UVic Mathematics Competition
September 26, 2023

University of Victoria

- No calculators, books or notes are allowed.
- Write solutions in the booklets provided. Clearly separate rough work from solutions.
- All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit.
- Partial credit will be given only for substantial progress toward a solution.
- Questions are of equal value.

Duration: 2 hours

Question 1. Evaluate $\displaystyle\int_0^{\pi/2} \sin(\sin^2 x)\cos(\cos^2 x)\,dx$.

Question 2. A unit circle is centred uniformly at random in the plane. What is the probability that this circle contains exactly three lattice points in its interior?

Question 3. Alice and Bob play a game in which they take turns choosing a vector $\mathbf{v}_i$ from a supply $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2n} \in \mathbb{R}^d$. Alice goes first, and the game stops when the supply is empty. A player is declared the winner if the sum of their n chosen vectors has a larger length in $\mathbb{R}^d$ than the other player's vector sum. Show that Alice has a strategy to ensure Bob never wins.

Question 4. Let α be an irrational real number with $\alpha > 1$. Show that every nonnegative integer n can be expressed in the form

$$n = d_0 + d_1\lfloor\alpha\rfloor + d_2\lfloor\alpha^2\rfloor + \cdots + d_k\lfloor\alpha^k\rfloor$$

for some integers $k \geq 0$ and $0 \leq d_i \leq \lceil\alpha\rceil$ with $d_i = \lceil\alpha\rceil$ for at most one value of i.