Topological aspects of real-valued logic

by

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Abstract

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We study interactions between general topology and the model theory of real-valued logic. This thesis includes both applications of topological ideas to obtain results in pure model theory, and a model-theoretic approach to the study of compacta via their rings of continuous functions viewed as metric structures.

We introduce an infinitary real-valued extension of first-order continuous logic for metric structures which is analogous to the discrete logic $\mathcal{L}_{\omega_1,\omega}$, and use topological methods to develop the model theory of this new logic. Our logic differs from previous infinitary logics for metric structures in that we allow the creation of formulas $\inf_n \varphi_n$ and $\sup_n \varphi_n$ for all countable sequences $(\varphi_n)_{n<\omega}$ of formulas. Our more general context allows us to axiomatize several important classes of structures from functional analysis which are not captured by previous logics for metric structures. We give a topological proof of an omitting types theorem for this logic, which gives a common generalization of the omitting types theorems of Henson and Keisler. Consequently, we obtain a strengthening of a result of Ben Yaacov and Iovino concerning separable quotients of Banach spaces. We show that continuous functions on separable metric structures are definable in our $\mathcal{L}_{\omega_1,\omega}$ if and only if they are automorphism invariant.

The second part of this thesis develops the model theory of the C*-algebras $C(X)$, for X a compact Hausdorff space. We describe all complete theories of these algebras for X a compact 0-dimensional space. We show that the complete theories of $C(X)$ (for X of any dimension) having quantifier elimination are exactly the theories of $\mathbb{C}$, $\mathbb{C}^2$, and $C(2^{\mathbb{N}})$, and that if the theory of $C(X)$ is model complete and X is connected then X is co-elementarily equivalent to the pseudoarc. We use model-theoretic forcing to answer a question of P. Bankston by showing that the pseudoarc is a co-existentially closed continuum.

Acknowledgements

Contents

Chapter 1

Introduction

This thesis is concerned with real-valued logic, that is, with a kind of logic where "truth values" lie in $\mathbb{R}$, rather in than in a two-element set as in traditional logic. Logics with more than two truth values were formalized by Łukasiewicz in the 1920's for three truth values [80] and later infinitely many truth values [81]. Pavelka added rational constant connectives to the real-valued version of Łukasiewicz logic and proved a completeness theorem for the resulting Łukasiewicz-Pavelka logic [92, 93, 94]. Later, Hájek, Paris, and Shepherdson proved that Łukasiewicz-Pavelka logic is a conservative extension of Łukasiewicz logic [58]. The reader interested in a survey of Łukasiewicz logic and its variants can consult [59].

Our interest is in using many-valued logic to study mathematical structures arising from functional analysis. The logic we will be using has the same expressive power as Łukasiewicz-Pavelka logic (see [25, Proposition 1.18]), though it is formally different and was motivated by different concerns. Initial applications of mathematical logic in analysis came by way of the ultrapower construction, which was first used by Krivine in his 1967 thèse d'état [72]. The ultrapower of a Banach space can be seen as a special case of the nonstandard hull construction of Luxemburg [82], which also applies to metric spaces in general. Ultrapowers of Banach spaces were later used by Dacunha-Castelle and Krivine to study Orlicz spaces [30] and more general Banach spaces and Banach algebras [31]; at approximately the same time, McDuff considered ultrapowers of tracial von Neumann algebras [84]. We recall here the definition of the ultrapower of a Banach space; a more general definition of ultraproducts of metric structures, which includes the von Neumann algebra case and many others, appears as Definition 2.1.7 below.

Definition 1.0.1. Let X be a (real or complex) Banach space, and let $\mathcal{U}$ be an ultrafilter on a set I. Let $\ell_\infty(X) = \{\,(x_i)_{i \in I} : \sup_{i \in I} \|x_i\| < \infty\,\}$, and let $c_{\mathcal{U}}(X) = \{\,(x_i)_{i \in I} \in l_\infty(X) : \lim_{i \to \mathcal{U}} x_i = 0\,\}$. The *ultrapower* of X by $\mathcal{U}$ is defined to be the Banach space obtained as the quotient:

$$X^{\mathcal{U}} = \ell_\infty(X)/c_{\mathcal{U}}(X).$$

Krivine, and later Stern, used ultrapowers to solve a variety of problems arising in functional analysis; see, for example, [75], [77], and [103]. Krivine also introduced a real-valued logic, and described connections between model theory and Banach spaces, in the papers [73], [74], and [76].

In 1966 Chang and Keisler [27] introduced a general framework for studying logics with truth values taken in a fixed compact Hausdorff space K. The motivating example for Chang and Keisler's work was the case $K = [0, 1]$. In another direction, in 1976 Henson [62] introduced the notion of approximate

satisfaction of positive bounded formulas for structures based on Banach spaces (see [63] for a survey of this approach). Recently there has been a considerable amount of activity in a $[0, 1]$-valued logic called *continuous first-order logic*, introduced by Ben Yaacov and Usvyatsov [16]. Continuous first-order logic is a reformulation of Henson's logic in the framework of Chang and Keisler, and is the basic logic we will be using throughout this thesis.

One significant difference between continuous first-order logic and the continuous logic of Chang and Keisler is that the latter used traditional structures with a distinguished equality relation, while the former considers structures without equality, but instead equipped with a distinguished metric. More precisely, the semantic objects for continuous first-order logic are *metric structures*, that is, bounded metric spaces with distinguished uniformly continuous functions and predicates (see Definition 2.1.1 below for the precise definition). The *formulas* are defined recursively in a manner analogous to the definition of formulas in first-order logic, but we allow as connectives all continuous functions $f : [0, 1]^n \to [0, 1]$, and we take inf and sup as replacements for the quantifiers $\exists$ and $\forall$. In this setting each formula defines a uniformly continuous $[0, 1]$-valued function on each metric structure, and this uniform continuity of formulas is closely related to the real-valued version of the compactness theorem of first-order logic. We give precise definitions and theorem statements for the results we will use from continuous first-order logic in Chapter 2, and we refer to [13] for a survey.

Throughout the thesis we focus on areas of interaction between topology and the model theory of continuous first-order logic and its extensions. These interactions manifest in two ways. First, in Chapter 3, we develop an infinitary extension of continuous first-order logic. Our main results in this section are purely model-theoretic statements, but we use tools from general topology in the proofs. On the other hand, Chapter 4 concerns the model theory of commutative unital C*-algebras. Such algebras arise as algebras of continuous complex-valued functions on compact Hausdorff spaces, so we view this study as an indirect model theory of compacta.

The first major goal of this thesis is to extend continuous first-order logic to an *infinitary* logic analogous to the classical logic $\mathcal{L}_{\omega_1, \omega}$. In the discrete setting, the logic $\mathcal{L}_{\omega_1, \omega}$ extends first-order logic by allowing as formulas the infinitary conjunction $\bigwedge_{i < \omega} \varphi_i$ and infinitary disjunction $\bigvee_{i < \omega} \varphi_i$, whenever the φ_i are $\mathcal{L}_{\omega_1, \omega}$ formulas with a total of finitely many free variables. Amongst logics extending first-order, the logic $\mathcal{L}_{\omega_1, \omega}$ has the most successfully developed model theory – see [69]. The model theory of this logic is significantly different from first-order model theory, in large part due to the failure of the compactness theorem.

A version of $\mathcal{L}_{\omega_1, \omega}$ for metric structures, which extends continuous first-order logic, was introduced by Ben Yaacov and Iovino [14]. In their logic formulas of the form $\sup_{i < \omega} \varphi_i$ and $\inf_{i < \omega} \varphi_i$ are permitted, provided that the total number of free variables remains finite, and the formulas φ_i have a common modulus of uniform continuity. Given the connection between uniform continuity and compactness, and the fact that an infinitary logic cannot be expected to satisfy the compactness theorem, it appears somewhat unnatural to insist on a common modulus of uniform continuity when creating infinitary formulas. In Chapter 3 we define a new candidate for $\mathcal{L}_{\omega_1, \omega}$ which does not have any restriction on the moduli of continuity of the constituents of an infinitary formula. A consequence of this is that formulas no longer necessarily define continuous functions on all metric structures, and structures need not be elementary substructures of their metric completions. The model theory of our logic therefore tends to have two aspects. First, we obtain results which are closely analogous to those in the classical setting. Second, we observe that if the formulas involved happen to lie in a *continuous fragment* of our $\mathcal{L}_{\omega_1, \omega}$,

then we can state versions of our results for metric structures based on *complete* metric spaces. The following theorem, which is the main result of Section 3.3, is a good illustration. The notion of *principal types* is an adaptation of the corresponding notion from discrete logic (see Definition 3.3.3).

Theorem 1.0.2. *Let T be a theory in a countable fragment L of $\mathcal{L}_{\omega_1,\omega}$. For each $n < \omega$, let Σ_n be a type consistent with T that is not principal over T. Then there is a separable model of T that omits every Σ_n.*

If L is a continuous fragment, and the types have the stronger property of not being metrically *principal, then the separable model omitting each Σ_n can be taken to be complete.*

Consequences of our Omitting Types Theorem include a description of when an $\mathcal{L}_{\omega_1,\omega}$ theory has prime models (Corollary 3.4.10), and the following version of Keisler's two-cardinal theorem (Theorem 3.4.4):

Theorem 1.0.3. *Let S be a two-sorted metric signature, and let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$. Let T be an L-theory and let $\mathcal{M} = \langle M, V, \dots \rangle$ be a model of T where M has density κ and V has density λ, with $\kappa > \lambda \geq \aleph_0$. Then there is a model $\mathcal{N} = \langle N, W, \dots \rangle \equiv_L \mathcal{M}$ with N of density $\aleph_1$ and W of density $\aleph_0$. Moreover, there is a model $\mathcal{M}_0 = \langle M_0, V_0, \dots \rangle$ such that $\mathcal{M}_0 \preceq_L \mathcal{M}, \mathcal{M}_0 \preceq_L \mathcal{N}$, and V_0 is dense in W.*

One of the most important features of the discrete $\mathcal{L}_{\omega_1,\omega}$ is Scott's theorem that every countable discrete structure is determined up to isomorphism by a single sentence. The analogous statement for $\mathcal{L}_{\omega_1,\omega}$ for metric structures is also true (with "countable" replaced by "separable"), as was shown by Ben Yaacov, Nies, and Tsankov [15]. In fact, the Scott sentence for a complete metric structure can be found in the earlier continuous version of $\mathcal{L}_{\omega_1,\omega}$. Nevertheless, we describe in Section 3.5 why our more general logic is needed to prove the following version of Scott's definability theorem (Theorem 3.2.3).

Theorem 1.0.4. *Let M be a separable complete metric structure. For any continuous function $P : M^n \to [0,1]$, the following are equivalent:*

1. *There is an $\mathcal{L}_{\omega_1,\omega}$ formula $\varphi(\vec{x})$ such that for all $\vec{a} \in M^n$,*

$$\varphi^M(\vec{a}) = P(\vec{a}).$$

2. *P is fixed by all automorphisms of M.*

We include in Chapter 3 a comparison of the various infinitary $[0,1]$-valued logics, as well as the framework of Metric Abstract Elementary Classes. Finally, in Section 3.6 we describe various important classes of Banach spaces which can be axiomatized in $\mathcal{L}_{\omega_1,\omega}$, and give the following application of Keisler's two-cardinal theorem to separable quotients of Banach spaces, which improves a result of Ben Yaacov and Iovino.

Theorem 1.0.5. *Let X and Y be infinite-dimensional Banach spaces with $\mathrm{density}(X) > \mathrm{density}(Y)$. Let $T : X \to Y$ be a surjective bounded linear function. Let L be a countable continuous fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, where S is a two-sorted signature, each sort of which is the signature of Banach spaces, together with a symbol to represent T. Then there are Banach spaces X', Y' with Y' separable and X' of density $\aleph_1$, and a surjective bounded linear function $T' : X' \to Y'$, such that $(X, Y, T) \equiv_L (X', Y', T')$.*

Chapter 4, which contains material from the joint papers [38], [39], and [40], is devoted to developing the model theory of commutative unital C*-algebras. In this chapter we work primarily in continuous first-order logic, though we also make brief use of the logic developed in Chapter 3. As mentioned above, the material of this chapter can be seen as an indirect model theory of compacta. Several other approaches to the model-theoretic study of compacta have been successfully used in the past, as we describe in Section 4.1.

In the particular case where the space X is 0-dimensional we show that many model-theoretic properties of the algebra $CL(X)$ of clopen subsets of X translate to $C(X)$. For example, in Section 4.2 we show that there are exactly $\aleph_0$ distinct complete theories of $C(X)$, corresponding to the complete theories of Boolean algebras in discrete logic.

Theorem 1.0.6. *Let X and Y be 0-dimensional compact Hausdorff spaces. Then $C(X) \equiv C(Y)$ in continuous first-order logic if and only if $CL(X) \equiv CL(Y)$ in discrete first-order logic. In particular, for any infinite ordinal α, $C(\alpha + 1) \equiv C(\beta\omega)$. Moreover, if α is a countable limit ordinal then $C(2^\omega) \equiv C(\beta\omega \setminus \omega) \equiv C(\beta\alpha \setminus \alpha)$.*

We show that saturation properties of the Boolean algebra $CL(X)$ translate to $C(X)$. For general 0-dimensional spaces we do not obtain a complete transfer, but rather show that when $CL(X)$ is $\aleph_1$-saturated then $C(X)$ is quantifier-free $\aleph_1$-saturated. On the other hand, when X has no isolated points, the correspondence is perfect, even when $\aleph_1$-saturation is weakened to the notion of degree-1 $\aleph_1$-saturation (see Definition 4.4.1). In Section 4.4 we prove results which imply:

Theorem 1.0.7. *Let X be a compact 0-dimensional space without isolated points. The following are equivalent:*

- $C(X)$ *is $\aleph_1$-saturated,*

- $C(X)$ *is quantifier-free $\aleph_1$-saturated,*

- $C(X)$ *is degree-1 $\aleph_1$-saturated,*

- $CL(X)$ *is $\aleph_1$-saturated.*

A key step in the above proof is showing that the continuous first-order theory of $C(2^\mathbb{N})$ has quantifier elimination. In Section 4.3 we go further and completely characterize those theories of commutative unital C*-algebras with quantifier elimination.

Theorem 1.0.8. *The theories of commutative unital C*-algebras with quantifier elimination are exactly the complete theories of $\mathbb{C}$, $\mathbb{C}^2$, and $C(2^\mathbb{N})$.*

Studying quantifier elimination also sheds light on the existentially closed models of various theories of commutative unital C*-algebras. In dual form, questions of this form had been considered by Bankston, who in particular asked if the pseudoarc is a co-existentially closed continuum. Rephrased in terms of C*-algebras, this question asks if $C(\text{pseudoarc})$ is an existentially closed model of the theory of commutative unital C*-algebras without minimal projections. We answer this question in the affirmative in Corollary 4.3.16.

Theorem 1.0.9. *The pseudoarc is a co-existentially closed continuum.*

As a consequence, we show that if there are any model complete theories of algebras $C(X)$, where X is a connected compact Hausdorff space, then the only such theory is the theory of $C(\text{pseudoarc})$.

Chapter 2

Continuous first-order logic for metric structures

This chapter is primarily an exposition of continuous first-order logic for metric structures. We do not attempt to be exhaustive, but rather provide a self-contained introduction containing the material we will need in the future chapters. Most of this chapter consists of well-known facts from model theory which were adapted to the metric setting in [13]. Some of the results of Section 2.3 are straightforward adaptations of well-known results from discrete logic, but to the best of our knowledge no explicit proofs have appeared in print. We take this opportunity to provide proofs of those results, though we claim originality of neither the results nor the methods of proof. Section 2.4 is the only section of this chapter containing new material.

2.1   Definitions

The basic semantic objects we will be considering throughout this thesis are *metric structures*[1]. Our definition of metric structures agrees with that of [16], except that we do not require the underlying metric spaces to be complete.

Definition 2.1.1. A *metric structure* is a bounded metric space (M, d^M), together with:

- A set $(f_i^M)_{i \in I}$ of uniformly continuous *functions* $f_i : M^{n_i} \to M$,

- a set $(P_j^M)_{j \in J}$ of uniformly continuous *predicates* $P_j : M^{m_j} \to [a_j, b_j]$ for some $a_j < b_j \in \mathbb{R}$,

- a set $(c_k^M)_{k \in K}$ of distinguished *elements* of M.

We place no restrictions on the index sets I, J, and K. We usually write metric structures as tuples $(M, d^M, (f_i^M)_{i \in I}, (P_j^M)_{j \in J}, (c_k^M)_{k \in K})$, and we often write M for both the structure and the underlying metric space.

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[1]In fact, much of the model theory of real-valued logic can be developed with under somewhat weaker conditions than "metric"; specifically, instead of being based on metric spaces the structures used can be based on uniform spaces where the uniformity is generated by a fixed family of pseudometrics. This approach is considered in [55]. For the applications we have in mind "metric" is sufficient, so we will not pursue the more general setting in this thesis.

Various special classes of metric structures have been considered in the literature. For example, metric structures with 1-Lipschitz functions and predicates are the structures used in Łukasiewicz-Pavelka logic [59]. In many treatments of continuous logic, such as [16], attention is restricted to metric structures based on complete metric spaces.

Many classes of structures from functional analysis can be described in the framework of metric structures, although often some work is necessary since most such structures are not *bounded* metric spaces. For structures based on Banach spaces, including Banach lattices, Banach algebras, C*-algebras, and von Neumann algebras, one can either restrict attention to the unit ball or use many-sorted structures with a sort for each closed ball of integer radius centred at 0. For our purposes it is largely unimportant which approach we choose, but for concreteness we adopt the convention that when we consider structures based on Banach spaces we are considering them as many-sorted structures.

When considering metric structures in general we assume that M has diameter 1 and all predicate symbols take values in $[0, 1]$, for notational simplicity. If M is a metric structure then the uniform continuity of the distinguished functions and predicates on M ensures that they can be extended to the metric completion of the underlying metric space of M. We denote the resulting structure by $\overline{M}$, and call it the *completion* of M. Similarly, we call a metric structure *complete* if the underlying metric space is complete.

We have the natural notions of a metric structure N being a *substructure* of another metric structure M of the same signature, denoted $N \subseteq M$. An isometric function $f : N \to M$ is an *embedding* if the image of N is a substructure of M.

Metric structures are the semantic objects we will be studying. On the syntactic side, we have *metric signatures*. By a *modulus of continuity* for a uniformly continuous function $f : M^n \to M$ we mean a function $\delta : \mathbb{Q} \cap (0, 1) \to \mathbb{Q} \cap (0, 1)$ such that such that for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$ and all $\epsilon \in \mathbb{Q} \cap (0, 1)$,

$$\sup_{1 \leq i \leq n} d(a_i, b_i) < \delta(\epsilon) \implies d(f(a_i), f(b_i)) \leq \epsilon.$$

Similarly, δ is a modulus of continuity for $P : M^n \to [0, 1]$ means that for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in M$,

$$\sup_{1 \leq i \leq n} d(a_i, b_i) < \delta(\epsilon) \implies |P(a_i) - P(b_i)| \leq \epsilon.$$

Definition 2.1.2. A *metric signature* consists of the following information:

- A set $(f_i)_{i \in I}$ of *function symbols*, each with an associated arity and modulus of uniform continuity,

- a set $(P_j)_{j \in J}$ of *predicate symbols*, each with an associated arity and modulus of uniform continuity,

- a set $(c_k)_{k \in K}$ of *constant symbols*.

When no ambiguity can arise, we say "signature" instead of "metric signature".

When S is a metric signature and M is a metric structure, we say that M is an S-*structure* if the distinguished functions, predicates, and constants of M match the requirements imposed by S. Given a signature S, the *terms* of S are defined recursively, exactly as in the discrete case.

Definition 2.1.3. Let S be a metric signature. The S-*formulas* of continuous first-order logic are defined recursively as follows.

1. If t_1 and t_2 are terms then $d(t_1, t_2)$ is a formula.

2. If $t_1, \ldots, t_n$ are S-terms, and P is an n-ary predicate symbol, then $P(t_1, \ldots, t_n)$ is a formula.

3. If $\varphi_1, \ldots, \varphi_n$ are formulas, and $f : [0,1]^n \to [0,1]$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is a formula.

4. If φ is a formula and x is a variable, then $\inf_x \varphi$ and $\sup_x \varphi$ are formulas.

We think of the (3) as constructing formulas by using connectives, and (4) as adding quantifiers. In particular, we say that an appearance of a variable in a formula is *free* if it is not under the scope of a sup or inf, and is *bound* otherwise. We write $\varphi(x_1, \ldots, x_n)$ to emphasize that the free variables appearing in the formula φ are a subset of $\{x_1, \ldots, x_n\}$. We often write $\vec{x}$ for a finite tuple of variables when the length is unimportant. Following model-theoretic convention, when M is a metric structure and $\vec{a} = (a_1, \ldots, a_n)$ is a tuple from M, we often write $\vec{a} \in M$ instead of $\vec{a} \in M^n$, particularly when we do not wish to specify the length of $\vec{a}$.

Given an S-structure M, an S-formula $\varphi(\vec{x})$, and a tuple $\vec{a} \in M$, there is a natural way to recursively define the value of φ in M when evaluated at $\vec{a}$, denoted by $\varphi^M(\vec{a})$. We have $\varphi^M(\vec{a}) \in [0,1]$.

