one obtains

$$\|\mathrm{Id} \oplus \hat{\Xi}^n_0\| \leqslant \prod_{k=0}^{n-2} \Big[1 + e^{N_*\mu} e^{-kN_*\tau}\Big]^{-1}.$$

Using the bound from below in lemma 3.4 for $\gamma(\hat{U}_0, \hat{F}_0)$ and the comparison estimate in lemma A.24, one gets

$$\gamma(\hat{A}^n_{-n}\hat{U}_{-n}, \hat{F}_0) \geqslant \frac{\gamma(\hat{U}_0, \hat{F}_0)}{\|\mathrm{Id} \oplus \hat{\Xi}^n_0\|} \geqslant \frac{(1-\theta_*)^d D^{-1}_{\mathrm{FI}}}{\widehat{C}^3_{\epsilon,d} C^{4d}_{\epsilon,d} K_d} e^{-N_*\mu} \prod_{k=0}^{n-2} \Big[1 + e^{N_*\mu} e^{-kN_*\tau}\Big]^{-1}. \ \square$$

We now explain how to choose θ_* so that N_* is the smallest possible. We use the following lemma whose proof is left to the reader. We will choose later $\alpha = 3d + 6$.

Lemma 3.9. *Let* $\alpha > 1$. *Then*

- $\theta_* := \frac{1}{1+\alpha} = \arg\max\{\theta(1-\theta)^\alpha : 1 < \theta < 1\}$,
- $\theta_*(1-\theta_*)^\alpha \geqslant \theta_*(1-\alpha\theta_*) = \frac{1}{(\alpha+1)^2}$.

We estimate the infinite product in lemma 3.8 using the following lemma. We will choose later $\rho = \mu/\tau$ and $a = e^{-N_*\tau}$.

Lemma 3.10. *Let* $a \in (0,1)$ *and* $\rho > 0$. *Then*

$$\prod_{n=0}^{+\infty} \big[1 + a^{n-\rho}\big] \leqslant \exp\Big(\frac{1+a}{1-a}\Big)\Big(\frac{1}{a}\Big)^{\rho(\rho+2)/2}.$$

Proof. We choose n_* such that $n_* \leqslant \rho < n_* + 1$. We split the infinite product in two parts. On the one hand

$$\begin{aligned}\prod_{n=0}^{n_*} \big[1 + a^{n-\rho}\big] &= \prod_{n=0}^{n_*} \big[a^{\rho-n} + 1\big]\Big(\frac{1}{a}\Big)^{\sum_{n=0}^{n_*}\rho-n}, \\ &\leqslant \exp\Big(\sum_{n=0}^{n_*} a^{\rho-n}\Big)\Big(\frac{1}{a}\Big)^{(n_*+1)\rho-n_*(n_*+1)/2} \leqslant \exp\Big(\frac{a^{\rho-n_*}}{1-a}\Big)\Big(\frac{1}{a}\Big)^{\rho(\rho+2)/2}.\end{aligned}$$

On the other hand

$$\prod_{n\geqslant n_*+1} \big[1 + a^{n-\rho}\big] \leqslant \exp\Big(\sum_{n\geqslant n_*+1} a^{n-\rho}\Big) \leqslant \exp\Big(\frac{a^{n_*+1-\rho}}{1-a}\Big).$$

Using the convexity of the function $\rho \in [n_*, n_*+1] \mapsto a^{n_*+1-\rho} + a^{\rho-n_*}$, we obtain $a^{n_*+1-\rho} + a^{\rho-n_*} \leqslant 1 + a$ and conclude the proof. ◻

Assumption 3.11. *Let* $\theta_* = \frac{1}{3d+7}$ *and* N_* *satisfy*

$$D_{SVG}\, e^{-N_*\tau} \leqslant \theta_*(1-\theta_*)^{3d+6} \frac{1-e^{-\tau}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d+5} K_d D_{FI}} < D_{SVG}\, e^{-N_*\tau} e^{\tau}, \qquad (8)$$

with $K_d := \bar{\Delta}_d(X)^{5d}$.

Proof of theorem 1.2, item 1. Using the estimate

$$(1-\theta_*)^d \geqslant 1 - \frac{d}{3d+7} = \frac{2d+7}{3d+7} \geqslant \frac{2}{3},$$

the second crucial step 3.8, lemma 3.10 with

$$a := e^{-N_* \tau}, \ \ \rho := \frac{\mu}{\tau}, \ \ e^{-N_* \mu} = a^\rho,$$

we obtain for every $n \geqslant 0$,

$$\gamma(\hat{A}^n_{-n}\hat{U}_{-n}, \hat{F}_0) \geqslant \tfrac{2}{3}\widehat{C}_{\epsilon,d}^{-3} C_{\epsilon,d}^{-4d} K_d^{-1} D_{\text{FI}}^{-1} \exp\Big(-\frac{1+a}{1-a}\Big) a^{\rho(\rho+4)/2}$$

with $K_d = \bar{\Delta}_d(X)^{5d}$. Using $a \leqslant \frac{1}{2}\theta_*$, we obtain

$$\frac{1+a}{1-a} \leqslant \frac{6d+15}{6d+13} \leqslant \frac{15}{13}, \quad \frac{2}{3}\exp\Big(-\frac{1+a}{1-a}\Big) \geqslant \frac{1}{5},$$

$$\gamma(\hat{A}^n_{-n}\hat{U}_{-n}, \hat{F}_0) \geqslant \frac{a^{\rho(\rho+4)/2}}{5\widehat{C}_{\epsilon,d}^{3} C_{\epsilon,d}^{4d} K_d D_{\text{FI}}}. \tag{9}$$

Using $a \geqslant \dfrac{(3d+7)^{-2}}{2\widehat{C}_{\epsilon,d}^{7} C_{\epsilon,d}^{8d+5} K_d D_{\text{FI}}} \dfrac{1-e^{-\tau}}{D_{\text{SVG}}\, e^{\tau}}$, we obtain

$$\gamma(\hat{A}^n_{-n}\hat{U}_{-n}, \hat{F}_0) \geqslant \frac{1}{5\widehat{C}_{\epsilon,d}^{3} C_{\epsilon,d}^{4d} K_d D_{\text{FI}}} \Big[\frac{(3d+7)^{-2}}{2\widehat{C}_{\epsilon,d}^{7} C_{\epsilon,d}^{8d+5} K_d D_{\text{FI}}} \frac{1-e^{-\tau}}{D_{\text{SVG}}\, e^{\tau}} \Big]^{\frac{\rho(\rho+4)}{2}}.$$

We conclude by using $\hat{A}^n_{-n}\hat{U}_{-n} \to \hat{E}_0$ and the comparison between the minimal gaps, $\gamma(E_0, F_0) \geqslant \gamma(\hat{E}_0, \hat{F}_0)/K'_d$ where the constant $K'_d = \bar{\Delta}_2(X)^{4d}\bar{\Delta}_d(X)^{3d}$ is given by lemma A.40. ◻

4 Proof of items 2 and 3 of theorem 1.2

We first show that property (FI) is related to a super-multiplicative sequence (10) $(f_m(k))_{m \geqslant 0}$. We use the notion of Jacobian of index d, introduced in definition in A.30 and denoted by $\Sigma_d(A)$. Proposition A.32 implies,

$$\prod_{i=1}^d \sigma_i(A) \leqslant \Sigma_d(A) = \prod_{i=1}^d \sigma''_i(A) \leqslant K_d \prod_{i=1}^d \sigma_i(A)$$

where $K_d = \bar{\Delta}_d(X)^{2d^2}$. In the Hilbert case $K_d = 1$ and $\Sigma_d(A) = \prod_{i=1}^d \sigma_i(A)$. Proposition A.34 shows that the Jacobian is sub-multiplicative,

$$\forall k \in \mathbb{Z}, \ \forall m_1, m_2 \geqslant 0, \quad \Sigma_d(k, m_1+m_2) \leqslant \Sigma_d(k, m_1)\Sigma_d(k+m_1, m_2),$$

where $\Sigma_d(k,n) := \Sigma_d(A(k,n))$. We define for every $k \in \mathbb{Z}$ and $m \geqslant 0$,

$$f_m(k) := \inf_{n \geqslant 0} \frac{\Sigma_d(k-m, m+n)}{\Sigma_d(k-m,m)\Sigma_d(k,n)}. \tag{10}$$

We have obviously $f_m(k) \leqslant K_d(X)^{-1} \leqslant 1$. We show in the following lemma that $f_m(k)$ is super-multiplicative and that the ratio appearing in property (FI) is comparable to $f_m(k)$.

Lemma 4.1. *For every $k \in \mathbb{Z}$,*

1. $\forall m_1, m_2 \geqslant 0, \quad f_{m_1+m_2}(k) \geqslant f_{m_1}(k) f_{m_2}(k-m_1)$ *and* $f_m(k) \leqslant 1$,
2. $K_d^{-2} \inf_{n \geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-m,m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} \leqslant f_m(k) \leqslant K_d \inf_{n \geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-m,m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)}$,
3. $\forall m,n \geqslant 0, \quad \prod_{i=1}^{d} \frac{\sigma_i(k-m,m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} \leqslant K_d^2$,

with $K_d = \bar{\Delta}_d(X)^{2d^2}$,

Proof of item 1. As $\Sigma_d(k-m_1-m_2, m_1+m_2) \leqslant \Sigma_d(k-m_1-m_2, m_2)\Sigma_d(k-m_1, m_1)$,

$$\frac{\Sigma_d(k-m_1-m_2, m_1+m_2+n)}{\Sigma_d(k-m_1-m_2, m_1+m_2)\Sigma_d(k,n)} \geqslant \frac{\Sigma_d(k-m_1-m_2, m_1+m_2+n)}{\Sigma_d(k-m_1-m_2, m_2)\Sigma_d(k-m_1, m_1+n)} \frac{\Sigma_d(k-m_1, m_1+n)}{\Sigma_d(k-m_1,m_1)\Sigma_d(k,n)}.$$

The first quotient is bounded from below by $f_{m_2}(k-m_1)$, the second by $f_{m_1}(k)$.

Proof of item 2 and 3. The proof follows the comparison between $\Sigma_d(k,n)$ and $\prod_{i=1}^{d} \sigma_i(k,n)$. □

In the following lemma we estimate a bound from below of $f_m(k)$ from partial information on $f_{mN_*}(k)$.

Lemma 4.2. *Let $N_* \geqslant 1$, $\alpha \geqslant 1$, and $(A_k)_{k \in \mathbb{Z}}$ be a sequence of operators satisfying property (FI). Then for every $k \in \mathbb{Z}$,*

$$\inf_{m \geqslant 1} f_m(k) \geqslant K_d^{-1} D_{FI}^{-2} e^{-(1+\alpha)N_* \mu} \inf_{m \geqslant 1,\ n \geqslant \alpha N_*} \frac{\Sigma_d(k-mN_*, mN_*+n)}{\Sigma_d(k-mN_*, mN_*)\Sigma_d(k,n)},$$

where $K_d = \bar{\Delta}_d(X)^{8d^2}$.

