FLEXIBILITY OF THE PRESSURE FUNCTION

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Abstract. We study the flexibility of the pressure function of a continuous potential (observable) with respect to a parameter regarded as the inverse temperature. The points of non-differentiability of this function are of particular interest in statistical physics, since they correspond to phase transitions. It is well known that the pressure function is convex, Lipschitz, and has an asymptote at infinity. We prove that in a setting of one-dimensional compact symbolic systems these are the only restrictions. We present a method to explicitly construct a continuous potential whose pressure function coincides with *any* prescribed convex Lipschitz asymptotically linear function starting at a given positive value of the parameter. In fact, we establish a multidimensional version of this result. As a consequence, we obtain that for a continuous observable the phase transitions can occur at a countable dense set of temperature values. We go further and show that one can vary the cardinality of the set of ergodic equilibrium states as a function of the parameter to be any number, finite or infinite.

1. Overview

Katok launched the flexibility program which has been described in a nutshell as follows: "there should be no restrictions on the dynamical characteristics apart from a few obvious ones". Hence, the flexibility program is geared towards an understanding of the most general constraints which define a common class of dynamical systems and the building of tools to readily change all other dynamical specifications within those constraints. This is a novel direction in dynamics which has been explicitly stated in [10]. At the same time however, the core problems are clear and accessible to a rather broad community of mathematicians working within the area and this has made the program develop at a rapid pace. Although Katok originally formulated the program for smooth dynamical systems, his perception is highly

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relevant for general topological dynamical systems on compact spaces. In this note we apply it to the topological pressure functional in the class of compact symbolic systems.

Within the last few years there has been a great deal of activity around Katok's ideas of flexibility. We briefly describe some of these works. For conservative Anosov flows on three-dimensional manifolds the basic flexibility problem involves realization of arbitrary pairs of numbers as values for topological and metric entropy subject only to the variational inequality. In [24] the authors consider smooth closed Riemannian surfaces of negative curvature and show that all the possible values for the topological and metric (with respect to the Liouville measure) entropies of the geodesic flow are realized within this class. In situations where the relevant invariant measure varies with the dynamics, one may be interested in the values of Lyapunov exponents. The flexibility of Lyapunov exponents was proven for expanding maps on a circle [22] and for Anosov area-preserving diffeomorphisms on tori [23]. Subsequently, the fundamental paper [10] outlines the program and provides flexibility results for volume-preserving systems with respect to the volume measure. These results have already been improved and extended in [15].

There are also applications of the flexibility paradigm in settings other than smooth flows on manifolds. The class of piecewise expanding unimodal maps is considered in [3]. The authors show that the only restrictions for the values of the topological and metric entropies in this class are that both are positive and the topological entropy is at most log 2. In [2] some maps arising in the study of Fuchsian groups are analyzed and it is proven that all possible values of the entropy are attained. Lastly, flexibility results are established for the values of polynomial slow entropy for rigid transformations [6] and homeomorphisms on a continuum [47].

The path to obtaining the full range of allowable parameters opened by Katok and then followed by others consists of starting from a map whose dynamics is well understood and studying what happens under perturbations. The main challenge of this approach is that the values of dynamical invariants can be precisely calculated in only a handful of cases. Moreover, there are not many methods available to perturb a system in a controlled manner.

Establishing flexibility calls for versatile constructions in large families to cover all possible values of dynamical quantities. This is precisely the route we take to gain total control over one of the most important objects in thermodynamic formalism. Our work asserts flexibility of a whole pressure function, rather than of finite number of values for

Lyapunov exponents or topological and metric entropies. In contrast to the perturbation methods described above, we build a dynamical system with the desired properties from the ground up. We remark that the pressure function can, in turn, be applied to obtain information about Lyapunov exponents, dimension, multifractal spectra, or natural invariant measures. We refer to [5, 43, 44, 48] for details and further references.

Our setting is one-dimensional compact symbolic systems. One of the many reasons that symbolic systems are important is that they serve as proxies for smooth systems. In many occasions it is more straightforward to identify properties of symbolic systems, which then can be transferred to smooth systems. We investigate the possible behavior of the topological pressure restricted to a linear span of a fixed finite set of continuous potentials. The pressure is then viewed as a function of the coefficients in the linear combinations of the potentials. Such multivariable pressure functions play a fundamental role in multifractal analysis, which studies level sets of asymptotically defined quantities such as Birkhoff averages and local entropies. A nice overview of the theory can be found in [9]. The pressure function is used as the main tool to compute the dimension spectra of the simultaneous level sets, see e.g. [8] and [17].

To be precise, let $\phi : X \to \mathbb{R}$ be a continuous potential associated with a symbolic dynamical system (X, σ) over a finite alphabet. The *topological pressure* of ϕ can be defined via the Variational Principle by

$$P_{\rm top}(\phi) = \sup \left\{ h(\mu) + \int \phi \, d\mu \right\} \tag{1}$$

where the supremum is taken over the set of all σ-invariant probability measures on X and $h(\mu)$ denotes the measure-theoretic entropy of the measure μ. The measures which realize the above supremum are called the *equilibrium states* of ϕ. Classical manuscripts about the pressure and equilibrium states are [12, 48, 54].

Fix m continuous potentials $\phi_1, ..., \phi_m$. For $(t_1, ...t_m) \in \mathbb{R}^m$ the *multivariable pressure function* is the map

$$(t_1, ..., t_m) \mapsto P_{\rm top}(t_1\phi_1 + ... + t_m\phi_m).$$

We now describe a few basic properties of this map. It is an immediate consequence of the Variational Principle that the pressure function is Lipschitz and convex. The defining characteristic of a convex function on $\mathbb{R}^m$ is that it has a supporting hyperplane at each point of its graph. It follows from the description of the equilibrium states as tangent functionals to the pressure given by Walters [55] that each such hyperplane arises as the graph of a function $(t_1, \ldots, t_m) \mapsto h(\mu) + \int (t_1\phi_1 + \ldots + t_m\phi_m)\, d\mu$ for an equilibrium state μ. The *vertical intercept* (i.e. the value of the function evaluated at $(0, \ldots, 0)$) of such a hyperplane is $h(\mu)$. The entropies of all invariant probability measures are bounded above by the topological entropy of the system (X, σ). Hence, if a real valued function of m variables is a pressure function then it is convex, Lipschitz, and the vertical intercepts of its supporting hyperplanes form a bounded set of nonnegative numbers. We prove that these conditions are *necessary and sufficient*.

Theorem 1. *Let $\alpha > 0$ and let $F(t_1, \ldots, t_m)$ be a convex Lipschitz function on $(\alpha, \infty)^m$ such that all the supporting hyperplanes to the graph of F intersect the vertical axis in a closed interval $[b, c] \subset [0, \infty)$. Then there exists a full shift on a finite alphabet and continuous potentials $\phi_1, \ldots, \phi_m$ such that $P_{\text{top}}(t_1\phi_1 + \ldots + t_m\phi_m) = F(t_1, \ldots, t_m)$ for all $(t_1, \ldots, t_m) \in (\alpha, \infty)^m$.*

Our proof is explicit and constructive. For an arbitrary function F satisfying these properties we build a set of m continuous potentials whose pressure function coincides with F. Theorem 1 falls in line with Katok's flexibility program. We identify the general constraints on the pressure function and provide a tool to acquire any pressure function within those constraints.

In the case when X is a transitive subshift of finite type and the potentials $\phi_1, ..., \phi_m$ are Hölder the pressure function $P_{\text{top}}(t_1\phi_1 + ... + t_m\phi_m)$ is analytic. This fact goes back to the results of Bowen [12] and Ruelle [48, 49]. Starting with an analytic function $F(t_1, ..., t_m)$ we obtain from Theorem 1 a set of continuous potentials for which the pressure function coincides with F. However, our potentials are not Hölder. This raises an interesting question of whether an analog of Theorem 1 holds in the case when the potentials are required to be Hölder, i.e. whether any analytic convex function is a pressure function for a set of Hölder continuous potentials.

We briefly outline the ideas which go into the proof of Theorem 1. For simplicity, we consider a one-parameter pressure function here, i.e. $m = 1$. We start with a convex Lipschitz function $F : (\alpha, \infty) \to \mathbb{R}$. By convexity, for each point on the graph of F there is at least one supporting line and, by our assumption, it intercepts the vertical axis in the interval $[b, c] \subset [0, \infty)$. Figure 1 below illustrates the setup. As was mentioned before, a supporting line to the pressure function $P(t\phi)$ at t must have vertical intercept $h(\mu_t)$ and slope $\int \phi\, d\mu_t$, where μ_t is one of the equilibrium states of the potential $t\phi$. Our goal is to construct ϕ

such that $F(t) = P(t\phi)$. The general idea is that the equilibrium states of $t\phi$ move among a sequence of disjointly supported subshifts when t changes. Hence, we need to find a family of subshifts whose entropies fill up the whole interval $[b, c]$. Good candidates for this purpose are the β-shifts.

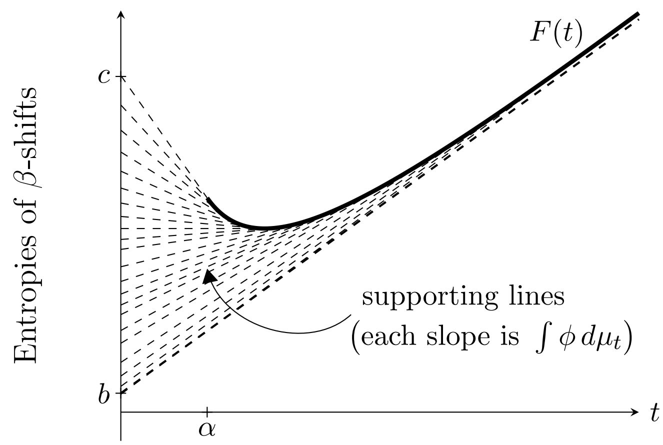

FIGURE 1. This figure illustrates the proof of Theorem 1.

The origin of β-shifts lies in the study of expansions of real numbers in an arbitrary real base $\beta > 1$, which were introduced by Renyi [46]. Roughly speaking, the β-shift X_β consists of the sequences of the coefficients in the β expansions of reals in $[0, 1)$. The measure-theoretic properties of β-shifts and their connection to these expansions were initially studied in [42, 33, 30]. It was shown that $\{X_\beta; \beta > 1\}$ is an increasing family of shift-invariant closed sets with $h_{\text{top}}(X_\beta) = \log \beta$ and X_β has a unique measure of maximal entropy.

The entropies of β-shifts have the properties we need for our construction. The next step would be to define the potential ϕ on each X_β as the constant equal to the slope of the corresponding supporting line. There is an obstacle, however: our subshifts X_β are nested. We avoid it by introducing an additional "dimension" in the following way. We take a product of each β-shift with a suitably chosen Sturmian shift. Sturmian shifts are low complexity systems with a variety of combinatorial properties useful for our analysis. They have been studied since the birth of symbolic dynamics [41], but modern interest was sparked by numerous applications in computer science [13, 16, 28, 51].

The low complexity of Sturmian shifts ensures that they do not contribute to the entropy of the product. We let β run from e^b to e^c and obtain a set of disjoint subshifts of an appropriate full shift which are products of β-shifts with Sturmian shifts and whose topological entropies fill the interval $[b, c]$ (recall Figure 1). We are in a position to define the potential ϕ on each such subshift to be the slope of the supporting line to $F(t)$ which crosses the vertical axis at $\log \beta$.

After we make this careful arrangement on all the subshifts, we still need to take care of all the other points in our full shift. The idea is to make ϕ drop off sharply and force the equilibrium measures to be supported on the products of β-shifts and Sturmian shifts. This is the most challenging part of the proof. We accomplish this by using a pin-sequence technique introduced in [4] to measurably split orbits of symbolic dynamical systems into finite segments.

Our results have implications for occurrences of phase transitions. A phase transition is observed when one follows an evolution of a system depending on a continuous external parameter and a sharp change of the behaviour of the system happens. Understanding the mechanism of this phenomenon is a fundamental goal in statistical physics. To achieve this, simplified mathematical models were proposed, the most well known one being the Ising model [34, 21, 25, 50], leading to the development of thermodynamic formalism. In this setting, the quantity $h(\mu) + \int \phi \, d\mu$ represents the negative free energy of the system in the state μ with respect to the observable ϕ. Hence, the pressure of ϕ is the minimum of free energies and the equilibrium states of ϕ characterize the equilibria of the system. The existence of more than one equilibrium state corresponds to a phase transition.

One way to change the equilibrium state of the system is by adding heat. The measure of temperature in thermodynamics is the absolute temperature T, which is always a positive number with the limit $T \to 0^+$ being absolute zero. Hence, a positive parameter $t = 1/T$ (the inverse temperature of the system) is introduced and one studies how the equilibrium states of $t\phi$ change with t, identifying the values of t for which the potential $t\phi$ has more than one equilibrium state. A classical result by Walters [55] is that non-differentiability of the pressure function $t \to P(t\phi)$ at t_0 is equivalent to the potential $t_0\phi$ having two equilibrium states with distinct entropies. Such points of non-differentiability are called *first-order* phase transitions. Points where the pressure function is differentiable, but not analytic, are termed *higher-order* phase transitions. Although non-uniqueness of equilibrium states may not appear at such points, they still indicate a sharp change in some property of the system.