Remark 2.1.4. Given an S-structure M and an S-formula $\varphi(x_1, \ldots, x_n)$, we have a function $\varphi^M : M^n \to [0,1]$ given by $\vec{a} \mapsto \varphi^M(\vec{a})$. Since S specifies a modulus of uniform continuity for each symbol, and the connectives are uniformly continuous, this function is uniformly continuous, and the modulus of uniform continuity depends only on φ, not on M. In Chapter 3 we will consider an extension of the logic described here in which formulas no longer define uniformly continuous functions.

We write $M \models \varphi(\vec{a})$ to mean $\varphi^M(\vec{a}) = 0$. A substructure N of a structure M is an *elementary substructure*, written $N \preceq M$, if for all formulas $\varphi(\vec{x})$ and all $\vec{a} \in N$, we have $\varphi^N(\vec{a}) = \varphi^M(\vec{a})$. Because of the inclusion of continuous functions as connectives, it is equivalent to ask only that $N \models \varphi(\vec{a})$ if and only if $M \models \varphi(\vec{a})$. An embedding $f : N \to M$ is an *elementary embedding* if the image of f is an elementary substructure of M.

It is useful to note that we can express weak inequalities as formulas. Particularly, suppose that $\varphi(\vec{x})$ and $\psi(\vec{y})$ are formulas in a common signature S. Then for any S-structure M, and any $\vec{a}, \vec{b} \in M_0$, we have

$$M \models \min\{\psi(\vec{b}) - \varphi(\vec{a}), 0\} \iff \varphi^M(\vec{a}) \leq \psi^M(\vec{b}).$$

In light of this observation, we write $\varphi \leq \psi$ as an abbreviation for $\min\{\psi - \varphi, 0\}$. Being able to express inequalities as formulas will be particularly useful when either φ or ψ is the constant formula r for some $r \in \mathbb{R}$.

We see that $M \models \min\{\varphi, \psi\}$ if and only if $M \models \varphi$ or $M \models \psi$, so we think of min as $\vee$, and occasionally write $\varphi \vee \psi$ instead of $\min\{\varphi, \psi\}$. For similar reasons we write $\varphi \wedge \psi$ for $\max\{\varphi, \psi\}$. We note that the usage of $\vee$ for min is opposite to the meaning of $\vee$ in the context of lattices, but since we will not perform any lattice calculations in this thesis, we trust that no confusion will arise. The quantifier $\sup_x$ behaves as $\forall x$, in that $M \models \sup_x \varphi$ if and only if $M \models \varphi(x)$ for all $x \in M$. The quantifier $\inf_x$ is not precisely $\exists$, for $M \models \inf_x \varphi$ if and only if for every $\epsilon > 0$ there is $x \in M$ such that $\varphi^M(x) < \epsilon$.

Unlike the other connectives from discrete logic, there is no connective in continuous logic which corresponds to negation. That is, given a formula φ, there may not exist a formula ψ such that for every metric structure M and $\vec{a} \in M$, $M \models \psi(\vec{a})$ if and only if $M \not\models \varphi(\vec{a})$. When a metric structure is based on a discrete metric space we can take $1 - \varphi$ to mean $\neg\varphi$, but in general this is not a classical negation. An important special case of this observation is that we can always express equality of two elements (or tuples) in a metric structure, since $a = b$ if and only if $M \models d(a, b)$, but we usually cannot express $a \neq b$.

Remark 2.1.5. Given any structure in the sense of classical first-order logic, we can equip the structure with the discrete metric and identify distinguished relations with their characteristic functions to obtain a (complete) metric structure. There is then a natural way to associate a formula $\tilde{\varphi}$ of continuous logic to each first-order formula φ:

- If φ is $t_1 = t_2$ then $\tilde{\varphi}$ is $d(t_1, t_2)$,

- if φ is $R(t_1, \ldots, t_n)$ then $\tilde{\varphi}$ is also $R(t_1, \ldots, t_n)$,

- if φ is $\neg\psi$ then $\tilde{\varphi}$ is $1 - \tilde{\psi}$,

- if φ is $\psi \wedge \theta$ then $\tilde{\varphi}$ is $\max\{\tilde{\psi} \wedge \tilde{\theta}\}$,

- if φ is $\exists x\psi$ then $\tilde{\varphi}$ is $\inf_x \tilde{\psi}$.

A straightforward induction on formulas, using that M has the discrete metric, shows that for any discrete formula $\varphi(\vec{x})$ and any $\vec{a} \in M$ we have $M \models \varphi(\vec{a})$ (as a discrete structure) if and only if $M \models \tilde{\varphi}(\vec{a})$ (as a metric structure). This translation from discrete to continuous logic is already present (in a slightly different setting) in [27], and is also discussed in detail in [13].

We adopt notation and terminology from the discrete setting. In particular, we call a formula *universal* (respectively, *existential*) if it is of the form $\sup_{\vec{x}} \varphi(\vec{x})$ (respectively, $\inf_{\vec{x}} \varphi(\vec{x})$) where φ is quantifier-free. We denote by $T_\forall$ (respectively, $T_\exists$) the set of universal (respectively, existential) consequences of a theory T. When $T = \mathrm{Th}(M)$ we write $T_\forall = \mathrm{Th}_\forall(M)$.

Remark 2.1.6. One unfortunate consequence of our definition of formulas is that even if $S = \emptyset$ there are $2^{\aleph_0}$ S-formulas, due to clause (3) of the definition. For some results, such as the Downward Löwenheim-Skolem theorem, it is important to have that the number of S-formulas is $|S| + \aleph_0$, as it is in discrete logic. To accomplish this we will implicitly assume that instead of taking *all* continuous $f : [0,1]^n \to [0,1]$ as connectives, we instead assume that for each n we have chosen some (fixed but unspecified) countable subset of these functions which is uniformly dense in the set of all continuous $f : [0,1]^n \to [0,1]$. In fact such countable dense sets can be generated from very few functions; see [16] and [25] for examples. For our purposes we will simply assume that whenever we explicitly write a formula the connectives it uses are in our fixed dense set.

We will make use of the *ultraproduct* of metric structures. The definition generalizes the definitions of ultraproducts of Banach spaces (given in Chapter 1), C*-algebras, and von Neumann algebras, when each of these types of spaces are viewed as metric structures. The metric ultraproduct also generalizes the classical discrete ultraproduct, which is obtained as the special case where all the structures are based on discrete metric spaces. Recall that if X is a topological space, I is an index set, $(a_i)_{i \in I}$ and a are elements of X, and $\mathcal{U}$ is an ultrafilter on I, then we say that a is the *ultrafilter limit of* $(a_i)_{i \in I}$ *along* $\mathcal{U}$, denoted $a = \lim_{i \to \mathcal{U}} a_i$, if for all open sets O around a we have $\{i \in I : a_i \in O\} \in \mathcal{U}$.

Definition 2.1.7. Fix a metric signature S, an index set I, and an ultrafilter $\mathcal{U}$ on I. For each $i \in I$, let M_i be an S-structure. The *ultraproduct* of the M_i's, denoted $\prod_{\mathcal{U}} M_i$, is the metric structure whose underlying set is $\prod_{i \in I} M_i / \sim$, where $(a_i) \sim (b_i)$ if and only if $\lim_{i \to \mathcal{U}} d(a_i, b_i) = 0$. The operations on $\prod_{\mathcal{U}} M_i$ are defined as follows:

- For any sequences (a_i) and (b_i), the distance between their equivalences classes $[a_i]$ and $[b_i]$ is computed as

$$d([a_i], [b_i]) = \lim_{i \to \mathcal{U}} d(a_i, b_i).$$

- For each n-ary function symbol f,

$$f^{\prod_{\mathcal{U}} M_i}([a_i]) = [b_i] \iff \lim_{i \to \mathcal{U}} d(f^{M_i}(a_i), b_i) = 0.$$

- For each n-ary predicate symbol P,

$$P^{\prod_{\mathcal{U}} M_i}([a_i]) = \lim_{i \to \mathcal{U}} P^{M_i}(a_i).$$

- For each constant symbol c,

$$c^{\prod_{\mathcal{U}} M_i} = [(c^{M_i})].$$

It follows from the fact that all of the structures are S-structures (and, in particular, that all of the interpretations of each function or predicate symbol satisfy a common modulus of uniform continuity) that the operations defined above are well-defined. The above definition also satisfy the uniform continuity restrictions imposed by S, so the ultraproduct of a sequence of S-structures is again an S-structure.

Remark 2.1.8. The ultraproduct of a sequence of metric structures along a countably incomplete ultrafilter is always based on a complete metric space, even if the index models are not. In fact, for any sequence $(M_i)_{i \in I}$ of metric structures, and any ultrafilter $\mathcal{U}$ on I, we have

$$\prod_{\mathcal{U}} M_i = \prod_{\mathcal{U}} \overline{M}_i.$$

One inclusion is clear. For the other, suppose that $(a_i)_{i \in I}$ is a sequence from $\prod_{i \in I} \overline{M}_i$. Let $I = I_0 \supseteq I_1 \supseteq \cdots$ be sets in $\mathcal{U}$ whose intersection is not in $\mathcal{U}$. For each $i \in I_n$ such that $i \notin I_{n+1}$, pick $b_i \in M_i$ such that $d(a_i, b_i) < \frac{1}{i}$ in $\overline{M}_i$. Then

$$\lim_{i \to \mathcal{U}} d(a_i, b_i) = 0,$$

so $[a_i]_{\mathcal{U}} = [b_i]_{\mathcal{U}}$, and $[b_i]_{\mathcal{U}} \in \prod_{\mathcal{U}} M_i$.

2.2 Basic model theory

Most of the main theorems of first-order model theory extend to the metric context, with the measure of size of a structure being its *density*, that is, the least cardinality of a dense subset. We limit our discussion to those results that will be useful for us in the following chapters. In the present setting it is straightforward to verify that for any metric structure M we have $M \preceq \overline{M}$, so in all of the model-construction theorems that follow we can take the resulting structure to be complete; this will not be true in Chapter 3.

Convention 2.2.1. For a metric structure M, we denote by $|M|$ the *cardinality* of the metric space M, and by $\|M\|$ the *density* of M.

To start, we have a version of Łoś' theorem for metric ultraproducts.

Theorem 2.2.2 (Łoś). *Let S be a metric signature, and let $(M_i)_{i \in I}$ be S-structures. For any ultrafilter $\mathcal{U}$ on I, any S-formula $\varphi(\vec{x})$, and tuples $\vec{a}_i \in M_i$,*

$$\varphi^{\prod_{\mathcal{U}} M_i}([\vec{a}_i]) = \lim_{i \to \mathcal{U}} \varphi^{M_i}(\vec{a}_i).$$

In particular, it follows that the diagonal embedding $a \mapsto [a, a, \ldots]$ of a structure M into its ultrapower $M^{\mathcal{U}}$ is an elementary embedding. If M is based on a *compact* metric space then the diagonal embedding is easily seen to be surjective, and in fact in this case M is the unique complete model of $\mathrm{Th}(M)$.

Next, we have the metric version of the compactness theorem.

Theorem 2.2.3 (Compactness)**.** *For any set of sentences T, the following are equivalent:*

1. *T is consistent,*

2. *every finite subset of T is consistent,*

3. *for every finite $\Delta \subseteq T$, and every $\epsilon > 0$, there is a structure M such that for all $\sigma \in T$, $\sigma^M < \epsilon$.*

As in the discrete case, the compactness theorem implies the Upward Löwenheim-Skolem theorem.

Theorem 2.2.4 (Upward Löwenheim-Skolem)**.** *Every metric structure M which is not totally bounded has an elementary extension of density κ for every $\kappa \geq \|M\|$.*

We also have the Downward Löwenheim-Skolem theorem, which we will generalize in Theorem 3.1.8.

Theorem 2.2.5 (Downward Löwenheim-Skolem)**.** *Let S be a metric signature, and M an S-structure. For any countable $A \subseteq M$ there is an S-structure $N \preceq M$ with $A \subseteq N$ and $|N| = |A| + |S| + \aleph_0$. We may also take N to be complete, in which case $N \preceq \overline{M}$ and $\|N\| = |A| + |S| + \aleph_0$.*

Detecting elementarity can be done using the Tarski-Vaught test.

Proposition 2.2.6 (Tarski-Vaught Test)**.** *Let S be a metric signature, and $M \subseteq N$ be S-structures. The following are equivalent:*

- *$M \preceq N$,*

- *For every S-formula $\varphi(\vec{x}, y)$, and every $\vec{a} \in M$,*

$$\inf_{b \in N} \varphi^N(\vec{a}, b) = \inf_{c \in M} \varphi^N(\vec{a}, c).$$

The concept of a *type* is defined exactly as in first-order logic, and as in that case the compactness theorem shows that if M is a structure and Σ is a type over a subset of M then there is some elementary extension of M which realizes Σ. For a cardinal κ we say a structure M is κ-*saturated* if M realizes all types over sets $A \subseteq M$ with $|A| < \kappa$. Again, the same arguments as in the discrete case show that every structure has a κ-saturated elementary extension for every κ. One easy consequence of saturation is that when M is $\aleph_0$-saturated the quantifier inf behaves exactly as $\exists$, rather than only approximately as described earlier. Of particular use to us later is that most ultraproducts, including all those where the ultrafilter is countably incomplete, have some saturation.

Theorem 2.2.7. *Let S be a metric signature, and $(M_i)_{i \in I}$ be a sequence of S-structures. For any countably incomplete ultrafilter $\mathcal{U}$ on I, the ultraproduct $\prod_{\mathcal{U}} M_i$ is $\aleph_1$-saturated.*

We will need some results concerning preservation of certain classes of structures by algebraic operations. Particularly, we will use the following standard facts. The first statement is proved in [13], while the other two appear in [45].

Lemma 2.2.8. *Let $\mathcal{K}$ be a class of S-structures for a fixed metric signature S.*

- *If $\mathcal{K}$ is elementary then $\mathcal{K}$ is closed under unions of elementary chains.*

- *$\mathcal{K}$ is universally axiomatizable if and only if it is elementary and closed under substructures.*

- *$\mathcal{K}$ is $\forall\exists$-axiomatizable if and only if it is elementary and closed under unions of chains.*

2.3   Quantifier elimination and model companions

We consider inf and sup as the analogues of the quantifiers $\exists$ and $\forall$, respectively. As in the discrete case, it is often the case that formulas without quantifiers are significantly easier to analyse than formulas with quantifiers. If a formula φ can be uniformly approximated by quantifier-free formulas then φ is essentially quantifier-free, so we take this as our definition of quantifier elimination.

Definition 2.3.1. An S-theory T has *quantifier elimination* if for every S-formula $\varphi(\vec{x})$ and every $\epsilon > 0$ there is an S-formula $\psi_\epsilon(\vec{x})$ such that

$$T \models |\varphi(\vec{x}) - \psi_\epsilon(\vec{x})| \leq \epsilon.$$

The standard tests for quantifier elimination from discrete logic apply in the metric setting as well. The only test we will need in this thesis is the following from [13, Proposition 13.6].

Proposition 2.3.2. *Let S be a metric signature, and T an S-theory. The theory T has quantifier elimination if and only if for every $M, N \models T$ (with $\|M\| \leq |S|$), and every finitely generated $A \subseteq M$, for each embedding $f : A \to N$ there is an elementary extension R of M and an embedding $i : N \to R$ such that the following diagram commutes:*
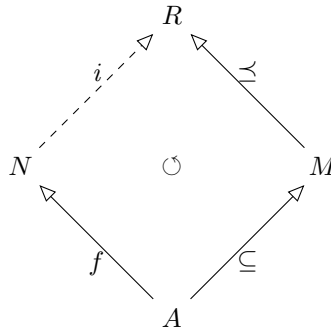
An equivalent statement is obtained if the elementary extension R is required to be an ultrapower of M.

Definition 2.3.3. A theory T is *model-complete* if whenever $M, N \models T$ then every embedding of M into N is an elementary embedding.

It is clear that quantifier elimination implies model-completeness, and by Lemma 2.2.8 model-completeness implies $\forall\exists$-axiomatizability. A particularly important kind of model-complete theory is one which is the model companion or completion of another theory. The results about model companions that we describe below are well-known to experts in the area (see, for instance, [43], [53], [54]), though to the best of our knowledge they have not appeared explicitly, so we take this opportunity to provide

proofs. In fact, the proofs we provide are only slight modifications of the arguments used in the discrete case; see [28, Section 3.5].

Definition 2.3.4. Let T and T^* be theories in the same signature. We say that T^* is the *model companion* of T^* if the following two conditions hold:

1. T^* is model-complete,

2. every model of T embeds into a model of T^*, and every model of T^* embeds into a model of T.

We say that T^* is the *model completion* of T if it is the model companion of T and the following additional property holds:

3. For every $M \models T$, $T^* \cup \Delta_M$ is a complete theory in the signature of M augmented with a new constant symbol for each element of M, where $\Delta_M = \{ \varphi(\vec{a}) : \varphi \text{ is quantifier-free, } \vec{a} \in M, \text{ and } M \models \varphi(\vec{a}) \}$

By Lemma 2.2.8, statement (2) in the definition of model companion is equivalent to saying $T_\forall = (T^*)_\forall$.

Lemma 2.3.5. *If a theory has a model companion then that companion is unique up to logical equivalence.*

Proof. Suppose that T^* and T^{**} are model companions of T, and pick a model $A_0 \models T^*$. Then by part (2) of the definition of model companion we can form a chain $A_0 \subseteq B_0 \subseteq A_1 \subseteq B_1 \subseteq \cdots$, where each $A_i \models T^*$ and $B_i \models T^{**}$. Let M be the union of this chain, which is also the union of the chain of A_i's and the union of the chain of B_i's. Since T^* and T^{**} are model complete the chains of A_i's and B_i's are both elementary, so we have on the one hand that $A_0 \preceq M$, and on the other that $M \models T^{**}$. Therefore every model of T^* is a model of T^{**}. Interchanging the roles of T^* and T^{**} completes the proof. $\square$

In Chapter 4 we will apply model companions and completions in the context where T is $\forall\exists$-axiomatizable. In this case model companions are closely related to existentially closed structures, which we now define.

Definition 2.3.6. A structure M is *existentially closed* in another structure N if $M \subseteq N$ and for every existential $\varphi(\vec{x})$ and every $\vec{a} \in M$, $\varphi^M(\vec{a}) = \varphi^N(\vec{a})$ (equivalently, for all existential $\varphi(\vec{x})$ and $\vec{a} \in M$, $M \models \varphi(\vec{a}) \iff N \models \varphi(\vec{a})$). A structure M is *existentially closed for T* if M is existentially closed in every model of T which contains it. Finally, M is an *existentially closed model of T* if M is existentially closed for T and $M \models T$.

Lemma 2.3.7. *For any structures $M \subseteq N$, the following are equivalent:*

1. *M is existentially closed in N,*

2. *there is a structure M' with $M \preceq M'$ and $N \subseteq M'$.*

Proof. First assume (1). Let eldiag_M denote the elementary diagram of M, that is, the collection of all $\varphi(\vec{a})$ in the language expanded with constants for each element of M, such that $\vec{a} \in M$ and $M \models \varphi(\vec{a})$. Let $T = \text{eldiag}_M \cup \Delta_N$. It suffices to show that T' is consistent, for then a model of T' is the desired M'. If T' is inconsistent then by compactness there is some formula $\varphi(\vec{a}, \vec{b}) \in \Delta_N$ (here $\vec{a} \in M$ and

$\vec{b} \in N \setminus M$) and an $\epsilon > 0$ such that $\text{eldiag}_M \models \varphi(\vec{a}, \vec{b}) \geq \epsilon$. As eldiag_M is a theory in a language without constants for $\vec{b}$, this is equivalent to $\text{eldiag}_M \models \sup_{\vec{y}} \varphi(\vec{a}, \vec{y}) \geq \epsilon$. Define

$$\sigma := \inf_{\vec{y}} (1 - \varphi(\vec{a}, \vec{y})).$$

The above argument shows that $\sigma^M \leq 1 - \epsilon$. On the other hand, $N \models \varphi(\vec{a}, \vec{b})$, so $\sigma^N = 1$, contradicting that M is existentially closed in N.

For the other direction, we only need recall that the value of an existential formula can only decrease when computed in a larger model. Thus for any existential $\varphi(\vec{x})$, and any $\vec{a} \in M$, we have

$$\varphi^{M'}(\vec{a}) \leq \varphi^N(\vec{a}) \leq \varphi^M(\vec{a}) = \varphi^{M'}(\vec{a}).$$

$\square$

Lemma 2.3.8 (Robinson's Test)**.** *For any theory T, the following are equivalent:*

1. *T is model complete,*

2. *every model of T is existentially closed for T,*

Proof. (1) implies (2) is clear from the definitions. For (2) implies (1), suppose that $M_0 \subseteq N_0$ are models of T. By Lemma 2.3.7 there is an elementary extension M_1 of M_0 such that $N_0 \subseteq M_1$. The hypothesis (2), together with Lemma 2.3.7, applied to N_0 and M_1 produces an elementary extension N_1 of N_0 such that $M_1 \subseteq N_1$. Continuing in this way, we produce a chain

$$M_0 \subseteq N_0 \subseteq M_1 \subseteq N_1 \subseteq \cdots ,$$

where each $M_i \preceq M_{i+1}$ and $N_i \preceq N_{i+1}$. Let R be the union of the chain. Then $R = \bigcup_{i<\omega} M_i$, so $M_0 \preceq R$. Similarly, $N_0 \preceq R$. Then for any formula $\varphi(\vec{x})$, and any $\vec{a} \in M$, we have

$$\begin{aligned} \varphi^{M_0}(\vec{a}) &= \varphi^R(\vec{a}) && \text{because } M_0 \preceq R \\ &= \varphi^{N_0}(\vec{a}) && \text{because } N_0 \preceq R \end{aligned}$$

So $M_0 \preceq N_0$ as required. $\square$

In discrete logic a theory T is model complete if and only if every formula is equivalent, modulo T, to a universal formula, and this characterization of model completeness is often used to prove the discrete version of Lemma 2.3.8. The standard proof of this equivalence, for example as found in [28], does not appear to adapt well to the $[0,1]$-valued setting. In fact we do not know if model completeness of a continuous theory T is equivalent to every formula being, modulo T, uniformly approximable by universal formulas.