Proof. We claim for every $m \geqslant 1$,

$$f_{mN_*}(k) \geqslant K_d^{-1} D_{\text{FI}}^{-1} e^{-\alpha N_* \mu} \inf_{n \geqslant \alpha N_*} \frac{\Sigma_d(k - mN_*, mN_* + n)}{\Sigma_d(k - mN_*, mN_*)\Sigma_d(k, n)}.$$

It is enough to bound from below in the definition of $f_{mN_*}(k)$,

$$\inf_{1 \leqslant n \leqslant \alpha N_*} \frac{\Sigma_d(k - mN_*, mN_* + n)}{\Sigma_d(k - mN_*, mN_*).\Sigma_d(k, n)}$$

Consider $1 \leqslant n \leqslant \alpha N_*$ and choose p such that $\alpha N_* \leqslant p$. Then

$$\Sigma_d(k - mN_*, mN_* + n)\Sigma_d(k + n, p - n) \geqslant \Sigma_d(k - mN_*, mN_* + p).$$

Dividing by $\Sigma_d(k - mN_*, mN_*)\Sigma_d(k, n)$ and rewriting in a different way, we obtain

$$\frac{\Sigma_d(k - mN_*, mN_* + n)}{\Sigma_d(k - mN_*, mN_*)\Sigma_d(k, n)} \geqslant \\ \Big[\frac{\Sigma_d(k - mN_*, mN_* + p)}{\Sigma_d(k - mN_*, mN_*)\Sigma_d(k, p)}\Big] \Big[\frac{\Sigma_d(k + n - n, p)}{\Sigma_d(k + n - n, n)\Sigma_d(k + n, p - n)}\Big].$$

The second bracket is bounded from below using property (FI) by

$$f_{k+n}(n) \geqslant {K'_d}^{-1} D_{\text{FI}}^{-1} e^{-n\mu} \geqslant {K'_d}^{-1} D_{\text{FI}}^{-1} e^{-\alpha N_* \mu},$$

where $K'_d = \bar{\Delta}_d(X)^{4d^2}$ is obtained from lemma 4.1. The claim is proved. We conclude by using the super-multiplicative property

$$\forall 0 \leqslant n \leqslant N_*, \ f_{mN_*+n}(k) \geqslant f_{mN_*}(k) f_n(k + mN_*) \geqslant f_{mN_*}(k) {K'_d}^{-1} D_{\text{FI}}^{-1} e^{-N_* \mu}. \ \square$$

Proof of theorem 1.2, item 2. Step 1. We use lemma A.44 to bound from below the ratio in property (FI) by the angle between the fast and slow local spaces,

$$\forall m, n \geqslant 0, \quad \prod_{i=1}^{d} \frac{\sigma_i(k - m, m + n)}{\sigma_i(k - m, m)\sigma_i(k, n)} \geqslant \widehat{C}_{\epsilon,d}^{-3} \, \gamma(\widehat{\widehat{U}}(k, m), \check{V}(k, n)).$$

Step 2. We show for every $n \geqslant (1 + \frac{\rho(\rho+4)}{2})N_*$ and $m \geqslant 1$,

$$\delta(\check{V}(k, n), \check{F}_k) \leqslant \frac{5}{2(3d + 7)} \gamma(\widehat{\widehat{U}}(k, mN_*), \check{F}_k).$$

From the definition of N_* in assumption 3.11, we obtain

$$D_{\text{SVG}}\, e^{-n\tau} \leqslant \theta'(1-\theta')^6 \frac{1-e^{-\tau}}{C_{\epsilon,d}^4}, \quad \theta' := \theta_* e^{-(n-N_*)\tau} \frac{(1-\theta_*)^{3d}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d+1} K_d D_{\text{FI}}}$$

with $K_d := \bar{\Delta}_d(X)^{5d}$. From notations 3.2 and lemma 2.4 and A.42,

$$\begin{aligned} F_k^\perp &= \text{Graph}(\Theta_k(n)^\perp) \quad \text{for some} \quad \Theta_k(n)^\perp : V(k,n)^\perp \to U(k,n)^\perp, \\ \breve{F}_k &= \text{Graph}(\widehat{\Theta}_k(n)) \quad \text{for some} \quad \widehat{\Theta}_k(n) : \breve{V}(k,n) \to \widehat{U}(k,n), \\ &\|\Theta_k(n)^\perp\| \leqslant \theta', \quad \|\widehat{\Theta}_k(n)\| \leqslant C_{\epsilon,d}^{2d} K_d' \theta'(1+\theta')^{d-1}. \end{aligned}$$

with $K_d' := \bar{\Delta}_d(X)^d$. Using $(1+\theta') \leqslant (1-\theta_*)^{-1}$ and lemma A.25, we obtain

$$\begin{aligned} \delta(\breve{V}(k,n), \breve{F}_k) \leqslant \|\widehat{\Theta}_k(n)\| &\leqslant \theta_*(1-\theta_*)^{2d+1} e^{-(n-N_*)\tau} \frac{K_d'}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{6d+1} K_d D_{\text{FI}}}, \\ &\leqslant \frac{(3d+7)^{-1}}{2} \frac{K_d' \big(e^{-N_*\tau}\big)^{\rho(\rho+4)/2}}{\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{6d+1} K_d D_{\text{FI}}}. \end{aligned}$$

On the other hand, using equation (9),

$$\gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k) \geqslant \frac{1}{5} \frac{\big(e^{-N_*\tau}\big)^{\rho(\rho+4)/2}}{\widehat{C}_{\epsilon,d}^3 C_{\epsilon,d}^{4d} K_d D_{\text{FI}}}$$

and using the bound $K_d' \leqslant C_{\epsilon,d}$, we conclude the proof of the claim,

$$\delta(\breve{V}(k,n), \breve{F}_k) \leqslant \frac{5}{2(3d+7)} \gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k).$$

Step 3. We conclude the proof of item 2 of theorem 1.2. Equations (A.16) imply

$$\begin{aligned} \gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{V}(k,n)) &\geqslant \frac{\gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k) - \delta(\breve{V}(k,n), \breve{F}_k)}{1 + \delta(\breve{V}(k,n), \breve{F}_k)}, \\ &\geqslant \frac{6d+9}{6d+19} \gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k) \geqslant \tfrac{3}{5} \gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k). \end{aligned}$$

Using lemma 4.2 with $\alpha = 1 + \frac{\rho(\rho+4)}{2}$, one gets

$$\inf_{m \geqslant 1} f_m(k) \geqslant \frac{3}{5} \inf_{m \geqslant 1} \gamma(\widehat{\widetilde{U}}(k, mN_*), \breve{F}_k) \frac{\big(e^{-N_*\mu}\big)^{2+\rho(\rho+4)/2}}{\widehat{C}_{\epsilon,d}^3 K_d'' D_{\text{FI}}^2},$$

where $K''_d = \bar{\Delta}_d(X)^{8d^2}$. Using

$$\mu = \tau\rho,\ \bar{\Delta}_d(X)^{8d^2+5d} \leqslant C_{\epsilon,d}^2,$$
$$\frac{\rho(\rho+4)}{2} + \rho\Big(2 + \frac{\rho(\rho+4)}{2}\Big) = \tfrac{1}{2}\rho(\rho^2+5\rho+8),$$

and item 2 of lemma 4.1, one obtains

$$\inf_{m\geqslant 0, n\geqslant 0} \prod_{i=1}^{d} \frac{\sigma_i(k-m, m+n)}{\sigma_i(k-m,m)\sigma_i(k,n)} \geqslant \frac{3}{25\widehat{C}_{\epsilon,d}^6 C_{\epsilon,d}^{6d} D_{\text{FI}}^3}\Big[\frac{(3d+7)^{-2}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d+5} K_d D_{\text{FI}}} \frac{1-e^{-\tau}}{D_{\text{SVG}}\, e^{\tau}}\Big]^{\rho(\rho^2+5\rho+8)/2}. \quad \square$$

Proof of theorem 1.2, item 3. We assume $n \geqslant (1 + \frac{\rho(\rho+4)}{2})N_*$ and write the assumptions 3.11 on θ_*, N_* in the form

$$D_{\text{SVG}}\, e^{-n\tau} \leqslant \theta'(1-\theta')^6 \frac{1-e^{-\tau}}{C_{\epsilon,d}^4}, \quad \theta' = \frac{\theta_*(1-\theta_*)^{3d}}{2\widehat{C}_{\epsilon,d}^7 C_{\epsilon,d}^{8d+1} K_d D_{\text{FI}}} e^{-(n-N_*)\tau}$$

with $K_d := \bar{\Delta}_d(X)^{5d}$. Notice that $\frac{1}{2}\theta_*(1-\theta_*)^{3d} \leqslant \frac{1}{20}$.

Part 1. We first estimate $\gamma(E_k, V(k,n))$ by $\gamma(E_k, F_k)$. Equation (A.16) gives,

$$\gamma(E_k, V(k,n)) \geqslant \frac{\gamma(E_k, F_k) - \delta(V(k,n), F_k)}{1 + \delta(V(k,n), F_k)}.$$

Item 1 of lemma 2.4 and $(n - N_*)\tau \geqslant \frac{\rho(\rho+4)}{2} N_*\tau$ gives

$$\delta(V(k,n), F_k) \leqslant \theta' \leqslant \tfrac{1}{20}\widehat{C}_{\epsilon,d}^{-7} C_{\epsilon,d}^{-8d-1} K_d^{-1} D_{\text{FI}}^{-1} (e^{-N_*\tau})^{\rho(\rho+4)/2}.$$

By taking $n \to +\infty$ in equation (9) and by using lemma A.40, one obtains,

$$\gamma(E_k, F_k) \geqslant K_d'^{-1} \gamma(\widehat{E}_k, \check{F}_k) \geqslant 5^{-1} \widehat{C}_{\epsilon,d}^{-3} C_{\epsilon,d}^{-4d} K_d'^{-1} K_d^{-1} D_{\text{FI}}^{-1} (e^{-N_*\tau})^{\rho(\rho+4)/2}.$$

where $K'_d = \bar{\Delta}_2(X)^{4d}\bar{\Delta}_d(X)^{3d}$. As $K'_d K_d = \bar{\Delta}_2(X)^{4d}\bar{\Delta}_d(X)^{8d} \leqslant C_{\epsilon,d}$, we have,

$$\delta(V(k,n), F_k) \leqslant \theta' \leqslant \tfrac{1}{4}\gamma(E_k, F_k), \quad \gamma(E_k, V(k,n)) \geqslant \tfrac{3}{5}\gamma(E_k, F_k).$$

Using item 4 of theorem A.35, we have for every $w \in E_k$,

$$
\begin{aligned}
\|A(k,n)w\| &\geqslant |\langle\tilde{\phi}|A(k,n)w\rangle|, \qquad (\forall\tilde{\phi} \in \tilde{V}(k+n,n)^{\perp},\ \|\tilde{\phi}\| = 1) \\
\|A(k,n)^*\tilde{\phi}\| &\geqslant C_{\epsilon,d}^{-1}\sigma_d(k,n)\|, \qquad \text{(item 2 of theorem 2.2)} \\
\|A(k,n)w\| &\geqslant \langle\frac{A(k,n)^*\tilde{\phi}}{\|A(k,n)^*\tilde{\phi}\|}|w\rangle\|A(k,n)^*\tilde{\phi}\|, \\
&\geqslant \sup\{|\langle\phi|w\rangle| : \phi \in V(k,n)^{\perp},\ \|\phi\| = 1\}C_{\epsilon,d}^{-1}\sigma_d(k,n) \\
&\geqslant \gamma(E_k, V(k,n))C_{\epsilon,d}^{-1}\sigma_d(k,n)\|w\|, \qquad \text{(equation (A.12))} \\
&\geqslant \tfrac{3}{5}\gamma(E_k,F_k)C_{\epsilon,d}^{-1}\sigma_d(k,n)\|w\|.
\end{aligned}
$$

