UVic Mathematics Competition

October 5, 2021

University of Victoria

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough work from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. Show that, for all sufficiently large integers k, it is possible to arrange k cubes (possibly of different sizes) to tile a single larger cube. With $k = 8$, for instance, 8 identical cubes can be arranged in a $2 \times 2 \times 2$ pattern to tile a single cube.

Question 2. For every finite nonempty set A of real numbers, let $\Pi(A)$ denote the product of all elements of A. Evaluate the sum $\sum \frac{1}{\Pi(A)}$, where the sum is taken over all nonempty subsets of the set $\{1, 2, \ldots, 2021\}$.

Question 3. Let $f(x) = e^x \cos(x^2) + x^2$ and $g(x) = e^x \sin(x) - x^3$. Show that there are infinitely many values of x such that $f(x) = g(x)$.

Question 4. A circle c is tangent to three mutually tangent circles of radii 1, 3 and 4, as shown. Determine the radius of c.