

UVIC MATHEMATICS COMPETITION 2022

- All necessary work to justify an answer and all the steps of a proof must be clearly shown to obtain full credit.
- Partial credit will only be awarded for substantial progress towards a solution.
- All questions are worth equal marks.

NO CALCULATORS, NOTES OR BOOKS ALLOWED

1) Show that there exist infinitely many natural numbers n such that the base 3 expansion of n has first and last digits 1, and the base 5 expansion of n has first and last digits 2.

2) For an integer n , write $\log^{(n)}(x)$ for $\log(\log(\cdots \log(x)))$, where \log denotes the natural logarithm, and the logarithm is applied n times. For $x > 0$ let $F(x) = x \log x \log(\log x) \cdots \log^{(n)}(x)$ where n is the smallest integer such that $\log^{(n)}(x) < e$.

Is the improper integral

$$\int_e^\infty \frac{1}{F(x)} dx$$

convergent or divergent?

3) Suppose that A and B are 2×2 real matrices such that $AB = BA$. Suppose that A has a non-real eigenvalue and B has a real eigenvalue. Show that B is a multiple of the identity.

4) Let ABC be a triangle such that length of the side BC is twice the length of the side BA . Let D be the midpoint of BC and E be the midpoint of BD . Prove that the line DA bisects the angle at A of the triangle AEC .