Part 2. We estimate $\gamma(F_k,\tilde{U}(k,n))$ by $\gamma(F_k,E_k)$. Using equation (A.16) and item 1 of lemma 2.7, we have

$$
\begin{aligned}
\gamma(F_k,\tilde{U}(k,n)) &\geqslant \frac{\gamma(F_k,E_k) - \delta(\tilde{U}(k,n),E_k)}{1+\delta(\tilde{U}(k,n),E_k)} \\
\delta(\tilde{U}(k,n),E_k) &\leqslant \theta' \leqslant \tfrac{1}{4}\gamma(E_k,F_k) \leqslant \tfrac{1}{2}\gamma(F_k,E_k) \\
\gamma(F_k,\tilde{U}(k,n)) &\geqslant \tfrac{1}{3}\gamma(F_k,E_k).
\end{aligned}
$$

Let $w \in F_k$, $w = u + v$ where $u \in U(k,n)$ and $v \in V(k,n)$. Then $\|v\| \leqslant C_{\epsilon,d}\|w\|$ thanks to item 3 of theorem 2.2,

$$
\begin{aligned}
A(k,n)w &= \tilde{u} + \tilde{v}, \quad \tilde{u} \in \tilde{U}(k+n,n),\ \tilde{v} \in \tilde{V}(k+n,n), \\
\|\tilde{v}\| &\leqslant C_{\epsilon,d}\sigma_{d+1}(k,n)\|v\| \leqslant C_{\epsilon,d}^2\sigma_{d+1}(k,n)\|w\|, \\
\|\tilde{v}\| &\geqslant \|A(k,n)w\|\,\gamma(F_{k+n},\tilde{U}(k+n,n)).
\end{aligned}
$$

Hence

$$
\|A(k,n)w\| \leqslant 3C_{\epsilon,d}^2\gamma(F_{k+n},E_{k+n})^{-1}\sigma_{d+1}(k,n)\|w\|. \qquad \square
$$

Appendices

The purpose of this appendix is to clarify the notion of *approximate singular value decomposition* of a bounded operator in a Banach space. We need two precise theorems A.35 and A.43. The first theorem is usually stated for compact selfadjoint operators in an Hilbert space (see [12]). In Hilbert spaces, for non compact operators, we did not find good references, although the results are certainly known by the specialists. In Banach spaces, we are not aware of any statements as in A.35 and A.43. Nevertheless quite similar ideas may be found in [1] and [8].

A Basic results in Banach spaces

Let $(X, \|\cdot\|)$ be a real Banach space. We do not assume X to be reflexive. We call X^* the topological dual space and denote by $\langle \eta | u \rangle$ the duality between $\eta \in X^*$ and $u \in X$. If X is an Hilbert space we identify $X^* = X$ and the duality $\langle \cdot | \cdot \rangle$ with the scalar product. If U is a closed (vector) subspace of X, U becomes a Banach space with the induced norm, U^* denotes the corresponding *dual space*, and $U^\perp$ denotes the *annihilator* of U, the subspace of linear forms of X^* vanishing on U. Conversely if $H \subset X^*$ is a subspace, the *pre-annihilator* of H is the subspace $H^{\perp\!\!\perp} := \{u \in X : \langle \eta | u \rangle = 0, \ \forall \eta \in H\}$. Write $\mathcal{B}(X)$ for the space of bounded linear operators on X. If $(Y, \|\cdot\|)$ is another Banach space, write $\mathcal{B}(X, Y)$ for the space of bounded linear operators from X to Y. If $U \subset X$ is a closed subspace of X, we denote by $A|U$ the restriction to U of $A \in \mathcal{B}(X, Y)$. We say that a splitting $X = U \oplus V$ of two closed subspaces is topological if the projector $\pi_{U|V}$ onto U parallel to V (or equivalently $\pi_{V|U}$) is a bounded operator. For a Bounded operator $A \in \mathcal{B}(X, Y)$, we call $A^* \in \mathcal{B}(Y^*, X^*)$ the dual operator.

A.1 Auerbach basis and distortion

The purpose of this section is to clarify the notion of a distortion of a Banach norm with respect to the best euclidean norm. We use the notion of Auerbach bases as a substitute for orthonormal bases. We begin by recalling the notion of Auerbach families.

Definition A.1. Let X be a Banach space, and $d \geqslant 1$.

- A family of vectors $(u_1, \ldots, u_d)$ in X is said to be *Auerbach* if

$$\forall j = 1, \ldots, d, \quad \|u_j\| = 1 \ \text{ and } \ \operatorname{dist}(u_j, \operatorname{span}(u_k : k \neq j)) = 1.$$

- If $(u_1, \ldots, u_d)$ are linearly independent in X, a *dual family* is any family of linear forms $(\eta_1, \ldots, \eta_d)$ of X^* satisfying $\langle \eta_i | u_j \rangle = \delta_{ij}$. Similarly if $(\eta_1, \ldots, \eta_d)$ are linearly independent in X^*, a *predual family* is any family of vectors $(u_1, \ldots, u_d)$ of X satisfying $\langle \eta_i | u_j \rangle = \delta_{ij}$.

If $\dim(X) = d$, dual bases and predual families do always exist and they are unique. We show in the following lemma that Auerbach families can be characterized by the existence of normalized dual families.

Lemma A.2. *Let X be a Banach space, and $d \geqslant 1$.*

1. *A family of vectors $(u_1, \ldots, u_d)$ of X is Auerbach if and only if $\|u_j\| = 1$ for every $j = 1, \ldots, d$ and there exists a dual family $(\eta_1, \ldots, \eta_d)$ of X^* satisfying $\|\eta_i\| = 1$ for every $j = 1, \ldots, d$.*

2. *Suppose* $\dim(X) = d$. *A family of linear forms* $(\eta_1, \dots, \eta_d)$ *of* X^* *is an Auerbach basis if and only if* $\|\eta_i\| = 1$ *and its unique predual family* $(u_1, \dots, u_d)$ *of* X *satisfies* $\|u_j\| = 1$ *for every* $j = 1, \dots, d$.

If $\dim(X) = +\infty$, an Auerbach family in X^* does not admit in general a predual Auerbach family. We will show in lemma A.11 that such predual families do exist if we relax a little the notion of Auerbach family. If X is an Hilbert space of finite dimension, an Auerbach family is an orthonormal family, and two families of vectors $(u_1, \dots, u_d)$ and $(\eta_1, \dots, \eta_d)$ are dual to each other if and only if they are equal.

The following lemma shows that Auerbach families exist in any Banach space. We will see that this notion is a key tool for the notion of singular values of bounded operators.

Lemma A.3. *Let* X, Y *be Banach spaces,* $\dim(X) = d \geqslant 1$, $A \in \mathcal{B}(X, Y)$ *injective, and* $\tilde{X} = AX$. *Let* $(u_1, \dots, u_d)$ *be vectors of* X *and* $(\tilde{\eta}_1, \dots, \tilde{\eta}_d)$ *be linear forms of* $\tilde{X}^*$ *realizing the supremum in*

$$\Sigma_d(A) := \sup \big\{ \det \big([\langle \tilde{\eta}_i | A u_j \rangle]_{1 \leqslant i,j \leqslant d} \big) \ : \ \tilde{\eta}_i \in \tilde{X}^*, \ u_j \in X, \ \|\tilde{\eta}_i\| = \|u_j\| = 1 \big\}.$$

Let η_i *be a Hahn-Banach extension to* Y *of* $\tilde{\eta}_i$ *with* $\|\eta_i\| = 1$. *Then* $(u_1, \dots, u_d)$ *is an Auerbach family of* X, $(\eta_1, \dots, \eta_d)$ *is an Auerbach family of* Y^*, *and*

$$\Sigma_d(A) = \sup \big\{ \det \big([\langle \zeta_i | A u_j \rangle]_{1 \leqslant i,j \leqslant d} \big) \ : \ \zeta_i \in Y^*, \ u_j \in X, \ \|\zeta_i\| = \|u_j\| = 1 \big\}.$$

Notice in the previous lemma that, in the case $X = Y$ and $A = \mathrm{Id}$, $(\eta_1, \dots, \eta_d)$ and $(u_1, \dots, u_d)$ are not a priori dual to each other. We call the particular constant $\Sigma_d(A)$ appearing in lemma A.3 when $A = \mathrm{Id}$, the *projective distortion*

$$\Sigma_d(X) := \sup \big\{ \det \big([\langle \eta_i | u_j \rangle]_{1 \leqslant i,j \leqslant d} \big) \ : \ \eta_i \in X^*, u_j \in X, \|\eta_i\| = \|u_j\| = 1 \big\}. \quad \text{(A.1)}$$

The name "projective distortion" is related to the notion of projective norm introduced in (A.20) and the estimate of the distortion of the canonical duality (A.21) and (A.22).

A Banach norm introduces a distortion in the volume of unit balls of finite-dimensional subspaces. This distortion may depend on the dimension of the subspace. In order to obtain optimal estimates when X is actually an Hilbert space, we introduce a notion of volume distortion that turn out to be trivial for Hilbert spaces.

Definition A.4. Let X be a Banach space and $d \geqslant 1$. The *volume distortion* is

$$\Delta_d(X) := \sup \Big\{ \frac{\| \sum_{j=1}^d \lambda_j u_j \|}{\big(\sum_{j=1}^d |\lambda_j|^2 \big)^{1/2}} \ : \ u \text{ is an Auerbach family and } \lambda \neq 0 \Big\} \quad \text{(A.2)}$$

where the supremum is realized over every $u = (u_1, \dots, u_d)$ Auerbach family of X and every non-zero $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$. If X is a Hilbert space $\Delta_d(X) = 1$. In general we have $1 \leqslant \Delta_d(X) \leqslant \sqrt{d}$. In order to simplify the estimates, we will use instead a *simplified volume distortion*

$$\bar{\Delta}_d(X) := \max(\Delta_d(X), \Delta_d(X^*), \Delta_d(X^{**})). \tag{A.3}$$

Although we do not intend to compute this constant for different Banach spaces, we give an exact estimate of $\Delta_d(X)$ for $X = \ell^p_d$ the space $\mathbb{R}^d$ endowed with the norm $\|x\|_p = \big(\sum_{n=1}^d |x_n|^p\big)^{1/p}$, $x = (x_1, \dots, x_d)$, with natural change for $p = +\infty$. Recall that the *Banach-Mazur distance* between two isomorphic spaces X and Y is the number

$$d_{BM}(X, Y) := \inf\{\|T\|\|T^{-1}\|,\ T : X \to Y \text{ linear bounded isomorphism}\}.$$

Proposition A.5. *For every $p \in [1, 2]$, $\Delta_d(\ell^p_d) = d_{BM}(\ell^p_d, \ell^2_d) = d^{|\frac{1}{p} - \frac{1}{2}|}$. Hence*

$$\lim_{p \to 2^-} \Delta_d(\ell^p_d) = 1.$$

If $U \subset X$ is a subspace of X, then $\Delta_d(U) \leqslant \Delta_d(X)$. We have for instance $\Delta_d(X) \leqslant \Delta_d(X^{**})$. By extending any Auerbach family $(\eta_1, \dots, \eta_d)$ of U^* by Hahn-Banach while keeping $\|\eta_i\| = 1$, we still obtain an Auerbach family in X^* and thus $\Delta_d(U^*) \leqslant \Delta_d(X^*)$. We show in the following lemma that $\Delta_d(X)$ and $\Delta_d(X^*)$ admit equivalent definitions in the case $\dim(X) = d$.

