Some Notes on the Negative Binomial Distribution

The negative binomial distribution can be defined in terms of the random variable $Y =$ number of failures in independent Bernoulli trials (with probability of “success, $p$) before the $r^{\text{th}}$ success.

The probability mass function for $Y$ is:

$$\Pr[Y = y] = \binom{r + y - 1}{y} p^r (1 - p)^y; \quad y = 0, 1, 2, \ldots; \quad r = 1, 2, \ldots; \quad 0 < p < 1. \quad (1)$$

The negative binomial distribution gets its name from the following relationship:

$$\binom{r + y - 1}{y} = (-1)^y \binom{-r}{y} = (-1)^y \frac{(-r)(-r-1) \cdots (-r-y+1)}{(y)(y-1)(y-2) \cdots (2)(1)}, \quad (2)$$

which defines the usual binomial coefficients in the case of negative integers.

Then, using the usual binomial expansion for a negative power (remember this from high school?), namely:

$$(1 + x)^{-r} = \sum_{i=0}^{\infty} \binom{-r}{i} x^i = \sum_{i=0}^{\infty} (-1)^i \binom{r+i-1}{i} x^i, \quad (3)$$

it follows immediately that

$$\sum_{y=0}^{\infty} \Pr[Y = y] = 1. \quad (4)$$

The mean of the distribution can be obtained as follows:

$$E(Y) = \sum_{y=0}^{\infty} \binom{r + y - 1}{y} p^r (1 - p)^y = \frac{\sum_{y=1}^{\infty} (r + y - 1)!}{y!(r-1)!} p^r (1 - p)^y$$

$$= \frac{r(1-p)}{p} \sum_{y=1}^{\infty} \binom{r + y - 1}{y-1} p^{r+1} (1 - p)^{y-1} = \frac{r(1-p)}{p} \sum_{k=0}^{\infty} \binom{r + 1 + k - 1}{k} p^{r+1} (1 - p)^k$$

$$= \frac{r(1-p)}{p} = r[(1/p) - 1] \quad (5)$$
In the same way, we can show that

\[ E(Y^2) = \frac{r(1-p)[1+r(1-p)]}{p^2}, \]

so that

\[ \text{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{r(1-p)}{p^2}. \] (6)

Some other characteristics of this distribution are as follows:

\[ \text{Skew}(Y) = \frac{(2-p)}{\sqrt{r(1-p)}} ; \text{ which is always positive} \] (8)

\[ \text{Excess Kurtosis}(Y) = \frac{6(1-p) + p^2}{r(1-p)} ; \text{ which is always positive} \] (9)

\[ \text{Mode}(Y) = \frac{(r-1)(1-p)}{p} ; \text{ if } r > 1 \text{ (otherwise the mode = 0)}, \] (10)

and the characteristic function for the negative binomial distribution is

\[ \phi_Y(t) = \left( \frac{p}{1-p e^{it}} \right)^r. \] (11)

You know that the Poisson is a limiting case of the Negative Binomial distribution. This comes about by re-parameterizing the latter distribution in terms of the mean, \( \mu = r[1/p] - 1 \), derived above. Then the probability mass function for the Negative Binomial distribution becomes

\[ \Pr[Y = y] = \frac{\lambda^y}{y!} \frac{(y+r-1)!}{(r-1)!} \frac{1}{(1+\lambda/r)^y}. \] (12)

Taking the limit as \( r \to \infty \), this p.m.f collapses to that for a Poisson-distributed random variable:

\[ \Pr[Y = y] = \frac{\lambda^y}{y!} \exp(-\lambda). \] (13)

The Geometric distribution is also a special case of the Negative Binomial distribution. In equation (1), set \( r = 1 \):

\[ \Pr[Y = y] = p(1-p)^y ; \quad y = 0, 1, 2, \ldots ; \quad 0 < p < 1. \] (13)