For symbolic systems the first systematic study of a family of potentials exhibiting a phase transition at some value t_0 was done by Lopes in [37, 38, 39] building upon the previous work of Hofbauer [29]. Consequently, phase transitions in thermodynamic formalism were examined using various approaches, see e.g. [7, 14, 18, 20, 33, 36]. We note that the main result of [36] deals with the shape of the pressure function after the transitional value: an example is built on a subshift of finite type where a phase transition occurs but the pressure is strictly convex. Prior to that paper, in all known examples with phase transitions, the pressure function was either flat after the transition, or there was at least some interval where the pressure was flat [33].

Another aspect concerns the number and frequency of phase transitions. In [35] we construct a continuous potential on $\{0,1\}^{\mathbb{Z}}$ whose first-order phase transitions occur at any given increasing sequence. Up to that point there were no examples in the literature of more than two phase transitions in the compact symbolic setting. Note that the convexity of the pressure implies that at most countably many points of non-differentiability are possible. Although we see from [35] that the case of infinitely many such points can indeed be realized for the pressure, the requirement for them to form an increasing sequence is actually quite restrictive. A convex function, in general, may have a dense set of points where its derivative does not exist. The question remained whether similar behavior is feasible for the pressure function. As a consequence of our main result we see that the answer is yes. We provide a method of obtaining continuous potentials whose phase transitions form any given countable set. In addition, the pressure function between the phase transitions can be made strictly convex, whereas the pressure in [35] is piecewise linear.

Finally we turn our attention to the type of phase transitions where the pressure function is analytic, but uniqueness of equilibrium states fails. The first example of a transitive system for which two equilibria co-exist despite the analyticity of the pressure was given in 2015 by Leplaideur [36]. His work made it clear that the hope for high regularity of the pressure function to ensure uniqueness of the equilibrium state was unfounded. We show that regularity of the pressure does not impose any limitations on the behavior of the equilibria of the system. At any smooth point of the pressure function the corresponding potential may have any number of ergodic equilibrium states, finite or infinite. Moreover, the cardinality of equilibrium states may change drastically when the values of the parameter change. The next theorem provides a flexible way of constructing systems of potentials with varying cardinalities of the equilibrium measures.

Theorem 2. *Let $f(t)$ be a strictly convex differentiable function on (α,∞) with support line intercepts lying in a bounded interval $[b,c] \subset [0,\infty)$. Then for any $\ell \in \mathbb{N}$ and any upper semi-continuous function $N\colon (\alpha,\infty) \to \{1,\dots,\ell,\infty\}$, there exists a full shift (X,σ) and a potential function ϕ such that*

- $P(t\phi) = f(t)$ *for all* $t \in (\alpha,\infty)$*;*
- *the cardinality of the set of ergodic equilibrium states for* $t\phi$ *is exactly* $N(t)$*.*

This result contrasts sharply with the case of Hölder potentials, where the pressure function is analytic and the equilibrium state is always unique. It also immediately provides examples of the types found in [33, 35, 36].

The paper is organized as follows. In Sections 2, 3, and 4 we introduce the terminology and prove preliminary lemmas concerning convex functions, beta-shifts and Sturmian shifts respectively. Section 5 is devoted to the proof of Theorem 1. In Section 6 we examine the one-parameter pressure function and establish a slight strengthening of Theorem 1 in this case. Also, here we supply a procedure for building potentials with a given countable set of first-order phase transitions. Lastly, Section 7 contains a discussion on the cardinality of equilibrium states and the proof of Theorem 2.

2. Convex analysis

Suppose $F : (\alpha,\infty)^m \to \mathbb{R}$ is a convex function of m variables. A vector $\mathbf{v} \in \mathbb{R}^m$ is a *subgradient* of F at $\boldsymbol{s} \in (\alpha,\infty)^m$ if for all $\boldsymbol{t} \in (\alpha,\infty)^m$ we have

$$F(\boldsymbol{t}) \geq F(\boldsymbol{s}) + \mathbf{v}\cdot(\boldsymbol{t}-\boldsymbol{s}).$$

Hence, $\mathbf{v}$ is a subgradient of F at $\boldsymbol{s}$ if the affine function $G(\boldsymbol{t}) = F(\boldsymbol{s}) + \mathbf{v}\cdot(\boldsymbol{t}-\boldsymbol{s})$ is a global underestimator of F. The graph of G is a hyperplane in $\mathbb{R}^{m+1}$ which is called a *supporting hyperplane of F at $\boldsymbol{s}$*. We refer to $G(0)$ as the *vertical axis intercept* of the hyperplane, so that the intercept is at $F(\boldsymbol{s}) - \mathbf{v}\cdot\boldsymbol{s}$. Under the assumptions of Theorem 1 this intercept must lie in the interval $[b,c]$. The set of all subgradients of F at $\boldsymbol{s}$ is called the *subdifferential* of F at $\boldsymbol{s}$ and is denoted by $\partial F(\boldsymbol{s})$. Since F is convex, for any $\boldsymbol{s} \in (\alpha,\infty)^m$ the set $\partial F(\boldsymbol{s})$ is nonempty, closed and convex. Moreover, $\partial F(\boldsymbol{s})$ is a singleton if and only if F is differentiable at $\boldsymbol{s}$.

Let F be as in Theorem 1. Denote by L the Lipschitz constant of F. We define

$$S = \mathrm{Cl}\left(\bigcup_{\boldsymbol{s}\in(\alpha,\infty)^m} \{(F(\boldsymbol{s}) - \mathbf{v}\cdot\boldsymbol{s}, \mathbf{v}) : \mathbf{v} \in \partial F(\boldsymbol{s})\} \right). \tag{2}$$

Then S is a bounded subset of $\mathbb{R}^{m+1}$. In fact, $S \subset [b, c] \times [-L, L]^m$.

Lemma 3. *Let F be as in Theorem 1 and let the set S be as defined above. For each $\boldsymbol{t} \in (\alpha, \infty)^m$,*

$$F(\boldsymbol{t}) = \sup_{(h,\mathbf{v})\in S} (h + \mathbf{v}\cdot\boldsymbol{t}).$$

Proof. If $\boldsymbol{s} \in (\alpha, \infty)^m$ and $\mathbf{v} \in \partial F(\boldsymbol{s})$ then for any $\boldsymbol{t} \in (\alpha, \infty)^m$, $F(\boldsymbol{t}) \geq h + \mathbf{v}\cdot\boldsymbol{t}$ where $h = F(\boldsymbol{s}) - \mathbf{v}\cdot\boldsymbol{s}$. By continuity, the same inequality holds for any $(h, \mathbf{v}) \in S$, so that

$$F(\boldsymbol{t}) \geq \sup_{(h,\mathbf{v})\in S} (h + \mathbf{v}\cdot\boldsymbol{t}).$$

Conversely, given $\boldsymbol{t} \in (\alpha, \infty)^m$, let $\mathbf{v} \in \partial F(\boldsymbol{t})$ and $h = F(\boldsymbol{t}) - \mathbf{v}\cdot\boldsymbol{t}$ so that $(h, \mathbf{v}) \in S$. Now

$$F(\boldsymbol{t}) = h + \mathbf{v}\cdot\boldsymbol{t} \leq \sup_{(h,\mathbf{v})\in \mathbf{S}} (h + \mathbf{v}\cdot\boldsymbol{t}),$$

establishing the reverse inequality. □

3. Beta-shifts

The β-shifts, which emerged from the notion of base β representation of real numbers [46], were first systematically studied as dynamical systems by Parry in [42]. For a fixed $\beta > 1$ every real $r \in [0, 1]$ has a β-expansion

$$r = \sum_{n=1}^{\infty} r_n \beta^{-n},$$

where r_n are from the set $\{0, 1, ..., \lfloor\beta\rfloor\}$. Here, and throughout the text, $\lfloor.\rfloor$ denotes the floor function, i.e. $\lfloor\beta\rfloor$ is the largest integer not exceeding β. The coefficients r_n of the β-expansion of r are defined using the *β-transformation* $T_\beta(r) = \beta r \pmod 1$; $r_n = \lfloor\beta T_\beta^{n-1}(r)\rfloor$.

Consider the set of all sequences of the coefficients in β-expansions of real numbers in $[0, 1)$. In the case where β is an integer, our convention is to include the point $(\beta 000\ldots)$. The *β-shift* X_β is defined to be the closure of the extension of this set to two sided sequences. Hence, X_β is a subshift of $\{0, ..., \lfloor\beta\rfloor\}^{\mathbb{Z}}$ with shift map σ. Renyi [46] gave a description of X_β in terms of the β-expansion of 1. Precisely, we define the *maximal* word w^β by $w_n^\beta = \lfloor \beta T_\beta^{n-1}(1) \rfloor$. The sequence $(r_n)_{n=1}^\infty$ corresponds to a β-expansion of some $r \in [0,1)$ if and only if for all $j \in \mathbb{N}$ the word $\sigma^j(r_1 r_2 ...)$ is smaller than w^β according to the lexicographical order.

It is well known that the topological entropy of X_β is $\log \beta$ (the proof can be found in [46, 33, 53]). In addition, results of Hofbauer [31] and Walters [53] show that β-shifts are intrinsically ergodic. The unique measure of maximal entropy of X_β is weak-mixing [42] and Bernoulli [52].

We need some facts about the language of a β-shift. As usual, let $\mathcal{L}_n(X_\beta)$ denote the set of words of length n forming sub-words of elements of X_β and let $\mathcal{L}(X_\beta) = \bigcup_n \mathcal{L}_n(X_\beta)$ be the *language* of X_β. Some of the calculations in the next lemma may also be found in Walters' book [54, page 178].

Lemma 4. *Let $\beta > 1$ and let X_β denote the β-shift. Then*

$$\beta^n \le |\mathcal{L}_n(X_\beta)| \le \frac{\beta}{\beta - 1}\beta^n.$$

Proof. Fix $\beta > 1$ and let $N_n = |\mathcal{L}_n(X_\beta)|$. Since X_β has entropy $\log \beta$, sub-multiplicativity of N_n implies that $N_n \ge \beta^n$.

For the opposite inequality let w^β be the maximal word for X_β. It follows from the description of X_β above that an arbitrary element of $\mathcal{L}(X_\beta)$ is a concatenation of *sub-prefixes* of w^β, where a sub-prefix is a word u of some length k such that $u_i = w_i^\beta$ for $i = 1, \dots, k-1$ and $u_k < w_k^\beta$ followed by a (possibly empty) initial segment of w^β. In particular, an element of $\mathcal{L}_n(X_\beta)$ is either the length n prefix of w^β, or it is a sub-prefix of some length $j < n$ followed by an arbitrary element of $\mathcal{L}_{n-j}(X_\beta)$. Finally, we observe that there are w_j^β sub-prefixes of w^β of length j. Hence we see

$$N_n = 1 + \sum_{j=1}^{n} w_j^\beta N_{n-j},$$

where N_0 is taken to be 1. Write $p_n = w_n^\beta / \beta^n$ (so that the p_n's sum to 1); and $m_n = \max_{j \le n} N_j / \beta^j$. Dividing the above equation through by β^n, we obtain

$$m_n \le \beta^{-n} + \sum_{j=1}^{n} p_j m_{n-1} \le \beta^{-n} + m_{n-1}.$$

Therefore, $m_n \le \sum_{j=0}^{\infty} \beta^{-j} = \frac{\beta}{\beta-1}$, so that $N_n \le \frac{\beta}{\beta-1}\beta^n$ as required. $\square$

The following strengthening of the nesting property of β-shifts is one of the ingredients in the proof of the main theorem. Although it can be found in the literature (see e.g. [33]), we give a short proof here for the sake of completeness.

Lemma 5. *Let $\beta > 1$. Then*

$$\bigcap_{\beta' > \beta} X_{\beta'} = X_\beta.$$

Proof. Let β and n be fixed. By definition of w^β, $T_\beta^j(1) < (w_j^\beta + 1)/\beta$ for $j = 0, \dots, n-1$. For $\beta' > \beta$, it is straightforward to see $T_{\beta'}^j(1) > T_\beta^j(1)$ provided $T_{\beta'}^i(1) < (w_i^\beta + 1)/\beta'$ for $i = 0, \dots, j-1$. Since the condition $T_{\beta'}^j(1) < (w_j^\beta + 1)/\beta'$ is satisfied on a small interval to the right of β, we see that for any $n \in \mathbb{N}$, there exists an interval $[\beta, \beta + \delta_n)$ on which $w_j^\beta = w_j^{\beta'}$ for $j = 0, \dots, n-1$. The conclusion follows. □

4. Sturmian shifts

Sturmian shifts were introduced by Morse and Hedlund in [41] as symbolic coding of geodesic trajectories on a flat torus. This makes them one of the earliest general classes of shift spaces studied in dynamics. The most interesting property of these shifts is, undoubtedly, their low complexity. A non-periodic Sturmian shift not only has zero entropy, it has the smallest growth rate of blocks possible for infinite shift spaces [19]. In addition, Sturmian shifts are minimal [27] and uniquely ergodic [11].

While Sturmian words are generally based on the alphabet $\{0, 1\}$, we allow Sturmian words with alphabet $\{\lfloor\gamma\rfloor, \lceil\gamma\rceil\}$ for any γ. We recall that $\lfloor\gamma\rfloor$ is the largest integer not exceeding γ, while $\lceil\gamma\rceil$ denotes the smallest integer greater than or equal to γ, and $\text{frac}(\gamma) = \lceil\gamma\rceil - \lfloor\gamma\rfloor$ is the fractional part of γ. Given $\gamma \in \mathbb{R}$, we first form the sequence $(y_i^\gamma)_{i=-\infty}^\infty$ by $y_i^\gamma = \lfloor(i+1)\gamma\rfloor - \lfloor i\gamma\rfloor$, with symbols $\lfloor\gamma\rfloor$ and $\lceil\gamma\rceil$. The Sturmian space Y_γ is the orbit closure of y^γ, that is $\text{Cl}(\{\sigma^n(y^\gamma) \colon n \in \mathbb{Z}\})$. A *Sturmian word* with *slope* γ is an element of $\mathcal{L}(Y_\gamma)$. A Sturmian word is an element of $\bigcup_{\gamma\in\mathbb{R}} \mathcal{L}(Y_\gamma)$.