Lemma A.6. *Let be $d \geqslant 1$ and X be a Banach space of dimension d. Then*

1. $\Delta_d(X^*) = \sup\Big\{ \dfrac{\big(\sum_{i=1}^d |\lambda_j|^2\big)^{1/2}}{\| \sum_{j=1}^d \lambda_j u_j\|} \ :\ u \text{ is an Auerbach basis of } X,\ \lambda \neq 0\}\Big\},$
2. $\Delta_d(X) = \sup\Big\{ \dfrac{\big(\sum_{i=1}^d |\lambda_i|^2\big)^{1/2}}{\| \sum_{i=1}^d \lambda_i \eta_i\|} \ :\ \eta \text{ is an Auerbach basis of } X^*,\ \lambda \neq 0\Big\},$
3. $\Delta_d(X) = \Delta_d(X^{**})$.

In particular we obtain an "explicit" bound between the Banach norm and the Euclidean norm either in U or in U^*.

Corollary A.7. *Let $d \geqslant 1$ and X be a Banach space of dimension d.*

1. *If $(u_1, \dots, u_d)$ is an Auerbach basis of X, then*

$$\forall \lambda \in \mathbb{R}^d,\ \frac{1}{\Delta_d(X^*)} \Big(\sum_{j=1}^d |\lambda_j|^2 \Big)^{1/2} \leqslant \| \sum_{j=1}^d \lambda_j u_j \| \leqslant \Delta_d(X) \ \Big(\sum_{j=1}^d |\lambda_j|^2 \Big)^{1/2}.$$

2. *If $(\eta_1, \ldots, \eta_d)$ is an Auerbach basis of X^*, then*

$$\forall \lambda \in \mathbb{R}^d,\ \frac{1}{\Delta_d(X)} \Big(\sum_{i=1}^d |\lambda_i|^2 \Big)^{1/2} \leqslant \| \sum_{i=1}^d \lambda_i \eta_i \| \leqslant \Delta_d(X^*) \ \Big(\sum_{i=1}^d |\lambda_i|^2 \Big)^{1/2}.$$

Every subspace $U \subset X$ of finite dimension d admits a topological complement (a closed subspace V such that $X = U \oplus V$). For instance, if $(u_1, \ldots, u_d)$ is an Auerbach basis of U, if $(\eta_1, \ldots, \eta_d)$ is an Auerbach basis in U^* dual to $(u_1, \ldots, u_d)$, that has been extended to X by Hahn-Banach as linear forms of norm one, then $(\eta_1, \ldots, \eta_d)$ is again an Auerbach family in X^*, and $V = \bigcap_{i=1}^d \ker(\eta_i)$ is a topological complement to U where the projector $\pi_{U|V}$ onto U parallel to V is given by

$$\pi_{U|V}(w) = \sum_{i=1}^d \langle \eta_i | w \rangle u_i, \quad \forall w \in X. \tag{A.4}$$

Notice that if $(u_1, \ldots, u_d)$ and $(\eta_1, \ldots, \eta_d)$ are dual to each other but not necessarily Auerbach, then in addition to (A.4), we have,

$$\begin{aligned} \pi_{V|U} &= \mathrm{Id} - \pi_{U|V} = \pi_d \circ \cdots \circ \pi_1, \quad \text{where} \\ \pi_k(w) &= w - \langle \eta_k | w \rangle u_k, \quad \forall w \in X. \end{aligned} \tag{A.5}$$

Definition A.8. Let X be a Banach space, $d \geqslant 1$, and $X = U \oplus V$ be a splitting such that $\dim(U) = d$. We say that the splitting is *orthogonal* if there exist Auerbach families $(u_1, \ldots, u_d)$ of X and $(\eta_1, \ldots, \eta_d)$ of X^* dual to each other such that

$$U = \operatorname{span}(u_1, \ldots, u_d) \ \text{ and } \ V = \bigcap_{i=1}^d \ker(\eta_i) = \operatorname{span}(\eta_1, \ldots, \eta_d)^{\perp\!\!\perp}.$$

If X is a Hilbert space, we recover the usual notion of orthogonal complements. In particular the two projectors $\pi_{V|U}$ and $\pi_{U|V}$ have norm one. In general if X is a Banach space, the norm of the projectors is not any more one. We give two results giving the bound of the norm of these projectors in terms of the volume distortion. We use the simplified volume distortion given in (A.3).

Lemma A.9. *Let X be a Banach space, $u \in X$, $\eta \in X^*$, such that $\langle \eta | u \rangle = 1$, and $\|\eta\| = 1$. Let $U = \operatorname{span}(u)$, $V = \ker(\eta)$, and $K_d := \bar{\Delta}_2(X)^3$. Then*

$$\|\pi_{U|V}\| = \|u\|, \ \text{ and } \ \|\pi_{V|U}\| \leqslant K_d \|u\|.$$

For any dimension, we obtain the following bound.

Lemma A.10. *Let X be a Banach space, $d \geqslant 1$, $\dim(U) = d$, and $X = U \oplus V$ be an orthogonal splitting. Let $K_d := \bar{\Delta}_2(X)^4 \bar{\Delta}_d(X)^2$. Then*

$$\forall u \in U,\ \forall v \in V, \quad \frac{1}{K_d}\sqrt{\|u\|^2 + \|v\|^2} \leqslant \|u+v\| \leqslant K_d\sqrt{\|u\|^2 + \|v\|^2}$$

In particular $\|\pi_{U|V}\| \leqslant K_d$ and $\|\pi_{V|U}\| \leqslant K_d$.

We are now able to extend item 2 of lemma A.2 to Banach spaces of infinite dimension.

Lemma A.11. *Let X be a Banach space and $d \geqslant 1$. Let be $K_d := \bar{\Delta}_2(X)^{3d}$. Then for every Auerbach family $(\eta_1, \ldots, \eta_d)$ of X^*, for every $\epsilon > 0$, there exist a predual family $(u_1, \ldots, u_d)$ in X satisfying*

$$1 \leqslant \operatorname{dist}(u_k, \operatorname{span}(u_l : l \neq k)) \ \text{ and } \ \|u_k\| \leqslant (1+\epsilon)K_d, \quad \forall k = 1, \ldots, d.$$

If X is a Hilbert space, $\epsilon = 0$, $K_d = 1$ and $(u_1, \ldots, u_d) = (\eta_1, \ldots, \eta_d)$.

The previous result suggests the following definition.

Definition A.12. Let X be a Banach space, $d \geqslant 1$ and $C \geqslant 1$. A family of vectors $(u_1, \ldots, u_d)$ is said to be a *C-Auerbach family* if

$$C^{-1} \leqslant \operatorname{dist}(u_k, \operatorname{span}(u_l : l \neq k)) \ \text{ and } \ \|u_k\| \leqslant C, \quad \forall k = 1, \ldots, d.$$

A splitting $X = U \oplus V$ where $\dim(U) = d$, is said to be *C-orthogonal* if there exist C-Auerbach families $(u_1, \ldots, u_d)$ of X and $(\eta_1, \ldots, \eta_d)$ of X^* dual to each other such that $U = \operatorname{span}(u_1, \ldots, u_d)$ and $V = \operatorname{span}(\eta_1, \ldots, \eta_d)^{\perp\!\!\perp}$.

Lemma A.11 shows that, if V is a subspace of X of codimension d, and $\epsilon > 0$, then there exists U such that $X = U \oplus V$ is a $(1+\epsilon)K_d$-orthogonal splitting.

If X is a Hilbert space, a 1-Auerbach family corresponds to an orthonormal family, a C-Auerbach family represents a distorted orthonormal family. We give in the following lemma several equivalent characterizations of C-Auerbach bases in the case X is a finite dimensional Hilbert space.

Lemma A.13. *Let $P = [P_{i,j}]_{1 \leqslant i,j \leqslant d}$ be a real matrix and $C \geqslant 1$. $\mathbb{R}^d$ is equipped with the standard euclidean norm $\|\cdot\|_2$. The following 3 conditions are equivalent.*

1. *The column vectors $\overrightarrow{C_j} := (P_{i,j})_{i=1}^d$ form a C-Auerbach basis.*
2. *The singular values of P satisfy $C \geqslant \sigma_1 \geqslant \cdots \geqslant \sigma_d \geqslant 1/C$.*

3. *For every* $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$,

$$\frac{1}{C}\Big(\sum_{j=1}^{d} |\lambda_j|^2\Big)^{1/2} \leqslant \|\sum_{j=1}^{d} \lambda_j \overrightarrow{C_j}\|_2 \leqslant C\Big(\sum_{j=1}^{d} |\lambda_j|^2\Big)^{1/2}.$$

In particular, since the singular values of P *and* P^* *coincide, the 3 conditions are also equivalent to*

4. *The row vectors* $\overrightarrow{R_i} := (P_{i,j})_{j=1}^d$ *form a* C*-Auerbach basis.*

5. *For every* $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$,

$$\frac{1}{C}\Big(\sum_{i=1}^{d} |\lambda_i|^2\Big)^{1/2} \leqslant \|\sum_{i=1}^{d} \lambda_i \overrightarrow{R_i}\|_2 \leqslant C\Big(\sum_{i=1}^{d} |\lambda_i|^2\Big)^{1/2}.$$

If X is a Banach space, many previous results involving Auerbach families can be extended to C-Auerbach families. The volume distortion of a C-Auerbach family can be expressed using the volume distortion defined in A.4.

Lemma A.14. *Let* X *be a Banach space,* $d \geqslant 1$, *and* $C \geqslant 1$. *Define* $K_d := \bar{\Delta}_d(X)^2$. *If* $(e_1, \dots, e_d)$ *is a* C*-Auerbach family, then for every* $(\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$,

$$\frac{1}{CK_d}\Big(\sum_{j=1}^{d} |\lambda_j|^2\Big)^{1/2} \leqslant \|\sum_{j=1}^{d} \lambda_j e_j\| \leqslant CK_d\,\Big(\sum_{j=1}^{d} |\lambda_j|^2\Big)^{1/2}$$

We extend lemma A.10 to C-Auerbach families.

Lemma A.15. *Let* X *be a Banach space,* $d \geqslant 1$ *and* $C \geqslant 1$. *Let* $X = U \oplus V$ *be a* C*-orthogonal splitting with* $\dim(U) = d$. *Define* $K_d := \bar{\Delta}_2(X)^4 \bar{\Delta}_d(X)^4$. *Then*

$$\forall u \in U,\ \forall v \in V, \quad \frac{1}{C^2K_d}\sqrt{\|u\|^2 + \|v\|^2} \leqslant \|u + v\| \leqslant C^2K_d\sqrt{\|u\|^2 + \|v\|^2}.$$

We also extend lemma A.2 to C-Auerbach families.

Lemma A.16. *Let* X *be a Banach space,* $C \geqslant 1$, $d \geqslant 1$, *and* $K_d := \bar{\Delta}_2(X)^{3d}\bar{\Delta}_d(X)^2$.

- *If* $(u_1, \dots, u_d)$ *is a* C*-Auerbach family of* X, *then there exists a* C*-Auerbach family* $(\eta_1, \dots, \eta_d)$ *of* X^* *dual to* $(u_1, \dots, u_d)$.

- *If* $(\eta_1, \dots, \eta_d)$ *is a* C*-Auerbach family of* X^*. *Then for every* $\epsilon > 0$, *there exists a* $CK_d(1+\epsilon)$*-Auerbach family of* X *predual to* $(\eta_1, \dots, \eta_d)$.

- *If* U *is a subspace of dimension* d, $(\tilde{\eta}_1, \dots, \tilde{\eta}_d)$ *is a* C*-Auerbach basis of* U^* *and* $(\eta_1, \dots, \eta_d)$ *is some Hahn-Banach extension such that* $\|\tilde{\eta}_i\| = \|\eta_i\|$, *then* $(\eta_1, \dots, \eta_d)$ *is again a* C*-Auerbach family and there exists a* C*-Auerbach basis* $(u_1, \dots, u_d)$ *in* U *predual to* $(\eta_1, \dots, \eta_d)$.