Lemma 2.3.9. *If T is $\forall\exists$-axiomatizable then every model of T can be extended to an existentially closed model of T.*

Proof. Fix $M \models T$, and let $(\sigma_\beta)_{\beta<\kappa}$ be an enumeration of all existential sentences with parameters from M. Form a chain $M = M_0 \subseteq M_1 \subseteq \cdots$ of length κ of models of T such that if there is a model of T extending M_β which satisfies σ_β, then $M_{\beta+1} \models \sigma_\beta$. At limit stages take unions, which again gives a

model of T by Lemma 2.2.8. Let $N_1 = \bigcup_{\alpha < \kappa} M_\alpha$, and note that N_1 has the property that any existential sentence with parameters from M which is satisfied by a model of T extending N_1 is already satisfied by N_1. Repeat this process ω times to form a chain $M = N_0 \subseteq N_1 \subseteq \cdots$, and let $R = \bigcup_{j < \omega} N_j$. Again by Lemma 2.2.8 we have $R \models T$. Any existential sentence with parameters from R in fact has parameters from some N_j, and hence if it is satisfied in some model of T extending N_j then it is satisfied in N_{j+1}, and hence in R. Therefore R is the desired existentially closed model of T extending M. $\qquad\square$

Lemma 2.3.10. *If M is existentially closed in N then for every $\forall\exists$-sentence σ, $\sigma^M \leq \sigma^N$. In particular, $M \models \mathrm{Th}_{\forall\exists}(N)$.*

Proof. Let $\sigma = \sup_{\vec{x}} \varphi(\vec{x})$, where φ is existential. For each $\vec{a} \in M$ we have $\varphi^M(\vec{a}) = \varphi^N(\vec{a})$ by definition of M being existentially closed in N. Therefore

$$
\begin{aligned}
\sigma^M &= \sup_{\vec{x} \in M} \varphi^M(\vec{x}) \\
&= \sup_{\vec{x} \in M} \varphi^N(\vec{x}) \\
&\leq \sup_{\vec{x} \in N} \varphi^N(\vec{x}) \\
&= \sigma^N.
\end{aligned}
$$

$\square$

The above lemma implies, in particular, that for $\forall\exists$-axiomatizable theories being existentially closed for T is the same as being an existentially closed model of T, so the following characterization of the model companion of a $\forall\exists$-axiomatizable theory could be equivalently stated for the class of models existentially closed for T, instead of the class of existentially closed models of T.

Proposition 2.3.11. *Let T be a $\forall\exists$-axiomatizable theory. Then T has a model companion if and only if the class of existentially closed models of T is the class of models of a theory T^*, in which case T^* is the model companion of T.*

Proof. Let $\mathcal{K}$ denote the class of all existentially closed models of T. Suppose that T has a model companion T^*; we first show that in this case $\mathcal{K}$ is the class of models of T^*. Given any $M \in \mathcal{K}$, find $N \models T^*$ and $R \models T$ such that $M \subseteq N \subseteq R$. Then M is existentially closed in R, and hence also in N. In particular, $M \models \mathrm{Th}_{\forall\exists}(N)$ by Lemma 2.3.10. As we observed earlier, model complete theories are $\forall\exists$-axiomatizable, so this implies that $M \models T^*$.

It remains to be shown that if $\mathcal{K}$ is the class of models of a theory T' then T' is the model companion of T. By Lemma 2.3.10 every model of T' is a model of T. Lemma 2.3.9 implies that every model of T can be extended to an existentially closed model of T, i.e., a model of T', so condition (1) of the definition of model companion is satisfied. Since models of T' are also models of T, we have that every model of T' is an existentially closed model of T'. It then follows by Robinson's test (Lemma 2.3.8) that T' is model complete. $\qquad\square$

The extra condition making a model companion into a model completion is thought of as saying that there is a kind of uniqueness to the ways models of T can be embedded into models of T^*. As in the discrete case, we have the following useful description of model completions, which we will use in Chapter 4.

Proposition 2.3.12. *Let T be a theory with model companion T^*. The following are equivalent:*

1. T^* *is the model completion of T,*

2. *T has the* amalgamation property*, i.e., whenever $A, B, C \models T$ and $f : A \to B$ and $g : A \to C$ are embeddings, then there exist $D \models T$ and embeddings $r : B \to D$ and $s : C \to D$ such that $rf = sg$.*

Conditions (1) and (2) are implied by the following condition, which is also an equivalence if T is universally axiomatizable:
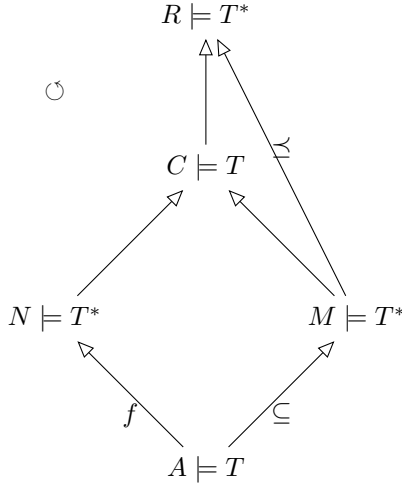
3. T^* *has quantifier elimination.*

Proof. Suppose that (1) holds, and let $A, B, C \models T$, $f : A \to B$, and $g : A \to C$ be as in the hypothesis of the amalgamation property. Find $B', C' \models T^*$ such that $B \subseteq B'$ and $C \subseteq C'$. Then $(B', f(a))_{a \in A}$ and $(C', g(a))_{a \in A}$ are each models of $T^* \cup \Delta_A$, which is a complete theory by definition of T^* being the model completion of T. Since complete theories have joint embedding, there is $(A', a^{A'})_{a \in A} \models T^* \cup \Delta_A$ which extends $(B', f(a))_{a \in A}$ and $(C', g(a))_{a \in A}$. Let $h : A \to A'$ be the map $a \mapsto a^{A'}$; we have that h an embedding, and there are maps $r : B \to A'$ and $s : C \to A'$ such that $rf = sg = h$. Also, $A' \models T^*$, so there is $D \models T$ such that $A' \subseteq D$, and this D, together with the maps r and s, shows that T has the amalgamation property.

Now suppose that (2) holds, and let $A \models T$. Let $(B, f(a))_{a \in A}$ and $(C, g(a))_{a \in A}$ be models of $T^* \cup \Delta_A$. Find $B', C' \models T$ such that $B \subseteq B'$ and $C \subseteq C'$, and then use the amalgamation property to find $A' \models T$ and embeddings $f' : B' \to A'$ and $g' : C' \to A'$ which amalgamate B' and C' over A. Now extend A' to some $D \models T^*$. By the model completeness of T^*, the embeddings of B and C into D are elementary, and so $(B, f(a))_{a \in A} \equiv (C, g(a))_{a \in A}$. That is, we have shown that $T^* \cup \Delta_A$ is a complete theory.

Suppose that (3) holds; we show (1). For any $A \models T$, it is clear that any two models of $T^* \cup \Delta_A$ satisfy the same quantifier-free sentences. By quantifier elimination for T^* such models satisfy all the same sentences, and hence T^* is the model completion of T.

Finally, suppose that T is universally axiomatizable and (2) holds. To show that T^* has quantifier elimination we use the quantifier elimination test of Proposition 2.3.2. So suppose that we have $M, N \models T^*$, and $\vec{a} \in M$, and let A be the substructure of M generated by $\vec{a}$. Fix an embedding $f : A \to N$. Since T^* is the model companion of T we have that $M, N \models (T^*)_\forall = T_\forall$, and T is universally axiomatizable, so $M, N \models T$. Also $A \subseteq M$ so $A \models T_\forall$, and hence $A \models T$ as well. By (2) we obtain $C \models T$ and embeddings of M and N into C; extending C to a model $R \models T^*$, we have the following commutative diagram:

$$R \models T^*$$

$$\circlearrowleft$$

$$C \models T \qquad \preceq$$

$$N \models T^* \qquad M \models T^*$$

$$f \qquad \subseteq$$

$$A \models T$$

By model completeness of T^* the embedding of M into R is elementary, so we have satisfied the conditions of the quantifier elimination test. $\qquad\square$

2.4 Saturation of ultraproducts

To conclude this chapter we consider when ultraproducts of separable structures have higher degrees of saturation than is given by Theorem 2.2.7. This section is the only section of the chapter containing original material.

In the discrete setting, an ultrafilter $\mathcal{U}$ on ω is called *saturating* if given any countable sequence $(M_i)_{i<\omega}$ of countable discrete structures of the same signature S with $|S| < 2^{\aleph_0}$, the ultraproduct $\prod_{\mathcal{U}} \mathcal{M}_i$ is $2^{\aleph_0}$-saturated (see [50]). It is well-known that if the Continuum Hypothesis holds then every non-principal ultrafilter on ω is saturating. The existence of saturating ultrafilters under Martin's Axiom was proved by Ellentuck and Rucker in [41]. Fremlin and Nyikos [50] gave neccessary and sufficient conditions for the existence of saturating ultrafilters. By cov(meagre) we denote the covering number of the meagre ideal, i.e., the least cardinal κ such that $\mathbb{R}$ is the union of κ meagre sets.

Theorem 2.4.1 ([50, Theorem 6]). *There exists a saturating ultrafilter if and only if* cov(meagre) $= 2^{<\mathfrak{c}} = \mathfrak{c}$.

We will show that the same ultrafilters which are saturating for discrete structures are also saturating for metric structures.

Definition 2.4.2. An ultrafilter $\mathcal{U}$ is *metric saturating* if given any countable sequence $(M_i)_{i<\omega}$ of separable metric structures in a signature of size $< 2^{\aleph_0}$, the ultraproduct $\prod_{\mathcal{U}} M_i$ is $2^{\aleph_0}$-saturated.

We will use the following combinatorial description of saturating ultrafilters.

Theorem 2.4.3 ([49, A3D]). *Let $\pi_1 : \omega \times \omega \to \omega$ be the projection onto the first coordinate. Let $\mathcal{U}$ be a non-principal ultrafilter on ω. Then $\mathcal{U}$ is saturating if and only if whenever $\mathcal{B} \subseteq \omega \times \omega$ is such that $|\mathcal{B}| < 2^{\aleph_0}$ and for all $F \in [\mathcal{B}]^{<\aleph_0}$ $\pi_1[\bigcap F] \in \mathcal{U}$ then there exists a function $f : \omega \to \omega$ such that $\pi_1[f \cap B] \in \mathcal{U}$ for every $B \in \mathcal{B}$.*

Theorem 2.4.4. *An ultrafilter on ω is saturating if and only if it is metric saturating.*

Proof. Suppose that $\mathcal{U}$ is metric saturating, and let $(A_i)_{i<\omega}$ be a sequence of discrete countable structures in a common signature S with $|S| < 2^{\aleph_0}$. Let M be the ultraproduct of the A_i's; since each A_i is discrete it does not matter whether we use the metric or classical definition of ultraproduct, and the resulting metric on M is again discrete. It then follows immediately from the $2^{\aleph_0}$-saturation of M as a metric structure and Remark 2.1.5 that M is also $2^{\aleph_0}$-saturated as a discrete structure.

Now suppose that $\mathcal{U}$ is saturating, and let $(M_i)_{i<\omega}$ be a sequence of separable metric structures in a signature of cardinality $< 2^{\aleph_0}$. Since the ultraproduct of a sequence of structures is the same as the ultraproduct of their metric completions, we may assume that each M_i is countable; we also assume that each M_i has ω as its underlying set. Let $\Sigma(x)$ be a type over $< 2^{\aleph_0}$ parameters from $M = \prod_{\mathcal{U}} M_i$. Since the signature also has size $< 2^{\aleph_0}$ we have $|\Sigma| < 2^{\aleph_0}$. Without loss of generality we may assume that Σ is closed under finite conjunctions (i.e., maxima).

For each $n < \omega$ and $\varphi \in \Sigma$, define

$$B_n(\varphi) = \left\{ (i,a) \in \omega \times \omega : \varphi^{M_i}(a) < \frac{1}{n} \right\}.$$

Let $\mathcal{B} = \{ B_n(\varphi) : \varphi \in \Sigma, n < \omega \}$, and note that $|\mathcal{B}| < 2^{\aleph_0}$. Our first task is to show that for all finite $\mathcal{B}_0 \subseteq \mathcal{B}$ we have $\pi_1[\mathcal{B}_0] \in \mathcal{U}$, where $\pi_1 : \omega \times \omega \to \omega$ is projection on the first coordinate. Let $\mathcal{B}_0 = \{ B_{n_0}(\varphi_0), \ldots, B_{n_k}(\varphi_k) \}$, and $n = \max\{n_0, \ldots, n_k\}$. Then $\bigcap \mathcal{B}_0 \supseteq B_n(\varphi_0 \wedge \cdots \wedge \varphi_k)$. Letting $\psi = \varphi_0 \wedge \cdots \wedge \varphi_k$, it is sufficient to show that $\pi_1[B_n(\psi)] \in \mathcal{U}$. But $\pi_1[B_n(\psi)] = \{ i < \omega : \exists a \in \omega(\psi^{M_i}(a) < \frac{1}{n}) \}$, and Σ is finitely satisfiable in M, so we have, in particular, that $M \models \inf_x \psi(x)$. By Łoś' Theorem (Theorem 2.2.2) we obtain $\pi_1[B_n(\psi)] \in \mathcal{U}$.

By Theorem 2.4.3 we obtain a function $f : \omega \to \omega$ such that $\{ i < \omega : (i, f(i)) \in B \} \in \mathcal{U}$ for all $B \in \mathcal{B}$. Therefore for all $n < \omega$ and all $\varphi \in \Sigma$ we have $\{ i < \omega : \varphi^{M_i}(f(i)) < \frac{1}{n} \} \in \mathcal{U}$; it follows from Łoś' Theorem that the image of $(f(i))_{i<\omega}$ in the ultraproduct M realizes Σ. $\square$

Corollary 2.4.5. *A metric saturating ultrafilter exists if and only if* $\mathrm{cov}(\text{meagre}) = 2^{<\mathfrak{c}} = \mathfrak{c}$.

Chapter 3

Infinitary $[0, 1]$-valued logic

This chapter is devoted to the development of the model theory of an infinitary $[0, 1]$-valued logic analogous to the discrete logic $\mathcal{L}_{\omega_1,\omega}$. In fact, several such logics have previously been introduced by a variety of authors (see [88], [14], and [99], as well as an early version of [29]) so after developing the model theory of our logic we include a discussion of how it compares to these other infinitary logics for metric structures. The chapter ends with applications of our infinitary logic to the study of Banach spaces.

The logic described in this chapter was introduced by the author in [36], which also contains the results of Sections 3.3, 3.4.1, and 3.6. Sections 3.2, 3.4.2, and 3.5 contain material which has not appeared elsewhere.

3.1 Definitions and basic properties

Our goal is to develop an infinitary logic for metric structures which is analogous to the discrete logic $\mathcal{L}_{\omega_1,\omega}$. Recall from Chapter 2 that in first-order logic for metric structures the connectives max and min behave as $\wedge$ and $\vee$, respectively. To add infinitary conjunctions and disjunctions we therefore allow the formation of $\sup_n \varphi_n$ and $\inf_n \varphi_n$ as formulas whenever each φ_n is a formula.

Definition 3.1.1. Let S be a signature for metric structures. We define the formulas of $\mathcal{L}_{\omega_1,\omega}(S)$ recursively, as follows:

1. All first-order formulas for the signature S are $\mathcal{L}_{\omega_1,\omega}(S)$ formulas,

2. whenever $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}_{\omega_1,\omega}(S)$ formulas and $f : [0,1]^n \to [0,1]$ is continuous then $f(\varphi_1, \ldots, \varphi_n)$ is an $\mathcal{L}_{\omega_1,\omega}(S)$ formula,

3. for every sequence $(\varphi_n)_{n<\omega}$ of $\mathcal{L}_{\omega_1,\omega}(S)$ formulas we have $\mathcal{L}_{\omega_1,\omega}(S)$ formulas $\sup_n \varphi_n$ and $\inf_n \varphi_n$,

4. for any $\mathcal{L}_{\omega_1,\omega}(S)$ formula φ, we have the $\mathcal{L}_{\omega_1,\omega}(S)$ formulas $\sup_x \varphi$ and $\inf_x \varphi$.

As in Chapter 2 we tacitly assume that a countable set of continuous functions from $[0,1]^n$ to $[0,1]$ which is dense in the topology of uniform convergence has been fixed, so that clause (2) of the definition does not increase the number of formulas. We will refer to these continuous functions as *finitary continuous connectives*. Keeping with the notation of Chapter 2, we sometimes denote $\sup_n \varphi_n$ by $\bigwedge \varphi_n$ and $\inf_n \varphi_n$ by $\bigvee \varphi_n$.

Remark 3.1.2. While it may seem that we have added only an *approximate* infinitary disjunction of formulas by adding $\inf_n \varphi_n$, we can in fact recover *exact* disjunction. Suppose that $\varphi_n(\vec{x})$ are formulas in the same finite tuple of free variables. Define

$$\theta(\vec{x}) = \inf_{n<\omega} \sup_{R\in\mathbb{N}} \min\{1, R\varphi_n(\vec{x})\}.$$

Then in any structure M, for any tuple $\vec{a}$, we have

$$M \models \theta(\vec{a}) \qquad \Longleftrightarrow \qquad M \models \varphi_n(\vec{a}) \text{ for some } n.$$

Using the actual disjunction, we can also show that our logic has a negation, making it much more expressive than finitary continuous logic. Given any formula $\varphi(\vec{x})$, define

$$\neg\varphi(\vec{x}) = \bigvee_{n<\omega} \left(\varphi(\vec{x}) \geq \frac{1}{n} \right),$$

where $\bigvee$ is the exact disjunction described above. Then for any structure M, and any $\vec{a} \in M$,

$$M \models \neg\varphi(\vec{a}) \iff (\exists n < \omega)M \models \varphi(\vec{a}) \geq \frac{1}{n}$$
$$\iff (\exists n < \omega)\varphi^M(\vec{a}) \geq \frac{1}{n}$$
$$\iff \varphi^M(\vec{a}) \neq 0$$
$$\iff M \not\models \varphi(\vec{a})$$

Unlike the formulas of first-order continuous logic, the formulas of $\mathcal{L}_{\omega_1,\omega}$ need not define continuous functions on structures. One consequence of this fact is that there are examples of structures which are not $\mathcal{L}_{\omega_1,\omega}$-elementary substructures of their metric completions. In fact, we give an example of of a structure which is not even $\mathcal{L}_{\omega_1,\omega}$-elementarily equivalent to its metric completion.

Example 3.1.3. Let S be the signature consisting of countably many constant symbols $(q_n)_{n<\omega}$. Consider the formula

$$\varphi(x) = \inf_{n<\omega} \sup_{R\in\mathbb{N}} \min\{1, Rd(x, q_n)\}.$$

For any a in a structure M we have $M \models \varphi(a)$ if and only if $a = q_n$ for some n. In particular, if M is a countable metric space which is not complete, and $(q_n)_{n<\omega}$ is interpreted as an enumeration of M, then

$$M \models \sup_x \varphi(x) \qquad \text{and} \qquad \overline{M} \not\models \sup_x \varphi(x).$$

3.1.1   Fragments of $\mathcal{L}_{\omega_1,\omega}$

It will be useful to consider subsets of the full set of $\mathcal{L}_{\omega_1,\omega}$ formulas, especially when they are sufficiently rich to be used as logics in their own right.

Definition 3.1.4. Let S be a metric signature. A *fragment* of $\mathcal{L}_{\omega_1,\omega}(S)$ is a set L of $\mathcal{L}_{\omega_1,\omega}(S)$ formulas with the following properties:

1. every first-order formula is in L,

2. L is closed under finitary continuous connectives,

3. L is closed under $\sup_x$ and $\inf_x$,

4. L is closed under subformulas,

5. L is closed under substituting terms for free variables.

It is clear that for any set K of $\mathcal{L}_{\omega_1,\omega}$ formulas there is a smallest fragment L such that $K \subseteq L$. Moreover, with our conventions regarding finitary continuous connectives in place, this smallest fragment satisfies $|L| = |K| + \aleph_0$. In particular, every formula of $\mathcal{L}_{\omega_1,\omega}$ generates a countable fragment.

We extend the basic definitions from model theory to arbitrary fragments of $\mathcal{L}_{\omega_1,\omega}$. For structures M and N of the appropriate signatures we write $M \equiv_L N$ to mean that $\sigma^M = \sigma^N$ for all $\sigma \in L$. Likewise we write $M \preceq_L N$ to mean $M \subseteq N$ and $\varphi^M(\vec{a}) = \varphi^N(\vec{a})$ for every $\varphi(\vec{x}) \in L$ and $\vec{a} \in M$. When we write $\equiv$ or $\preceq$ without specifying a fragment, we mean the first-order fragment. To determine when $M \preceq_L N$, it is useful to have a version of the Tarski-Vaught test. We omit the proof, which is a routine induction on the complexity of formulas.