Our terminology comes from the following geometric interpretation of a Sturmian sequence, which we illustrate in Figure 2. For $\gamma \in \mathbb{R}$ we draw a line with slope γ through the origin on a square grid. Moving from left to right we record the number of times our line intercepts the horizontal grid lines in each strip between two consecutive vertical grid lines. These numbers form the corresponding sequence y^γ. Hence, we can "read off" the Sturmian word y^γ from the graph of the line with slope γ.

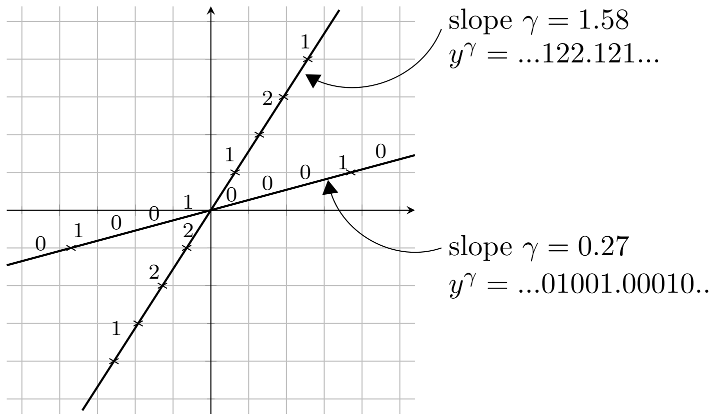

FIGURE 2. Geometric interpretation of Sturmian words.

In the next lemma we characterize the elements of $\mathcal{L}_n(Y_\gamma)$ and Y_γ using a vertical axis intercept of the line with slope γ.

Lemma 6. *Let $\gamma \in \mathbb{R}$ be fixed. A word $y_0 \dots y_{n-1}$ belongs to $\mathcal{L}_n(Y_\gamma)$ if and only if there exists $a \in [0, 1)$ such that $y_i = \lfloor (i+1)\gamma + a \rfloor - \lfloor i\gamma + a \rfloor$ for $i = 0, \dots, n-1$. In particular if $y_0 \dots y_{n-1}$ belongs to $\mathcal{L}_n(Y_\gamma)$, then $y_0 + \dots + y_{n-1}$ is either $\lfloor n\gamma \rfloor$ or $\lceil n\gamma \rceil$.*

A sequence $(y_i)_{i=-\infty}^{\infty}$ belongs to Y_γ if and only if

(a) there exists $a \in [0, 1)$ such that $y_i = \lfloor \gamma(i + 1) + a \rfloor - \lfloor \gamma i + a \rfloor$ for each $i \in \mathbb{Z}$; or

(b) there exists $a \in (0, 1]$ such that $y_i = \lceil \gamma(i + 1) + a \rceil - \lceil \gamma i + a \rceil$ for each $i \in \mathbb{Z}$.

Clearly if $a + i\gamma \notin \mathbb{Z}$ for all i, then the two sequences described in the lemma are equal.

Proof. If γ is rational, y^γ is periodic and there is nothing to prove so we suppose γ is irrational.

We first establish the characterization of words. If there exists a such that $y_i = \lfloor \gamma(i + 1) + a \rfloor - \lfloor \gamma i + a \rfloor$ for $i = 0, \dots, n - 1$, then let k be such that $a < \mathrm{frac}(k\gamma) < a + \min\{1 - \mathrm{frac}(a + \gamma i) \colon 0 \le i \le n\}$ (such a k exists since the multiples of γ are dense modulo 1). One can then check $y_i = y^\gamma_{k+i}$ for $i = 0, \dots, n - 1$. The converse is immediate. Now if y is of this form, $y_0 + \dots + y_{n-1} = \lfloor a + n\gamma \rfloor - \lfloor a \rfloor$, which is either $\lfloor n\gamma \rfloor$ or $\lceil n\gamma \rceil$ as required.

We then establish the characterization of Y_γ. First suppose $y = \lim_{k\to\infty} \sigma^{n_k} y^\gamma$. By refining the subsequence if necessary, we may assume that we are in one of the two cases (i) $\operatorname{frac}(n_k\gamma)$ is a non-increasing sequence converging to some $a \in [0,1)$; or (ii) $\operatorname{frac}(n_k\gamma)$ is a strictly increasing sequence converging to some $a \in (0,1]$. For case (i), we note that $\lfloor\cdot\rfloor$ is right continuous, so that for each i, $\lfloor(n_k+i+1)\gamma\rfloor - \lfloor(n_k+i)\gamma\rfloor \to \lfloor(i+1)\gamma+a\rfloor - \lfloor i\gamma+a\rfloor$, establishing (a). In the second case, the fact that $\operatorname{frac}(n_k\gamma)$ is strictly increasing implies that each n_k only appears once, so that $|n_k| \to \infty$. It follows that for any i, $\lfloor(n_k+i+1)\gamma\rfloor - \lfloor(n_k+i)\gamma\rfloor = \lceil(n_k+i+1)\gamma\rceil - \lceil(n_k+i)\gamma\rceil$ for all sufficiently large k (the only integer multiple of γ is 0). Then since $\lceil\cdot\rceil$ is left continuous, a similar argument to the one above ensures that y satisfies (b).

For the converse, if y satisfies (a), then let (n_k) be chosen so that $\operatorname{frac}(n_k\gamma)$ decreases to a. Then $\sigma^{n_k}y^\gamma$ converges to y. Similarly, if y satisfies (b), then choosing (n_k) such that $\operatorname{frac}(n_k\gamma)$ increases to a ensures $\sigma^{n_k}y^\gamma$ converges to y. □

The *weight* of a Sturmian word $y_0\dots y_{j-1}$ is $y_0+\dots+y_{j-1}$. We need the following crude bound on the number of Sturmian words of a given weight with a fixed length.

Lemma 7. *For any j and n, there are at most $j(j+1)$ Sturmian words of length j and weight n.*

Proof. By Lemma 6, a Sturmian word is parameterized by an intercept a and a slope γ. To satisfy the constraint on the weight, we require $0 \le a < 1$ and $n \le a+j\gamma < n+1$. That is, one is looking for a straight line joining a point $(0,a)$ to a point $(j, n+b)$ with $0 \le a, b < 1$. Such lines all lie within the parallelogram $y - \frac{n}{j}x \in [0,1)$, $x \in [0,j]$. We refer the reader to the sketch in Figure 3.

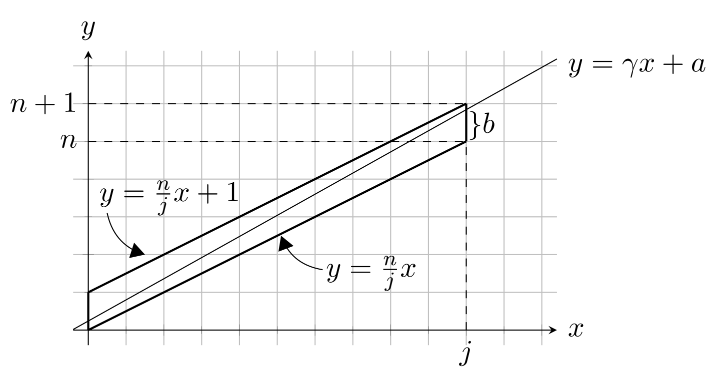

FIGURE 3. This figure illustrates the proof of Lemma 7.

For each i in $0,\dots,j$, there is precisely one integer lattice point in that vertical line within the closure of the parallelogram, namely $(i,\lceil \frac{n}{j}i\rceil)$; or two points $(i,\frac{n}{j}i)$ and $(i,\frac{n}{j}i+1)$ if $\frac{n}{j}i$ is an integer. Given a Sturmian sequence $y_0\dots y_{j-1}$ of weight n, let a and γ satisfy

$$y_i=\lfloor (i+1)\gamma+a\rfloor-\lfloor i\gamma+a\rfloor \text{ for } i=0,\dots,j-1. \tag{3}$$

One may then reduce a keeping (3) satisfied until the line $y=a+\gamma x$ first hits one of the lattice points, (i,k) say, with $i\in\{0,1,\dots,j\}$. One may then rewrite the equation of the line as $y=k+\gamma(x-i)$, and then reduce γ until the line hits another lattice point in the parallelogram. The Sturmian word is determined by i together with the x-coordinate of the second lattice point. There are $(j+1)\times j$ such choices, so at most $j(j+1)$ Sturmian words of length j and weight n. □

We now describe the points appearing in the closure of the union of the sets of all Sturmian words over a range of γ.

Lemma 8. *Let Y_γ denote the Sturmian sequence space with slope γ as above. Suppose $(\gamma^{(n)})_{n\in\mathbb{N}}$ is a sequence of real numbers and $(y^{(n)})_{n\in\mathbb{N}}$ is a sequence of points with $y^{(n)}\in Y_{\gamma^{(n)}}$. Suppose further that $y^{(n)}\to y$. Then the sequence $\gamma^{(n)}$ is convergent to some $\gamma\in\mathbb{R}$. Either $y\in Y_\gamma$; or γ is rational and y is an aperiodic sequence that is the concatenation of two periodic semi-infinite words.*

Proof. Let $\ell\in\mathbb{N}$. Since $y^{(n)}\to y$, there exists an n_0 such that for all $n\ge n_0$, the terms of $y^{(n)}$ in coordinates $-\ell$ to $\ell-1$ agree with those of y. Hence for $n,n'\ge n_0$ the words $y^{(n)}_{-\ell}\dots y^{(n)}_{\ell-1}$ and $y^{(n')}_{-\ell}\dots y^{(n')}_{\ell-1}$ have the same weight, which must lie simultaneously in the intervals $(2\ell\gamma^{(n)}-1,2\ell\gamma^{(n)}+1)$ and $(2\ell\gamma^{(n')}-1,2\ell\gamma^{(n')}+1)$ by Lemma 6. It follows that $|\gamma^{(n)}-\gamma^{(n')}|<\frac{1}{\ell}$. Since ℓ is arbitrary, we see that $(\gamma^{(n)})$ is Cauchy. Let γ be the limit of $(\gamma^{(n)})$.

Passing to a subsequence, we may assume that $a^{(n)}\to a$. Each $y^{(n)}$ may be expressed either in the form (a) or (b) of Lemma 6 with parameters $a^{(n)}$ and $\gamma^{(n)}$. We may further assume that either each term of the subsequence is expressed in the form (a); or each term is expressed in the form (b). Note that for those $i\in\mathbb{Z}$ where $a+i\gamma$ is not an integer, the sequences $\lfloor a^{(n)}+i\gamma^{(n)}\rfloor$ and $\lceil a^{(n)}+i\gamma^{(n)}\rceil$ eventually stabilize to $\lfloor a+i\gamma\rfloor$ and $\lceil a+i\gamma\rceil$ respectively.

We deal first with the case where γ is irrational. In this case, there is at most one $i_0\in\mathbb{Z}$ such that $a+i_0\gamma\in\mathbb{Z}$. If there is no such i_0, then $\lfloor a^{(n)}+i\gamma^{(n)}\rfloor\to\lfloor a+i\gamma\rfloor$ for each i and a similar statement is true for the ceilings. It follows that $y_i=\lim y_i^{(n)}=\lfloor a+(i+1)\gamma\rfloor-\lfloor a+i\gamma\rfloor=\lceil a+(i+1)\gamma\rceil-\lceil a+i\gamma\rceil$ for each i.

If there exists i_0 such that $a + i_0\gamma \in \mathbb{Z}$, then we consider three cases: (i) $a^{(n)} + i_0\gamma^{(n)} = a + i_0\gamma$ infinitely often; (ii) we can pass to a subsequence such that $a^{(n)}+i_0\gamma^{(n)}$ is strictly increasing; (iii) we can pass to a subsequence such that $a^{(n)} + i_0\gamma^{(n)}$ is strictly decreasing. In case (i), along the subsequence where $a^{(n)}+i_0\gamma^{(n)} = a+i_0\gamma$, $\lfloor a^{(n)}+i\gamma^{(n)}\rfloor \to \lfloor a+i\gamma\rfloor$ for all i and $\lceil a^{(n)} + i\gamma^{(n)}\rceil \to \lceil a+i\gamma\rceil$ for all i, so that $y \in Y_\gamma$. In case (ii), along the subsequence, $\lfloor a^{(n)} + i\gamma^{(n)}\rfloor \to \lceil a + i\gamma\rceil - 1$ for all i and $\lceil a^{(n)} + i\gamma^{(n)}\rceil \to \lceil a+i\gamma\rceil$. In case (iii), along the subsequence $\lfloor a^{(n)} + i\gamma^{(n)}\rfloor \to \lfloor a+i\gamma\rfloor$ for all i and $\lceil a^{(n)} + i\gamma^{(n)}\rceil \to \lfloor a + i\gamma\rfloor + 1$ for all i. In all cases, we see $y \in Y_\gamma$.