A.2 Grassmannian, gaps, and graphs

The geometry of Grassmannian spaces is a well studied object in the case of Hilbert spaces. For Banach spaces, the notion of angle is not canonically well-defined and several equivalent definition could be used. The *d-dimensional Grassmannian space* is the set, $\text{Grass}(d, X)$, of all subspaces of X of dimension $d \geqslant 1$. The *d-dimensional coGrassmannian space* is the set, $\text{coGrass}(d, X)$, of all closed subspaces of X of codimension d. We denote by S_X the unit sphere of X. We first recall two estimates (see also Kato [10], chapter 4, section 2.3); for every closed non trivial subspace N of X,

$$\begin{aligned} \text{dist}(u, N) &= \sup\{\langle \phi | u \rangle : \phi \in N^{\perp},\ \|\phi\| = 1\}, \quad \forall u \in X, \\ \text{dist}(\phi, N^{\perp}) &= \sup\{\langle \phi | u \rangle : u \in N,\ \|u\| = 1\}, \quad \forall \phi \in X^*. \end{aligned} \tag{A.6}$$

Definition A.17. Let X be a Banach space and M, N be two closed non-trivial subspaces of X. The *maximal gap between M and N* is

$$\begin{aligned} \delta(M, N) &:= \sup\big\{\text{dist}(u, N) : u \in M,\ \|u\| = 1\big\}, \\ &= \sup\big\{\langle \phi | u \rangle : u \in M,\ \phi \in N^{\perp},\ \|u\| = \|\phi\| = 1\big\}. \end{aligned} \tag{A.7}$$

We also define another equivalent distance

$$d(M, N) := \sup\big\{\text{dist}(u, S_N) : u \in M,\ \|u\| = 1\big\}, \tag{A.8}$$

and observe that d satisfies the triangle inequality and the estimate

$$\delta(M, N) \leqslant d(M, N) \leqslant 2\delta(M, N). \tag{A.9}$$

The notion of maximal gap between subspaces $\delta(M, N)$ was introduced by Gohberg and Marcus [6], (see also Kato [10], chapter 4, section 2.1), under the name *opening* or *aperture*. We use mainly $\delta(M, N)$ in two cases: either for $\dim(M) = \dim(N) < +\infty$ or for $\text{codim}\,(M) = \text{codim}\,(N) < +\infty$. We recall the duality identity (see equation (2.19) in Kato [10], chapter 4, section 2.3)

$$\delta(M, N) = \delta(N^{\perp}, M^{\perp}), \quad \forall M, N \text{ closed subspace of } X. \tag{A.10}$$

In general the maximal gap is not symmetric, but for finite-dimensional subspaces of equal dimension we have (see [9], Lemma 213)

$$\dim M = \dim N < +\infty \ \Rightarrow\ \delta(M, N) \leqslant \frac{\delta(N, M)}{1 - \delta(N, M)}. \tag{A.11}$$

We use another estimate which enables us to recover the standard estimate in the Hilbert case.

Lemma A.18. *Let X be a Banach space and* $d \geqslant 1$. *Define*

$$K_2 := \min(2, \Delta_2(X)^2 \Delta_2(X^*)^2).$$

For every subspaces M, N *of* X, *if* $\dim M = \dim N = d$, *then*

$$\delta(M, N) \leqslant K_2 \delta(N, M).$$

In particular, if X is a Hilbert space, $\delta(M, N) = \delta(N, M)$.

For complementary subspaces we use another notion called the minimal gap (see Kato [10], chapter 4, section 4.1).

Definition A.19. Let X be a Banach space and M, N be two closed non trivial subspaces of X. The*minimal gap* is

$$\gamma(M, N) := \inf \big\{ \mathrm{dist}(u, N) : u \in M,\ \|u\| = 1 \big\}. \tag{A.12}$$

A similar notion has been introduced in [4]

$$\perp(M, N) := \inf \big\{ \|u - v\| : u \in M,\ v \in N,\ \|u\| = \|v\| = 1 \big\}. \tag{A.13}$$

The second definition is more symmetric and equivalent to the first one

$$\gamma(M, N) \leqslant \perp(M, N) \leqslant 2\gamma(M, N). \tag{A.14}$$

The notion of minimal gap is equivalent to the notion of minimal angle $\theta(M, N)$ that is used in Gohberg and Krein [7] (chapter VI, section 5.1) where

$$\theta(M, N) := \arcsin \gamma(M, N), \quad \theta \in [0, \pi/2],$$

We use mainly the notion of minimal gap for complementary subspaces $X = M \oplus N$ where M and N are closed. The norm of the projector onto M parallel to N is not necessarily bounded. Whether it is bounded or not, we have (see equation (4.7) in Kato [10], chapter 4, section 4.1),

$$X = M \oplus N \Rightarrow \gamma(M, N) = \|\pi_{M|N}\|^{-1}. \tag{A.15}$$

Notice that lemma A.15 shows that, if the splitting $X = M \oplus N$, with $\dim(M) = d$, is C-orthogonal, then $\gamma(M, N) \geqslant 1/(C^2 K_d)$. If X is an Hilbert space, $\gamma(M, M^\perp) = 1$. If two closed subspaces N and N' are complementary with respect to the same M, $X = M \oplus N = M \oplus N'$, then their minimal gaps are comparable (see equation (4.34) in Kato [10], chapter 4, section 4.5) provided $\delta(N, N')$ is small enough

$$\gamma(M, N') \geqslant \frac{\gamma(M, N) - \delta(N', N)}{1 + \delta(N', N)}, \ \ \gamma(N', M) \geqslant \frac{\gamma(N, M) - \delta(N, N')}{1 + \delta(N, N')}. \tag{A.16}$$

The duality identity (A.10) is also valid for the minimal gap (see equation (4.14) Kato [10], chapter 4, section 4.2)

$$X = M \oplus N \Rightarrow \gamma(N^{\perp}, M^{\perp}) = \gamma(M, N). \tag{A.17}$$

The minimal gap can also be computed using duality between subspaces of complementary dimension. Let $M \subset X$, $\Xi \subset X^*$, such that $\dim(M) = d$ and $\dim(\Xi) = d$. Define

$$\langle \Xi | M \rangle := \sup \big\{ \det([\langle \xi_i | u_j \rangle]_{1 \leqslant i,j \leqslant d}) : \xi_i \in \Xi,\ u_j \in M,\ \|\xi_i\| = \|u_j\| = 1 \big\}. \tag{A.18}$$

Notice that

$$\Sigma_d(X) = \sup\{\langle \Xi | M \rangle : M \subset X,\ \Xi \subset X^*,\ \dim(M) = \dim(\Xi) = d\}.$$

Lemma A.20. *Let X be a Banach space, $d \geqslant 1$, M and N be two closed subspaces such that $X = M \oplus N$ and $\dim M = d$. Define $K_d := \bar{\Delta}_d(X)^{2d}$ and $K'_d := \bar{\Delta}_2(X)^{3d^2} \bar{\Delta}_d(X)^{2d}$. Then*

$$(K'_d)^{-1} \gamma(M, N)^d \leqslant \langle N^{\perp} | M \rangle \leqslant K_d\, \gamma(M, N).$$

The topology on the Grassmannian space $\mathrm{Grass}(d, X)$ and coGrassmannian space $\mathrm{coGrass}(d, X)$ is given by a fundamental system of open neighborhoods.

Definition A.21. Let X be a Banach space and V_0 be a subspace of X of finite dimension or codimension. The *basic neighborhood complementary to* V_0 is the subset

$$\mathcal{N}(V_0) = \{U \subset X : U \text{ is a closed subspace and } X = U \oplus V_0 \text{ is topological}\}.$$

The set $\{\mathcal{N}(V_0) : \mathrm{codim}\,(V_0) = d\}$ defines a topology of $\mathrm{Grass}(d, X)$; similarly the set $\{\mathcal{N}(U_0) : \dim(U_0) = d\}$ defines a topology of $\mathrm{coGrass}(d, X)$.

Each basic neighborhood is modeled on a Banach space. The following construction shows that $\mathcal{N}(U_0)$ is bijectively mapped to $\mathcal{B}(V_0, U_0)$.

Definition A.22. Let $X = U_0 \oplus V_0$ be a topological splitting of closed subspaces.

1. If $\Theta \in \mathcal{B}(V_0, U_0)$, the *graph* of Θ is the closed subspace

$$\mathrm{Graph}(\Theta) := \{v + \Theta v : v \in V_0\} \in \mathcal{N}(U_0).$$

2. Conversely every $V \in \mathcal{N}(U_0)$ is the graph of some operator $\Theta \in \mathcal{B}(V_0, U_0)$.

Notice that $V \in \mathcal{N}(U_0)$ if and only if $V^\perp = \mathrm{Graph}(\Theta^\perp) \in \mathcal{N}(U_0^\perp)$ for some $\Theta^\perp \in \mathcal{B}(V_0^\perp, U_0^\perp)$.

Lemma A.23. *Let X be a Banach space, $d \geqslant 1$, and $X = U_0 \oplus V_0$ be a splitting of closed subspaces of X where $\dim(U_0) = d$. Assume $U_0 = \mathrm{span}(u_1, \ldots, u_d)$ and $V_0 = \mathrm{span}(\eta_1, \ldots, \eta_d)^{\perp\!\!\!\perp}$. Let $V \in \mathcal{N}(U_0)$, $\Theta \in \mathcal{B}(V_0, U_0)$ such that $V = \mathrm{Graph}(\Theta)$, and $\Theta^\perp \in \mathcal{B}(V_0^\perp, U_0^\perp)$ such that $V^\perp = \mathrm{Graph}(\Theta^\perp)$. Then*

- $\forall v \in V,\ \Theta(v) = -\sum_{i=1}^d \langle \Theta^\perp \eta_i | v \rangle u_i,$
- $\Theta^\perp = -\pi^*_{V_0|U_0} \circ \Theta^* \circ \rho^*_{U_0}$

where $\rho_{U_0} : U_0 \to X$ is the canonical injection.

In the following lemma, we show that the norm of $\mathrm{Id} \oplus \Theta$ and the minimal gap $\gamma(U, V_0)$ are inverse proportional. We interpret

$$\mathrm{Id} \oplus \Theta : U_0 \to U = \mathrm{Graph}(\Theta), \quad \Theta \in \mathcal{B}(U_0, V_0), \tag{A.19}$$

as an isomorphism between U_0 and U and call it the *canonical isomorphism between U_0 and U parallel to V_0*. Notice that $(\mathrm{Id} \oplus \Theta)^{-1} = (\pi_{U_0|V_0}|U)$.

Lemma A.24. *Let X be a Banach space and $X = U_0 \oplus V_0$ be a topological splitting of X of subspaces of finite dimension or codimension. Then for every $U \in \mathcal{N}(V_0)$ and $\Theta \in \mathcal{B}(U_0, V_0)$ such that $U = \mathrm{Graph}(\Theta)$,*

$$\gamma(U_0, V_0) \leqslant \gamma(U, V_0) \|\mathrm{Id} \oplus \Theta\| \leqslant 1.$$

The following lemma shows that the maximal gap between two subspaces U and U' of $\mathcal{N}(V_0)$ sufficiently close to some fixed $U_0 \in \mathcal{N}(V_0)$ is equivalent to the distance $\|\Theta - \Theta'\|$.