Proposition 3.1.5. *Let S be a metric signature, and L a fragment of $\mathcal{L}_{\omega_1,\omega}(S)$. For any S-structures M and N, the following are equivalent:*

1. *$M \preceq_L N$,*

2. *$M \subseteq N$, and for every L-formula $\varphi(\vec{x}, y)$, and every $\vec{a} \in M$, $\inf_{b \in M} \varphi^M(\vec{a}, b) = \inf_{c \in N} \varphi^M(\vec{a}, c)$.*

Some fragments of $\mathcal{L}_{\omega_1,\omega}$ contain only formulas which define continuous functions on all structures.

Definition 3.1.6. A fragment L of $\mathcal{L}_{\omega_1,\omega}(S)$ is *continuous* if for every formula $\varphi(x_1, \ldots, x_n) \in L$ and every S-structure M, the function $\varphi^M : M^n \to [0,1]$ is continuous.

Our main use of the notion of continuous fragment comes from the following easy observation.

Proposition 3.1.7. *For any metric signature S, and any continuous fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, every S-structure is an L-elementary substructure of its metric completion.*

Proof. Let M be an L-structure with completion $\overline{M}$. For any L-formula $\varphi(\vec{x}, y)$ and any $\vec{a} \in M$, it follows from the continuity of φ and the density of M in $\overline{M}$ that we have

$$\inf_{b \in M} \varphi(\vec{a}, b) = \inf_{c \in \overline{M}} \varphi(\vec{a}, c).$$

This completes the proof by the Tarski-Vaught test (Proposition 3.1.5). $\square$

The next result is a Downward Löwenheim-Skolem theorem for countable fragments of $\mathcal{L}_{\omega_1,\omega}$. In addition to being quite useful, its statement exemplifies a theme that will be present several times in the remainder of this chapter: In arbitrary fragments models of a certain kind can be constructed, and if the fragment is continuous the model may additionally be chosen to be complete.

Proposition 3.1.8 (Downward Löwenheim-Skolem)**.** *Let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, and let M be an S-structure. For any countable set $A \subseteq M$ there is a countable L-structure N such that $A \subseteq N \preceq_L M$. If the fragment L is continuous then we may choose $A \subseteq N \preceq_L \overline{M}$ with N separable and complete.*

Proof. The proof of the first statement is the same as the proof in the first order fragment, which is itself a straightforward modification of the proof from discrete logic; see [16, Proposition 7.3].

Now let us see that the second statement follows from the first. Indeed, suppose we have a countable L-structure N such that $A \subseteq N \preceq_L M$, and the fragment L is continuous. Then $M \preceq_L \overline{M}$, and $N \preceq_L \overline{M}$, from which it follows that $\overline{N} \preceq_L \overline{M}$; the model $\overline{N}$ is then the desired separable complete elementary substructure of $\overline{M}$. $\square$

In Chapter 4 we will make use of the fact that satisfaction of formulas for $\mathcal{L}_{\omega_1,\omega}$ is not changed by forcing (in the sense of set theory). The following result appears in [40]. Our conventions for forcing agree with those of [78], except that we call the ground model V.

Proposition 3.1.9. *Let M be a metric structure, $\varphi(\vec{x})$ be an $\mathcal{L}_{\omega_1,\omega}$ formula, and $\vec{a} \in M$. Let $\mathbb{P}$ be any notion of forcing. Then the value $\varphi^M(\vec{a})$ is the same whether computed in the ground model V or in the forcing extension $V[G]$.*

Proof. We first observe that M remains a metric structure in $V[G]$. Indeed, the distance function on M remains a real-valued function on M satisfying the properties of being a metric, so (M, d) is still a metric space in the forcing extension. A uniformly continuous function on M^n (taking values either in M or $\mathbb{R}$) remains uniformly continuous in the forcing extension because the definition of uniform continuity is equivalent to the version where ϵ and δ are *rational*, and forcing preserves $\mathbb{Q}$. We note that even if M is complete in V it may not be complete in $V[G]$, since some partial orders $\mathbb{P}$ will add new Cauchy sequences to M. However, in this thesis we have taken a relaxed definition of "metric structure" which does not require completeness, so the above is sufficient to see that M is still a metric structure in $V[G]$.

The remainder of the proof is by induction on the complexity of formulas; the key point is that we consider the structure M in $V[G]$ as the same set as it is in V. The base case of the induction is the atomic formulas, which are of the form $P(\vec{x})$ for some distinguished predicate P. In this case since the structure M is the same in V and in $V[G]$, the value of $P^M(\vec{a})$ is independent of whether it is computed in V or $V[G]$.

The next case is to handle the case where φ is $f(\psi_1, \ldots, \psi_n)$, where each ψ_i is a formula and $f : [0,1]^n \to [0,1]$ is continuous. By induction hypothesis each $\psi_i^M(\vec{a})$ can be computed either in V or $V[G]$, and so the same is true of $\varphi^M(\vec{a}) = f(\psi_1^M(\vec{a}), \ldots, \psi_n^M(\vec{a}))$. A similar argument applies to the case when φ is $\sup_n \psi_n$ or $\inf_n \psi_n$.

Finally, we consider the case where $\varphi(\vec{x}) = \inf_y \psi(\vec{x}, y)$ (the case with sup instead of inf is similar). Here we have that for every $b \in M$, $\psi^M(\vec{a}, b)$ is independent of whether computed in V or $V[G]$ by induction. In both V and $V[G]$ the infimum ranges over the same set M, and hence $\varphi^M(\vec{a})$ is also the same whether computed in V or $V[G]$. $\square$

The above Proposition shows only that the value of formulas is not changed by forcing. It is, however, possible for forcing to produce *new* formulas. Indeed, if $\mathbb{P}$ adds a new sequence $(\varphi_1, \varphi_2, \ldots)$ of formulas, then the formula $\inf_n \varphi_n$ will exist in the forcing extension but not in the ground model.

3.1.2 The logic topology

We describe a topological space associated to each fragment of $\mathcal{L}_{\omega_1,\omega}$. This topology will be used heavily in Section 3.3 below. In fact, most of the material in this section can be carried out in the more general setting of abstract model theory; see [12, Chapters I and II] for abstract model theory, and [22], [23],

[24] for uses of topology in that setting. Definitions from abstract model theory were adapted to the real-valued context in [25]. For our purposes it will be sufficient to remain in the context of fragments of $\mathcal{L}_{\omega_1,\omega}$.

Definition 3.1.10. Let S be a metric signature, and L a fragment of $\mathcal{L}_{\omega_1,\omega}(S)$. We denote by $\mathrm{Str}(S)$ the class of all S-structures. For any L-theory T we denote by $\mathrm{Mod}_L(T)$ the class of S-structures M such that $M \models T$. When σ is an L-sentence we write $\mathrm{Mod}_L(\sigma)$ for $\mathrm{Mod}_L(\{\sigma\})$.

The *(L-)logic topology* is the topology on $\mathrm{Str}(S)$ whose closed classes are given by $\mathrm{Mod}_L(T)$.

It is straightforward to verify that the definition above does define a topology on Str_L, and that the classes of the form $\mathrm{Mod}_L(\sigma)$, for σ an L-sentence, form a base of closed classes.

Remark 3.1.11. Our definition of $\mathrm{Str}(S)$ raises certain foundational issues. The logic topology is defined as a collection of proper classes, and thus is problematic from the point of standard axiomatizations of set theory, such as ZFC. There are two natural ways to overcome this difficulty. The first is to replace the class of all L-structures by the set of all complete L-theories. Informally, this is equivalent to working with the quotient $\mathrm{Str}(S)/ \equiv_L$. This approach also makes the logic topology Hausdorff, which is not the case for the definition given above (see Proposition 3.1.12 below). An alternative approach is to notice that in all of our uses of this topology we only need to consider structures of cardinality at most $2^{2^{\aleph_0}}$. We could therefore use Scott's trick (see e.g. [67, 9.3]) to select one representative from each isomorphism class of S-structures of cardinality at most $2^{2^{\aleph_0}}$, and then replace the class of all S-structures by the set of these chosen representatives. In this thesis we will use $\mathrm{Str}(S)$ as originally presented, as the reader will have no difficulty translating our arguments into either of these two approaches.

Proposition 3.1.12. *For every fragment L, the L-logic topology on $\mathrm{Str}(S)$ is completely regular, but not T_0.*

Proof. To see that the logic topology is not T_0, we need only note that two structures M and N are topologically indistinguishable if and only if $M \equiv_L N$.

Now we prove that the logic topology is completely regular. For any L-sentence σ, the function from $\mathrm{Str}(S)$ to $[0,1]$ defined by $\mathcal{M} \mapsto \sigma^{\mathcal{M}}$ is continuous. This is because for each $r \in \mathbb{Q} \cap (0,1)$ we have that $\sigma \leq r$ and $r \leq \sigma$ are conditions expressible as L-sentences. We therefore have $\sigma^{-1}([r,s]) = \mathrm{Mod}(\sigma \geq r \wedge \sigma \leq s)$ for every $r, s \in \mathbb{Q} \cap [0,1]$. It follows immediately from the definition of the logic topology that functions defined by sentences in this way are sufficient to separate points from closed classes. $\square$

Call a fragment L of $\mathcal{L}_{\omega_1,\omega}$ *compact* if whenever every finite subset of an L-theory has a model then the whole theory has a model. Then we have:

Proposition 3.1.13. *A fragment L is compact if and only if the L-logic topology on $\mathrm{Str}(S)$ is compact.*

Proof. Suppose the logic topology is compact, and let T be an L-theory such that every finite subset of T has a model. Define $\mathcal{F} = \{\, \mathrm{Mod}_L(\Delta) : \Delta \text{ is a finite subset of } T \,\}$. The intersection of any finite subcollection of $\mathcal{F}$ is again a member of $\mathcal{F}$, and by hypothesis is non-empty. The compactness of the logic topology then implies that there is some $M \in \bigcap \mathcal{F}$; this M is a model of T.

Conversely, suppose that L is compact, and that $\mathcal{F}$ is a centred family of closed classes. Let $\mathcal{F}'$ be a set of L-theories such that $\mathcal{F} = \{\, \mathrm{Mod}_L(\Sigma) : \Sigma \in \mathcal{F}' \,\}$, and let $T = \bigcup \mathcal{F}'$. Since $\mathcal{F}$ is centred every finite subset of T has a model, and hence T also has a model. Any model of T is a member of $\bigcap \mathcal{F}$. $\square$

We note that the above proposition gives a straightforward topological proof of the compactness theorem for the first-order fragment (Theorem 2.2.3). Indeed, Łoś' theorem shows that for any family $(M_i)_{i \in I}$ of S-structures and any ultrafilter $\mathcal{U}$ on I, the ultraproduct $\prod_{\mathcal{U}} M_i$ is the ultrafilter limit of the M_is along $\mathcal{U}$ in the logic topology for the first-order fragment. In particular, all ultrafilter limits exist, so the logic topology for the first-order fragment is compact. This proof of compactness should be compared with the proof of the Omitting Types Theorem for arbitrary countable fragments, Theorem 3.3.4 below, in which only some ultrafilter limits exist (see, in particular, Claim 3.3.11.1).

Proposition 3.1.13 is included only to illustrate how topological properties of Str_L relate to model-theoretic properties of L. In fact, Caicedo [21] has shown a Lindström-type theorem characterizing the first-order fragment as the maximally expressive logic for metric structures satisfying the Downward Löwenheim-Skolem and Compactness theorems.

3.2 Scott Isomorphism and Definability

The most striking feature of the discrete logic $\mathcal{L}_{\omega_1,\omega}$ is the theorem of Scott [98] that every countable discrete structure is determined up to isomorphism by a single sentence of $\mathcal{L}_{\omega_1,\omega}$. A sentence which determines a structure up to isomorphism is known as a *Scott sentence* for that structure. Scott's immediate use of his isomorphism theorem was to prove a definability theorem, namely that a relation on a countable discrete structure is automorphism invariant if and only if it is $\mathcal{L}_{\omega_1,\omega}$-definable.

In this short section we prove a definability theorem for metric structures which is analogous to Scott's definability theorem. The existence of Scott sentences for metric structures was first observed in several infinitary logics different from the one presented here; we will define these logics precisely in Section 3.5 below. The first proof of the existence of Scott sentences for metric structures was by Sequeira in [99], in a continuous infinitary logic with an extra distinguished operation ρ. Subsequently Coskey and Lupini in [29] obtained Scott sentences in the logic introduced by Ben Yaacov and Iovino in [14], for structures whose underlying metric space is the Urysohn sphere and where all symbols in the language have a common modulus of uniform continuity. Very soon after, Ben Yaacov, Nies, and Tsankov [15] proved the existence of Scott sentences for general separable complete metric structures.

Theorem 3.2.1 ([15, Corollary 2.2]). *For each separable complete metric structure M there is an $\mathcal{L}_{\omega_1,\omega}$ sentence σ such that for every other separable complete metric structure N of the same signature,*

$$\sigma^N = \begin{cases} 0 & \text{if } M \cong N \\ 1 & \text{otherwise} \end{cases}$$

We can reformulate this result to apply to incomplete metric structures, but then we obtain uniqueness only at the level of the completion.

Corollary 3.2.2. *For each separable metric structure M there is an $\mathcal{L}_{\omega_1,\omega}$ sentence σ such that for every other separable metric structure N of the same signature,*

$$\sigma^N = \begin{cases} 0 & \text{if } \overline{M} \cong \overline{N} \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let σ be the Scott sentence for $\overline{M}$, as in Theorem 3.2.1. Theorem 3.2.1 was proved in the logic of [14], which is a continuous fragment of $\mathcal{L}_{\omega_1,\omega}$ (see Section 3.5 below). Therefore structures and their completions agree on the value of σ, and so

$$\sigma^N = \sigma^{\overline{N}} = \begin{cases} 0 & \text{if } \overline{M} \cong \overline{N} \\ 1 & \text{otherwise} \end{cases}.$$

$\square$

The existence of Scott sentences is the basis for our definability theorem.

Theorem 3.2.3. *Let M be a separable complete S-structure for some fixed countable signature S. For any continuous function $P : M^n \to [0,1]$, the following are equivalent:*

1. *There is an $\mathcal{L}_{\omega_1,\omega}(S)$ formula $\varphi(\vec{x})$ such that for all $\vec{a} \in M^n$,*

$$\varphi^M(\vec{a}) = P(\vec{a}).$$

2. *P is fixed by all automorphisms of M.*

Proof. The direction $(1) \implies (2)$ being clear, we prove $(2) \implies (1)$.

Fix a countable dense subset $D \subseteq M$. For each $\vec{a} \in D$, let $\theta_{\vec{a}}(\vec{x})$ be the formula obtained by replacing each occurrence of $\vec{a}$ in the Scott sentence of $(M, \vec{a})$ by $\vec{x}$. The Scott sentence is obtained from Theorem 3.2.1. Observe that this formula has the following property, for all $\vec{b} \in M^n$:

$$\theta_{\vec{a}}^M(\vec{b}) = \begin{cases} 0 & \text{if there is } \Phi \in \operatorname{Aut}(M) \text{ with } \Phi(\vec{b}) = \vec{a} \\ 1 & \text{otherwise} \end{cases}$$

For each $\epsilon > 0$, define:

$$\sigma_\epsilon(\vec{x}) = \inf_{\vec{y}} \max \left\{ d(\vec{x}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\}.$$

Each $\sigma_\epsilon(\vec{x})$ is a formula of $\mathcal{L}_{\omega_1,\omega}(S)$.

Claim 3.2.3.1. *Consider any $\epsilon \in \mathbb{Q} \cap (0,1)$ and any $\vec{b} \in M^n$.*

(a) *If $M \models \sigma_\epsilon(\vec{b})$ then $P(\vec{b}) \leq \epsilon$.*

(b) *If $P(\vec{b}) < \epsilon$ then $M \models \sigma_\epsilon(\vec{x})$.*

Proof. (a) Suppose that $M \models \sigma_\epsilon(\vec{b})$. Fix $\epsilon' > 0$, and pick $0 < \delta < 1$ such that if $d(\vec{b}, \vec{y}) < \delta$ then $\left| P(\vec{b}) - P(\vec{y}) \right| < \epsilon'$. This exists because we assumed that P is continuous. Now from the definition of $M \models \sigma_\epsilon(\vec{b})$ we can find $\vec{y} \in M^n$ such that

$$\max \left\{ d(\vec{b}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\} < \delta.$$

In particular, we have that $d(\vec{b}, \vec{y}) < \delta$, so $\left| P(\vec{b}) - P(\vec{y}) \right| < \epsilon'$. On the other hand, $\inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) < \delta$, and $\theta_{\vec{a}}(\vec{y}) \in \{0, 1\}$ for all $\vec{a} \in D^n$, so in fact there is $\vec{a} \in D^n$ with $P(\vec{a}) < \epsilon$ and $\theta_{\vec{a}}(\vec{y}) = 0$. For

such an $\vec{a}$ there is an automorphism of M taking $\vec{y}$ to $\vec{a}$, and hence by (2) we have that $P(\vec{y}) < \epsilon$ as well. Combining what we have,

$$P(\vec{b}) = \left|P(\vec{b})\right|$$
$$\leq \left|P(\vec{b}) - P(\vec{y})\right| + |P(\vec{y})|$$
$$< \epsilon' + \epsilon$$

Taking $\epsilon' \to 0$ we conclude $P(\vec{b}) \leq \epsilon$.

(b) Suppose that $P(\vec{b}) < \epsilon$, and again fix $\epsilon' > 0$. Using the continuity of P, find δ sufficiently small so that if $d(\vec{b}, \vec{y}) < \delta$ then $P(\vec{y}) < \epsilon$. The set D is dense in M, so we can find $\vec{y} \in D^n$ such that $d(\vec{b}, \vec{y}) < \min\{\delta, \epsilon'\}$. Then $P(\vec{y}) < \epsilon$, so choosing $\vec{a} = \vec{y}$ we have

$$\inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) = 0.$$

Therefore

$$\max \left\{ d(\vec{b}, \vec{y}), \inf_{\substack{\vec{a} \in D^n \\ P(\vec{a}) < \epsilon}} \theta_{\vec{a}}(\vec{y}) \right\} = d(\vec{b}, \vec{y}) < \epsilon',$$

and so taking $\epsilon' \to 0$ shows that $M \models \sigma_{\epsilon(\vec{b})}$.

$$\dashv \text{ - Claim 3.2.3.1}$$

Consider now any $\vec{a} \in M^n$. By (a) of the claim $P(\vec{a})$ is a lower bound for $\{\, \epsilon \in \mathbb{Q} \cap (0,1) : M \models \sigma_\epsilon(\vec{a}) \,\}$. If α is another lower bound, and $\alpha > P(\vec{a})$, then there is $\epsilon \in \mathbb{Q} \cap (0,1)$ such that $P(\vec{a}) < \epsilon < \alpha$. By (b) of the claim we have $M \models \sigma_\epsilon(\vec{a})$ for this ϵ, contradicting the choice of α. Therefore

$$P(\vec{a}) = \inf \{\, \epsilon \in \mathbb{Q} \cap (0,1) : M \models \sigma_\epsilon(\vec{a}) \,\}.$$

Now for each $\epsilon \in \mathbb{Q} \cap (0,1)$, define a formula

$$\psi_\epsilon(\vec{x}) = \max \left\{ \epsilon, \sup_{m \in \mathbb{N}} \min \{\, m\sigma_\epsilon(\vec{x}), 1 \,\} \right\}.$$

Then for any $\vec{a} \in M^n$,

$$\psi_\epsilon^M(\vec{a}) = \begin{cases} \epsilon & \text{if } \sigma_\epsilon^M(\vec{a}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\varphi(\vec{x}) = \inf_{\epsilon \in \mathbb{Q} \cap (0,1)} \psi_\epsilon(\vec{x})$. Then

$$\varphi^M(\vec{a}) = \inf \{\, \epsilon : \sigma_\epsilon^M(\vec{a}) = 0 \,\} = P(\vec{a}).$$

$$\square$$

The theorem immediately yields a version where parameters from a countable set are allowed in the definitions:

Corollary 3.2.4. *Let M be a separable complete S-structure for some fixed countable signature S, and fix a countable $A \subseteq M$. For any continuous function $P : M^n \to [0,1]$, the following are equivalent:*

1. *There is an $\mathcal{L}_{\omega_1,\omega}(S)$ formula $\varphi(\vec{x})$ with parameters from A such that for all $\vec{a} \in M^n$,*

$$\varphi^M(\vec{a}) = P(\vec{a}),$$

2. *P is fixed by all automorphisms of M that fix A pointwise,*

3. *P is fixed by all automorphisms of M that fix $\overline{A}$ pointwise,*

Proof. The equivalence of (2) and (3) follows from the fact that $\mathrm{Aut}_A(M) = \mathrm{Aut}_{\overline{A}}(M)$. For the equivalence of (1) and (2), apply Theorem 3.2.3 to the structure obtained from M by adding a new constant symbol for each element of A. $\square$

Our definability theorems are unusual in that they are results about complete separable metric structures and continuous functions, but the proofs make considerable use of formulas which are *not* continuous. In fact, as we will see in Section 3.5 below, the use of discontinuous formulas is in a certain sense necessary for these results.

3.3 Omitting Types

In this section we consider methods for constructing models. We do not have the compactness theorem in infinitary logic, but the main result of this section shows that we do have an omitting types theorem. Our proof is topological, using the logic topology described in Section 3.1.2. In particular, a comparison of this proof with the topological proof of the compactness theorem (see Proposition 3.1.13) shows that this omitting types theorem is a weakening of the compactness theorem in the same way that for topological spaces, satisfying the Baire Category Theorem is weaker than being compact. To state our Omitting Types Theorem, we need some definitions. In the following definitions T is an L-theory, where L is a fragment of $\mathcal{L}_{\omega_1,\omega}(S)$ for some metric signature S.

