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Economics 203

Review of Utility Functions

What follows is a brief overview of the four types of utility functions you have/will encounter in Economics 203: Cobb-Douglas; perfect complements, perfect substitutes, and quasi-linear. In each case, the steps used for solving the consumer's utility-maximization problem are outlined, and any shortcuts are pointed out. Please come to office hours if you have any questions about this material. I strongly recommend not only knowing these steps, but practicing them incessantly.

1. Cobb-Douglas (CD)

General Form:
$U = x^a y^b$, where a and b are constants (number).

Examples:
$U = x^{0.5}y^{0.5}$
$U = xy$
$U = x^2 y$
$U = x^{\frac{1}{3}}y^{\frac{2}{3}}$

MRS:
Take the partial derivative of U with respect to x and the partial derivative of U with respect to y and put the partial derivative U with respect to x over the partial derivate of U with respect to y and simplify. For the general form U with respect to y (U_y) equals $bx^a y^{b-1}$. For the partial derivative of U with respect to x, $ax^{a-1}y^b$. Putting one over the other yields $x^{a-1}y^b/bx^a y^{b-1}$. Simplifying: $\frac{ax^{a-1-a}y^{b-b+1}}{b} = \frac{ax^{-1}y^1}{b} = \frac{ay}{bx}$.

Short cut for finding the MRS:
For CD utility function, it is always true that MRS = ay/bx. The indifference curves are always strictly convex (that is, the MRS diminishes).

Solving for the consumer's utility maximizing consumption bundle:
Set MRS = px/py. Use this equation and the equation for the budget line to solve for x and y.

Tips & Tricks:
With CD utility functions you never need to worry about so-called corner solutions.[1] Corner solutions can only ever arise if indifference curves cut either the x or y axis. It is easy to see that this isn't the case with CD functions, as utility equals zero if either x or y are zero.

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[1] A corner solution is one where the utility-maximizing consumption bundles involves either x or y being equal to zero. That is, we are at either the vertical or horizontal intercept of the budget line.

A nice trick with CD utility functions is that we can calculate directly from the exponents the share of the consumer's income spent on each good. In particular the consumer always spends a/(a+b) of her income on good x and b/(a+b) of her income on good y. So we know that expenditure on good x (that is, $p_x x$) equals a/(a+b) of her income m and expenditure on good y (that is, $p_y y$) equals b/(a+b) of her income m. To solve for good x, set $MRS = \frac{p_x}{p_y} \rightarrow \frac{ay}{bx} = \frac{p_x}{p_y} \rightarrow ap_y y = bp_x x$. Using the fact $p_x x = \frac{b}{a+b} m$ and re-arranging for x, the optimal bundle of good x is given by $x^* = \frac{am}{(a+b)p_x}$. Using the same steps, we can solve for good y: $y^* = \frac{bm}{(a+b)p_y}$.

So, if you know what the values for a, b, m, p_x, and p_y are you can just plug them into these formulas. This shortcut works no matter what the values of a and b are in utility function! Nice.

Please make certain that for exams you are however able to solve CD utility maximization problems from first principles (that is, by showing me you can derive the MRS and know to set it equal to the price ratio and use the equation for the budget line to solve). In instances where I ask you to show these steps, you will not receive credit for using the shortcut.

2. Perfect complements

General form:
U = min{ax,by}, where a and b are constants.

Examples:
U = min{x,y}
U = min{2x,y}
U = min{2x,4y}
U = min{(1/3)x,(1/4)y}

MRS:
The utility function is not differentiable, so we cannot use calculus techniques to find the MRS. However we know that the ICs are "L"-shaped, so all we need to know in order to draw them is the point at which they are kinked. It is always the case that these kinked points of the IC lie on a line whose equation you can derive by setting ax = by. That is, they lie on the line whose equation is y = (a/b)x.

Solving for the consumer's utility maximizing consumption bundle:
We know that the consumer will always choose a bundle where ax = by (that is, a bundle with the "correct" proportions of each good). So you just need find the point where the line y = (a/b)x crosses the budget line and you have solved for the utility maximizing choice of x and y for the consumer. Just as was true in the case of CD preferences, the consumer will get zero utility if either x or y equals zero, so we do not need to worry about corner solutions.

Tips & Tricks:
Sometimes you might find perfect complement preferences described in words, rather than through a utility function. For instance, you might be told that good x is sugar and good y is coffee, and that a particular consumer always like 2 sugars (i.e., 2 nits of x) with each coffee (i.e., for every 1 y). Given the method for finding the kinked points of the ICs it sometimes seems very tempting to try to apply what looks like this method in this context. So you might be tempted to think that if the consumer wants 2x for every 1y, then the ICs in this case are kinked on the line y = 2x. However tempting this may seem, its wrong! Think about which bundles have the "right" proportions of the goods in this case. We know that if the consumer has 1 coffee (1y) she wants 2 sugars (2x). If she has 2 coffees (2x) she wants 4 sugars (4y). And so on.

If you plot these points in a diagram in (x,y) space you will see that in fact you are plotting points on the line y = 1/2x, not y=2x! Number of coffees is half the number of sugars, right? So make sure you think very carefully in situations where perfect complement preferences are described verbally, or else you might fall into this trap (and so get the wrong answer).