Lemma A.25. *Let X be a Banach space, $X = U_0 \oplus V_0$ be a topological direct sum of subspaces of X of finite dimension or codimension. For every $\Theta, \Theta' \in \mathcal{B}(U_0, V_0)$ define $U := \mathrm{Graph}(\Theta)$ and $U' := \mathrm{Graph}(\Theta')$. Then*

1. *if $\delta(U, U_0) < \gamma(V_0, U_0)$, then* $\displaystyle \|\Theta\| \leqslant \frac{\delta(U, U_0)}{\gamma(V_0, U_0) - \delta(U, U_0)},$
2. *if $\delta(U, U_0) < \gamma(V_0, U_0)$ and $\delta(U', U) < \gamma(V_0, U)$, then*

$$\|\Theta' - \Theta\| \leqslant \Big[\frac{\gamma(V_0, U_0)}{\gamma(V_0, U_0) - \delta(U, U_0)}\Big] \frac{\delta(U', U)}{\gamma(V_0, U) - \delta(U', U)},$$

3. $\delta(U_0, U) \leqslant \|\Theta\|, \quad \Big[1 + \dfrac{\delta(U, U_0)}{\gamma(V_0, U_0)}\Big]^{-1} \delta(U, U') \leqslant \|\Theta - \Theta'\|$.

Let $X = U_0 \oplus V_0 = U \oplus V$ be two splittings of X by closed subspaces where $\dim(U_0) = d$ and $\dim(U) = d$. Assume $U_0 \in \mathcal{N}(V)$ or $U \in \mathcal{N}(V_0)$. The following lemma shows that the minimal gap $\gamma(U_0, V)$ or $\gamma(U, V_0)$ can be measured by a d-dimensional determinant adapted to $(V^\perp, U_0)$ or $(V_0^\perp, U)$ that are both of dimension d.

Lemma A.26. *Let X be a Banach space, $d \geqslant 1$, $C_0 \geqslant 1$, and $X = U_0 \oplus V_0$ be a C_0-orthogonal splitting with $\dim U_0 = d$. Let $(e_1, \ldots, e_d)$ and $(\phi_1, \ldots, \phi_d)$ be C_0-Auerbach bases dual to each other generating U_0 and $V_0^\perp$. Let $K_d := \bar{\Delta}_d(X)^{2d}$.*

1. *Let $\Theta^\perp \in \mathcal{B}(V_0^\perp, U_0^\perp)$, $\|\Theta^\perp\| \leqslant 1$, $V = \mathrm{Graph}(\Theta^\perp)^{\perp\!\!\perp}$ and $(\psi_1, \ldots, \psi_d)$ be a C-Auerbach basis of $V^\perp$. Then*
$$(C_0 C)^d \langle V^\perp | U_0 \rangle \geqslant \big| \det([\langle \psi_i | e_j \rangle]_{ij}) \big| \geqslant \frac{1}{K_d} \Big(\frac{1 - \|\Theta^\perp\|}{C_0 C} \Big)^d.$$

2. *Let $\Theta \in \mathcal{B}(U_0, V_0)$, $\|\Theta\| \leqslant 1$, $U = \mathrm{Graph}(\Theta)$ and $(f_1, \ldots, f_d)$ be a C-Auerbach basis of U. Then*
$$(C_0 C)^d \langle V_0^\perp | U \rangle \geqslant \big| \det([\langle \phi_i | f_j \rangle]_{ij}) \big| \geqslant \frac{1}{K_d} \Big(\frac{1 - \|\Theta\|}{C_0 C} \Big)^d.$$

A.3 Singular values decomposition

The notion of singular values for operators in Banach spaces is not canonically well-defined. Our starting definition is the following.

Definition A.27. Let X, Y be Banach spaces, $A \in \mathcal{B}(X, Y)$, and $d \geqslant 1$. We define the *singular value of A of index d* by
$$\sigma_d(A) := \sup_{\dim(U) = d} \inf \Big\{ \frac{\|Aw\|}{\|w\|} \ : \ w \in U \backslash \{0\} \Big\},$$
where the supremum is realized over every subspace U of X of dimension d.

We recall some elementary properties.

Lemma A.28. *Let X, Y be Banach spaces, $A \in \mathcal{B}(X, Y)$, and $d \geqslant 1$. Then*

1. *$\sigma_d(A) \geqslant \sigma_{d+1}(A)$,*

2. *$\sigma_d(AB) \leqslant \|A\| \sigma_d(B)$, $\sigma_d(AB) \leqslant \sigma_d(A) \|B\|$,*

3. $\sigma_d(A) > 0$ *and* $\sigma_{d+1}(A) = 0 \Longleftrightarrow \operatorname{codim}(\ker(A)) = d$.

Another definition could be used instead of $\sigma_d(A)$. It coincides with the first one when X and Y are Hilbert spaces.

Definition A.29. Let $A \in \mathcal{B}(X, Y)$. For every $d \geqslant 1$, define

$$\sigma'_d(A) := \inf_{\operatorname{codim}(V)=d-1} \sup \Big\{ \frac{\|Aw\|}{\|w\|} \ : \ w \in V \backslash \{0\} \Big\},$$

where the infimum is realized over every closed subspace V of codimension $d-1$.

It will be convenient to introduce a third notion of singular values using the notion of Jacobian.

Definition A.30. Let $A \in \mathcal{B}(X, Y)$. The *Jacobian of A of index d* is defined by,

$$\Sigma_d(A) := \sup \Big\{ \det \big([\langle \zeta_i | A u_j \rangle]_{1 \leqslant i,j \leqslant d} \big) \ : \ \zeta_i \in Y^*, \ u_j \in X, \ \|\zeta_i\| = \|u_j\| = 1 \Big\},$$

By convention $\Sigma_0(A) = 1$. Notice that, if $\dim(U) = d$,

$$\Sigma_d(A|U) = 0 \ \Leftrightarrow \ \dim(AU) < d \ \Leftrightarrow A \ \text{ is not injective on } U.$$

We may choose in the previous definition $\tilde\eta_i \in \overline{\operatorname{Im}(A)}^{\,*}$ and take ζ_i an extension of $\tilde\eta_i$ to Y^* by the Hahn-Banach theorem. If U is a closed subspace of X, we define the *Jacobian of A restricted to U of index d*, denoted $\Sigma_d(A|U)$, to be the Jacobian of $A|U \in \mathcal{B}(U, Y)$. If U has finite dimension and $A|U$ is injective, the supremum is attained by vectors $u_j \in U$ and linear forms $\tilde\eta_i \in \tilde U^*$, $\tilde U = AU$, of norm one. Both $(u_1, \ldots, u_d)$ and $(\tilde\eta_1, \ldots, \tilde\eta_d)$ are Auerbach bases by lemma A.3.
The third definition of singular values is based on the notion of Jacobian.

Definition A.31. Let $A \in \mathcal{B}(X, Y)$, define (assuming by convention $\Sigma_0(A) = 1$),

$$\sigma''_d(A) := \frac{\Sigma_d(A)}{\Sigma_{d-1}(A)} \ \text{ if } \ \Sigma_{d-1}(A) \neq 0, \quad \sigma''_d(A) = 0 \ \text{ if } \ \Sigma_{d-1}(A) = 0.$$

If U is a closed subspace of X, we define similarly $\sigma''_d(A|U)$ of the restriction of $(A|U) \in \mathcal{B}(U, Y)$.

The three definitions $\sigma_d(A), \sigma'_d(A)$ and $\sigma''_d(A)$ are comparable in Banach spaces, and equal in Hilbert spaces.

Proposition A.32. *Let X, Y be Banach spaces,* $d \geqslant 1$, *and* $K_d := [\Delta_d(Y^*)\Delta_d(X)]^d$. *Then for every* $A \in \mathcal{B}(X, Y)$,

$$\sigma_d(A) \leqslant \sigma'_d(A) \leqslant \sigma''_d(A) \leqslant K_d \, \sigma_d(A).$$

It may not be true that the singular values of A and A^* coincide. On the other hand the Jacobian admits a very symmetric definition using the identity

$$\langle\tilde{\eta}|Au\rangle = \langle A^*\tilde{\eta}|u\rangle, \quad \forall u \in X,\ \forall\tilde{\eta} \in Y^*.$$

Proposition A.32 and the following proposition shows that $\sigma_d(A)$ and $\sigma_d(A^*)$ are comparable modulo a constant depending only on the Banach norm of X. This constant is 1 for Hilbert spaces.

Proposition A.33. *Let X, Y be Banach spaces, $A \in \mathcal{B}(X, Y)$, $d \geqslant 1$, and $K_d := \max(\bar{\Delta}_d(X), \bar{\Delta}_d(Y))^{2d}$. Then*

1. $\Sigma_d(A) = \Sigma_d(A^*)$,
2. $K_d^{-1}\sigma_d(A) \leqslant \sigma_d(A^*) \leqslant K_d\sigma_d(A)$.

The following lemma shows that the projective distortion $\Sigma_d(X)$, equation (A.1), may not be equal to one and that the Jacobian may not be multiplicative. This anomaly disappears when the spaces are Hilbert.

Proposition A.34. *Let X, Y, Z be Banach spaces, $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(Y, Z)$, $d \geqslant 1$, and $K_d := \bar{\Delta}_d(X)^d$. Then*

1. $1 \leqslant \Sigma_d(X) \leqslant K_d$,
2. $\Sigma_d(BA) \leqslant \Sigma_d(B)\Sigma_d(A)$,
3. *if U is a subspace of dimension d,* $\Sigma_d(B|AU)\Sigma_d(A|U) \leqslant \Sigma_d(X)\Sigma_d(BA)$.

In the case X, Y are Hilbert spaces, the previous inequalities are equalities.

The following theorem is the main result of this appendix. The existence of singular vectors depends on a small parameter $\epsilon > 0$ that can be as small as we want. We do not assume that the operators are compact nor asymptotically compact, and there is thus no reason to find true eigenvectors even in Hilbert spaces. The parameter ϵ measures the discrepancy between a true and an approximate eigenvector. The estimates depend moreover in Banach spaces on the volume distortion introduced in the definition A.4. Although the following result is certainly well known to specialists, we did not find a good reference adapted to our needs.