Definition 3.3.1. A *type of T* is a set $\Sigma(\vec{x})$ of L-formulas with a common finite set of free variables such that there is $M \models T$ and $\vec{a} \in M$ with $\varphi^M(\vec{a}) = 0$ for all $\varphi \in \Sigma$.

Definition 3.3.2. Let $\Sigma(\vec{x})$ be a type of T. We say that a model $M \models T$ *realizes* Σ if there is $\vec{a} \in M$ such that $\varphi^M(\vec{a}) = 0$ for all $\varphi \in \Sigma$; in the same situation we say that $\vec{a}$ *realizes* Σ in M. If a model M does not realize Σ, then we say M *omits* Σ.

Definition 3.3.3. A type $\Sigma(\vec{x})$ of T *principal* over T is there is an L-formula $\varphi(\vec{x})$, terms $t_1(\vec{y}), \ldots, t_n(\vec{y})$ (where n is the length of $\vec{x}$), and $r \in \mathbb{Q} \cap (0,1)$ such that the following hold:

- $T \cup \{ \varphi(\vec{y}) \}$ is satisfiable, and

- $T \cup \{ \varphi(\vec{y}) \leq r \} \models \Sigma(t_1(\vec{y}), \ldots, t_n(\vec{y}))$.

In this case we say that φ and r *witness* the fact that Σ is principal over T.

The rest of this section is devoted to proving:

Theorem 3.3.4. *Let S be a metric signature, and let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$. Let T be an L-theory. For each $n < \omega$, let Σ_n be a type of T that is not principal over T. Then there is a separable model of T that omits every Σ_n.*

3.3.1 Topological preliminaries

The proof of Theorem 3.3.4 is topological, so we recall here some notions from general topology.

The most important notion from topology for us is the notion of Baire category. Recall that if X is a topological space and $A \subseteq X$, then A is *nowhere dense* if $\text{int}(\overline{A}) = \emptyset$. A space X is *Baire* if whenever $(A_n)_{n<\omega}$ is a sequence of closed nowhere dense subsets of X, then $X \setminus \left(\bigcup_{n<\omega} A_n \right)$ is dense in X. The classical Baire Category Theorem states that locally compact Hausdorff spaces and completely metrizable spaces are Baire. The logic topology is neither locally compact nor metrizable, but we will see that a relevant subspace has a more general property, which we shall now describe.

Definition 3.3.5. Let X be a completely regular space. A *complete sequence of open covers of X* is a sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of open covers of X with the following property: If $\mathcal{F}$ is a centred family of closed subsets of X such that for each $n < \omega$ there is $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ such that $F_n \subseteq U_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

A completely regular space X is *Čech-complete* if there exists a complete sequence of open covers of X.

If the space X is completely regular and Hausdorff then X is Čech-complete if and only if X is a G_δ subspace of some (equivalently, every) compactification. For metrizable spaces, being Čech-complete is equivalent to being completely metrizable. It follows from these two facts that if X is either locally compact Hausdorff or completely metrizable then X is Čech-complete. The following result states the two key facts about Čech-complete spaces that we will use in the proof of Theorem 3.3.4. These facts are stated and proved in [42] for completely regular Hausdorff spaces, but the proof does not use the Hausdorff condition.

Lemma 3.3.6. *Let X be a completely regular space.*

1. *If X is Čech-complete then X is Baire.*

2. *If X is Čech-complete and $F \subseteq X$ is a closed subspace, then F is Čech-complete.*

Remark 3.3.7. Under additional set-theoretic and topological assumptions, the first part of Lemma 3.3.6 can be improved, as follows. For an infinite cardinal κ, a space X is *κ-Baire* if the intersection of fewer than κ dense open subsets of X is dense in X. In this terminology our previous definition of Baire corresponds to $\aleph_1$-Baire. Recall that a space X has the *countable chain condition* if every family of pairwise disjoint open subsets of X is at most countable. Tall [104, Theorem 2.3] observed that Martin's Axiom implies that Čech-complete spaces with the countable chain condition are $2^{\aleph_0}$-Baire. Essentially the same proof shows that Martin's Axiom restricted to countable partial orders implies that any Čech-complete space with a countable base is $2^{\aleph_0}$-Baire. See [49] for details about Martin's Axiom.

The proof in [104] assumes the Hausdorff condition, but the result for Čech-complete Hausdorff spaces implies the same result for arbitrary Čech-complete spaces. If X is a Čech-complete space and $\equiv$ is the relation of topological indistinguishability, then $X/\equiv$ is a Čech-complete Hausdorff space. It is routine to check that for any cardinal κ, if $X/\equiv$ is κ-Baire then so is X.

3.3.2 Conventions

We fix, for the entirety of this section, a metric signature S and a countable fragment L of $\mathcal{L}_{\omega_1,\omega}(S)$. If C is any set of new constant symbols, we denote by L_C the smallest fragment of $\mathcal{L}_{\omega_1,\omega}(S\cup C)$ containing L. Note that if C is countable then L_C is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S\cup C)$. The sentences of L_C are exactly those sentences of the form $\varphi(\vec{c})$ for some $\vec{c}\in C$ and $\varphi(\vec{x})\in L$. If D is a set of constant symbols with $C\subseteq D$ and T is an L_C-theory, we write $\operatorname{Mod}_{L_D}(T)=\{\,M\in\operatorname{Str}(S\cup D):M\models T\,\}$ and $\operatorname{Mod}_{L_C}(T)=\{\,M\in\operatorname{Str}(S\cup C):M\models T\,\}$ when necessary to avoid ambiguity. If M is an S-structure and $\vec{a}=\{\,a_i:i<\omega\,\}$ is a set of elements of M, then the $(S\cup C)$- structure obtained from M by interpreting c_i as a_i is denoted by $\langle M,\vec{a}\rangle$.

We now fix a countable set $C=\{\,c_0,c_1,\dots\,\}$ of new constant symbols and an enumeration $\{\,\varphi_0(x),\varphi_1(x),\dots\,\}$ of the L_C-formulas in exactly one free variable x. We will primarily work in the following subspace of $\operatorname{Str}(S\cup C)$:

$$\mathcal{W}=\bigcap_{i<\omega}\ \bigcap_{r\in\mathbb{Q}\cap(0,1)}\left(\operatorname{Mod}_{L_C}\left(\inf_x\varphi_i(x)>0\right)\cup\bigcup_{j<\omega}\operatorname{Mod}_{L_C}(\varphi_i(c_j)<r)\right).$$

The following remark states the main property of $\mathcal{W}$ that we will use.

Remark 3.3.8. If $\langle M,\vec{a}\rangle\in\mathcal{W}$ and $M\models\inf_x\varphi(x)$, then for each $\epsilon\in\mathbb{Q}\cap(0,1)$ there is $j<\omega$ such that $\langle M,\vec{a}\rangle\models\varphi(c_j)\le\epsilon$. More generally, it follows from the fact that we can express inequalities in our formulas that if $(\inf_x\varphi(x))^{\langle M,\vec{a}\rangle}<r$ then there exists $r'\in\mathbb{Q}\cap(0,r)$ and $j<\omega$ such that $\langle M,\vec{a}\rangle\models\varphi(c_j)\le r'$.

Lemma 3.3.9. *If $\langle M,\vec{a}\rangle\in\mathcal{W}$, then $M\restriction\langle\vec{a}\rangle\preceq_L M$, where $M\restriction\langle\vec{a}\rangle$ is the substructure of M generated by $\vec{a}$.*

Proof. Immediate from Remark 3.3.8 and Proposition 3.1.5. □

We note that $\mathcal{W}$ is non-empty, since given any countable S-structure M we may interpret C as an enumeration $\vec{a}$ of M to obtain $\langle M,\vec{a}\rangle\in\mathcal{W}$.

There are two parts to the proof of the Omitting Types Theorem. First, in Section 3.3.3 we show that $\mathcal{W}$ is Čech-complete. Then in Section 3.3.4 we relate the model-theoretic notion of principal types to Baire category in $\mathcal{W}$, and use this to prove the Omitting Types Theorem.

3.3.3 Čech-completeness of $\mathcal{W}$

Fix an enumeration $\{\,\sigma_0,\sigma_1,\dots\,\}$ of the L_C-sentences such that σ_0 is an atomic sentence. To prove that $\mathcal{W}$ is Čech-complete we must show that it has a complete sequence of open covers (see Definition 3.3.5). In fact there are many such sequences; the following lemma gives the existence of a sequence with the properties we will need. By an *open rational interval* in $[0,1]$, we mean an interval $I\subseteq[0,1]$ with rational endpoints that is open in the subspace topology on $[0,1]$.

Lemma 3.3.10. *There exists a sequence $\langle\mathcal{U}_n:n<\omega\rangle$ of open covers of $\mathcal{W}$ with the following properties:*

1. *For every n and every $\epsilon>0$ there is $l\ge n$ such that for each $U\in\mathcal{U}_l$ there is a rational open interval I_U with $\operatorname{length}(I_U)\le\epsilon$ such that for all $\mathcal{N}\in U$, $\sigma_n^{\mathcal{N}}\in I_U$.*

2. *For every n, if $k \leq n$ is such that $\sigma_k = \inf_{i<\omega} \chi_i$, then for each $U \in \mathcal{U}_n$ there is a rational open interval I in $[0,1]$, and a $j < \omega$, such that for all $\mathcal{N} \in U$, $(\inf_{i<\omega} \chi_i)^{\mathcal{N}} \in I$ and $\chi_j^{\mathcal{N}} \in I$.*

3. *For every n, if $k \leq n$ is such that $\sigma_k = \inf_x \varphi$, then for each $U \in \mathcal{U}_n$ there is a rational open interval I in $[0,1]$ and a $j < \omega$ such that for all $\mathcal{N} \in U$, $(\inf_x \varphi)^{\mathcal{N}} \in I$ and $\varphi(c_j)^{\mathcal{N}} \in I$.*

Proof. We first define a sequence $(\mathfrak{I}_n)_{n<\omega}$ of open covers of $[0,1]$, the nth of which corresponds to splitting $[0,1]$ into n rational open intervals in $[0,1]$ with small overlap. To do this, for each $n < \omega$ let $\epsilon_n = \frac{1}{2^{n+2}}$. For each n, define an open cover of $[0,1]$ as follows:

$$\mathfrak{I}_n = \left\{ \left[0, \frac{1}{n+2} + \epsilon_n\right), \left(\frac{1}{n+2} - \epsilon_n, \frac{2}{n+2} + \epsilon_n\right), \cdots, \left(\frac{n+1}{n+2} - \epsilon_n, 1\right] \right\}.$$

For a sentence σ and a rational open interval $I \subseteq [0,1]$, we temporarily abuse notation to write

$$\operatorname{Mod}(\sigma \in I) = \left\{ M \in \mathcal{W} : \sigma^M \in I \right\}.$$

We construct the sequence $\langle \mathcal{U}_n : n < \omega \rangle$ recursively, so that the following properties hold:

(a) Each $\mathcal{U}_n$ is an open cover of $\mathcal{W}$,

(b) Each $U \in \mathcal{U}_n$ is of the form $U = \bigcap O_U$, where O_U is a finite collection of open classes such that:

    (i) Each element of O_U is of the form $\operatorname{Mod}(\theta \in J)$, where θ is a sentence and $J \in \mathfrak{I}_n$,

    (ii) For each $k \leq n$ there is $J_k \in \mathfrak{I}_n$ such that $\operatorname{Mod}(\sigma_k \in J_k) \in O_U$,

    (iii) If $\operatorname{Mod}(\inf_{i<\omega} \chi_i \in J) \in O_U$, then there exists $j < \omega$ such that $\operatorname{Mod}(\chi_j \in J) \in O_U$,

    (iv) If $\operatorname{Mod}(\inf_x \varphi) \in O_U$ then there exists $j < \omega$ such that $\operatorname{Mod}(\varphi(c_j) \in J) \in O_U$.

It is clear that a sequence $\langle \mathcal{U}_n : n < \omega \rangle$ satisfying (a) and (b) will satisfy $(1) - (3)$.

For the base case, define

$$\mathcal{U}_0 = \{ \operatorname{Mod}(\sigma_0 \in I) : I \in \mathfrak{I}_0 \}.$$

Since the intervals in $\mathfrak{I}_0$ are open, $\mathcal{U}_0$ is an open cover, and the conditions in (b) are satisfied trivially.

Suppose that $\mathcal{U}_n$ is defined satisfying (a) and (b). We first refine $\mathcal{U}_n$ to a cover $\widetilde{\mathcal{U}}_n$ as follows. For each function $f : \mathfrak{I}_n \to \mathfrak{I}_{n+1}$, and each $U \in \mathcal{U}_n$, let $O_U^f = \{ \operatorname{Mod}(\theta \in f(J)) : \operatorname{Mod}(\theta \in J) \in O_U \}$, and let $U^f = \bigcap O_U^f$. Then let

$$\widetilde{\mathcal{U}}_n = \left\{ U^f : U \in \mathcal{U}_n, f : \mathfrak{I}_n \to \mathfrak{I}_{n+1} \right\}.$$

If σ_{n+1} is not an infinitary disjunction and is not of the form $\inf_x \varphi$, then define

$$\mathcal{U}_{n+1} = \left\{ U \cap \operatorname{Mod}(\sigma_{n+1} \in I) : U \in \widetilde{\mathcal{U}}_n, I \in \mathfrak{I}_{n+1} \right\}.$$

Note that $\mathcal{U}_{n+1}$ is a cover of $\mathcal{W}$ since $\widetilde{\mathcal{U}}_n$ is a cover of $\mathcal{W}$ and $\mathfrak{I}_{n+1}$ is a cover of $[0,1]$. If σ_{n+1} is the infinitary disjunction $\inf_{i<\omega} \chi_i$, then define

$$\mathcal{U}_{n+1} = \left\{ U \cap \operatorname{Mod}(\sigma_{n+1} \in I) \cap \operatorname{Mod}(\chi_j \in I) : U \in \widetilde{\mathcal{U}}_n, I \in \mathfrak{I}_{n+1}, j < \omega \right\}.$$

Finally, if σ_{n+1} is of the form $\inf_x \varphi$, define

$$\mathcal{U}_{n+1} = \left\{ U \cap \mathrm{Mod}(\sigma_{n+1} \in I) \cap \mathrm{Mod}(\varphi(c_j) \in I) : U \in \widetilde{\mathcal{U}}_n, I \in \mathcal{I}_{n+1}, j < \omega \right\}.$$

It is easy to see that (b) is preserved, so we only need to observe that $\mathcal{U}_{n+1}$ is a cover of $\mathcal{W}$. This follows from Remark 3.3.8 and the fact that $\widetilde{\mathcal{U}}_n$ is a cover. $\qquad\square$

Proposition 3.3.11. *The space $\mathcal{W}$ is Čech-complete.*

Proof. Let $\langle \mathcal{U}_n : n < \omega \rangle$ be a sequence of open covers as given by Lemma 3.3.10. Let $\mathcal{F}$ be a centred family of closed sets such that for each $n < \omega$ there is $F_n \in \mathcal{F}$ and $U_n \in \mathcal{U}_n$ such that $F_n \subseteq U_n$. To show that $\langle \mathcal{U}_n : n < \omega \rangle$ is a complete sequence of open covers, we must show that $\bigcap \mathcal{F} \neq \emptyset$. It is easy to check, using (1) from Lemma 3.3.10, that $\bigcap \mathcal{F} = \bigcap_{n<\omega} F_n$.

For each $n < \omega$, choose $M_n \in F_0 \cap \cdots \cap F_n$. Let $\mathcal{D}$ be a non-principal ultrafilter on ω. We will show that $\prod_{\mathcal{D}} M_n \in \mathcal{W} \cap \bigcap_{n<\omega} F_n$.

Claim 3.3.11.1. *For any L_C-sentence σ, $\sigma^{\prod_{\mathcal{D}} M_n} = \lim_{n \to \mathcal{D}} \sigma^{M_n}$.*

Proof of Claim 3.3.11.1. The proof is by induction on the complexity of σ. The case where σ is an atomic sentence follows directly from the definition of the ultraproduct (Definition 2.1.7), and the case where σ is the result of applying a finitary connective follows from the continuity of the finitary connectives and the definition of ultrafilter limits, so we only need to deal with the infinitary disjunction and $\inf_x \varphi$ cases.

$\underline{\sigma = \inf_{i<\omega} \chi_i:}$
It is sufficient to show that for each $a \in \mathbb{Q} \cap (0,1)$, $a > \sigma^{\prod_{\mathcal{D}} M_n}$ if and only if $\left\{ n < \omega : \sigma^{M_n} < a \right\} \in \mathcal{D}$.

Suppose $a > \sigma^{\prod_{\mathcal{D}} M_n}$. Then

$$\inf_{i<\omega} \chi_i^{\prod_{\mathcal{D}} M_n} < a.$$

Hence there is some $j < \omega$ such that

$$\chi_j^{\prod_{\mathcal{D}} M_n} < a.$$

So by the inductive hypothesis, $\lim_{n\to\mathcal{D}} \chi_j^{M_n} < a$. That is,

$$\left\{ n < \omega : \chi_j^{M_n} < a \right\} \in \mathcal{D}.$$

We have $\chi_j^{M_n} \geq \sigma^{M_n}$ for each n, so $\left\{ n < \omega : \sigma^{M_n} < a \right\} \supseteq \left\{ n < \omega : \chi_j^{M_n} < a \right\}$, and hence

$$\left\{ n < \omega : \sigma^{M_n} < a \right\} \in \mathcal{D}.$$

Now assume $\left\{ n < \omega : \sigma^{M_n} < a \right\} \in \mathcal{D}$. Note that, by the inductive hypothesis, it suffices to find $j < \omega$ such that $\left\{ n < \omega : \chi_j^{M_n} < a \right\} \in \mathcal{D}$. Find $l < \omega$ such that $\sigma = \sigma_l$. Find $k \geq l$ such that $\sigma^{M_k} < a$ and for all $\mathcal{N} \in U_k$, $\sigma^{\mathcal{N}} < a$ (by (1) of Lemma 3.3.10). By (2) of Lemma 3.3.10, there is some $j < \omega$ such that for all $\mathcal{N} \in U_k$, $\chi_j^{\mathcal{N}} < a$. In particular, for all $n \geq k$, $\chi_j^{M_n} < a$. Thus for cofinitely many n we have $\chi_j^{M_n} < a$, and j is as desired.

$\underline{\sigma = \inf_x \varphi(x).}$
Suppose that $\left\{ n < \omega : (\inf_x \varphi)^{M_n} < a \right\} \in \mathcal{D}$. As in the previous case, by (1) of Lemma 3.3.10 we can

find $k < \omega$ such that $(\inf_x \varphi)^{\mathcal{N}} < a$ for all $\mathcal{N} \in U_k$. By (3) of Lemma 3.3.10 we get $j < \omega$ such that $\varphi(c_j)^{\mathcal{N}} < a$ for all $\mathcal{N} \in U_k$. For all $n \geq k$ we have $\varphi(c_j)^{M_n} < a$, and hence $\lim_{n \to \mathcal{D}} \varphi(c_j)^{M_n} < a$. By the inductive hypothesis we have $\varphi(c_j)^{\prod_{\mathcal{D}} M_n} < a$, and therefore $(\inf_x \varphi)^{\prod_{\mathcal{D}} M_n} < a$ as well.