3. Perfect substitutes

General form:
U = ax + by, where a and b are constants.

Examples:
U = x + y
U = 2x + y
U = 2x + 3y
U = (1/2)x + (3/4)y

MRS:
The MRS is always equal to a/b, a constant. That is, ICs are straight lines with slope (negative) a/b.

Solving for the consumer's utility maximizing consumption bundle:
These cases can look "tricky" because the MRS is a constant and more often than not will not equal the price ratio. Thus if you have simply memorized a bunch of math steps rather than understood the economics, you will have nowhere to go in this kind of situation. To solve for the consumer's utility maximizing choice, you need to find the MRS and *compare* it to the price ratio, rather than *set it equal* to the price ratio. If MRS > price ratio, the consumer's MB of x always exceeds the MC of x, and hence the consumer will buy all x and no y. Conversely, if the MRS < price ratio the consumer will buy all y and no x. Lastly, if the MRS happens to equal the price ratio, then all points on the budget line yield the same utility, and hence all maximize utility. That is, MB = MC no matter what consumption bundle is under consideration.

Tips & Tricks:
Draw a diagram! It makes it very clear which point on the budget line maximizes utility if you draw the budget line, then figure out whether the ICs are steeper or flatter than that budget line, then think about which point on that budget line puts the consumer on the highest indifference curve. It also helps to minimize confusion if you use a different colored pen for the budget line than that used for the indifference curve.

4. Quasi-linear

General form:
U = ax + f(y) OR U = f(x) + ay, where a is a constant.

Examples:
U = 2x + ln y
U = 0.5x + y0.5
U = 6 ln x + y
U = 2x0.5 + (3/4)y

MRS:
No shortcut for finding the MRS: you will need to take the ratio of partial derivatives in every case. Just as was true for CD utility functions, the MRS is diminishing (i.e., the ICs are strictly convex) but in this case it will only be a function of one of the goods. In particular, the good that enters the utility function linearly will not appear in the MRS (as this variable drops out when you take the partial derivative).

Thus the MRS will only be a function of the good that enters non-linearly. In the examples above, the MRS will only be a function of y for the first two utility functions, and will only be a function of x for the last two utility functions.

Solving for the consumer's utility maximizing consumption bundle:
With quasi-linear utility functions, indifference curves can cross the axes, so we do need to worry about corner solutions. These turn out to be the trickiest utility functions to be confronted with.

To solve the utility maximization problem, begin by setting the MRS = price ratio. Note that – because the MRS is function of just one good – this will yield an answer for that good immediately. This makes it look like you don't need to use the equation for the budget line to solve, but hopefully the economist in you realizes that this can't really be true, since ignoring the budget constraint would mean ignoring the very important fact that consumers have limited resources!!

To see how you can run into trouble here, take the utility function U = 2x + ln y and suppose that prices are px = $20 and py = $2. The MRS = 2y. Setting this MRS equal to the price ratio yields y = 5. This means that the consumer's IC is tangent to a BL whose slope is (negative) 10 at the point where y = 5.

But you need to be very careful, *precisely because* you have thus far ignored the budget constraint. You have no idea – without referring to a budget constraint – whether or not the consumer can actually afford these 5 units of good y. Given the price of y, the consumer would have to have income of at least \$10 in order to afford this quantity, so always check that this is the case. If it is, you are home free and can then calculate the consumption of the other good (good x in this case) by figuring out how much money the consumer has left over after buying the desired quantity of good y. That leftover money will all be spent on units of the other good (good x).

Without wanting to labor the point too much, suppose in this example that the consumer only has \$8 in income. If you blindly set the MRS equal to the price ratio you'll conclude that $y = 5$. If you then blindly use the equation for the budget line to solve for the consumption of good x, you will get $x = -(1/10)$. Of course, this makes absolutely no sense at all, as we can't have negative consumption. So what is going on? This is where thinking about the economics helps. One way to think about this problem is as follows: we know that given the prices for these goods, the consumer would like to consume $y = 5$, but she can't afford this. Draw a diagram to convince yourself that she should simply buy as much y as she can afford (4 units if her income is just 8), and buy no x at all. If you evaluate her MRS at $y = 4$ you will find that the MRS < the price ratio. This means that the MB of units of y exceeds the MC of units of y, so the consumer would like to purchase more y and less x, if that is feasible. But it is not feasible, since she is already consuming no x and spending all her income on good y.

Note that this would not have been a problem if the consumer had, say, \$30 in income. Then she could buy her 5 units of y, and have \$20 left over. What should she do with this \$20? Spend it in good x! So in this case the consumer would buy 5y and 2x.

Tips & Tricks:
Students sometimes have difficulty identifying quasi-linear utility functions, as they can take on many different forms. A failsafe way to identify a quasi-linear utility function is to recognize that it is always the case that one good enters the utility function linearly, while the other good enters non-linearly. So the utility function is "sort of" linear; hence the name quasi-linear.

Exercises:
Problem set material (Problem sets 2 and 3 in particular) will give you practice identifying and working with the different types of utility functions.