Now suppose that γ is rational, say $\gamma = \frac{p}{q}$. If infinitely many $\gamma^{(n)}$ are equal to γ then, since Y_γ is a finite set, we see that one element of Y_γ appears infinitely often in the sequence $y^{(n)}$, so that sequence is the limit and $y \in Y_\gamma$. Otherwise, we may take a sequence so that $\gamma^{(n)}$ converges strictly monotonically to γ and $a^{(n)}$ converges to a limit a. If $a+i\gamma \notin \mathbb{Z}$ for each i (or equivalently $a+i\gamma \notin \mathbb{Z}$ for $i = 0,\dots,q-1$), then the argument given above in the irrational case shows $y \in Y_\gamma$.

In the remaining case, there exists $i_0 \in \{0,\dots,q-1\}$ such that $a + i\gamma$ is an integer for each $i \in i_0 + q\mathbb{Z}$ (and $a + i\gamma$ is not an integer for other i's). We may then pass to a further subsequence so that the sequence $j^{(n)} = (a^{(n)} - a)/(\gamma - \gamma^{(n)})$ is monotonic. If $\gamma^{(n)}$ is increasing, $a^{(n)} + i\gamma^{(n)} < a + i\gamma$ when $i > j^{(n)}$ and $a^{(n)} + i\gamma^{(n)} > a + i\gamma$ when $i < j^{(n)}$; the situation is reversed if $\gamma^{(n)}$ is decreasing.

For the remainder of the proof, we focus on the case where $\gamma^{(n)}$ is increasing. If $j^{(n)} \to -\infty$, then for each $i \in \mathbb{Z}$, $\lceil a^{(n)}+i\gamma^{(n)}\rceil \to \lceil a+i\gamma\rceil$ and $\lfloor a^{(n)} + i\gamma^{(n)}\rfloor \to \lceil a + i\gamma\rceil - 1$. Hence we see that whether the sequence $y^{(n)}$ is expressed in form (a) or form (b), $y_i = \lceil a+(i+1)\gamma\rceil - \lceil a + i\gamma\rceil$ for all $i \in \mathbb{Z}$. Similarly if $j^{(n)} \to \infty$, then for each $i \in \mathbb{Z}$, $\lfloor a^{(n)} + i\gamma^{(n)}\rfloor \to \lfloor a + i\gamma\rfloor$ and $\lceil a^{(n)} + i\gamma^{(n)}\rceil \to \lfloor a + i\gamma\rfloor + 1$, so that $y_i = \lfloor a + (i+1)\gamma\rfloor - \lfloor a + i\gamma\rfloor$.

We now consider the case $j^{(n)} \to j^*$. We have

$$\lim_{n\to\infty} \lfloor a^{(n)} + \gamma^{(n)} i\rfloor = \begin{cases} \lfloor a + \gamma i\rfloor & \text{if } i < j^* \text{ or } i \notin i_0 + q\mathbb{Z}; \\ \lfloor a + \gamma i\rfloor - 1 & \text{if } i > j^* \text{ and } i \in i_0 + q\mathbb{Z}. \end{cases}$$

and

$$\lim_{n\to\infty} \lceil a^{(n)} + \gamma^{(n)} i\rceil = \begin{cases} \lceil a + \gamma i\rceil & \text{if } i > j^* \text{ or } i \notin i_0 + q\mathbb{Z}; \\ \lceil a + \gamma i\rceil + 1 & \text{if } i < j^* \text{ and } i \in i_0 + q\mathbb{Z}. \end{cases}$$

If $j^* \in i_0 + q\mathbb{Z}$, then $\lfloor a^{(n)} + \gamma^{(n)} j^*\rfloor$ converges to one of $\lfloor a + \gamma j^*\rfloor$ and $\lfloor a + \gamma j^*\rfloor - 1$; and $\lceil a^{(n)} + \gamma^{(n)} j^*\rceil$ converges to one of $\lceil a + \gamma j^*\rceil$ and $\lceil a + \gamma j^*\rceil + 1$.

Hence y, the difference sequence of one of $\left(\lim_{n\to\infty}\lfloor a^{(n)}+\gamma^{(n)}i\rfloor\right)_i$ or $\left(\lim_{n\to\infty}\lceil a^{(n)}+\gamma^{(n)}i\rceil\right)_i$, is the concatenation of two semi-infinite periodic words, as claimed. In the case where $\gamma^{(n)}$ is decreasing, an almost identical argument applies. □

5. Main Theorem

We present the proof of Theorem 1. Let the alphabet of the shift be

$$A = \{0,1\ldots,\lfloor e^c\rfloor\}\times\{\lfloor b\rfloor,\ldots,\lceil c\rceil\}\times\{\lfloor -L\rfloor,\ldots,\lceil L\rceil\}^m$$

and let σ denote the shift map on $A^{\mathbb{Z}}$. We construct the potential functions as follows. For each vector $\boldsymbol{\gamma}=(\gamma_0,\gamma_1,\ldots,\gamma_m)\in S$ let

$$Z_{\boldsymbol{\gamma}} = X_{e^{\gamma_0}}\times Y_{\gamma_0}\times Y_{\gamma_1}\times\ldots\times Y_{\gamma_m},$$

where $X_{e^{\gamma_0}}$ is the β-shift with parameter $\beta=e^{\gamma_0}$ and for $k=0,\ldots,m$ Y_{γ_k} is the Sturmian system with angle γ_k (so the alphabet of $X_{e^{\gamma_0}}$ is $\{0,1,\ldots,\lfloor e^{\gamma_0}\rfloor\}$ and the alphabet of Y_{γ_k} is $\{\lfloor\gamma_k\rfloor,\lceil\gamma_k\rceil\}$). In particular, $Z_{\boldsymbol{\gamma}}\subset A^{\mathbb{Z}}$. For $z=(x,y^0,\ldots,y^m)\in A^{\mathbb{Z}}$ the projections of z onto each coordinate are defined by $\pi_x(z)=x$ and $\pi_k(z)=y^k$, $k=0,\ldots,m$.

Let $\mathcal{L}_n(Z_{\boldsymbol{\gamma}})$ denote the collection of n-words in $Z_{\boldsymbol{\gamma}}$. For $z\in A^{\mathbb{Z}}$, we set $j_{\boldsymbol{\gamma}}(z)=\max\{l\colon z_{-(l-1)}\ldots z_{l-1}\in\mathcal{L}_{2l-1}(Z_{\boldsymbol{\gamma}})\}$, where $j_\gamma(z)$ is taken to be 0 if $z_0\notin\mathcal{L}_0(Z_{\boldsymbol{\gamma}})$. For $\boldsymbol{\gamma}=(\gamma_0,\gamma_1,\ldots,\gamma_m)\in S$ and $k=1,\ldots,m$ we define

$$\phi_{k,\boldsymbol{\gamma}}(z)=\gamma_k-\delta_{j_{\boldsymbol{\gamma}}(z)},$$

where δ_j is given by

$$\delta_j=\frac{c+2L+14+9\log j}{j\min\{\alpha,1\}} \tag{4}$$

with $\delta_0=\delta_1+2L$ and $\delta_\infty=0$. Notice that (δ_j) form a decreasing sequence, converging to 0.

If $d(z,z')\le 2^{-l}$, then the two words $z_{-(l-1)}\ldots z_{l-1}$ and $z'_{-(l-1)}\ldots z'_{l-1}$ are equal. Either both lie in $\mathcal{L}_{2l-1}(\boldsymbol{\gamma})$, in which case $|\phi_{k,\boldsymbol{\gamma}}(z)-\phi_{k,\boldsymbol{\gamma}}(z')|\le\delta_l$ or neither do, in which case $\phi_{k,\boldsymbol{\gamma}}(z)=\phi_{k,\boldsymbol{\gamma}}(z')$. Hence we have shown that for each $k=1,\ldots,m$ the family $\{\phi_{k,\boldsymbol{\gamma}}\colon\boldsymbol{\gamma}\in S\}$ is uniformly equicontinuous. The potentials ϕ_k on $A^{\mathbb{Z}}$ are then defined by

$$\phi_k(z)=\sup_{\boldsymbol{\gamma}\in S}\phi_{k,\boldsymbol{\gamma}}(z).$$

The uniform equicontinuity ensures that each ϕ_k is continuous.

The proof of the theorem splits into two parts. First, we restrict our considerations to the set $Z=\bigcup_{\boldsymbol{\gamma}\in S}Z_{\boldsymbol{\gamma}}$. On this set we show that the potentials $\phi_1,\ldots,\phi_n$ have the property we are looking for. Namely, for given values of parameters $t_k\ge\alpha$ the pressure of $t_1\phi_1+\ldots+t_m\phi_m$, when restricted to $\mathrm{Cl}(Z)$, coincides with the value of $F(t_1,\dots,t_m)$. Afterwards, we demonstrate that the values of ϕ_k outside of the set Z do not contribute to the pressure.

We start by describing the behavior of the potentials $\phi_1,\dots,\phi_k$ on Z.

Lemma 9. *Let $\boldsymbol{\gamma}=(\gamma_0,\dots,\gamma_m)\in S$. If $z\in Z_{\boldsymbol{\gamma}}$, then $\phi_k(z)=\gamma_k$ for each $k=1,\dots,m$.*

Proof. Let $\boldsymbol{\gamma}=(\gamma_0,\dots,\gamma_m)\in S$ and let $z=(x,y^0,\dots,y^m)\in Z_{\boldsymbol{\gamma}}$. Fix $1\le k\le m$. From the definition, we see $\phi_{k,\boldsymbol{\gamma}}(z)=\gamma_k-\delta_\infty=\gamma_k$. We will show that for any $\boldsymbol{\gamma}'=(\gamma_0',\dots,\gamma_m')\in S$ we have $\phi_{k,\boldsymbol{\gamma}'}(z)\le\gamma_k$.

If $\gamma_k'\le\gamma_k$, then $\phi_{k,\boldsymbol{\gamma}'}(z)\le\gamma_k'\le\gamma_k$. If $\gamma_k'>\gamma_k$, then we set $j=\lceil 1/(\gamma_k'-\gamma_k)\rceil$. Since $z\in Z_{\boldsymbol{\gamma}}$, we know that $y^k\in Y_{\gamma_k}$ and hence by Lemma 6 there exists $a\in[0,1)$ such that $y_i^k=\lfloor (i+1)\gamma_k+a\rfloor-\lfloor i\gamma_k+a\rfloor$ for each $i=-j,\dots,j-1$. In particular, $y_{-j}^k+\ldots+y_{j-1}^k=\lfloor j\gamma_k+a\rfloor-\lfloor -j\gamma_k+a\rfloor$, so that $y_{-j}^k+\ldots+y_{j-1}^k\in(2j\gamma_k-1,2j\gamma_k+1)$ and this holds for any $y_{-j}^k\dots y_{j-1}^k\in\mathcal{L}_{2j}(Y_{\gamma_k})$.

If the block $y_{-j}^k\dots y_{j-1}^k$ were also in $\mathcal{L}_{2j}(Y_{\gamma_k'})$, then similar to the argument above we would have $y_{-j}^k+\ldots+y_{j-1}^k\in(2j\gamma_k'-1,2j\gamma_k'+1)$. However, by the choice of j, $2j\gamma_k-1\le 2j\gamma_k'+1$, so that $\mathcal{L}_{2j}(Y_{\gamma_k})$ and $\mathcal{L}_{2j}(Y_{\gamma_k'})$ are disjoint. It follows that $z_{-j}\dots z_{j-1}\notin\mathcal{L}_{2j}(Z_{\boldsymbol{\gamma}'})$ and hence $j_{\boldsymbol{\gamma}'}(z)\le j$.

Using the facts that $j=\lceil 1/(\gamma_k'-\gamma_k)\rceil$ and $\gamma_k,\gamma_k'\in[-L,L]$ when $\boldsymbol{\gamma},\boldsymbol{\gamma}'\in S$ we obtain

$$j\le\frac{1}{\gamma_k'-\gamma_k}+1=\frac{1+\gamma_k'-\gamma_k}{\gamma_k'-\gamma_k}\le\frac{1+2L}{\gamma_k'-\gamma_k}.$$

Therefore,

$$\phi_{k,\boldsymbol{\gamma}'}(z)=\gamma_k'-\delta_{j_{\boldsymbol{\gamma}'}(z)}\le\gamma_k'-\delta_j<\gamma_k'-\frac{1+2L}{j}\le\gamma_k.$$

□

We now turn our attention to the invariant measures on Z.

Lemma 10. *Let $Z=\bigcup_{\boldsymbol{\gamma}\in S}Z_{\boldsymbol{\gamma}}$ as above. Then any ergodic invariant measure supported on $\mathrm{Cl}(Z)$ is supported on $Z_{\boldsymbol{\gamma}}$ for some $\boldsymbol{\gamma}\in S$.*

Proof. We first describe points of $\mathrm{Cl}(Z)$. Let $\bar z\in\mathrm{Cl}(Z)$. Then $\bar z$ is the limit of a sequence of points $z(n)$, where each $z(n)$ belongs to some $Z_{\boldsymbol{\gamma}(n)}$ with $\boldsymbol{\gamma}(n)\in S$. Write $z(n)=(x(n),y^0(n),\dots,y^m(n))$ and $\bar z=(\bar x,\bar y^0,\dots,\bar y^m)$. Since $y^k(n)\to\bar y^k$, it follows from Lemma 8 that $\gamma_k(n)$ converges to some limit for each $k=0,\dots,m$. Let $\bar{\boldsymbol{\gamma}}=\lim_{n\to\infty}\boldsymbol{\gamma}(n)$, so that $\bar{\boldsymbol{\gamma}} \in S$. By Lemma 8, for each $0 \le k \le m$, $\bar{y}^k$ belongs either to $Y_{\bar{\gamma}_k}$, or is the (non-periodic) concatenation of two semi-infinite periodic words.