Now suppose that $\left\{ n < \omega : (\inf_x \varphi)^{M_n} < a \right\} \notin \mathcal{D}$. In order to prove that $(\inf_x \varphi)^{\prod_{\mathcal{D}} M_n} \geq a$, we consider two cases. The case $\left\{ n < \omega : (\inf_x \varphi)^{M_n} > a \right\} \in \mathcal{D}$ is handled in the same way as the previous paragraph. For the other case, suppose that $\left\{ n < \omega : (\inf_x \varphi)^{M_n} = a \right\} \in \mathcal{D}$. Then for each $\epsilon \in \mathbb{Q} \cap (0,1)$ such that $\epsilon < \min \{ a, 1 - a \}$, we also have

$$\left\{ n < \omega : \left(\inf_x \varphi \right)^{M_n} \in (a - \epsilon, a + \epsilon) \right\} \in \mathcal{D}.$$

As in the preceding cases, this implies that $(\inf_x \varphi)^{\prod_{\mathcal{D}} M_n} \in (a - \epsilon, a + \epsilon)$ for each such ϵ. Taking $\epsilon \to 0$ we obtain $(\inf_x \varphi)^{\prod_{\mathcal{D}} M_n} = a$.

$$\dashv - \text{Claim 3.3.11.1}$$

For each $F \in \mathcal{F}$, let T_F be a theory such that $F = \mathrm{Mod}(T_F)$. Then Claim 3.3.11.1 implies that $\prod_{\mathcal{D}} M_n \models T_{F_m}$ for every $m < \omega$, so it only remains to check that $\prod_{\mathcal{D}} M_n \in \mathcal{W}$. The proof is essentially the same as the last case of the claim. Suppose that $\varphi(x)$ is an L_C-formula in one free variable, and that $(\inf_x \varphi)^{\prod_{\mathcal{D}} M_n} = 0$. Fix $r \in \mathbb{Q} \cap (0,1)$. We need to find j such that $\varphi(c_j)^{\prod_{\mathcal{D}} M_n} < r$. By Claim 3.3.11.1 we have $\lim_{n \to \mathcal{D}} (\inf_x \varphi)^{M_n} = 0$, so

$$\left\{ n < \omega : \left(\inf_x \varphi \right)^{M_n} < r \right\} \in \mathcal{D}.$$

Using (1) and (3) of Lemma 3.3.10 we can find k and j such that $\varphi(c_j)^{\mathcal{N}} < r$ for all $\mathcal{N} \in U_k$. Hence $\left\{ n < \omega : \varphi(c_j)^{M_n} < r \right\} \in \mathcal{D}$, and by Claim 3.3.11.1 we have $\varphi(c_j)^{\prod_{\mathcal{D}} M_n} < r$.   □

Corollary 3.3.12. *Let T be a consistent L-theory. Then $\mathcal{W} \cap \mathrm{Mod}_{L_C}(T)$ is non-empty and is Baire.*

Proof. Since T is consistent it has a countable model M, by Downward Löwenheim-Skolem (Proposition 3.1.8). If $\vec{a}$ is an enumeration of M, then $\langle M, \vec{a} \rangle \in \mathcal{W} \cap \mathrm{Mod}_{L_C}(T) \neq \emptyset$. The fact that $\mathcal{W} \cap \mathrm{Mod}_{L_C}(T)$ is Baire follows immediately from Lemma 3.3.6 and Proposition 3.3.11.   □

3.3.4   Proof of Omitting Types

We now connect the model-theoretic notions in the statement of Theorem 3.3.4 to the topology of the space $\mathcal{W}$. The connection between Baire spaces and the Omitting Types Theorem in classical logic is well-known. We give a proof in our $[0,1]$-valued setting which closely follows the arguments of [25]. For simplicity we present the details of the proof in the case where the signature S contains no function symbols. After the proof is complete we will describe the modifications necessary for the general case. When S has no function symbols, the definition of a type of T being principal (Definition 3.3.3) takes the following simplified form:

Definition 3.3.13. Let T be an L-theory in a signature without function symbols. A type $\Sigma(\vec{x})$ of T *principal* over T is there is an L-formula $\varphi(\vec{x})$ such that $T \cup \varphi(\vec{x})$ is satisfiable, and for some $r \in \mathbb{Q} \cap (0,1)$ we have $T \cup \{ \varphi(\vec{x}) \leq r \} \models \Sigma(\vec{x})$.

We can now give the connection between principality of types and Baire spaces.

Lemma 3.3.14. *Let $\Sigma(\vec{x})$ be a type of an L-theory T, and let $\vec{c}$ be new constant symbols. Then $\Sigma(\vec{x})$ is principal if and only if $\mathrm{Mod}_{L_{\vec{c}}}(T \cup \Sigma(\vec{c}))$ has nonempty interior in $\mathrm{Mod}_{L_{\vec{c}}}(T)$.*

Proof. Assume that $\Sigma(\vec{x})$ is principal, and let $\varphi(\vec{x}) \in L$ and $r \in \mathbb{Q} \cap (0,1)$ witness the principality of Σ. Then $T \cup \{\varphi(\vec{x})\}$ is satisfiable, and hence $\mathrm{Mod}_{L_{\vec{c}}}(T \cup \varphi(\vec{c})) \neq \emptyset$. If $r' \in \mathbb{Q} \cap (0,r)$, then $\mathrm{Mod}_{L_{\vec{c}}}(T) \cap \mathrm{Mod}_{L_{\vec{c}}}(\varphi(\vec{c}) < r')$ is a nonempty open subclass of $\mathrm{Mod}_{L_{\vec{c}}}(T \cup \Sigma(\vec{c}))$.

Conversely, suppose that $\mathrm{Mod}_{L_{\vec{c}}}(T \cup \Sigma(\vec{c}))$ has nonempty interior in $\mathrm{Mod}_{L_{\vec{c}}}(T)$, so it contains a basic open class. That is, there is an $L_{\vec{c}}$-sentence $\varphi(\vec{c})$ such that

$$\emptyset \neq \mathrm{Mod}_{L_{\vec{c}}}(T) \cap \mathrm{Mod}_{L_{\vec{c}}}(\varphi(\vec{c}) < 1) \subseteq \mathrm{Mod}_{L_{\vec{c}}}(T \cup \Sigma(\vec{c})).$$

It follows that there exists $s \in \mathbb{Q} \cap (0,1)$ such that $T \cup \{\varphi(\vec{x}) \leq s\}$ is satisfiable. Our choices of φ and s give us that

$$T \cup \{\varphi(\vec{x}) \leq s\} \models T \cup \{\varphi(\vec{x}) < 1\} \models \Sigma(\vec{x}).$$

It is easy to check that if $r \in \mathbb{Q} \cap (s,1)$ then the formula $\max\{s - \varphi, 0\}$ and the rational $1 - r$ witness that Σ is principal. $\square$

Lemma 3.3.15. *Let T be an L-theory. For any $\mathbf{i} = \langle i_1, i_2, \ldots, i_n \rangle \in \omega^{<\omega}$, let $R_{T,\mathbf{i}} : \mathcal{W} \cap \mathrm{Mod}_{L_C}(T) \to \mathrm{Mod}_{L_{\{c_{i_1}, \ldots, c_{i_n}\}}}(T)$ be the natural projection defined by*

$$\langle M, \vec{a} \rangle \mapsto \langle M, a_{i_1}, \ldots, a_{i_n} \rangle.$$

Then $R_{T,\mathbf{i}}$ is continuous, open, and surjective.

Proof. To keep the notation as simple as possible, we will give the proof only in the case where $\mathbf{i} = \langle 0 \rangle$ – the general case is similar. To see that $R_{T,\mathbf{i}}$ is continuous, observe that if σ is any L_{c_0}-sentence then σ is also an L_C-sentence, and the pre-image of the basic closed class $\mathrm{Mod}_{L_{c_0}}(\sigma)$ under $R_{T,\mathbf{i}}$ is the closed class $\mathrm{Mod}_{L_C}(\sigma)$.

Now suppose that $\varphi(c_0, \ldots, c_m)$ is an L_C-sentence (with possibly some of the c_i's, including c_0, not actually appearing). Define the L_{c_0}-sentence $\theta(c_0)$ by

$$\sup_{x_1} \cdots \sup_{x_m} \varphi(c_0, x_1, \ldots, x_m).$$

To finish the proof it suffices to show that $R_{T,\mathbf{i}}$ maps $(\mathcal{W} \cap \mathrm{Mod}_{L_C}(T)) \setminus \mathrm{Mod}_{L_C}(\varphi(c_0, \ldots, c_m))$ onto $\mathrm{Mod}_{L_{c_0}}(T) \setminus \mathrm{Mod}_{L_{c_0}}(\theta(c_0))$.

Suppose that $\langle M, \vec{a} \rangle \in (\mathcal{W} \cap \mathrm{Mod}_{L_C}(T)) \setminus \mathrm{Mod}_{L_C}(\varphi(c_0, \ldots, c_m))$. Then $\langle M, \vec{a} \rangle \not\models \varphi(c_0, \ldots, c_m)$, so there is $r \in \mathbb{Q} \cap (0,1)$ such that

$$\langle M, \vec{a} \rangle \models \varphi(c_0, \ldots, c_m) \geq r.$$

Then clearly

$$\langle M, a_0 \rangle \models \theta(c_0) \geq r.$$

It follows that $\langle M, a_0 \rangle \in \mathrm{Mod}_{L_{c_0}}(T) \setminus \mathrm{Mod}_{L_{c_0}}(\theta(c_0))$.

Now suppose that $\langle M, a_0 \rangle \in \mathrm{Mod}_{L_{c_0}}(T) \setminus \mathrm{Mod}_{L_{c_0}}(\theta(c_0))$. As above, find $r \in \mathbb{Q} \cap (0,1)$ such that $\langle M, a_0 \rangle \models \theta(c_0) \geq r$, and pick $r' \in (0,r)$. Then by definition of θ there are elements $a_1, \ldots, a_m \in M$

such that

$$\langle M, a_0, a_1, \ldots, a_m \rangle \models \varphi(c_0, c_1, \ldots, c_m) \geq r'.$$

By Downward Löwenheim-Skolem (Proposition 3.1.8) we can find a countable $M_0 \preceq_L M$ containing $a_0, a_1, \ldots, a_m$. Using the remaining constant symbols to enumerate M_0 as $\vec{a}$, we have

$$\langle M, \vec{a} \rangle \in (\mathcal{W} \cap \text{Mod}_{L_C}(T)) \setminus \text{Mod}_{L_C}(\varphi(c_0, \ldots, c_m)),$$

and $R_{T,\mathbf{i}}(\langle M, \vec{a} \rangle) = \langle M, a_0 \rangle$.   □

We now have all of the ingredients necessary to prove Theorem 3.3.4.

Theorem 3.3.16 (Omitting Types)**.** *Let T be an L-theory and let $\{ \Sigma_j(\vec{x}_j) \}_{j < \omega}$ be a countable set of types of T that are not principal over T. Then there is a model of T that omits each Σ_j.*

Proof. For each $j < \omega$, write $\vec{x}_j = (x_0, \ldots, x_{n_j-1})$. Then for $\mathbf{i} \in \omega^{n_j}$, define

$$\mathcal{C}_{T,j,\mathbf{i}} = R_{T,\mathbf{i}}^{-1}\left(\text{Mod}_{L_{\left\{ c_{i_0}, \ldots, c_{i_{n_j-1}} \right\}}}(T \cup \Sigma_j(c_{i_0}, \ldots, c_{i_{n_j-1}})) \right) \subseteq \mathcal{W} \cap \text{Mod}_{L_C}(T).$$

By Lemmas 3.3.14 and 3.3.15, each $\mathcal{C}_{T,j,\mathbf{i}}$ is closed with empty interior. Hence $\bigcup_{j<\omega, \mathbf{i} \in \omega^{n_j}} \mathcal{C}_{T,j,\mathbf{i}}$ is meagre in $\mathcal{W} \cap \text{Mod}_{L_C}(T)$. Since $\mathcal{W} \cap \text{Mod}_{L_C}(T)$ is Baire (Lemma 3.3.12), there exists

$$\langle M, \vec{a} \rangle \in (\mathcal{W} \cap \text{Mod}_{L_C}(T)) \setminus \bigcup_{j<\omega, \mathbf{i} \in \omega^{n_j}} \mathcal{C}_{T,j,\mathbf{i}}.$$

For such $\langle M, \vec{a} \rangle$ we have by definition of the $\mathcal{C}_{T,j,\mathbf{i}}$'s that for every $j < \omega$ no subset of $\vec{a}$ is a realization of Σ_j. Since we are in the case where there are no function symbols, $\vec{a}$ is the universe of a structure M_0. By Lemma 3.3.9, $M_0 \preceq_L M$. Thus $M_0 \models T$ and omits every Σ_j.   □

The preceding proof generalizes in a straightforward way to the case where the signature contains function symbols, but it is necessary to give a stronger definition of principal type. The only difficulty is that when there are function symbols present not every subset of a structure is the universe of a substructure, so in the proof of Theorem 3.3.4 we need to take M_0 to be $M \upharpoonright \langle \vec{a} \rangle$. The proof of Lemma 3.3.9 works even with function symbols present, so we still have that $M_0 \preceq M$, but we now need to prove that no subset of $M \upharpoonright \langle \vec{a} \rangle$ realizes any of the Σ_j. For this we need the more general definition of principality given in Definition 3.3.3, which we recall here.

Definition 3.3.17. Let T be an L-theory. A type $\Sigma(\vec{x})$ of T *principal* over T is there is an L-formula $\varphi(\vec{x})$, terms $t_1(\vec{y}), \ldots, t_n(\vec{y})$ (where n is the length of $\vec{x}$), and $r \in \mathbb{Q} \cap (0, 1)$ such that the following hold:

- $T \cup \{ \varphi(\vec{y}) \}$ is satisfiable, and

- $T \cup \{ \varphi(\vec{y}) \leq r \} \models \Sigma(t_1(\vec{y}), \ldots, t_n(\vec{y}))$.

The modification of principality to include terms was used by Keisler and Miller [68] in the context of discrete logic without equality, in [88] for an infinitary logic for metric structures (see Section 3.5), and independently by Caicedo and Iovino [25] for $[0,1]$-valued logic. Taking Definition 3.3.17 as the definition of principality, we may assume that whenever $\Sigma(\vec{x})$ is a type we wish to omit, and $t_1(\vec{y}), \ldots, t_n(\vec{y})$ are

terms, then $\Sigma(t_1(\vec{y}), \ldots, t_n(\vec{y}))$ is also one of the types to be omitted. Then we have that no subset of $M \restriction \langle \vec{a} \rangle$ realizes any of the types we wish to omit since elements of $M \restriction \langle \vec{a} \rangle$ are obtained from $\vec{a}$ by applying terms. $\qquad \square$

Remark 3.3.18. By assuming additional set-theoretic axioms it is possible to extend Theorem 3.3.4 to allow a collection of fewer than $2^{\aleph_0}$ non-principal types to be omitted. To do this, observe that $\mathcal{W}$ has a countable base, so Martin's Axiom restricted to countable partial orders implies that $\mathcal{W}$ is $2^{\aleph_0}$-Baire (see Remark 3.3.7). Then the same proof as above can be applied to a collection of fewer than $2^{\aleph_0}$ non-principal types. If T is a theory in a countable fragment L of $\mathcal{L}_{\omega_1,\omega}$ then there are at most $2^{\aleph_0}$ types of T. There are theories in which every model realizes a non-isolated type (see [28, Example 2.3.1] for an example in the discrete case), so it is not generally possible to omit $2^{\aleph_0}$ non-prinicipal types. Thus under the Continuum Hypothesis it is not always possible to omit $\aleph_1$ non-principal types. These observations show that the extension of Theorem 3.3.4 to omitting $\aleph_1$ non-principal types is undecidable on the basis of ZFC. In fact, similar observations show that in ZFC it is possible to omit $< \mathrm{cov}(\mathrm{meagre})$ non-principal types.

3.3.5   Omitting Types in Complete Structures

In applications of $[0,1]$-valued logics it is sometimes desirable to be able to produce metric structures based on *complete* metric spaces. There are two issues that need to be addressed in order to be able to take the metric completion of the structure obtained from Theorem 3.3.4. First, there are some types that may be omitted in a structure but not in its metric completion (such as the type of the limit of a non-convergent Cauchy sequence), so we need a stronger notion of principal type. Second, because of the infinitary connectives, it may not be the case that every structure is elementarily equivalent to its metric completion.

To resolve the first issue, we use the notion of *metrically principal* types from [25]. If $\Sigma(x_1, \ldots, x_n)$ is a type, then for each $\delta \in \mathbb{Q} \cap (0,1)$ we define:

$$\Sigma^\delta = \left\{ \inf_{y_1} \ldots \inf_{y_n} \left(\bigwedge_{k \leq n} d(x_k, y_k) \leq \delta \wedge \sigma(y_1, \ldots, y_n) \right) : \sigma \in \Sigma \right\}.$$

We think of Σ^δ as a thickening of Σ, since if M is a structure and $a_1, \ldots, a_n \in M$ realize Σ, then every n-tuple in the closed δ-ball around $(a_1, \ldots, a_n)$ realizes Σ^δ.

Definition 3.3.19. We say that a type $\Sigma(\vec{x})$ of T is *metrically principal* over T if for every $\delta > 0$ the type $\Sigma^\delta(\vec{x})$ is principal over T.

Proposition 3.3.20. *Let L be a countable fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, and let T be an L-theory. For each $n < \omega$, suppose that Σ_n is a type that is not metrically principal over T. Then there is $M \models T$ such that the metric completion of M omits each Σ_n.*

Proof. For each $n < \omega$, let $\delta_n > 0$ be such that $\Sigma_n^{\delta_n}$ is non-principal. Using Theorem 3.3.4 we get $M \models T$ that omits each $\Sigma_n^{\delta_n}$. Fix $n < \omega$; we show that $\overline{M}$, the metric completion of M, omits Σ_n. Suppose otherwise, and let $\vec{a} \in \overline{M}$ be a realization of Σ_n in $\overline{M}$. By definition of the metric completion there are $\vec{a}_1, \vec{a}_2, \ldots$ from M converging (coordinatewise) to $\vec{a}$. For k sufficiently large we then have that $\vec{a}_k$ is in

the δ_n-ball around $\vec{a}$. As we observed earlier, this implies that $\vec{a}_k$ satisfies $\Sigma_n^{\delta_n}$, contradicting that $\Sigma_n^{\delta_n}$ is not realized in M. □

The final problem to be resolved in order to have a satisfactory Omitting Types Theorem for complete structures is that we may not have $M \equiv_L \overline{M}$ (see Example 3.1.3). This problem arises because if $\varphi(x)$ is a formula of $\mathcal{L}_{\omega_1,\omega}$ and M is a structure, then the function from M to $[0,1]$ given by $a \mapsto \varphi^M(a)$ may not be continuous. Recall that a fragment L of $\mathcal{L}_{\omega_1,\omega}(S)$ is *continuous* if $a \mapsto \varphi^M(a)$ is a continuous function for every S-structure M and every L-formula φ. Applying Proposition 3.3.20 we therefore have:

Theorem 3.3.21 (Omitting Types for Complete Structures). *Let L be a countable continuous fragment of $\mathcal{L}_{\omega_1,\omega}$, and let T be a satisfiable L-theory. For each $n < \omega$ let Σ_n be a type that is not metrically principal. Then there is $M \models T$ such that M is based on a complete metric space and M omits each Σ_n.*

3.4 Applications of Omitting Types

In this section we apply the Omitting Types Theorem to obtain a $[0,1]$-valued version of Keisler's two-cardinal theorem (see [69, Theorem 30]). We then discuss the existence of prime models of theories in countable fragments.

3.4.1 Keisler's two-cardinal theorem

We begin with an easy lemma about metric spaces.

Lemma 3.4.1. *Let (M,d) be a metric space of density λ, where λ has uncountable cofinality. Then there is $R \in \mathbb{Q} \cap (0,1)$ and a set $D \subseteq M$ with $|D| = \lambda$ such that for all $x, y \in D$, $d(x,y) \geq R$, and for all $x \in M$ there exists $y \in D$ with $d(x,y) < R$.*

Proof. Build a sequence $\{\, x_\alpha : \alpha < \lambda \,\}$ in M recursively, starting from an arbitrary $x_0 \in M$. Given $\{\, x_\alpha : \alpha < \beta \,\}$, with $\beta < \lambda$, we have that $\{\, x_\alpha : \alpha < \beta \,\}$ is not dense in M. Hence there exists $x_\beta \in M$ and $R_\beta \in \mathbb{Q} \cap (0,1)$ such that $d(x_\beta, x_\alpha) \geq R_\beta$ for all $\alpha < \beta$. Then since $\mathrm{cof}(\lambda) > \omega$ there is $R \in \mathbb{Q} \cap (0,1)$ and $S \in [\lambda]^\lambda$ such that $R = R_\alpha$ for every $\alpha \in S$. Then $D = \{\, x_\alpha : \alpha \in S \,\}$ can be extended to the desired set. □

The above lemma does not always apply if the condition that λ has uncountable cofinality is dropped, as the following example (pointed out to us by Daniel Soukup) shows.

Example 3.4.2. Let $M = \aleph_\omega$. For each $n < \omega$ let $A_n = [\omega_n, \omega_{n+1})$, with $A_0 = [0, \omega)$, so $M = \bigcup_{n<\omega} A_n$. Define a metric on $\aleph_\omega$ as follows, for distinct $\alpha, \beta \in M$:

$$d(\alpha, \beta) = \begin{cases} \frac{1}{n+1} & \text{if } \alpha, \beta \in A_n, \\ 1 & \text{otherwise.} \end{cases}$$

This metric induces the discrete topology on M, so the density of M is $\lambda = \aleph_\omega$. Given any set $D \subseteq M$ of cardinality $\aleph_\omega$ there are arbitrarily large n such that $|D \cap A_n| \geq 2$, and so there is no n such that $d(\alpha, \beta) \geq 1/n$ for all $\alpha, \beta \in D$.

It will be important for us that certain predicates take values only in $\{0, 1\}$, and that this fact can be expressed in our logic. For any formula $\varphi(x)$, we define the formula $\mathrm{Discrete}(\varphi)$ to be $\min\{\varphi, 1 - \varphi\}$. It

is clear that if $\mathcal{M} \models \forall x \, \mathrm{Discrete}(\varphi(x))$, then $\varphi^{\mathcal{M}}(a) \in \{0,1\}$ for every $a \in \mathcal{M}$; in this case we say that φ is *discrete* in $\mathcal{M}$. Note that if $\varphi(x)$ is discrete in models of a theory T then we can relativize quantifiers to $\{\, x : \varphi(x) = 0 \,\}$ in models of T. We emphasize that discreteness of φ only means that φ takes values in $\{0,1\}$, *not* that the metric is discrete on $\{\, x : \varphi(x) = 0 \,\}$.

Definition 3.4.3. If S is a metric signature with a distinguished unary predicate U, and κ, λ are infinite cardinals, then we say that an S-structure $\mathcal{M} = \langle\, M, U, \ldots \,\rangle$ is *of type* (κ, λ) if the density of M is κ and the density of $\{\, a \in M : U(a) = 0 \,\}$ is λ.