Theorem A.35 (Approximate singular value decomposition)**.** *Let X, Y be Banach spaces, $A \in \mathcal{B}(X, Y)$, and $d \geqslant 1$. Assume $\sigma_d(A) > 0$ and choose $\epsilon > 0$. Define*

$$\Delta_d = \max(\bar{\Delta}_d(X), \bar{\Delta}_d(Y)), \quad C_{\epsilon,d}(X, Y) := (1+\epsilon)\Delta_d^{6d^2+15d+4}\ \Delta_2^{3d^2+4d+4}.$$

Then A admits an approximate singular value decomposition of index d and distortion $C_{\epsilon,d} = C_{\epsilon,d}(X, Y)$, defined in the following way:

- *there exist two $C_{\epsilon,d}$-orthogonal splittings $X = U \oplus V$, $Y = \tilde{U} \oplus \tilde{V}$,*
- *there exist $C_{\epsilon,d}$-Auerbach bases, $(e_1, \dots, e_d)$ of U and $(\phi_1, \dots, \phi_d)$ of $V^\perp$ dual to each over, such that $U = \mathrm{span}(e_1, \dots, e_d)$ and $V = \mathrm{span}(\phi_1, \dots, \phi_d)^{\perp\!\!\perp}$,*
- *there exist $C_{\epsilon,d}$-Auerbach bases, $(\tilde{e}_1, \dots, \tilde{e}_d)$ of $\tilde{U}$ and $(\tilde{\phi}_1, \dots, \tilde{\phi}_d)$ of $\tilde{V}^\perp$ dual to each over, such that $\tilde{U} = \mathrm{span}(\tilde{e}_1, \dots, \tilde{e}_d)$ and $\tilde{V} = \mathrm{span}(\tilde{\phi}_1, \dots, \tilde{\phi}_d)^{\perp\!\!\perp}$,*

satisfying the following properties, for every $i = 1, \dots, d$,

1. $AU = \tilde{U}$, $AV \subset \tilde{V}$, $A^*\tilde{V}^\perp = V^\perp$, $A^*\tilde{U}^\perp \subset U^\perp$, $\dim(U) = \dim(\tilde{U}) = d$,
2. $Ae_i = \sigma_i(A)\tilde{e}_i$, $A^*\tilde{\phi}_i = \sigma_i(A)\phi_i$,
3. $C_{\epsilon,d}^{-1}\sigma_i(A) \leqslant \sigma_i(A|U) \leqslant \sigma_i(A)$,
4. $C_{\epsilon,d}^{-1}\sigma_i(A) \leqslant \sigma_i(A^*|\tilde{V}^\perp) \leqslant \sigma_i(A)$,
5. $\sigma_{d+1}(A) \leqslant \|A|V\| \leqslant C_{\epsilon,d}\,\sigma_{d+1}(A)$
6. $\sigma_{d+1}(A) \leqslant \|A^*|\tilde{U}^\perp\| \leqslant C_{\epsilon,d}\,\sigma_{d+1}(A)$,
7. $\gamma(U,V),\ \gamma(V,U),\ \gamma(\tilde{U},\tilde{V}),\ \gamma(\tilde{V},\tilde{U}) \geqslant C_{\epsilon,d}^{-1}$.

If X is a Hilbert space, one may choose $C_{\epsilon,d} = 1+\epsilon$. If X, Y are of finite dimension, one may choose $\epsilon = 0$. If X, Y are Hilbert spaces of finite dimension, one may choose $V = U^\perp$, $\tilde{V} = \tilde{U}^\perp$, $C_{\epsilon,d} = 1$, $e_i = \phi_i$, $\tilde{e}_i = \tilde{\phi}_i$, $(e_1, \dots, e_d)$ and $(\tilde{e}_1, \dots, \tilde{e}_d)$ are orthonormal bases.

A.4 Exterior product

The algebraic exterior product $\bigwedge^d X$ is defined canonically of the following procedure. We first consider the space of almost null functions of $X^d \to \mathbb{R}$,

$$\mathcal{F} := \Big\{ \sum_{w \in X^d} \lambda_w \delta_w : \lambda_w \in \mathbb{R},\ \mathrm{card}\{w : \lambda_w \neq 0\} < +\infty \Big\}$$

where $\delta_w : X^d \to \mathbb{R}$ is the Dirac function at $w \in X^d$. We next consider the subspace $\mathcal{G}$ of $\mathcal{F}$ defined by

$$\begin{aligned}\mathcal{G} := \mathrm{span}\Big\{ &\delta_{(\lambda w_1 + \mu w_1', w_2, \dots, w_d)} - \lambda\delta_{(w_1, w_2, \dots, w_d)} - \mu\delta_{(w_1', w_2, \dots, w_d)}, \\ &\delta_{(w_1, \dots, w_{i-1}, w_i', w_{i+1}', w_{i+2}, \dots, x_d)} + \delta_{(w_1, \dots, w_{i-1}, w_{i+1}', w_i', w_{i+2}, \dots, w_d)} : \\ &\qquad 1 \leqslant i \leqslant d-1,\ w_1, \dots, w_d, w_1', \dots, w_d' \in X^d,\ \lambda, \mu \in \mathbb{R} \Big\}.\end{aligned}$$

The *algebraic exterior product* the vector space of equivalent classes

$$\bigwedge^d X := \mathcal{F}/\mathcal{G} = \{w + \mathcal{G} : w \in \mathcal{F}\}$$

We define the *canonical injection* $X^d \to \bigwedge^d X$ into the quotient space by

$$(w_1, \dots, w_d) \in X^d \mapsto w_1 \wedge \dots \wedge w_d := \delta_{(w_1,\dots,w_d)} + \mathcal{G} \in \bigwedge^d X$$

It is then easy to check that $\bigwedge^d X$ is spanned by *simple vectors*, vectors of the form $w_1 \wedge \dots \wedge w_d$. The canonical map $(w_1, \dots, w_d) \mapsto w_1 \wedge \dots \wedge w_d$ is multilinear alternating, and its image generates $\bigwedge^d X$. Moreover $\bigwedge^d X$ satisfies the universal property: every multilinear and alternating function $f : X^d \to Y$, where Y is any vector space, factorizes uniquely through a linear map $F : \bigwedge^d X \to Y$ by $F(w_1 \wedge \dots \wedge w_d) = f(w_1, \dots, w_d)$.

Several norms may be chosen for the exterior product. In the case where X is a Banach space, we choose the projective norm defined in the following way. Every $w \in \bigwedge^d X$ is a finite sum of vectors of the form $w_1^\alpha \wedge \dots \wedge w_d^\alpha$ where α is an index. As this representation is not unique, we introduce the *projective norm* of $\|w\|$ defined by

$$\|w\| := \inf \Big\{ \sum_\alpha \prod_{i=1}^d \|w_i^\alpha\| : w = \sum_\alpha w_1^\alpha \wedge \dots \wedge w_d^\alpha \Big\}. \tag{A.20}$$

It is easy to check that $\| \cdot \|$ is a genuine norm: $w \neq 0 \Rightarrow \|w\| \neq 0$. In the case X is a Hilbert space, we choose instead the *Euclidean norm* associated to the scalar product defined by extending by bilinearity to $\bigwedge^d X \times \bigwedge^d X$

$$\langle w_1 \wedge \dots \wedge w_d | w'_1 \wedge \dots \wedge w'_d \rangle := \det([\langle w_i | w'_j \rangle]_{1 \leqslant i,j \leqslant d}).$$

The projective norm and the Euclidean norm are not equal in general when X is a Hilbert space. We call the completion of the algebraic exterior product with respect to the chosen norm, the *normed exterior product*, and we denote it by $\bigwedge^d X$. We point out that $\bigwedge^d (X^*)$ denotes the normed exterior product of X^* and not the dual of $\bigwedge^d X$. If X is a Hilbert space, $X^* = X$ and $\bigwedge^d (X^*) = \bigwedge^d X = (\bigwedge^d X)^*$.

We define a *canonical duality* between $\bigwedge^d (X^*)$ and $\bigwedge^d X$ by extending by linearity for every $\theta_i \in X^*$ and $w_j \in X$,

$$\langle \theta_1 \wedge \dots \wedge \theta_d | w_1 \wedge \dots \wedge w_d \rangle := \det \big([\langle \theta_i | w_j \rangle]_{1 \leqslant i,j \leqslant d} \big). \tag{A.21}$$

We notice that the canonical linear map $\bigwedge^d (X^*) \to (\bigwedge^d X)^*$ is injective but may have a norm $\Sigma_d(X)$ greater than one (see A.34 for a bound from above of $\Sigma_d(X)$),

$$\begin{aligned} &\forall \theta \in \bigwedge^d (X^*),\ \forall w \in \bigwedge^d X, \quad |\langle \theta | w \rangle| \leqslant \Sigma_d(X) \|\theta\| \|w\|, \\ &\forall w_j \in X, \quad \sup_{\|\theta_i\|=1} \langle \textstyle\bigwedge_{i=1}^d \theta_i | \bigwedge_{j=1}^d w_j \rangle \geqslant \| \bigwedge_{j=1}^d w_j \|. \end{aligned} \tag{A.22}$$

In particular, for every Auerbach family $(u_1, \ldots, u_d)$ of X,

$$\Sigma_d(X)^{-1} \leqslant \|u_1 \wedge \ldots \wedge u_d\| \leqslant 1. \tag{A.23}$$

Let $(u_1, \ldots, u_d)$ be a linearly independent family of X, $U = \operatorname{span}(u_1, \ldots, u_d)$, and $1 \leqslant r \leqslant d$. For every sequence $I = (i_1, \ldots, i_r)$ of r ordered elements in $\{1, \ldots, d\}$, we denote $u_I := u_{i_1} \wedge \ldots \wedge u_{i_r}$. Then $\{u_I\}_I$ is a basis of $\bigwedge^r X$ spanning $\bigwedge^r U$. The following lemma gives an estimate on the volume distortion of this basis in $\bigwedge^r X$.

Lemma A.36. *Let X be a Banach space, $1 \leqslant r \leqslant d$, $(u_1, \ldots, u_d)$ be a C-Auerbach family of X dual to a C-Auerbach family $(\eta_1, \ldots, \eta_d)$ of X^*. Then $\{u_I\}_I$ and $\{\eta_I\}_I$ are a $C^r\Sigma_r(X)$-Auerbach families dual to each other of $\bigwedge^r X$ and $\bigwedge^r X^*$ respectively.*

Let $0 \leqslant r \leqslant d$. We denote by $(w, w') \in \bigwedge^r X \times \bigwedge^{d-r} X \mapsto w \wedge w' \in \bigwedge^d X$ the canonical bilinear map extending

$$(w_1 \wedge \ldots \wedge w_r) \wedge (w_{r+1} \wedge \ldots \wedge w_d) = w_1 \wedge \ldots \wedge w_d.$$

Lemma A.37. *If X is a Banach space and $\| \cdot \|$ is the projective norm, or if X is a Hilbert space and $\| \cdot \|$ is the Euclidean norm, then for every $0 \leqslant r \leqslant d$*

$$\forall w \in \textstyle\bigwedge^r X,\ \forall w' \in \bigwedge^{d-r} X, \quad \|w \wedge w'\| \leqslant \|w\| \|w'\|.$$

The following lemma extends the volume distortion estimate of lemma A.36.

Lemma A.38. *Let X be a Banach space, $d \geqslant 1$, $C \geqslant 1$, $X = U \oplus V$ be a C-orthogonal splitting of closed subspaces with $\dim(U) = d$. Let $(u_1, \ldots, u_d)$ and $(\eta_1, \ldots, \eta_d)$ be C-Auerbach bases dual to each other spanning U and $V^\perp$. Let $V' \subset V$ be a subspace of V of dimension $d' \geqslant 0$ and $X' := U \oplus V'$. Define*

$$K_d := \Sigma_d(X) \bar{\Delta}_{\binom{d+d'}{d}}(\textstyle\bigwedge^d X)^2 \displaystyle\max_{0 \leqslant r \leqslant d'} \Big(\Sigma_r(X) \bar{\Delta}_{\binom{d'}{r}}(\textstyle\bigwedge^r X)^2 \Big) \bar{\Delta}_2(X)^{8d} \bar{\Delta}_d(X)^{8d}.$$

Then every $w \in \bigwedge^d X'$ admits a unique decomposition $w = \sum_I u_I \wedge v_I$ where the summation is realized over every ordered sequence $I = (i_1, \ldots, i_r)$ of $\{1, \ldots, d\}$, $u_I = u_{i_1} \wedge \cdots \wedge u_{i_r}$, $v_I \in \bigwedge^{d-r} V'$ is any vector, and $0 \leqslant r \leqslant d$. Moreover

$$C^{-2d} K_d^{-1} \Big(\sum_I \|v_I\|^2 \Big)^{1/2} \leqslant \|w\| \leqslant C^{2d} K_d \Big(\sum_I \|v_I\|^2 \Big)^{1/2}.$$

Non-zero simple vectors in $\bigwedge^d X$ are in one-to-one correspondence with subspaces of X of dimension d. We introduce the following notations to clarify this correspondence.