We claim that $\bar{x} \in X_{e^{\bar{\gamma}_0}}$. To see this, first notice that since $z(n) \in Z_{\boldsymbol{\gamma}(n)}$, we have $x(n) \in X_{e^{\gamma_0(n)}}$. For any $\beta > \gamma_0$ there is $n_0 \in \mathbb{N}$ such that $\gamma_0(n) < \beta$ for $n \ge n_0$. It follows that $x(n) \in X_{e^\beta}$ for all $n \ge n_0$ and hence $\bar{x} \in X_{e^\beta}$. Since $\beta > \gamma_0$ is arbitrary, $\bar{x}$ lies in $\bigcap_{\beta > \gamma_0} X_{e^\beta} = X_{e^{\gamma_0}}$ by Lemma 5.

Let C denote the (countable) collection of non-periodic concatenations of two semi-infinite periodic points with symbols in the set $\{\min(\lfloor b \rfloor, \lfloor -L \rfloor), \ldots, \max(\lceil c \rceil, \lceil L \rceil)\}$. We have shown that any point of $\mathrm{Cl}(Z)$ either lies in some $Z_{\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma} \in S$ or one of its Sturmian coordinates lies in C.

Let μ be an ergodic invariant measure supported on $\mathrm{Cl}(Z)$. Suppose for a contradiction that μ is supported on $\mathrm{Cl}(Z) \setminus Z$. Then for μ-a.e. $z = (x, y^0, \ldots, y^m)$, there exists a $0 \le k \le m$ such that $y^k \in C$. Since μ is ergodic, there exists a k such that for μ-a.e. $(x, y^0, \ldots, y^m)$, $y^k \in C$. In particular, the projection of μ onto the kth Sturmian factor is supported on a countable set. But this is a contradiction as countable sets of aperiodic words do not support any finite invariant measures. Hence μ is supported on Z.

It is left to show that μ is supported on some $Z_{\boldsymbol{\gamma}}$. Fix $k \in \{0, \ldots, m\}$ and consider the projection map $f_k(z) = y^k_0$ where $z = (x, y^0, \ldots, y^m) \in Z$. Since μ is ergodic and f is continuous, there is $\gamma_k \in \mathbb{R}$ such that

$$\frac{1}{N} \sum_{i=0}^{N-1} f(\sigma^i z) = \gamma_k \quad \text{for } \mu\text{-almost all } z \in Z.$$

Suppose that $z \in Z_{\boldsymbol{\gamma}'}$ for some $\boldsymbol{\gamma}' \in S$ and satisfies the above. Then an application of Lemma 6 gives

$$\frac{1}{N} \sum_{i=0}^{N-1} f(\sigma^i z) = \frac{y^k_0 + \ldots + y^k_{N-1}}{N} = \frac{\lfloor N\gamma'_k + a \rfloor}{N}$$

for some $a \in [0, 1)$. Hence, $\gamma'_k = \gamma_k$ and μ is supported on Z_γ with $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_m)$. □

Corollary 11. *Let $\boldsymbol{t} = (t_1, \ldots, t_m) \in (\alpha, \infty)^m$ and μ be an ergodic shift-invariant measure supported on* $\mathrm{Cl}(Z)$*. Then*

$$h(\mu) + \int (t_1\phi_1 + \ldots + t_m\phi_m)\, d\mu \le F(t_1, \ldots, t_m).$$

Further there exists an ergodic measure $\mu_{\boldsymbol{t}}$ supported on Z such that

$$h(\mu_{\boldsymbol{t}}) + \int (t_1\phi_1 + \ldots + t_m\phi_m)\, d\mu_{\boldsymbol{t}} = F(t_1, \ldots, t_m).$$

Proof. Let μ be as in the statement of the corollary. Note that by Lemma 10, μ is supported on $Z_{\boldsymbol{\gamma}}$ for some $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_m) \in S$. Then $h(\mu) \leq h_{\text{top}}(Z_{\boldsymbol{\gamma}}) = h_{\text{top}}(X_{e^{\gamma_0}}) + h_{\text{top}}(Y_{\gamma_0}) + \ldots + h_{\text{top}}(Y_{\gamma_m}) = \gamma_0$ and $\int \phi_k\, d\mu = \gamma_k$ by Lemma 9, so that

$$h(\mu) + \int (t_1\phi_1 + \ldots + t_m\phi_m)\, d\mu \leq \gamma_0 + t_1\gamma_1 + \ldots + t_m\gamma_m.$$

Since $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_m) \in S$, the last term is bounded by $F(t_1, \ldots, t_m)$ by Lemma 3, proving the inequality in the statement.

For the equality in the statement, let $\mathbf{v} = (\mathrm{v}_1, \ldots, \mathrm{v}_m)$ be any subgradient of F at $\boldsymbol{t} = (t_1, \ldots, t_m)$. We set $\gamma_k = \mathrm{v}_k$ for $k = 1, \ldots, m$ and $\gamma_0 = F(\boldsymbol{t}) - \mathbf{v} \cdot \boldsymbol{t}$. Then $\boldsymbol{\gamma} = (\gamma_0, \ldots, \gamma_m) \in S$. Let $\nu_{\boldsymbol{t}}$ be the measure on $Z_{\boldsymbol{\gamma}}$ that is the product of the measure of maximal entropy on the β-shift $X_{e^{\gamma_0}}$ and $m+1$ Sturmian measures supported on each of the components $Y_{\gamma_0}, \ldots, Y_{\gamma_m}$ respectively and let $\mu_{\boldsymbol{t}}$ be an ergodic component. Since $\mu_{\boldsymbol{t}}$ projects in the first factor onto the measure of maximal entropy on $X_{e^{\gamma_0}}$, we see $h(\mu_{\boldsymbol{t}}) \geq \gamma_0$. On the other hand, since $\mu_{\boldsymbol{t}}$ is supported on $Z_{\boldsymbol{\gamma}}$, we see $h(\mu_{\boldsymbol{t}}) \leq h_{\text{top}}(Z_{\boldsymbol{\gamma}}) = \gamma_0$.

Each potential ϕ_k takes the value γ_k on the support of $\mu_{\boldsymbol{t}}$, so that

$$\begin{aligned} h(\mu_{\boldsymbol{t}}) + \int t_1\phi_1 + \ldots + t_m\phi_m\, d\mu_{\boldsymbol{t}} &= \gamma_0 + t_1\gamma_1 + \ldots + t_m\gamma_m \\ &= F(\boldsymbol{t}) - \mathbf{v} \cdot \boldsymbol{t} + \mathbf{v} \cdot \boldsymbol{t} \\ &= F(\boldsymbol{t}) \end{aligned}$$

as required. □

Hence we have shown that for each $(t_1, \ldots, t_m) \in (\alpha, \infty)^m$,

$$F(t_1, \ldots, t_m) = \sup_{\mu} \left\{ h(\mu) + \int (t_1\phi + \ldots + t_m\phi_m)\, d\mu \right\},$$

where μ runs over all ergodic invariant measures supported on $\mathrm{Cl}(Z)$. Recall that in the Variational Principle, the pressure is attained if the supremum is taken only over ergodic invariant measures. In order to complete the proof of the theorem it suffices to show that for all $(t_1, \ldots, t_m) \in (\alpha, \infty)^m$ and for each ergodic shift-invariant measure μ on $A^{\mathbb{Z}}$ such that $\mu(\mathrm{Cl}(Z)^c) \neq 0$, one has

$$h(\mu) + \int (t_1\phi_1 + \ldots + t_m\phi_m)\, d\mu < F(t_1, \ldots, t_m). \tag{5}$$

To see this, we use a technique introduced by Antonioli in [4] of *pinning sequences*. This is part of a more general set of ideas described in the notes [45] on “Coupling and Splicing”.

The *pinning space* is a closed subshift Ω of $A^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ defined by the following conditions. Let $(u,v) \in A^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$. Then $(u,v) \in \Omega$ if and only if

(1) if $i < j$ and $v_{i+1} = \ldots = v_j = 0$, then $u_i \ldots u_j \in \mathcal{L}(Z)$; (Note: there is no requirement that $v_i = 0$).
(2) if $i < j$ and $v_i = v_j = 1$, then $u_i \ldots u_j \notin \mathcal{L}(Z)$.

We denote the shift map on Ω by $\bar{\sigma}$ to distinguish it from the shift σ on $A^{\mathbb{Z}}$. We refer to v in the pair $(u,v) \in \Omega$ as a *pinning sequence* for $u \in A^{\mathbb{Z}}$.

Clearly if $u \in Z$, then $(u, \mathbf{0}) \in \Omega$, where $\mathbf{0}$ is the sequence of all 0's. For a fixed $u \in A^{\mathbb{Z}}$, the set of v such that $(u,v) \in \Omega$ corresponds to the set of all greedy partitions of u into words in $\mathcal{L}(Z)$: each such word corresponds to a maximal string in v of the form $10\ldots0$.

In case $u \in A^{\mathbb{Z}} \setminus Z$, one may obtain a pinning sequence v such that $(u,v) \in \Omega$ by a limit of greedy algorithms as follows: for each n, let $k_0^{(n)} = -n$ and let $k_{i+1}^{(n)}$ be the smallest integer greater than $k_i^{(n)}$ such that $u_{k_i^{(n)}} \ldots u_{k_{i+1}^{(n)}} \notin \mathcal{L}(Z)$ (where the sequence terminates if there is no such $k_{i+1}^{(n)}$). Then define a sequence $(v^{(n)})$ by

$$v_j^{(n)} = \begin{cases} 1 & \text{if } j \in \{k_i^{(n)} : i \geq 0\}; \\ 0 & \text{otherwise.} \end{cases}$$

Any subsequential limit v of the $v^{(n)}$ sequences satisfies $(u,v) \in \Omega$. In particular, $A^{\mathbb{Z}}$ is a factor of the shift on Ω by the projection onto the first coordinate.

Given an ergodic invariant measure μ on $A^{\mathbb{Z}}$, we now build a suitable lift to Ω. Denote by $\boldsymbol{\delta}_p$ the Dirac measure supported on the point p. If μ is supported on Z, then clearly $\bar{\mu} = \mu \times \boldsymbol{\delta}_{\mathbf{0}}$ is a suitable lift. If not, let u be a generic point of $A^{\mathbb{Z}}$ for μ, and let v be its pinning sequence so that $(u,v) \in \Omega$. We then let $\bar{\nu}$ be a subsequential limit of the sequence $\frac{1}{n}(\boldsymbol{\delta}_{(u,v)} + \ldots + \boldsymbol{\delta}_{\bar{\sigma}^{n-1}(u,v)})$. By the μ-genericity of u, the projection of $\bar{\nu}$ onto the $A^{\mathbb{Z}}$ coordinate is μ. Since $\bar{\nu}$ may fail to be ergodic, we consider the ergodic components of $\bar{\nu}$. By ergodicity of μ, almost every ergodic component of $\bar{\nu}$ is supported on Ω and projects onto μ. We let $\bar{\mu}$ be any such ergodic component and call $\bar{\mu}$ a *lift* of μ to Ω.

If μ is an ergodic invariant measure supported on $A^{\mathbb{Z}} \setminus \mathrm{Cl}(Z)$, there is a word $w \notin \mathcal{L}(Z)$ such that $\mu([w]) > 0$. By ergodicity μ-a.e. u contains infinitely many copies of the word w, so that if $(u,v) \in \Omega$, then over each occurrence of w in u, there is at least one pin (1 in the corresponding pinning sequence, v). By the greedy property, a single 1 in v, together with u determines all subsequent terms of v. It follows that there are at most $|w|$ different v's such that $(u, v) \in \Omega$. In particular, the projection map from the pinning space to $A^{\mathbb{Z}}$ is $\bar{\mu}$-almost surely finite-to-one. It follows that $h(\bar{\mu}) = h(\mu)$. Since $\bar{\mu}$ projects to μ, we see immediately that $\int f \circ \pi \, d\bar{\mu} = \int f \, d\mu$ for any Borel function f on $A^{\mathbb{Z}}$, where π is the projection of Ω onto the first coordinate, $A^{\mathbb{Z}}$.