Theorem 3.4.4. *Let S be a metric signature with a distinguished unary predicate symbol U, and let L be a countable fragment of $\mathcal{L}_{\omega_1, \omega}(S)$. Let T be an L-theory such that $T \models \forall x \, \mathrm{Discrete}(U(x))$, and let $\mathcal{M} = \langle\, M, V, \ldots \,\rangle$ be a model of T of type (κ, λ) where $\kappa > \lambda \geq \aleph_0$. Then there is a model $\mathcal{N} = \langle\, N, W, \ldots \,\rangle \equiv_L \mathcal{M}$ of type $(\aleph_1, \aleph_0)$. Moreover, there is a model $\mathcal{M}_0 = \langle\, M_0, V_0, \ldots \,\rangle$ such that $\mathcal{M}_0 \preceq_L \mathcal{M}, \mathcal{M}_0 \preceq_L \mathcal{N}$, and V_0 is dense in W.*

Proof. By Downward Löwenheim-Skolem, we may assume that $\mathcal{M}$ is of type (κ^+, κ) for some $\kappa \geq \aleph_0$. Our first step is to expand $\mathcal{M}$ into a structure in a larger language that includes an ordering of a dense subset of $\mathcal{M}$ in type κ^+. To do this we expand the signature S to a new signature S' by adding a unary predicate symbol L, a binary predicate symbol $\trianglelefteq$, a constant symbol c, and a unary function symbol f. Let M' be the disjoint union of M and κ^+. Extend the metric d from M to a metric d' on M' by making d' the discrete metric on κ^+ and setting $d'(m, \alpha) = 1$ for every $m \in M, \alpha \in \kappa^+$. We interpret L as $L(x) = 0$ if and only if $x \in \kappa^+$. Interpret c as κ, and let $\trianglelefteq$ be the characteristic function of the ordinal ordering on κ^+, and arbitrary elsewhere.

Find $D \subseteq M$ of size κ^+, and $R \in \mathbb{Q} \cap (0,1)$, as in Lemma 3.4.1. Define $f : M' \to M'$ so that below κ the function f is an enumeration of a dense subset of V, from κ to κ^+ f is an enumeration of D, and f is arbitrary otherwise. This gives a metric structure $\mathcal{M}' = \langle\, M', V, \ldots, \kappa^+, \trianglelefteq, \kappa, f \,\rangle$.

Now let $\mathcal{M}'_0 = \langle\, M'_0, V_0, \ldots, L_0, \trianglelefteq_0, c_0, f_0 \,\rangle$ be a countable elementary substructure of $\mathcal{M}'$. Add countably many new constant symbols d_l, $l \in L$, and another constant symbol d^*. Let T be the elementary diagram of $\mathcal{M}'_0$, together with the sentences $\{\, d_l \triangleleft d^* : l \in L \,\}$. Define

$$\Sigma(x) = \{\, L(x) \,\} \cup \{\, U(f(x)) \,\} \cup \{\, d(x, d_l) = 1 : l \triangleleft c \,\}.$$

We note that a model of T that omits Σ corresponds to a elementary extension of $\mathcal{M}'_0$ in which V_0 is dense in the interpretation of U. The extension is proper because the interpretation of d^* will satisfy $d(f(d^*), f(d_l)) \geq R$ for every l, and $f(d^*) \notin L$, while every $m \in M'_0 \setminus L$ satisfies $d(m, d_l) < R$ for some l. We have V_0 dense in the interpretation of U because the image of f on elements of L below c is dense in U, and omitting Σ ensures that no new such elements are added.

Claim 3.4.4.1. $\Sigma(x)$ *is non-principal over* T.

Proof. We note first that if t is a term that is not a variable symbol or a constant symbol then $T \models \forall x \neg U(t(x))$. It therefore suffices to show that if $\psi(x)$ is a formula consistent with T and $r \in \mathbb{Q} \cap (0,1)$, then $T \cup \{\, \psi(x) \leq r \,\} \not\models \Sigma(x)$.

Now suppose that $\psi(x)$ is consistent with T. Let us write $\psi(x, d)$ to emphasize that the new constant symbol d may appear. If either $\psi(x, d) \wedge \neg L(x)$ or $\psi(x, d) \wedge L(x) \wedge \neg U(f(x))$ is consistent with T then by

definition of Σ, $T \cup \{\, \psi(x,d)\,\} \not\models \Sigma(x)$ and we are done. So we may assume that $\psi(x,d) \wedge L(x) \wedge U(f(x))$ is consistent with T. It follows from the definition of T that

$$\mathcal{M}_0' \models \forall z \in L \inf_{y \in L} \inf_{x \in L} (z \trianglelefteq y \wedge U(f(x)) \wedge \psi(x,y)).$$

By elementary equivalence, $\mathcal{M}'$ is also a model of this sentence. Pick $q \in \mathbb{Q} \cap (0, r)$. For each $\alpha \in \kappa^+$, find $x_q^\alpha \in \kappa^+$ such that

$$\mathcal{M}' \models \inf_{y \in L} (\alpha \trianglelefteq y \wedge U(f(x_q^\alpha)) \wedge \psi(x_q^\alpha, y)) \le q.$$

This implies that $\mathcal{M}' \models U(f(x_q^\alpha))$, so by our choice of f we have that $x_q^\alpha < \kappa$. Since κ^+ is regular there exists x_q such that for all sufficiently large α, $x_q = x_q^\alpha$. We thus have

$$\mathcal{M}' \models \forall z \in L \inf_{y \in L} (z \trianglelefteq y \wedge U(f(x_q)) \wedge \psi(x_q, y)) \le q.$$

By elementary equivalence,

$$\mathcal{M}_0' \models \inf_{x \in L} \forall z \in L \inf_{y \in L} (z \trianglelefteq y \wedge U(f(x)) \wedge \psi(x,y)) \le q.$$

Now pick $r' \in \mathbb{Q} \cap (0,1)$ such that $q < r' < r$. Then there exists $x_{r'}$ such that

$$\mathcal{M}_0' \models \forall z \in L \inf_{y \in L} (z \trianglelefteq y \wedge U(f(x_{r'})) \wedge \psi(x_{r'}, y)) \le r'.$$

This implies that $\mathcal{M}_0' \models U(f(x_{r'})) = 0$, so there is some l such that $x_{r'} = d_l \trianglelefteq c_0$. Thus, using that the metric d is discrete in L_0,

$$\mathcal{M}_0' \models \forall z \in L \inf_{y \in L} (z \trianglelefteq y \wedge \inf_{x \in L} (\psi(x,y) \le r' \wedge d(x, d_l) = 0).$$

We therefore have that $\psi(x,d) \le r' \wedge d(x, d_l) = 0$ is consistent with T. Since $d(x, d_l) = 1$ appears in Σ, this shows that $\psi(x,d) \ge r' \not\models \Sigma(x)$, and hence $\psi(x,d) \ \le \ r \not\models \Sigma(x)$.

$$\dashv - \text{Claim 3.4.4.1}$$

By Claim 3.4.4.1 and the Omitting Types Theorem (Theorem 3.3.4) there is $\mathcal{M}_1' \models T$ that omits Σ. Repeating the above argument ω_1 times we get an elementary chain $(\mathcal{M}_\alpha')_{\alpha < \omega_1}$. For each $\alpha < \omega_1$ let $\mathcal{M}_\alpha$ denote the reduct of $\mathcal{M}_\alpha'$ to S. Then $\mathcal{N} = \bigcup_{\alpha < \omega_1} \mathcal{M}_\alpha$ is the desired model. $\qquad\square$

We note that instead of using a discrete predicate U, we could instead have used a two-sorted language, with only notational differences in the proof. We give an application of the two-cardinal theorem to separable quotients of Banach spaces in Section 3.6.

3.4.2 Prime Models

We use the Omitting Types Theorem for $\mathcal{L}_{\omega_1, \omega}$ to give conditions for the existence of prime models. Throughout the section we fix a metric signature S, a countable fragment L of $\mathcal{L}_{\omega_1, \omega}(S)$, and a (not necessarily complete) L-theory T.

Definition 3.4.5. A model of T is *prime* if it L-elementarily embeds into every model of T.

The key to producing prime models is the notion of *complete conditions*.

Definition 3.4.6. Let $\varphi(\vec{x})$ be an L-formula. For $r \in \mathbb{Q} \cap (0,1)$, we say that the condition $\varphi(\vec{x}) \leq r$ is *complete* (with respect to T) if $\varphi(\vec{x})$ is consistent with T, and for every L-formula $\psi(\vec{x})$ and every $s \in \mathbb{Q} \cap (0,1)$, either

$$T \cup \{\, \varphi(\vec{x} \leq r \,\} \models \psi(\vec{x}) \leq s,$$

or

$$T \cup \{\, \varphi(\vec{x} \leq r \,\} \models \psi(\vec{x}) \geq s.$$

In the terminology of Definition 3.3.3 this says that $\varphi \leq r$ isolates a complete type.

Lemma 3.4.7. *Suppose that $\varphi(\vec{x}) \leq r$ is a complete condition. Then $T \cup \{\, \varphi(\vec{x}) \leq r \,\} \models \varphi(\vec{x}) = 0$.*

Proof. For any $q \in \mathbb{Q} \cap (0,1)$ we cannot have $T \cup \{\, \varphi(\vec{x}) \leq r \,\} \models \varphi(\vec{x}) \geq r$, since this would contradict the assumption that φ is consistent with T. By completeness we therefore have $T \cup \{\, \varphi(\vec{x}) \leq r \,\} \models \varphi(\vec{x}) \leq q$ for all $q \in \mathbb{Q} \cap (0,1)$ $\hfill\square$

Lemma 3.4.8. *For a complete type $\Sigma(\vec{x})$, the following are equivalent:*

1. *Σ is non-principal,*

2. *Σ contains no complete conditions,*

3. *for each complete condition $\varphi(\vec{x}) \leq r$, there exists $\epsilon > 0$ such that Σ does not contain $\varphi(\vec{x}) \leq r + \epsilon$.*

Proof. If Σ contains the complete condition $\varphi(\vec{x}) \leq r$, then the complete type isolated by $\varphi(\vec{x}) \leq r$ must be Σ, so Σ is principal. This shows (1) implies (2).

For (2) implies (3), if there is a complete condition $\varphi(\vec{x}) \leq r$ such that for every $\epsilon > 0$ we have $(\varphi(\vec{x}) \leq r + \epsilon) \in \Sigma$, then by the completeness of Σ, $(\varphi(\vec{x}) \leq r) \in \Sigma$.

Finally, suppose that (3) holds and that Σ is principal. Then there is a formula $\varphi(\vec{x})$ consistent with T, and an $r \in \mathbb{Q} \cap (0,1)$, such that $T \cup \{\, \varphi(\vec{x}) \leq r \,\} \models \Sigma$. By the completeness of Σ we have $(\varphi(\vec{x}) \leq r + \epsilon) \in \Sigma$ for every $\epsilon > 0$, so $\varphi(\vec{x}) \leq r$ cannot be complete by (3), contradicting the fact that $\varphi(\vec{x}) \leq r$ isolates a complete type. $\hfill\square$

Theorem 3.4.9. *Let T be a complete L-theory. A model $M \models T$ is prime if and only if M is countable and every finite tuple of elements of M satisfies a complete condition in M.*

Proof. Suppose that M is prime. By the Downward Löwenheim-Skolem theorem there is a countable model of T, so M must be countable. Consider any finite tuple $\vec{a} \in M$, and let $\Sigma = \operatorname{tp}_L^M(\vec{a})$. If Σ contains no complete conditions then by Lemma 3.4.8 Σ is non-principal, so by Theorem 3.3.4 there is a model of T which omits Σ. The model M cannot be L-elementarily embedded into such a model, contradicting the assumption that M is prime. Therefore Σ contains a complete condition, and hence $\vec{a}$ satisfies a complete condition in M.

Now suppose that M is countable and every finite tuple satisfies a complete condition. Pick any model $N \models T$. Enumerate $M = \{a_n : n < \omega\}$, and for each $n < \omega$ let $\varphi_n(x_0, \ldots, x_n) \leq r_n$ be a complete condition satisfied by $(a_0, \ldots, a_n)$. By Lemma 3.4.7 we actually have $\varphi_0^M(a_0) = 0$, so $T \models \inf_{x_0} \varphi_0(x_0)$. Let $b_0 \in N$ be such that $\varphi_0^N(b_0) \leq r_0/2$. Now $T \cup \{\, \varphi_0(x_0) \leq r_0 \,\} \models \inf_{x_1} \varphi_1(x_0, x_1)$ by the completeness of φ_1 and Lemma 3.4.7. Choose $b_1 \in N$ such that $\varphi_1^N(b_0, b_1) \leq r_1/2$. Continue in this manner to

produce a sequence $(b_n)_{n<\omega}$ in N such that for all n, $\varphi_n^N(b_0, \ldots, b_n) \leq r_n/2$. Since each $\varphi_n \leq r_n$ is a complete condition, this implies $\mathrm{tp}_L^M(a_0, \ldots, a_n) = \mathrm{tp}_L^N(b_0, \ldots, b_n)$ for all $n < \omega$, so the map $a_n \mapsto b_n$ is an L-elementary embedding of M into N. $\square$

Corollary 3.4.10. *A complete L-theory T has a prime model if and only if there are no incompletable conditions with respect to T.*

Proof. Suppose first that T has a prime model M, and the condition $\varphi(\vec{x}) \leq r$ is incompletable with respect to T. Since T is complete there is $\vec{a} \in M$ such that $\varphi^M(\vec{a}) \leq r$. By Theorem 3.4.9 $\vec{a}$ satisfies a complete condition, contradicting that $\varphi \leq r$ is incompletable.

Now suppose that there are no incompletable conditions with respect to T. For each $n < \omega$, define

$$\Phi_n(x_1, \ldots, x_n) = \left\{ \varphi(x_1, \ldots, x_n) \geq \frac{r}{2} : \varphi \leq r \text{ is a complete condition} \right\}.$$

If we construct a model omitting each Φ_n then in that model every tuple will satisfy a complete condition, and hence be prime by Theorem 3.4.9. By Theorem 3.3.4 it suffices to show that each Φ_n is nonprincipal.

Suppose that $T \cup \{ \psi(\vec{x}) \leq s \} \models \Phi_n(\vec{x})$, where ψ is consistent with T. By hypothesis the condition $\psi \leq s$ is completable, so let $\varphi(\vec{x}) \leq r$ be a complete condition such that $T \cup \{ \varphi(\vec{x}) \leq r \} \models \psi(\vec{x}) \leq s$. Then $T \cup \{ \varphi(\vec{x}) \leq r \} \models \Phi_n(\vec{x})$. By definition we have $(\varphi \geq \frac{r}{2}) \in \Phi_n$, so $T \cup \{ \varphi \leq r \} \models \varphi \geq \frac{r}{2}$, contradicting Lemma 3.4.7. $\square$

3.5 Comparison of infinitary $[0, 1]$-valued logics

As mentioned above, the logic $\mathcal{L}_{\omega_1, \omega}$ presented in this chapter is not the only infinitary logic for metric structures that has been considered in the literature. Here we define the other infinitary logics for metric structures which have been proposed, and compare their properties. The names we give to these logics are not standard.

The first infinitary logic for metric structures appears to be the infinitary logic L_A considered by Ortiz in his thesis [88]. Ortiz's work was inspired by Keisler (and later Fajardo and Keisler)'s use of infinitary logic in the study of neocompact families (see [70] for an overview of this work). The logic L_A is an extension of Henson's logic of positive bounded formulas with approximate satisfaction, and hence is not precisely an infinitary generalization of the $[0, 1]$-valued continuous logic presented in Chapter 2. Since we have not presented Henson's logic in this thesis, we will give a somewhat informal discussion of L_A.

The logic L_A is formed from the positive bounded formulas by allowing the operations of countable conjunction and disjunction, negation, and the use of countable strings of quantifiers. Remark 3.1.2 shows that our $\mathcal{L}_{\omega_1, \omega}$ is at least as expressive as the part of L_A which uses only finite strings of quantifiers. In [89] Ortiz shows the full logic L_A satisfies a compactness theorem. Hence the finite quantifier part of L_A is not equivalent to our $\mathcal{L}_{\omega_1, \omega}$.

One of the main results of [88] (and also, in an expanded form with applications to Banach spaces, [91]) is an omitting types result for L_A. This omitting types theorem gives conditions under which a sentence σ of L_A has a model satisfying

$$\forall \vec{y} \varphi(\vec{y}),$$

where $\vec{y}$ is a finite tuple of variables, and $\varphi(\vec{y})$ is of the form

$$\forall \vec{\mathbf{x}}\theta(\vec{\mathbf{x}}),$$

with $\vec{\mathbf{x}}$ a (possibly infinite) tuple of variables, and θ quantifier-free. There are no restrictions on the quantifier complexity of σ. While our omitting types theorem does not apply to formulas involving infinitely many variables, it does apply to formulas of higher quantifier complexity than the formula φ above.

We now turn our attention to $[0,1]$-valued logics which extend continuous first-order logic.

Definition 3.5.1 ([14]). The logic $\mathcal{L}^C_{\omega_1,\omega}$ is defined in the same manner as our $\mathcal{L}_{\omega_1,\omega}$, except that when forming $\sup_n \varphi_n$ and $\inf_n \varphi_n$ it is required that the formulas φ_n satisfy a common modulus of uniform continuity.

We note that $\mathcal{L}^C_{\omega_1,\omega}$ is an example of a continuous fragment of our $\mathcal{L}_{\omega_1,\omega}$.

Definition 3.5.2 ([99]). The logic $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ is obtained from $\mathcal{L}^C_{\omega_1,\omega}$ by the addition of a new symbol ρ, which is interpreted in a structure M as

$$\rho^M(x,\varphi) = \inf\{d(x,y) : \varphi^M(y) = 0\}.$$

The main theorem of [14] is an omitting types theorem for $L^C_{\omega_1,\omega}$, proved using the framework of model-theoretic forcing. Our omitting types theorem (Theorem 3.3.4 above) yields this omitting types theorem as a special case, and indeed our work in this area was initially inspired by [14].

In [99] Sequeira proves the existence of Scott sentences for complete separable metric structures in $\mathcal{L}^C_{\omega_1,\omega}(\rho)$. The proof is an adaptation of the classical Scott analysis (see, for example, [69, Chapter 1]).

The main result of [15] is a metric version of the López-Escobar theorem characterizing the isomorphism invariant bounded Borel functions on a space of codes for structures as exactly those functions of the form $M \mapsto \sigma^M$ for an $\mathcal{L}^C_{\omega_1,\omega}$-sentence σ. The existence of Scott sentences in $\mathcal{L}^C_{\omega_1,\omega}$ for complete separable metric structures (see Theorem 3.2.1 above) is a corollary of this López-Escobar theorem. As a consequence, we obtain the unexpected result that the logics $\mathcal{L}_{\omega_1,\omega}, \mathcal{L}^C_{\omega_1,\omega}$, and $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ all have the same elementary equivalence relation for complete separable metric structures.

Proposition 3.5.3. *Let S be a metric signature. For any complete separable S-structures M and N, the following are equivalent:*

1. *$M \cong N$,*

2. *$M \equiv_{\mathcal{L}_{\omega_1,\omega}} N$,*

3. *$M \equiv_{\mathcal{L}^C_{\omega_1,\omega}} N$,*

4. *$M \equiv_{\mathcal{L}^C_{\omega_1,\omega}(\rho)} N$,*

Proof. It is clear that (1) implies each of the other statements. Since each $\mathcal{L}^C_{\omega_1,\omega}$ formula is also an $\mathcal{L}_{\omega_1,\omega}$ formula and an $\mathcal{L}^C_{\omega_1,\omega}(\rho)$ formula we have that (2) and (4) each imply (3). Finally, the implication from (3) to (1) is due to the existence of Scott sentences in $\mathcal{L}^C_{\omega_1,\omega}$. $\qquad\square$

Our proof of Theorem 3.2.3 relied on the fact that our logic $\mathcal{L}_{\omega_1,\omega}$ has the desirable property that whenever $\vec{c}$ is a finite tuple of constant symbols, the formula obtained from an $\mathcal{L}_{\omega_1,\omega}$ formula by replacing every occurrence of $\vec{c}$ by new variables $\vec{x}$ is again an $\mathcal{L}_{\omega_1,\omega}$ formula. We note that this is not true in $\mathcal{L}_{\omega_1,\omega}^C$ or $\mathcal{L}_{\omega_1,\omega}^C(\rho)$:

Example 3.5.4. Let M be a complete separable metric structure with an element a such that the $\mathrm{Aut}(M)$-orbit $\mathcal{O}_a$ of a is not all of M. Let $\theta_a(x)$ be the $\mathcal{L}_{\omega_1,\omega}$ formula obtained by replacing a by x in the Scott sentence of (M,a). Then for any $b \in X$,

$$\theta_a^M(b) = \begin{cases} 0 & \text{if } b \in \mathcal{O}_a, \\ 1 & \text{otherwise.} \end{cases}$$

Since M is connected and the image of θ_a^M is $\{0,1\}$, the function θ_a^M is not continuous. Therefore θ_a is not an $\mathcal{L}_{\omega_1,\omega}^C$ formula or an $\mathcal{L}_{\omega_1,\omega}^C(\rho)$ formula.

A consequence of this observation is that the method used in our proof of the Scott definability theorem does not obviously generalize to either of the other infinitary logics. Any version of Theorem 3.2.3 for $\mathcal{L}_{\omega_1,\omega}^C$ will require the stronger hypothesis that the predicate to be defined is *uniformly* continuous, since the formulas of that logic always define uniformly continuous functions.

For discrete structures the most prominent generalization of infinitary logic is Shelah's framework of *abstract elementary classes* (see [100]). A corresponding notion of *metric abstract elementary classes* was introduced by Hirvonen and Hyttinen in [66].