Definition A.39. Let X be a vector space and $d \geqslant 1$.

1. If U is a subspace of X of dimension d, we call

$$\widehat{U} := \operatorname{span}\{\textstyle\bigwedge_{i=1}^{d} w_i : \forall\, i,\ w_i \in U\} \subset \bigwedge^d X.$$

2. If V is a subspace of codimension d, we call

$$\check{V} := \operatorname{span}\{\textstyle\bigwedge_{i=1}^{d} w_i : \exists\, i,\ w_i \in V,\ \forall i,\ w_i \in X\} \subset \bigwedge^d X.$$

Then $\dim(\widehat{U}) = 1$ and $\operatorname{codim}(\check{V}) = 1$.

If $X = U \oplus V$ with $\dim(U) = d$, then $\bigwedge^d X = \widehat{U} \oplus \check{V}$. If $(\eta_1, \ldots, \eta_d)$ are linearly independent and $V = \operatorname{span}(\eta_1, \ldots, \eta_d)^{\perp\!\!\perp}$, then $\check{V}$ is the kernel of a simple linear form of $\bigwedge^d X$,

$$\check{V} = \{w \in \textstyle\bigwedge^d X : \langle \eta_1 \wedge \ldots \wedge \eta_d | w\rangle = 0\} = \operatorname{span}(\bigwedge_{i=1}^{d} \eta_i)^{\perp\!\!\perp}.$$

The following lemma compares the angle between U and V and the angle between $\widehat{U}$ and $\check{V}$. Using equation (A.15), we also obtain a comparison between $\|\pi_{U|V}\|$ and $\|\pi_{\widehat{U}|\check{V}}\|$, (see (A.4) for the definition of $\pi_{U|V}$).

Lemma A.40. *Let X be a Banach space, $d \geqslant 1$, $X = U \oplus V$ be a splitting of closed subspaces with* $\dim(U) = d$ *and $K_d := \bar{\Delta}_2(X)^4 \bar{\Delta}_d(X)^3$. Then $\bigwedge^d X = \widehat{U} \oplus \check{V}$ and*

$$K_d^{-d} \gamma(\widehat{U}, \check{V}) \leqslant \gamma(U, V) \leqslant K_d \gamma(\widehat{U}, \check{V})^{1/d},$$
$$K_d^{-1} \|\pi_{\widehat{U}|\check{V}}\|^{1/d} \leqslant \|\pi_{U|V}\| \leqslant K_d^d \|\pi_{\widehat{U}|\check{V}}\|.$$

In the case the splitting $X = U \oplus V$ is C-orthogonal, using lemma A.9, the norm of the two projectors admits a simpler estimate.

Lemma A.41. *Let X be a Banach space, $d \geqslant 1$, $C \geqslant 1$, $X = U \oplus V$ be a C-orthogonal splitting with* $\dim U = d$ *and $K_d := \bar{\Delta}_2(\bigwedge^d X)^3$. Then*

$$\|\pi_{\widehat{U}|\check{V}}\| \leqslant \Sigma_d(X) C^{2d}, \ \text{ and } \ \|\pi_{\check{V}|\widehat{U}}\| \leqslant \Sigma_d(X) K_d C^{2d}.$$

Angles between subspaces can also be measured by the norm of some graphs over a reference splitting as in lemma A.24. Consider a splitting $X = U_0 \oplus V_0$ with $\dim(U_0) = d$ and a subspace $V \in \mathcal{N}(U_0)$. Then $V = \operatorname{Graph}(\Theta)$ for some operator $\Theta \in \mathcal{B}(V_0, U_0)$ or equivalently, as explained in lemma A.23, $V^\perp = \operatorname{Graph}(\Theta^\perp)$ for some $\Theta^\perp \in \mathcal{B}(V_0^\perp, U_0^\perp)$. Lemma A.40 implies

$$\textstyle\bigwedge^d X = \widehat{U}_0 \oplus \check{V}_0 = \widehat{U}_0 \oplus \check{V},$$

and in particular $\check{V} \in \mathcal{N}(\hat{U}_0)$ is equal to the graph of some $\widehat{\Theta} \in \mathcal{B}(\check{V}_0, \hat{U}_0)$. The following lemma gives an estimate of $\|\widehat{\Theta}\|$ with respect to $\|\Theta^\perp\|$.

Lemma A.42. *Let X be a Banach space, $d \geqslant 1$, $C \geqslant 1$, and $X = U_0 \oplus V_0$ be a C-orthogonal splitting of closed subspaces with $\dim(U_0) = d$. Let $(u_1, \ldots, u_d)$ and $(\eta_1, \ldots, \eta_d)$ be C-Auerbach families in X and X^* respectively, dual to each over, such that $U_0 = \mathrm{span}(u_1, \ldots, u_d)$ and $V_0 = \mathrm{span}(\eta_1, \ldots, \eta_d)^{\perp\!\!\perp}$.*

Let $\Theta^\perp \in \mathcal{B}(V_0^\perp, U_0^\perp)$ and $V = \mathrm{Graph}(\Theta^\perp)^{\perp\!\!\perp}$. Then

- $\check{V} = \mathrm{span}(\bigwedge_{i=1}^d (\mathrm{Id} \oplus \Theta^\perp)\eta_i)^{\perp\!\!\perp} = \mathrm{Graph}(\widehat{\Theta})$ *for some* $\widehat{\Theta} \in \mathcal{B}(\check{V}_0, \widehat{U}_0)$,
- $\forall w \in \check{V}_0, \quad \widehat{\Theta}(w) = -\langle \bigwedge_{i=1}^d (\eta_i + \Theta^\perp \eta_i) | w \rangle \bigwedge_{i=1}^d u_i$,
- $\|\widehat{\Theta}\| \leqslant C^{2d} \Sigma_d(X) \|\Theta^\perp\| (1 + \|\Theta^\perp\|)^{d-1}$.

The next theorem shows that the approximate singular value decomposition of index d of a bounded operator $A \in \mathcal{B}(X, Y)$ admits a particular form when the operator is considered in the exterior product. Let

$$\widehat{A} := \bigwedge^d A \in \mathcal{B}(\bigwedge^d X, \bigwedge^d Y).$$

Theorem A.43. *Let X, Y be Banach spaces, $d \geqslant 1$, $\epsilon > 0$, and $A \in \mathcal{B}(X, Y)$ satisfying $\sigma_d(A) > 0$. Let $X = U \oplus V$ and $Y = \tilde{U} \oplus \tilde{V}$, be the approximate singular value decomposition of index d and distortion $C_{\epsilon,d}$ given in theorem A.35. Let*

$$\widehat{C}_{\epsilon,d} := C_{\epsilon,d}^{17d} \Sigma_d(X) (\bar{\Delta}_{\binom{2d}{d}}(\bigwedge^d X))^2 \max_{0 \leqslant r \leqslant d} \Big(\Sigma_r(X) (\bar{\Delta}_{\binom{d}{r}}(\bigwedge^r X))^2 \Big) \bar{\Delta}_2(X)^{24d} \bar{\Delta}_d(X)^{28d}.$$

Then

1. *$(\bigwedge_{i=1}^d e_i)$ and $(\bigwedge_{i=1}^d \phi_i)$ are $\widehat{C}_{\epsilon,d}$-orthogonal bases dual to each over,*
$$\widehat{U} = \mathrm{span}(\bigwedge_{i=1}^d e_i), \quad \check{V} = \mathrm{span}(\bigwedge_{i=1}^d \phi_i)^{\perp\!\!\perp},$$
2. *$(\bigwedge_{i=1}^d \tilde{e}_i)$ and $(\bigwedge_{i=1}^d \tilde{\phi}_i)$ are $\widehat{C}_{\epsilon,d}$-orthogonal bases dual to each over,*
$$\widehat{\tilde{U}} = \mathrm{span}(\bigwedge_{i=1}^d \tilde{e}_i), \quad \check{\tilde{V}} = \mathrm{span}(\bigwedge_{i=1}^d \tilde{\phi}_i)^{\perp\!\!\perp},$$
3. *$\bigwedge^d X = \widehat{U} \oplus \check{V}$, $\bigwedge^d Y = \widehat{\tilde{U}} \oplus \check{\tilde{V}}$, $\dim(\widehat{U}) = \dim(\widehat{\tilde{U}}) = 1$,*
4. *$\widehat{A}\widehat{U} = \widehat{\tilde{U}}$, $\widehat{A}\check{V} \subset \check{\tilde{V}}$, $\widehat{A}^* \check{\tilde{V}}^\perp = \check{V}^\perp$, $\widehat{A}^* \widehat{\tilde{U}}^\perp \subset \widehat{U}^\perp$,*
5. *$\widehat{C}_{\epsilon,d}^{-1} \prod_{i=1}^d \sigma_i(A) \leqslant \|\widehat{A}|\widehat{U}\| \leqslant \|\widehat{A}\| \leqslant \widehat{C}_{\epsilon,d} \prod_{i=1}^d \sigma_i(A)$,*
6. *$\widehat{C}_{\epsilon,d}^{-1} \prod_{i=1}^d \sigma_i(A) \leqslant \|\widehat{A}^*|\check{\tilde{V}}^\perp\| \leqslant \|\widehat{A}^*\| \leqslant \widehat{C}_{\epsilon,d} \prod_{i=1}^d \sigma_i(A)$,*

7. $\sigma_2(\widehat{A}) \leqslant \|\widehat{A}|\check{V}\| \leqslant \widehat{C}_{\epsilon,d}\,\sigma_1(A)\ \cdots\ \sigma_{d-1}(A)\sigma_{d+1}(A)$,

8. $\sigma_2(\widehat{A}) \leqslant \|\widehat{A}^*|\check{\widetilde{U}}^{\perp}\| \leqslant \widehat{C}_{\epsilon,d}\,\sigma_1(A)\ \cdots\ \sigma_{d-1}(A)\sigma_{d+1}(A)$,

9. $\gamma(\widehat{U},\check{V}) \geqslant \widehat{C}_{\epsilon,d}^{-1},\ \gamma(\check{V},\widehat{U}) \geqslant \widehat{C}_{\epsilon,d}^{-1}$.

In the following lemma we consider a product BA of two operators and the relative position of the approximate singular value decomposition of A and B.

Lemma A.44. *Let X, Y, Z be three Banach spaces, $A \in \mathcal{B}(X,Y)$, $B \in \mathcal{B}(Y,Z)$, $d \geqslant 1$, and $\epsilon > 0$. Assume $\sigma_d(A) > 0$ and $\sigma_d(B) > 0$. Let*

$$\bigwedge^d X = \widehat{U}_A \oplus \check{V}_A, \quad \bigwedge^d Y = \widehat{\widetilde{U}}_A \oplus \check{\widetilde{V}}_A = \widehat{U}_B \oplus \check{V}_B, \quad \bigwedge^d Z = \widehat{\widetilde{U}}_B \oplus \check{\widetilde{V}}_B,$$

be the two approximate singular value decompositions of index 1 and distortion $\widehat{C}_{\epsilon,d}$ of $\widehat{A}$ and $\widehat{B}$ obtained in theorem A.43. Then

$$\prod_{i=1}^{d} \frac{\sigma_i(BA)}{\sigma_i(A)\sigma_i(B)} \geqslant \widehat{C}_{\epsilon,d}^{-3}\,\gamma(\widehat{\widetilde{U}}_A, \check{V}_B).$$

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