Let $P = \{(u, v) \in \Omega\colon v_0 = 1\}$ and let $\bar{\phi}_k(u, v) = \phi_k(u)$ for $k = 1, \ldots, m$. It follows from the above that to prove (5) it suffices to show that for any $(t_1, \ldots, t_m) \in (\alpha, \infty)^m$, and any ergodic measure $\bar{\mu}$ on Ω such that $\bar{\mu}(P) > 0$,

$$h(\bar{\mu}) + \int (t_1\bar{\phi}_1 + \ldots + t_m\bar{\phi}_m) \, d\bar{\mu} < F(t_1, \ldots, t_m). \tag{6}$$

Let $\bar{\mu}$ be an ergodic measure on Ω such that $\bar{\mu}(P) > 0$. Let $\tau_P(u, v) = \min\{i \geq 1 : \bar{\sigma}^i(u, v) \in P\}$ be the first return time to P. Let $\bar{\sigma}_P$ denote the induced map of $\bar{\sigma}$ on P with invariant measure $\bar{\mu}_P(\cdot) = \bar{\mu}(\cdot \cap P)/\bar{\mu}(P)$, i.e. $\bar{\sigma}_P(u, v) = \bar{\sigma}^{\tau(u,v)}(u, v)$. By Abramov's formula [1] the relation between the entropy of the measure $\bar{\mu}$ on $(\Omega, \bar{\sigma})$ and the entropy of the induced measure $\bar{\mu}_P$ on $(P, \bar{\sigma}_P)$ is

$$h(\bar{\mu}_P) = \frac{1}{\bar{\mu}(P)} h(\bar{\mu}).$$

We introduce three countable partitions of P. Let $\mathcal{Q} = \{Q_1, \ldots\}$ be the partition of P according to the return time to P. Here

$$Q_j = \{(u, v) \in P \, : \, \tau_P(u, v) = j\}.$$

We let $\mathcal{R}$ to be a subpartition of $\mathcal{Q}$ according to the weights in each of the Sturmian components. Precisely, given $(u, v) \in Q_j$ we write out the components of u so that $(u, v) = (x, y^0, \ldots, y^m, v)$. For each $j \in \mathbb{N}$ and a tuple $\boldsymbol{n} = (n_0, \ldots, n_m) \in \mathbb{Z}^{m+1}$ we define

$$R_{j,\boldsymbol{n}} = \{(x, y^0, \ldots, y^m, v) \in Q_j \, : \, y_0^k + \ldots + y_{j-1}^k = n_k \text{ for } k = 0, \ldots, m\}.$$

We denote by N_j the set of tuples $\boldsymbol{n}$ for which the set $R_{j,\boldsymbol{n}}$ is not empty. Then $\mathcal{R}$ is the partition $\{R_{j,\boldsymbol{n}}\colon j \in \mathbb{N}, \ \boldsymbol{n} \in N_j\}$. Finally, let $\mathcal{P}$ denote the partition of P in which each Q_j is refined into cylinder sets of length j. In particular, $\mathcal{P}$ is a generating partition under $\bar{\sigma}_P$; the partitions $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{P}$ are successive refinements. We introduce the notation

$$q_j = \bar{\mu}_P(Q_j) \quad \text{and} \quad r_{j,\boldsymbol{n}} = \bar{\mu}_P(R_{j,\boldsymbol{n}}). \tag{7}$$

We have a set of equalities which will be extensively used in what follows:

$$\sum_{j=1}^{\infty} q_j = 1 \tag{8}$$

$$\sum_{\boldsymbol{n}\in N_j} r_{j,\boldsymbol{n}} = q_j \tag{9}$$

$$\sum_{j=1}^{\infty} j q_j = 1/\bar\mu(P) \quad \text{(Kac's lemma)} \tag{10}$$

First, we establish a connection between the elements of the partition $\mathcal{R}$ and subshifts forming Z.

Lemma 12. *Suppose $(u,v)\in R_{j,\boldsymbol{n}}$ with $\boldsymbol{n}=(n_0,\dots,n_m)$ and let $x=\pi_x(u)$ and $y^k=\pi_k(u)$ for $k=0,\dots,m$ so that $u=(x,y^0,\dots,y^m)$. Then there is $\boldsymbol{\gamma}\in S$ such that for each $k=0,\dots,m$, $\gamma_k\in(\frac{n_k-1}{j},\frac{n_k+1}{j})$, $x_0\dots x_{j-1}\in\mathcal{L}_j(X_{e^{\gamma_0}})$ and $y^k_0\dots y^k_{j-1}\in\mathcal{L}_j(Y_{\gamma_k})$.*

Proof. Let $(u,v)\in R_{j,\boldsymbol{n}}$, $x=\pi_x(u)$ and $y^k=\pi_k(u)$ for $k=0,1,\dots,m$. Since $u\in\mathcal{L}_j(Z)$, it follows that $u\in\mathcal{L}_j(Z_{\boldsymbol{\gamma}})$ for some $\boldsymbol{\gamma}=(\gamma_0,\dots,\gamma_m)\in S$. This implies that $x_0\dots x_{j-1}\in\mathcal{L}_j(X_{e^{\gamma_0}})$ and $y^k_0\dots y^k_{j-1}\in\mathcal{L}_j(Y_{\gamma_k})$ for $0\le k\le m$. It remains to show that $\gamma_k\in(\frac{n_k-1}{j},\frac{n_k+1}{j})$ where $\boldsymbol{n}=(n_0,\dots,n_m)$.

Since $(x,y^0,\dots,y^m,v)\in R_{j,\boldsymbol{n}}$ for each $k=0,\dots,m$ we have $n_k=y^k_0+\dots+y^k_{j-1}$. On the other hand, $y^k_0\dots y^k_{j-1}\in\mathcal{L}_j(Y_{\gamma_k})$ implies by Lemma 6 that $\lfloor j\gamma_k\rfloor\le n_k\le\lceil j\gamma_k\rceil$ so that for each $0\le k\le m$, $j\gamma_k\in(n_k-1,n_k+1)$ as required. □

Next we obtain an upper bound on the entropy of the measure $\bar\mu$ on Ω via the entropy of the corresponding induced measure $\bar\mu_P$. We use the notation introduced in (7).

Lemma 13. *Suppose $\bar\mu$ is an ergodic invariant measure on Ω such that $\bar\mu(P)>0$. Then*

$$h(\bar\mu_P)\le c+6+(2L+2)m+(3m+6)\sum_{j=1}^{\infty}q_j\log j+\sum_{j=1}^{\infty}\sum_{\boldsymbol{n}\in N_j}n_0 r_{j,\boldsymbol{n}}.$$

Proof. We use the simple bound $h(\bar\mu_P)\le H_{\bar\mu_P}(\mathcal{P})$, where $H_{\bar\mu_P}(\mathcal{P})$ is the entropy of the partition $\mathcal{P}$ with respect to the measure $\bar\mu_P$. We then estimate $H_{\bar\mu_P}(\mathcal{P})$ using conditional entropy:

$$H_{\bar\mu_P}(\mathcal{P})=H_{\bar\mu_P}(\mathcal{Q})+H_{\bar\mu_P}(\mathcal{R}|\mathcal{Q})+H_{\bar\mu_P}(\mathcal{P}_P|\mathcal{R}).$$

We have $H_{\bar\mu_P}(\mathcal{Q}) = \sum_{j=1}^{\infty} -q_j \log q_j$, which we separate into two parts as

$$\sum_{j=1}^{\infty} -q_j \log q_j = -\sum_{q_j<1/j^2} q_j \log q_j - \sum_{q_j\geq 1/j^2} q_j \log q_j.$$

The first term is at most $\frac{1}{e} - \sum_{j=2}^{\infty} \frac{1}{j^2} \log \frac{1}{j^2}$ (which we obtained using the fact that $-t\log t$ is increasing on $[0, \frac{1}{e}]$ and bounded above by $\frac{1}{e}$ on $[0,1]$). The second term is bounded above by $2\sum_{j=1}^{\infty} q_j \log j$, so that

$$H_{\bar\mu_P}(\mathcal{Q}) \leq 3 + 2\sum_{j=1}^{\infty} q_j \log j.$$

We now turn to $H_{\bar\mu_P}(\mathcal{R}|\mathcal{Q})$, which is given by

$$H_{\bar\mu_P}(\mathcal{R}|\mathcal{Q}) = \sum_{Q\in\mathcal{Q}} \bar\mu_P(Q)\left(-\sum_{R\in\mathcal{R}} \frac{\bar\mu(R\cap Q)}{\bar\mu_P(Q)} \log \frac{\bar\mu(R\cap Q)}{\bar\mu_P(Q)}\right).$$

We bound the term in parentheses by the logarithm of the number of elements in $\mathcal{R}$ into which the set Q is partitioned. Recall that $\mathcal{Q} = \{Q_j\}_{j\in\mathbb{N}}$ and each $Q_j = \bigcup_{\boldsymbol{n}\in N_j} R_{j,\boldsymbol{n}}$. If $\boldsymbol{n} = (n_0,\dots,n_m) \in N_j$ then $R_{j,\boldsymbol{n}}$ is not empty and hence by Lemma 12 there is $\boldsymbol{\gamma} = (\gamma_0,\dots,\gamma_m) \in S$ such that $\lfloor j\gamma_k\rfloor \leq n_k \leq \lceil j\gamma_k\rceil$ for each $k = 0,\dots,m$. Since $S \subset [b,c]\times[-L,L]^m$, there are at most $\lceil c\rceil j\cdot(2\lceil L\rceil+1)^m j^m$ tuples $\boldsymbol{n}$ in N_j. Now using (7), (8) and the fact that $\log\lceil c\rceil \leq c$ we obtain

$$\begin{aligned} H_{\bar\mu_P}(\mathcal{R}|\mathcal{Q}) &\leq \sum_{j=1}^{\infty} q_j \log\left(\lceil c\rceil(2\lceil L\rceil+1)^m j^{m+1}\right) \\ &\leq \sum_{j=1}^{\infty} q_j(c + (2L+1)m + (m+1)\log j) \\ &= c + (2L+1)m + (m+1)\sum_{j=1}^{\infty} q_j \log j \end{aligned}$$

Finally, we estimate $H_{\bar\mu_P}(\mathcal{P}|\mathcal{R})$. Similar to the above, we use a crude bound via the number of j-cylinders forming each $R_{j,\boldsymbol{n}} \in \mathcal{R}$. Recall that each element of $\mathcal{P}$ is a cylinder set generated by an element of $\mathcal{L}(Z)$. We separately estimate the number of projections of those words forming $R_{j,\boldsymbol{n}}$ onto each coordinate. Suppose $(u,v) \in R_{j,\boldsymbol{n}}$. Write $\boldsymbol{n} = (n_0,\dots,n_m)$, $x = \pi_x(u)$ and $y^k = \pi_k(u)$ for $0 \leq k \leq m$.

There are at most $j(j+1) \leq 2j^2$ choices for $y_0^k \dots y_{j-1}^k$ by Lemma 7, since each such Sturmian word must have the same weight n_k. By Lemma 12, and using the fact that $X_\beta \subset X_{\beta'}$ if $\beta < \beta'$, we see

$x_0 \dots x_{j-1} \in X_{e^{(n_0+1)/j}}$. By Lemma 4, the number of such choices of $x_0 \dots x_{j-1}$ is at most

$$\frac{e^{(n_0+1)/j}}{e^{(n_0+1)/j}-1}e^{n_0+1} = \frac{1}{1-e^{-(n_0+1)/j}}e^{n_0+1} \\ \leq \frac{1}{1-e^{-1/j}}e^{n_0+1} \leq je^2e^{n_0},$$

using the fact that $1/(1-e^{-1/j}) \leq ej$.

Multiplying the estimates, we see that the number of j-cylinders making up each $R_{j,\boldsymbol{n}}$ is at most $(2j^2)^{m+1}je^2e^{n_0}$. Therefore,

$$\begin{aligned} H_{\bar\mu_P}(\mathcal{P}|\mathcal{R}) &\leq \sum_{R\in\mathcal{R}} \bar\mu_P(R)\log\left(2^{m+1}j^{2m+3}e^2e^{n_0}\right) \\ &= \sum_{j=1}^\infty \sum_{\boldsymbol{n}\in N_j} r_{j,\boldsymbol{n}}\big((m+1)\log 2 + 2 + (2m+3)\log j + n_0\big) \\ &\leq \sum_{j=1}^\infty q_j(m+3) + (2m+3)\sum_{j=1}^\infty q_j \log j + \sum_{j=1}^\infty \sum_{\boldsymbol{n}\in N_j} n_0 r_{j,\boldsymbol{n}} \\ &= m+3+(2m+3)\sum_{j=1}^\infty q_j\log j + \sum_{j=1}^\infty\sum_{\boldsymbol{n}\in N_j} n_0 r_{j,\boldsymbol{n}}, \end{aligned}$$

where we used (9) in the second line and (8) in the third. Combining the above estimates we establish that

$$H_{\bar\mu_P}(\mathcal{P}) \leq c+6+(2L+2)m+(3m+6)\sum_{j=1}^\infty q_j\log j + \sum_{j=1}^\infty\sum_{\boldsymbol{n}\in N_j} n_0 r_{j,\boldsymbol{n}}.$$

□

Lastly, we estimate $\int \phi_k \, d\mu$ for $0 \leq k \leq m$. We define a version of $\bar\phi_k$ on the induced system by

$$\bar\phi_k^P(u,v) = \sum_{i=0}^{\tau_P(u,v)-1} \phi_k(\sigma^i u),$$

so that $\int \bar\phi_k^P \, d\bar\mu_P = \frac{1}{\bar\mu(P)}\int \bar\phi_k \, d\bar\mu = \frac{1}{\bar\mu(P)}\int \phi_k \, d\mu$. We continue using the notation from (7).

Lemma 14. *Suppose $\bar\mu$ is an ergodic invariant measure on Ω such that $\bar\mu(P) > 0$. Then for $k = 1, \dots, m$ we have*

$$\int \bar\phi_k^P \, d\bar\mu_P \leq 3 + \sum_{j=1}^\infty \sum_{\boldsymbol{n}\in N_j} n_k r_{j,\boldsymbol{n}} - \sum_{j=1}^\infty jq_j\delta_j,$$

where δ_j is defined as in (4).

Proof. We fix $1 \le k \le m$ and estimate $\bar{\phi}_k^P(u,v)$ for $(u,v) \in R_{j,\boldsymbol{n}}$. From the construction of $R_{j,\boldsymbol{n}}$ we know that $u_0 \dots u_j \notin \mathcal{L}(Z)$. It follows that whenever $\boldsymbol{\gamma} = (\gamma_0, \dots, \gamma_m) \in S$

$$\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \gamma_k - \delta_j$$

for each $i = 0, \dots, j-1$.

