The Solution!

(a) $$y_i = \beta x_i + \varepsilon_i$$

Let $\hat{\beta} = \sum_i a_i y_i$ be any linear estimator.

So, $E(\hat{\beta}) = \sum_i a_i E(y_i) = \beta \sum_i a_i x_i$, and

$$\text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta = \beta\left[\sum_i a_i x_i - 1\right].$$

Similarly,

$$\text{var.}(\hat{\beta}) = \sum_i a_i^2 \,\text{var}(y_i) = \sigma^2 \sum_i a_i^2.$$

So, $MSE(\hat{\beta}) = M = \sigma^2 \sum_i a_i^2 + \beta^2 \left[\sum_i a_i x_i - 1\right]^2.$

$$\frac{\partial M}{\partial a_j} = 2\sigma^2 a_j + 2\beta^2\left[\sum_i a_i x_i - 1\right] x_j = 0 \;\; ; \;\forall j. \quad (1)$$

multiply by y_j & add over all j:

$$2\sigma^2 \sum_j a_j y_j + 2\beta^2\left[\sum_i a_i x_i - 1\right]\sum_j x_j y_j = 0$$

$$\text{or: } \sigma^2 \hat{\beta} + \cancel{2}\beta^2\left[\sum_i a_i x_i - 1\right]\sum_j x_j y_j. \quad \text{— (2)}$$

Also, multiply (1) by x_j & sum over all j:

$$2\sigma^2 \sum_j a_j x_j + 2\beta^2\left[\sum_i a_i x_i - 1\right]\sum_j x_j^2 = 0$$

$$\Rightarrow \sum_i a_i x_i = \left[\frac{\beta^2 \sum_i x_i^2}{\sigma^2 + \beta^2 \sum x_i^2}\right] \qquad \left(\sum_i = \sum_j\right)$$

Substituting into (2):

$$\partial H / \partial a_j = 2 h a_j + 2(1-h) x_j \left[\Sigma a_i x_i - 1 \right] = 0.$$

Multiply by y_j & sum :

$$h \Sigma a_j y_j + (1-h) \Sigma x_j y_j \left[\Sigma a_i x_i - 1 \right] = 0,$$

or $$h \hat{\beta} + (1-h) \Sigma x_i y_i \left[\Sigma a_i x_i - 1 \right] = 0.$$

Sum over j after multiplying by x_j —

$$h \Sigma a_j x_j + (1-h) \Sigma x_j^2 \left[\Sigma a_i x_i - 1 \right] = 0$$

$$\Rightarrow \Sigma a_i x_i \left[h + (1-h) \Sigma x_i^2 \right] = (1-h) \Sigma x_i^2$$

$$\Rightarrow \Sigma a_i x_i = \frac{(1-h) \Sigma x_i^2}{h + (1-h) \Sigma x_i^2}$$

& so, $$h \hat{\beta} + (1-h) \Sigma x_i y_i \left[\frac{(1-h) \Sigma x_i^2}{h + (1-h) \Sigma x_i^2} - 1 \right] = 0$$

$$\Rightarrow h \hat{\beta} + (1-h) \Sigma x_i^2 b \left[\frac{(1-h) \Sigma x_i^2 - h - (1-h) \Sigma x_i^2}{h + (1-h) \Sigma x_i^2} \right] = 0$$

$$\Rightarrow \hat{\beta} = \left[\frac{1}{h + (1-h) \Sigma x_i^2} \right] (1-h) \Sigma x_i^2 b$$

$$= \left[\frac{(1-h) \Sigma x_i^2}{h + (1-h) \Sigma x_i^2} \right] b.$$

This estimator is operational for any $h \in (0,1)$, and if $h = 0$, or $h = 1$, $\hat{\beta} = b$, or $\hat{\beta} = 0$, respectively.