On the other hand, using Lemma 12 we can find $\boldsymbol{\gamma}' \in S$ such that $u_0 \dots u_{j-1} \in \mathcal{L}_j(Z_{\boldsymbol{\gamma}'})$ and $\gamma'_k \in \left(\frac{n_k-1}{j}, \frac{n_k+1}{j}\right)$. Consider any other $\boldsymbol{\gamma} = (\gamma_0, \dots, \gamma_m) \in S$. We have the following possibilities for the location of γ_k with respect to γ'_k: γ_k is less than $\gamma'_k + \frac{2}{j}$; γ_k is in one of the intervals $\left[\gamma'_k + \frac{2}{\ell}, \gamma'_k + \frac{2}{\ell-1}\right)$ where $\ell \in \{2, \dots, j\}$; or γ_k is at least $\gamma'_k + 2$.

If $\gamma_k < \gamma'_k + \frac{2}{j}$ then we see that $\gamma_k < \frac{n_k}{j} + \frac{3}{j}$ since $\gamma'_k \in \left(\frac{n_k-1}{j}, \frac{n_k+1}{j}\right)$. Therefore,

$$\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \gamma_k - \delta_j < \frac{n_k}{j} + \frac{3}{j} - \delta_j.$$

Now suppose that $\gamma_k \in \left[\gamma'_k + \frac{2}{\ell}, \gamma'_k + \frac{2}{\ell-1}\right)$ for some $\ell \in \{2, \dots, j\}$. We claim that $\mathcal{L}_\ell(Y_{\gamma_k})$ and $\mathcal{L}_\ell(Y_{\gamma'_k})$ are disjoint. To see this, note that any element of $\mathcal{L}_\ell(Y_{\gamma_k})$ has weight at least $\lfloor \ell\gamma_k \rfloor$ while any element of $\mathcal{L}_\ell(Y_{\gamma'_k})$ has weight at most $\lceil \ell\gamma'_k \rceil$. Since $\gamma_k \ge \gamma'_k + \frac{2}{\ell}$. it follows that $\ell\gamma_k \ge \ell\gamma'_k + 2$, so that $\lfloor \ell\gamma_k \rfloor > \lceil \ell\gamma'_k \rceil$. It follows that $\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \gamma_k - \delta_{\ell-1}$ for each $i = 0, \dots, j-1$. Using that $\gamma_k \in \left[\gamma'_k + \frac{2}{\ell}, \gamma'_k + \frac{2}{\ell-1}\right)$ and that $\left(\frac{2}{j} - \delta_j\right)$ is an increasing sequence we obtain

$$\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \gamma_k - \delta_{\ell-1} \le \gamma'_k + \frac{2}{\ell-1} - \delta_{\ell-1} < \frac{n_k+1}{j} + \frac{2}{j} - \delta_j.$$

Finally, let $\gamma_k \ge \gamma'_k + 2$. Since in this case $\lfloor \ell\gamma_k + a \rfloor > \lfloor \ell\gamma'_k + a' \rfloor$ for all $a, a' \in [0,1)$ we see that $y_i^k \notin \mathcal{L}_1(Y_{\gamma_k})$ for $i = 0, \dots, j-1$. Hence,

$$\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \gamma_k - \delta_0 \le 2L + \gamma'_k - \delta_0 < \frac{n_k+1}{j} - \delta_j,$$

since $\delta_0 - 2L \ge \delta_1 > \delta_j$ be definition.

We have shown that for all $\boldsymbol{\gamma} \in S$, for all $(u,v) \in R_{j,\boldsymbol{n}}$ and all $i = 0, \dots, j-1$ we have $\phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \frac{n_k}{j} + \frac{3}{j} - \delta_j$. Hence, $\phi_k(\sigma^i u) = \sup_{\boldsymbol{\gamma} \in S} \phi_{k,\boldsymbol{\gamma}}(\sigma^i u) \le \frac{n_k}{j} + \frac{3}{j} - \delta_j$ and

$$\bar{\phi}_k^P(u,v) = \sum_{i=0}^{j-1} \phi_k(\sigma^i u) \le j\left(\frac{n_k}{j} + \frac{3}{j} - \delta_j\right) = n_k + 3 - j\delta_j.$$

Now integrating and applying (9), we see

$$\begin{aligned}\int \bar{\phi}_k^P \, d\bar{\mu}_P &\leq \sum_{j=1}^{\infty} \sum_{\boldsymbol{n}\in N_j} (n_k + 3 - j\delta_j)\bar{\mu}_P(R_{j,\boldsymbol{n}}) \\ &= 3 + \sum_{j=1}^{\infty} \sum_{\boldsymbol{n}\in N_j} n_k r_{j,\boldsymbol{n}} - \sum_{j=1}^{\infty} j q_j \delta_j,\end{aligned}$$

as required. □

We are now ready to establish (5). Fix the values of the parameters $(t_1, ..., t_m) \in (\alpha, \infty)^m$. Suppose μ is an ergodic σ-invariant measure on $A^{\mathbb{Z}}$ whose support is not contained in $\mathrm{Cl}(Z)$. Then its lift $\bar{\mu}$ is an ergodic $\bar{\sigma}$-invariant measure on Ω such that $\bar{\mu}(P) > 0$ and we can induce on P. Combining the estimates in Lemma 13 and Lemma 14, we see that

(11)

$$\begin{aligned}&h(\bar{\mu}_P) + \int (t_1\bar{\phi}_1^P + \ldots + t_m\bar{\phi}_m^P)\, d\bar{\mu}_P \\ &\leq c + 6 + (2L+2)m + (3m+6)\sum_{j=1}^{\infty} q_j \log j + \sum_{j=1}^{\infty} \sum_{\boldsymbol{n}\in N_j} n_0 r_{j,\boldsymbol{n}} \\ &+ \sum_{j=1}^{\infty} \sum_{\boldsymbol{n}\in N_j} (t_1 n_1 + \ldots + t_m n_m) r_{j,\boldsymbol{n}} + (t_1 + \ldots + t_m)\left(3 - \sum_{j=1}^{\infty} j q_j \delta_j\right).\end{aligned}$$

We estimate the terms containing $r_{j,\boldsymbol{n}}$ first. Let $j \in \mathbb{N}$ and let $\boldsymbol{n} = (n_0, ..., n_m) \in N_j$. Since the set $R_{j,\boldsymbol{n}}$ is not empty, by Lemma 12 for each such j and $\boldsymbol{n}$ we can find some $\boldsymbol{\gamma} = (\gamma_0, ..., \gamma_m) \in S$ satisfying $n_k < j\gamma_k + 1$ for $k = 0, \ldots, m$. Therefore,

$$n_0 + t_1 n_1 + \ldots + t_m n_m \leq j(\gamma_0 + t_1\gamma_1 + \ldots + t_m\gamma_n) + 1 + t_1 + \ldots + t_m.$$

Since $\boldsymbol{\gamma} \in S$, Lemma 3 implies that $\gamma_0 + t_1\gamma_1 + \ldots + t_m\gamma_n \leq F(t_1, \ldots, t_m)$. Writing $\boldsymbol{t} = (t_1, ..., t_m)$ and using (9) we get

$$\begin{aligned}\sum_{\boldsymbol{n}\in N_j} (n_0 + t_1 n_1 \ldots + t_m n_m) r_{j,\boldsymbol{n}} &\leq \sum_{\boldsymbol{n}\in N_j} (jF(\boldsymbol{t}) + 1 + t_1 + \ldots + t_m) r_{j,\boldsymbol{n}} \\ &= [jF(\boldsymbol{t}) + 1 + t_1 + \ldots + t_m] q_j.\end{aligned}$$

Recall from (8) and (10) that $\sum_j q_j = 1$ and $\sum_j jq_j = \frac{1}{\bar{\mu}(P)}$. Hence, summing over j gives

$$\sum_{j=1}^{\infty} \sum_{\boldsymbol{n} \in N_j} (n_0 + t_1 n_1 \ldots + t_m n_m) r_{j,\boldsymbol{n}} \leq \sum_{j=1}^{\infty} [jF(\boldsymbol{t}) + 1 + t_1 + \ldots + t_m] q_j$$
$$= \frac{F(\boldsymbol{t})}{\bar{\mu}(P)} + 1 + t_1 + \ldots + t_m.$$

Substituting the bound we just obtained for the terms containing $r_{j,\boldsymbol{n}}$ into (11) and applying $\sum q_j = 1$ in the last line we get

$$h(\bar{\mu}_P) + \int (t_1 \bar{\phi}_1^P + \ldots + t_m \bar{\phi}_m^P) \, d\bar{\mu}_P$$
$$\leq (c + 2L + 9)m + 9m \sum_{j=1}^{\infty} q_j \log j + (t_1 + \ldots + t_m) \left(4 - \sum_{j=1}^{\infty} jq_j \delta_j \right) + \frac{F(\boldsymbol{t})}{\bar{\mu}(P)}$$
$$= \sum_{j=1}^{\infty} q_j \big[(c + 2L + 9)m + 9m \log j - (t_1 + \ldots + t_m)(j\delta_j - 4) \big] + \frac{F(\boldsymbol{t})}{\bar{\mu}(P)}.$$

Since

$$j\delta_j > \frac{c + 2L + 9 + 9 \log j}{\alpha} + 4$$

and $t_1 + ... + t_m > m\alpha$ we observe that the bracketed expression is negative. This gives

$$h(\bar{\mu}_P) + \int (t_1 \bar{\phi}_1^P + \ldots + t_m \bar{\phi}_m^P) \, d\bar{\mu}_P < \frac{F(t_1, \ldots, t_m)}{\bar{\mu}(P)}.$$

As mentioned above $h(\mu) = h(\bar{\mu})$; Abramov's formula then implies that $h(\mu) = \bar{\mu}(P) h(\bar{\mu}_P)$. Also, $\int \phi_k \, d\mu = \bar{\mu}(P) \int \bar{\phi}_k^P \, d\bar{\mu}_P$ so that

$$h(\mu) + \int (t_1 \phi_1 + \ldots + t_m \phi_m) \, d\mu < F(t_1, \ldots, t_m),$$

as required. This completes the proof of Theorem 1.

Remark 15. *Note that allowing α to be zero in the statement of the theorem makes it false. Any pressure function on a full shift Σ intercepts the vertical axis at $h_{\text{top}}(\Sigma)$. Hence, if the value of F at the origin is not equal to the logarithm of an integer greater than 1, F cannot be a pressure function on $[0, \infty)^m$ for any full shift. However, for any $\alpha > 0$ we can still match F to a pressure function on $(\alpha, \infty)^m$.*

6. One-parameter pressure function

Of particular interest is a one-parameter pressure function $t \mapsto P(t\phi)$ since then t can be interpreted as the inverse temperature of the system. It follows immediately from the variational principle that the pressure function is convex, Lipschitz and asymptotically linear at infinity. As a consequence of Theorem 1 we see that these are the only restrictions. Furthermore, in the one-parameter situation boundedness of the vertical axis intercepts of the supporting lines implies both the Lipschitz condition and the existence of a slant asymptote. Indeed, we have

Lemma 16. *Let $\alpha > 0$ and let $f(t)$ be a convex function on (α, ∞) such that the support lines to f at each $t \in (\alpha, \infty)$ have vertical axis intercepts in a closed interval $[b, c] \subset [0, \infty)$. Then $f(t)$ is Lipschitz.*

Proof. We first show that the derivatives are uniformly bounded above. Note that for a single-variable function f its subdifferential at $t \in (\alpha, \infty)$ is the interval $\partial f(t) = [f'(t^-), f'(t^+)]$, where $f'(t^-)$ and $f'(t^+)$ denote the left and right derivatives of f at t respectively. As in the multi-variable case, the subdifferential of f is characterized by the property that $v \in \partial f(t)$ if and only if $f(t)+v(s-t) \leq f(s)$ for all $s \in (\alpha, \infty)$. Given $t \in (\alpha, \infty)$ and $v \in \partial f(t)$, the *intercept* of the sub-tangent line with slope v is $F(t) - vt$, which is the intercept of the sub-tangent line $\ell(s) = f(t) + v(s - t)$ with the vertical axis.

Fix $\beta > \alpha$. Let $t \in (\alpha, \infty)$ be arbitrary and let $v \in \partial f(t)$. Let $\iota \in [b, c]$ be the corresponding intercept. Then $f(\beta) \geq \iota + v\beta \geq b + v\beta$. It follows that $v \leq (f(\beta) - b)/\beta$, giving a uniform upper bound on derivatives of f.

For a lower bound on the derivatives, consider a sub-tangent line at t with the slope v and intercept ι, so that $f(t) = \iota + vt$. In particular, we see $v \geq (f(t) - c)/t$. Since $\partial f(s) \leq \partial f(t)$ whenever $s \leq t$, it suffices to show that $\lim_{s \to \alpha^+} \partial f(s)$ is finite. By convexity, $f(\alpha^+)$ exists and is at least $f(\beta) - u(\beta - \alpha)$ where $u \in \partial f(\beta)$, so that $f(\alpha^+) \in (-\infty, \infty]$. Hence the inequality above shows that $\partial f(\alpha^+) \geq (f(\alpha^+) - c)/\alpha$ giving the required lower bound. □

Corollary 17. *Let $\alpha > 0$ and let $f(t)$ be a convex function on (α, ∞) such that the support lines to f at each $t \in (\alpha, \infty)$ have vertical axis intercepts in a closed interval $[b, c] \subset [0, \infty)$. Then there exists a full shift on a finite alphabet and a continuous potential ϕ such that $P_{\text{top}}(t\phi) = f(t)$ for all $t \in (\alpha, \infty)$.*

Proof. Note that $f(t)$ is Lipschitz by the above lemma and then apply Theorem 1 with $m = 1$. □

We point out that contrary to the one-parameter situation, the multi-parameter pressure function is no longer asymptotically linear. Furthermore, in higher dimensions the fact that the intercepts are bounded does not imply that the convex function is Lipschitz. We give an example to illustrate this.

Example 18. *This is an example of a convex non-Lipschitz function F such that the set of vertical-axis intercepts of all the support planes to the graph of F is bounded.*

Proof. For $(t_1, t_2) \in (0, \infty)^2$, set

$$F(t_1, t_2) = \sup_{s \in [0,\infty)} (t_1 s - t_2 s^2).$$

Notice that for a fixed s, $t_1 s - t_2 s^2$ is a linear function of (t_1, t_2) so that F is convex. For fixed $(t_1, t_2) \in (0, \infty)^2$,

$$t_1 s - t_2 s^2 = \frac{t_1^2}{4t_2} - t_2 \left(s - \frac{t_1}{2t_2} \right)^2 ,$$

so that $F(t_1, t_2) = \frac{t_1^2}{4t_2}$. Since $F(t_1, t_2)$ is the supremum of a collection of linear functions of (t_1, t_2), it is convex. Since each of the linear functions in the collection has intercept 0, the collection of intercepts is bounded. However F clearly fails to be Lipschitz. □

We finish this section with an application of our result to describe feasible occurrences of first-order phase transitions. Let $\alpha > 0$ and let (z_j) be an arbitrary (possibly finite) sequence of terms in (α, ∞). Let $S = \{z_j\}$. We define a function ϕ as follows. First let $g\colon (\alpha, \infty) \to \mathbb{R}$ be given by

$$g(s) = \sum_{\{j\colon z_j \le s\}} \frac{\alpha}{2^j z_j^2}.$$

Then define

$$f(t) = 3 + \int_0^t g(s)\, ds.$$

Notice that since $\alpha/(2^j z_j^2)$ are summable, g is continuous everywhere except on S, where it jumps upwards, guaranteeing that f is differentiable precisely on $(\alpha, \infty) \setminus S$.

We claim that f satisfies the hypotheses of Corollary 17. The vertical axis intercept of the support line at $t \in (\alpha, \infty)$ (or the support line with the largest gradient if $t \in S$) is given by

$$\begin{aligned} f(t) - tg(t) &= 3 - \int_0^t \big(g(t) - g(s)\big)\, ds \\ &= 3 - \int_0^t \sum_{\{j\colon s<z_j\le t\}} \frac{\alpha}{2^j z_j^2}\, ds \\ &\ge 3 - \int_0^t \sum_j \frac{\alpha}{2^j \max(\alpha, s)^2}\, ds \\ &= 3 - \int_0^t \frac{\alpha}{\max(\alpha, s)^2}\, ds \\ &= 3 - \int_0^\alpha \frac{1}{\alpha}\, ds - \int_\alpha^t \frac{\alpha}{s^2}\, ds \ge 1. \end{aligned}$$

The theorem shows that we are able to construct a potential ϕ whose pressure function has an arbitrary countable collection of first order phase transitions.

7. Cardinality of equilibrium states

In this section, we briefly outline a strategy for showing that not only is one free to specify the pressure function, but there is also a lot of freedom in controlling the cardinality of the set of ergodic equilibrium states. In particular, we prove Theorem 2.

In the proof of Theorem 1, for each function $F\colon (\alpha, \infty)^m \to \mathbb{R}$ satisfying the conditions of the theorem, we constructed a full shift and a family of potentials $(\phi_i)_{i=1}^m$ such that $F(t_1, \dots, t_m) = P(t_1\phi_1 + \dots + t_m\phi_m)$ for each $\mathbf{t} \in (\alpha, \infty)^m$. It is natural to ask about the cardinality of the set of equilibrium states for these $t_1\phi_1 + \dots + t_m\phi_m$. The proof establishes that the ergodic equilibrium states for $t_1\phi_1 + \dots + t_m\phi_m$ are precisely the measures of maximal entropy supported on the $Z_{\boldsymbol{\gamma}}$ such that $\boldsymbol{\gamma} \in \partial F(\mathbf{t})$.

For instance, if $\mathbf{t}$ is not a point of differentiability for F, then there are multiple (uncountably many) ergodic equilibrium states for $t_1\phi_1 + \dots + t_m\phi_m$. For $\mathbf{t}$ that are differentiability points of F, there is exactly one element $\boldsymbol{\gamma}$ of $\partial F(\mathbf{t})$. However the space $Z_{\boldsymbol{\gamma}} = X_{e^{\gamma_0}} \times Y_{\gamma_0} \times \dots \times Y_{\gamma_m}$ may still support uncountably many measures of maximal entropy if there is a rational relationship between $\gamma_0, \gamma_1, \dots, \gamma_m$ (more specifically, if there exists a non-trivial integer combination of irrational γ_j's taking an integer value).

For one-parameter pressure functions we can modify our construction slightly and obtain uniqueness of the equilibrium states everywhere except for the points of non-differentiability. The main difference is that instead of parameterizing by the supporting hyperplanes, S, we can parameterize simply by the intercept of the support line with the vertical axis. It is necessary to use a single Sturmian component rather than the two Sturmian components in order to avoid possible rational dependencies between gammas as described above.

When $m = 1$ in Theorem 1 we have $Z_{\boldsymbol{\gamma}} = X_{e^{\gamma_0}} \times Y_{\gamma_0} \times Y_{\gamma_1}$. We express γ_1 as a function of γ_0, which makes the factor Y_{γ_1} redundant. Recall from Section 3 that $X_{e^{\gamma_0}}$ has a unique measure of maximal entropy which is weak-mixing and Bernoulli. Also Y_{γ_0} is uniquely ergodic, so that $X_{e^{\gamma_0}} \times Y_{\gamma_0}$ supports a unique measure of maximal entropy [26]. This measure is then the only equilibrium state of $t\phi$ in the case when the pressure function is differentiable at t and γ_0 is the vertical intercept of its tangent line at t.

To make the above precise, we fix a convex function $f(t)$ on (α, ∞) such that the support lines to f at each $t \in (\alpha, \infty)$ have vertical axis intercepts in a closed interval $[b, c] \subset [0, \infty)$. For $\gamma \in [b, c]$ we define he function

$$s(\gamma) = \sup\{\mathrm{v} \colon \gamma + t\mathrm{v} \le f(t) \text{ for } t \in (\alpha, \infty)\}.$$

We show that the function $s(\gamma)$ is non-increasing and Lipschitz with Lipschitz constant $\frac{1}{\alpha}$. Let $\gamma < \gamma'$ and let v be such that $\gamma' + t\mathrm{v} \le f(t)$ for all $t \in (\alpha, \infty)$, then $\gamma + t\mathrm{v} \le f(t)$ for all $t \in (\alpha, \infty)$, so that $s(\gamma) \ge s(\gamma')$. Next, observe that if $\gamma + t\mathrm{v} \le f(t)$ for all $t \in (\alpha, \infty)$, then $\gamma' + t(\mathrm{v} - \frac{\gamma' - \gamma}{\alpha}) \le \gamma + t\mathrm{v} \le f(t)$ for all $t \in (\alpha, \infty)$, so that $s(\gamma') \ge \mathrm{v} - \frac{\gamma' - \gamma}{\alpha}$ and $|s(\gamma') - s(\gamma)| \le \frac{|\gamma' - \gamma|}{\alpha}$ as required.

It is easy to verify that for each $t \in (\alpha, \infty)$

$$f(t) = \sup_{\gamma \in [b,c]} (\gamma + s(\gamma)t).$$

Hence, we let the alphabet $A = \{0, 1, ..., \lfloor e^c \rfloor\} \times \{\lfloor b \rfloor, ..., \lceil c \rceil\}$ and for $z \in A^{\mathbb{Z}}$ define

$$\phi_\gamma(z) = s(\gamma) - \delta_{j_\gamma(z)} \quad \text{and} \quad \phi(z) = \sup_{\gamma \in [b,c]} \phi_\gamma(z),$$

where $j_\gamma(z)$ and $\delta_{j_\gamma(z)}$ are as in (4) with $Z_\gamma = X_{e^\gamma} \times Y_\gamma$. The uniform equicontinuity of the family $\{\phi_\gamma \colon \gamma \in [b, c]\}$ ensures that ϕ is continuous.

We still need to confirm that $\phi(z) = s(\gamma)$ whenever $z = (x, y) \in Z_\gamma$. Since $s(\gamma)$ is non-decreasing, $\phi_{\gamma'} \le s(\gamma)$ for $\gamma' \le \gamma$. For $\gamma' > \gamma$ we choose $j = \lceil 1/(\gamma - \gamma') \rceil$ and by looking at the weight of the word $y_{-j}...y_{j-1}$ conclude that it is not in $\mathcal{L}_{2j}(Y'_\gamma)$. It follows that $\phi_{\gamma'}(z) \le s(\gamma') - \delta_j$. Since $j \le \frac{1+c}{\gamma - \gamma'}$ and $s(\gamma)$ is Lipschitz with constant $\frac{1}{\alpha}$, we see that

$$\phi_{\gamma'}(z) \le s(\gamma') - \delta_j \le s(\gamma') - \frac{1+c}{\alpha j} \le s(\gamma') - \frac{\gamma-\gamma'}{\alpha} \le s(\gamma).$$

The rest is a verbatim repetition of the proof of Theorem 1 with $\gamma = \gamma_0$ and $m = 0$. The only minor adjustment is in Lemma 14, where the integral estimate becomes

$$\int \bar{\phi}^P \, d\bar{\mu}_P \le \tfrac{3}{\alpha} + \sum_{j,n} j r_{j,n} s\big(\tfrac{n}{j}\big) - \sum_j j q_j \delta_j.$$

The reason is that the value of the potential ϕ on each $R_{j,n}$ is approximately $s\big(\frac{n}{j}\big)$, which follows from Lemma 12 and the fact that $s(\gamma)$ is Lipschitz.

We have established the initial construction where the potential $t\phi$ has a unique equilibrium state for each t where $f(t)$ is differentiable. Now we are in position to add equilibrium states to $t\phi$ at various points $t \in (\alpha, \infty)$ as we see fit. The key idea is to replace the sets Z_γ which support the equilibrium states for $t\phi$ by

$$Z_\gamma = X_{e^\gamma} \times Y_\gamma \times D_\gamma$$

where D_γ is a *decoration factor*. Suppose for $t \in (\alpha, \infty)$ we would like the potential $t\phi$ to have precisely $N(t)$ ergodic equilibrium states. Then we impose the following conditions on the family $\{D_\gamma : \gamma \in [b, c]\}$:

(i) for $\gamma \in \partial f(t)$ the subshift Z_γ supports exactly $N(t)$ ergodic measures of maximal entropy;
(ii) $h_{\text{top}}(\text{Cl}(\bigcup_{\gamma\in[b,c]} D_\gamma)) = 0$;
(iii) For any $\gamma \in (b, c)$, any invariant measure supported on the set $\bigcap_{\varepsilon>0} \text{Cl}\big(\bigcup_{\gamma'\in(\gamma-\varepsilon,\gamma+\varepsilon)} Z_{\gamma'}\big)$ is supported on Z_γ.

Condition (ii) gives an additional term in Lemma 13, which has to be compensated for in the definition of δ_j. The fact that the additional factor has zero topological entropy ensures that δ_j still converges to 0. Condition (iii) is a mild extension of Lemma 10 to this context.

Theorem 2 is an application of our technique, which illustrates the flexibility of cardinalities of equilibrium measures. Note that the first implication of the statement of the theorem follows from Corollary 17. To prove the second implication, we parameterize by the intercept of the tangent line with the vertical axis as outline above: for each intercept, γ, the line $\gamma + s(\gamma)t$ is tangent to $f(t)$. We let the point of tangency be $\tau(\gamma)$. The function τ is a homeomorphism from (b, c), the interior of the set of intercepts, to (α, ∞).

We use the following choices:

$$D_\gamma = \begin{cases} \{\bar{i}\colon 1 \le i \le N(\tau(\gamma))\} & \text{if } N(\tau(\gamma)) \text{ is finite;} \\ \mathrm{Cl}\left(\bigcup_{t\in[1,\ell]} Y_t\right) & \text{if } N(\tau(\gamma)) = \infty, \end{cases}$$

where $\bar{i}$ denotes the fixed point $\ldots iii{\cdot}iii\ldots$ of the full shift on ℓ symbols.

As we pointed out before, $X_{e^\gamma} \times Y_\gamma$ supports a unique measure of maximal entropy. If $N(\tau(\gamma)) = k$, then D_γ consists of k fixed points, so that it is evident that Z_γ supports exactly k ergodic measures of maximal entropy. If $N(\tau(\gamma)) = \infty$, then D_γ supports uncountably many ergodic measures of maximal entropy, and so does Z_γ. This establishes condition (i).

Condition (ii) follows from a theorem of Mignosi [40]; or from Lemma 7. (Lemma 7 implies that the number of words of length n is at most $(\ell n + 1)n(n + 1)$).

To establish condition (iii), notice that the upper semi-continuity of $N(t)$ ensures that for all γ,

$$\bigcap_{\varepsilon>0} \mathrm{Cl}\Big(\bigcup_{|\gamma'-\gamma|<\epsilon} D_{\gamma'}\Big) = D_\gamma. \tag{12}$$

In particular, if μ is an ergodic measure supported on $\bar{Z} := \mathrm{Cl}\big(\bigcup Z_\gamma\big)$, by Lemma 10, the projection of μ on its first two factors is supported on some $X_{e^\gamma} \times Y_\gamma$. By (12), the only points in $\bar{Z}$ projecting to $X_{e^\gamma} \times Y_\gamma$ are points in Z_γ. This completes the proof of Theorem 2.

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