Definition 3.5.5. Let S be a metric signature (possibly with only continuous, rather than uniformly continuous, functions and predicates), and let $\mathcal{K}$ be a class of complete S-structures. Let $\preceq_{\mathcal{K}}$ be a binary relation on $\mathcal{K}$. The pair $(\mathcal{K}, \preceq_{\mathcal{K}})$ is called a *metric abstract elementary class* if it satisfies the following properties:

1. $\mathcal{K}$ and $\preceq_{\mathcal{K}}$ are closed under isomorphism,

2. $\preceq_{\mathcal{K}}$ is a partial order refining the substructure relation,

3. if $M, N, R \in \mathcal{K}$, $M \preceq_{\mathcal{K}} R$, $N \preceq_{\mathcal{K}} R$, and $M \subseteq N$, then $M \preceq_{\mathcal{K}} N$,

4. if $(M_\alpha)_{\alpha<\gamma}$ is a $\preceq_{\mathcal{K}}$-increasing chain in $\mathcal{K}$ then $\overline{\bigcup_{\alpha<\gamma} M_\alpha} \in \mathcal{K}$, and $M_\alpha \preceq_{\mathcal{K}} \overline{\bigcup_{\alpha<\gamma} M_\alpha}$ for all $\alpha < \gamma$,

5. there is a cardinal $LS(\mathcal{K})$ such that if $M \in \mathcal{K}$ and $A \subseteq M$ then there is $N \in \mathcal{K}$ with $A \subseteq N$, $|N| = |A| + LS(\mathcal{K})$, and $N \preceq_{\mathcal{K}} M$.

The basic example of an abstract elementary class of discrete structures is the class of models of a first-order theory with the first-order elementary substructure relation, and the analogous statement is also true of metric abstract elementary classes. In the discrete setting an important example of an abstract elementary class is the class of models of an $\mathcal{L}_{\omega_1,\omega}$ sentence σ, with the L-elementary equivalence relation, for L a countable fragment containing σ. In the metric setting the (complete) models of an $\mathcal{L}_{\omega_1,\omega}$ sentence σ do *not* always form a metric abstract elementary class. Condition (4) of the definition can fail, because while the union of an L-elementary chain of models of σ is again a model of σ, taking the completion does not always result in an elementarily equivalent structure (see Example 3.1.3). We note that this issue can arise only when the indexing ordinal of the models has cofinality ω, since otherwise the union of a chain of complete structures is already complete.

Discrete abstract elementary classes can be described in terms of the infinitary logics $\mathcal{L}_{\kappa,\omega}$. Indeed, Shelah's Presentation Theorem from [100] shows that given an abstract elementary class $\mathcal{K}$, there is an expansion S' of the signature of $\mathcal{K}$ such that $\mathcal{K}$ is the class of reducts of a model of a first-order S'-theory omitting certain types. By the well-known fact that the property of omitting a type can be written as a sentence of infinitary logic, it follows that every discrete abstract elementary class is the class of reducts of an $\mathcal{L}_{\kappa,\omega}(S')$ sentence for some κ.

In the metric case it still true that every metric abstract elementary class is the class of reducts of a continuous first-order theory omitting certain types (see [105, Theorem 1.2.7]). It is possible to express type omission in our $\mathcal{L}_{\omega_1,\omega}$ (or the natural generalization to higher $\mathcal{L}_{\kappa,\omega}$). The presentation theorem for metric abstract elementary classes, due to Zambrano in [105], relied on functions which are continuous, but possibly not uniformly continuous, but very recently Zambrano improved this result in [106] to show that only uniformly continuous functions are necessary. He also asked whether our $\mathcal{L}_{\omega_1,\omega}$ can be used to give a direct proof of the result of Boney [18] that metric abstract elementary classes have Hanf numbers. In general, it seems that the following question, originally asked of us by Boney, has not been completely resolved:

Question 3.5.6. What is the precise relation between metric abstract elementary classes, $\mathcal{L}_{\omega_1,\omega}$, and $\mathcal{L}^C_{\omega_1,\omega}$ (as well as their natural generalizations to $\mathcal{L}_{\kappa,\omega}$ for other uncountable κ)?

3.6 Applications to Banach spaces

To conclude this chapter we illustrate the use of $\mathcal{L}_{\omega_1,\omega}$ with connections to Banach spaces. First, we give a partial list of classes of Banach spaces of interest in analysis which can be axiomatized in $\mathcal{L}_{\omega_1,\omega}$.

- All classes of structures axiomatizable in finitary continuous logic. In the signature of lattices the class of Banach lattices isomorphic to $L_p(\mu)$ for a fixed $1 \leq p < \infty$ and measure μ is axiomatizable, by results from [20], [31]. The class of Banach spaces isometric to $L_p(\mu)$ is also axiomatizable in the signature of Banach spaces (see [62]), as is the class of Banach spaces isometric to $C(K)$ for a fixed compact Hausdorff space K (see [61]). Further examples are described in [63, Chapter 13]. More recent examples include subclasses of the class of Nakano Banach spaces (see [96], [97]).

- In any signature with countably many constants $(c_i)_{i<\omega}$, the statement that the constants form a dense set can be expressed by the following sentence:

$$\forall x \bigvee_{i<\omega} (d(x, c_i) = 0).$$

  In particular, this sentence implies that the structure is separable, and hence this example cannot be expressed in the first-order fragment, because the Upward Löwenheim-Skolem theorem (Theorem 2.2.4) implies that every separable infinite-dimensional Banach space has elementary extensions of arbitrarily large density.

- In the signature of normed spaces with countably many new constants $(c_i)_{i<\omega}$, the following formula $\varphi(x)$ expresses that $x \in \overline{\text{span}}\{c_i : i < \omega\}$:

$$\varphi(x) : \bigvee_{n<\omega} \bigvee_{a_0 \in \mathbb{Q} \cap (0,1)} \cdots \bigvee_{a_{n-1} \in \mathbb{Q} \cap (0,1)} \left(\left\| x - \sum_{i<n} a_i c_i \right\| = 0 \right).$$

We can express that $(c_i)_{i<\omega}$ is a λ-basic sequence for a fixed λ with the sentence σ_λ:

$$\sigma_\lambda : \bigwedge_{N<\omega} \bigwedge_{a_0\in\mathbb{Q}\cap(0,1)} \cdots \bigwedge_{a_{N-1}\in\mathbb{Q}\cap(0,1)} \left(\max_{n\leq N} \left\| \sum_{j=1}^n a_j c_j \right\| \leq \lambda \left\| \sum_{j=1}^N a_j c_j \right\| \right).$$

We can therefore express that $(c_i)_{i<\omega}$ is a Schauder basis:

$$(\forall x \varphi(x)) \wedge \bigvee_{\lambda\in\mathbb{Q}} \sigma_\lambda.$$

Note that this cannot be expressed in the first-order fragment of $\mathcal{L}_{\omega_1,\omega}$, since having a Schauder basis implies separability.

The same ideas as in the above example allow us to express that X (or equivalently, X^*) is not super-reflexive – see [95, Theorem 3.22].

- In the signature of normed spaces with an additional predicate $|\!|\!|\cdot|\!|\!|$, we can express that $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent by the axioms for $|\!|\!|\cdot|\!|\!|$ being a norm, plus the sentence:

$$\bigvee_{C\in\mathbb{Q}} \bigvee_{D\in\mathbb{Q}} \forall x \left(C \|x\| \leq |\!|\!|x|\!|\!| \leq D \|x\| \right).$$

- In the signature of Banach spaces augmented with two new sorts Y, Z for closed (infinite-dimensional) subspaces, the following expresses that Y and Z witness the failure of hereditary indecomposability (see [2, Proposition 1.1]):

$$\bigvee_{\delta\in\mathbb{Q}\cap(0,1)} \forall y \in Y \forall z \in Z \left(\|y - z\| \geq \delta \|y + z\| \right).$$

- Failures of reflexivity can be expressed as follows. Beginning with a two-sorted signature, each sort being the signature for Banach spaces, add countably many constants $(c_i)_{i<\omega}$ to the first sort, and $(c_i^*)_{i<\omega}$ to the second sort. Let S be the signature obtained by then adding a relation symbol F for the natural pairing on $X \times X^*$. Then in structures (X, X^*), the following expresses that the constants witness the non-reflexivity of X (see [95, Theorem 3.10]):

$$\bigvee_{\theta\in\mathbb{Q}\cap(0,1)} \bigwedge_{j<\omega} \left(\bigwedge_{i<j} F(c_i, c_j^*) = 0 \right) \wedge \left(\bigwedge_{j\leq i<\omega} F(c_i, c_j^*) = \theta \right).$$

This example cannot be expressed in the first-order fragment of $\mathcal{L}_{\omega_1,\omega}$, since it is known that there are reflexive Banach spaces with non-reflexive ultrapowers.

- The failure of a Banach space to be stable, in the sense of Krivine and Maurey [77], can be axiomatized in the signature of normed spaces with constants $(c_i)_{i<\omega}$ and $(d_i)_{i<\omega}$ as follows:

$$\bigvee_{\epsilon\in\mathbb{Q}\cap(0,1)} \bigvee_{j<\omega} \bigvee_{i<j} |\|c_i - d_j\| - \|c_j - d_i\|| \geq \epsilon.$$

More generally, we may replace $\|x - y\|$ with any formula $\varphi(x, y)$ to express that φ is not stable (see [16]). It is well-known that stability is not first-order axiomatizable.

Several of the examples above show that negations of well-known properties in Banach spaces can be expressed as single sentences of $\mathcal{L}_{\omega_1,\omega}$. In light of 3.1.2, the positive versions can also be expressed as $\mathcal{L}_{\omega_1,\omega}$ sentences. Many of the sentences above are expressed in language including constant symbols for the elements of a fixed sequences. We could avoid changing the language if we allowed infinite strings of quantifiers in our logic - such a logic would be the analogue of $\mathcal{L}_{\omega_1,\omega_1}$ for metric structures (this approach is taken by Ortiz in [88]).

We give an application of the two-cardinal theorem from Section 3.4.1 to separable quotients of Banach spaces. The well-known *separable quotient problem* asks whether every infinite-dimensional Banach space has an infinite-dimensional separable quotient; equivalently, whether whenever X is infinite-dimensional there is an infinite-dimensional separable Y and a surjective bounded linear map from X to Y. This conjecture has been verified in several cases - see [87] for a survey. More recent results include that every dual space has a separable quotient [1], and that it is consistent with ZFC that every Banach space of density $\geq \aleph_\omega$ has a separable quotient [34].

While we do not give any new answer to the separable quotient problem, we do show that in a sense countable continuous fragments of $\mathcal{L}_{\omega_1,\omega}$ cannot distinguish between arbitrary quotients and separable ones, so any property of a space which implies it does *not* have a separable quotient cannot be expressed in such a fragment. Our result generalizes a theorem of Ben Yaacov and Iovino [14] for the first-order fragment. We will need the following lemma from [14]; the proof sketched there has an error[1], so for the convenience of the reader we provided a detailed proof here.

Lemma 3.6.1 ([14, Proposition 5.1]). *Let X, Y, X', Y' be Banach spaces, and let $T : X \to Y$ and $T' : X' \to Y'$ be bounded linear functions such that $(X, Y, T) \equiv (X', Y', T')$. Then T is surjective if and only if T' is surjective.*

Proof. Throughout the proof we write $B_A(r)$ to denote the open ball in the space A centred at 0 and of radius r. We actually prove that T is surjective if and only if the following statement holds:

$$(\exists \delta > 0)\, (\forall \epsilon > 0)\, (\forall y \in \overline{B_Y(\delta)})\, (\exists x \in \overline{B_X(1)})\, \|T(x) - y\| \leq \epsilon. \qquad (3.6.1)$$

For a fixed δ, the remaining part of Statement 3.6.1 can be expressed as a sentence of (finitary) continuous logic, so proving that surjectivity is equivalent to Statement 3.6.1 is sufficient.

Suppose that T is surjective. By the Open Mapping Theorem T is an open map, so in particular there is a $\delta > 0$ such that $B_Y(\delta) \subseteq T[B_X(1)]$, which clearly implies Statement 3.6.1.

Now for the converse direction we will show that T is an open map. Surjectivity then follows because $T[X]$ will be a subspace of Y containing a ball around 0. By the linearity of T it actually suffices just to show that there is an open ball around 0 included in $T[B_X(1)]$.

We first note that Statement 3.6.1 implies that there is an $r > 0$ such that $B_Y(r) \subseteq \overline{T[B_X(1)]}$. Indeed, set $r = \delta/2$, and pick $y \in B_Y(r)$. Then $2y \in B_Y(\delta)$, so by hypothesis we have $x \in \overline{B_X(1)}$ with $\|T(x) - 2y\| \leq \epsilon/2$. We have $\frac{x}{2} \in B_X(1)$, and

$$\left\| T\left(\frac{x}{2}\right) - y \right\| = \frac{1}{2}\|T(x) - 2y\| \leq \epsilon/2,$$

---

[1] Specifically, their version of Statement 3.6.1 had δ depending on ϵ, while ours gives a uniform δ.

so the ϵ-neighbourhood of $T[B_X(1)]$ includes $B_Y(r)$. It follows that $B_Y(r) \subseteq \overline{T[B_X(1)]}$.

The remainder of the proof is exactly a portion of the proof of the Open Mapping Theorem, which we reproduce here almost verbatim from [48].

By the linearity of T we have $B_Y(r2^{-n}) \subseteq \overline{T[B_X(2^{-n})]}$. Suppose we are given $y \in Y$ with $\|y\| < r/2$. Then $y \in \overline{T[B_X(1/2)]}$, so there is $x_1 \in B_X(1/2)$ such that $\|y - T(x_1)\| < r/4$. Repeating this process we find $x_n \in B_X(2^{-n})$ such that $\left\|y - \sum_{j=1}^{n} T(x_j)\right\| < r2^{-n-1}$. The completeness of X implies that the series $\sum_{j=1}^{\infty} x_j$ converges to some $x \in X$. Then $\|x\| < \sum_{j=1}^{\infty} 2^{-n} = 1$, and $y = T(x)$. Therefore $B_Y(r/2) \subseteq T[B_X(1)]$. $\qquad\square$

Theorem 3.6.2. *Let X and Y be infinite-dimensional Banach spaces with* $\mathrm{density}(X) > \mathrm{density}(Y)$*. Let $T : X \to Y$ be a surjective bounded linear function. Let L be a countable continuous fragment of $\mathcal{L}_{\omega_1,\omega}(S)$, where S is a two-sorted signature, each sort of which is the signature of Banach spaces, together with a symbol to represent T. Then there are Banach spaces X', Y' with Y' separable and X' of density $\aleph_1$, and a surjective bounded linear function $T' : X' \to Y'$, such that $(X, Y, T) \equiv_L (X', Y', T')$.*

Proof. By Theorem 3.4.4 we get normed linear spaces $\widetilde{X}, \widetilde{Y}$ and a bounded linear map $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$ with the desired properties. Since L is a continuous fragment we may take completions to get the desired spaces X', Y' and function T'. Surjectivity of T' follows from Lemma 3.6.1 and the fact that every fragment contains the finitary part of $\mathcal{L}_{\omega_1,\omega}$. $\qquad\square$

We note that if the space Y in the statement of Corollary 3.6.2 is already separable then the Downward Löwenheim-Skolem Theorem suffices to obtain a stronger result:

Proposition 3.6.3. *Fix a continuous countable fragment L of $\mathcal{L}_{\omega_1,\omega}$. Then every infinite-dimensional separable quotient of a non-separable Banach space X is also a quotient of a Banach space X', where X' has density $\aleph_1$, and $X' \preceq_L X$.*

Proof. Let $D \subseteq Y$ be countable and dense, and use Downward Löwenheim-Skolem to find $(X', Y', T') \preceq_L (X, Y, T)$ of density $\aleph_1$ with $D \subseteq Y'$. By the continuity of the fragment L, we may assume that X' and Y' are complete. It therefore suffices to observe $Y' = Y$. Indeed, we have $D \subseteq Y' \subseteq Y$, with D dense in Y, so Y' is also dense in Y. Since Y' is complete it is closed in Y, and hence $Y' = Y$. $\qquad\square$

Chapter 4

Model theory of commutative C*-algebras

This chapter contains applications of the continuous first-order logic described in Chapter 2 to the study of commutative unital C*-algebras. By the Gelfand-Naimark theorem (Theorem 4.1.2 below) our present work can be viewed as a model-theoretic approach to the study of compact Hausdorff spaces (see Section 4.1.1 below for a discussion of previous model-theoretic approaches to compact spaces). The results of this chapter were obtained in collaboration with several co-authors, and appear in the preprints [38], [39], and [40]. For brevity we refer to compact Hausdorff spaces as *compacta*, and connected compacta as *continua*.

4.1   Commutative C*-algebras

We recall here some notions from the theory of C*-algebras that will be useful later in the chapter. We also describe how C*-algebras can be viewed as metric structures in the sense of Definition 2.1.1.

Definition 4.1.1. A *commutative unital C*-algebra* is a complex Banach space A together with a commutative multiplication $\cdot : A^2 \to A$ making A into an associative algebra with multiplicative unit 1, and an adjoint operation $^* : A \to A$, such that the following hold for all $a, b \in A$:

- $\|ab\| \leq \|a\| \, \|b\|$,

- $(a^*)^* = a$,

- $(a + b)^* = a^* + b^*$ and $(ab)^* = b^* a^*$

- for all $\lambda \in \mathbb{C}$, $(\lambda a)^* = \bar{\lambda} a^*$,

- $a \in A$, $\|aa^*\| = \|a\|^2$.

C*-algebras being, by definition, based on complete metric spaces, throughout this chapter we consider only metric structures based on complete metric spaces. We will be working in first-order continuous logic, so we will not encounter any of the complications that arose in Chapter 3 regarding taking metric completions.

The primary notion of morphism between unital C*-algebras is the *(unital) *-homomorphism*, that is, an algebraic homomorphism which also preserves the multiplicative unit and the adjoint operation. Such morphisms are automatically 1-Lipschitz, and are isometric if they are injective (see [32, Theorem 1.5.5]). By a **-isomorphism* we mean a bijective *-homomorphism.

Given a compact Hausdorff space X, it is straightforward to see that $C(X)$, the set of continuous complex-valued functions on X, is a commutative unital C*-algebra with the operations of pointwise sum, multiplication, scalar multiplication, and complex conjugation, and equipped with the supremum norm. The converse to this statement is the following well-known theorem of Gelfand and Naimark.

Theorem 4.1.2 ([32, Theorem I.3.1]). *Every commutative unital C*-algebra is *-isomorphic to $C(X)$ for some compact space X. The association between X and $C(X)$ is a contravariant equivalence of categories from the category of compact Hausdorff spaces with continuous maps to the category of commutative unital C*-algebras with *-homomorphisms.*

Remark 4.1.3. In general topology it is more common to study rings of *real-valued* functions than the complex-valued functions we use here. There are two reasons for our choice to use complex-valued functions. The first is that the Gelfand-Naimark theorem provides a convenient abstract characterization of those Banach algebras which are isometrically isomorphic to $C(X)$ for some compact X. An abstract characterization also exists in the real-valued setting (see [65, Theorem 6]), but we will benefit from the fact that an appropriate framework for studying commutative unital C*-algebras in continuous logic has already been developed, beginning in [46]. The second reason for preferring complex-valued functions is that there has recently been considerable interest in the general model-theoretic study of C*-algebras (see [45]), and our present work is a contribution to the commutative case of this larger project.

An element $f \in C(X)$ is *self-adjoint* if $f^* = f$, equivalently, $f[X] \subseteq \mathbb{R}$. The function f is *positive* if there is a $g \in C(X)$ such that $f = g^*g$, equivalently, $f[X] \subseteq [0, \infty)$. Importantly, a *projection* is a function $f \in C(X)$ such that $f = f^* = f^2$, or equivalently, $f[X] \subseteq \{0, 1\}$. A projection is *non-trivial* if it is neither identically 0 nor 1, that is, if $f[X] = \{0, 1\}$. Projections are exactly the indicator functions of clopen subsets of X. A *minimal* projection is the indicator function of a connected clopen set.

In order to treat model-theoretic aspects of commutative C*-algebras, we must view these objects as metric structures. In [46] a modified version of continuous first-order logic was presented for use with C*-algebras, based on their notion of *domains of quantification*. This is essentially equivalent to our convention of viewing a C*-algebra as a many-sorted structure, with a sort for each closed ball of rational radius centred at 0. Each sort is equipped with addition, multiplication, scalar multiplication, and adjoint functions with appropriate sorts as the codomains. Inclusion maps between the sorts are also included in the language.

The class of C*-algebras "should" be universally axiomatizable, since it is closed under ultraproducts and substructures. Unfortunately, in the framework described above, some $\forall\exists$-axioms are required to ensure that in each structure the interpretations of the sorts are as intended. This problem can be overcome by adding predicate symbols for each formula of the form $\|P(\vec{x})\|$, where P is a $*$-polynomial. As these predicates were already given by formulas, this change is harmless. See [46] for details.

In the appropriate language, then, the class of C*-algebras is universally axiomatizable. Commutativity of multiplication is clearly expressed by the universal sentence $\sup_{x,y} \|xy - yx\|$. We will later be interested in those algebras of the form $C(X)$ where X is a continuum. The space X is a continuum if

and only if $C(X)$ has no non-trivial projections, and this can be expressed by the universal axiom

$$\sup_{\|f\|=1} \left(\|1 - ff^*\| \leq \frac{1}{2} \right) \vee \left(\|ff^* - (ff^*)^2\| \geq \frac{1}{4} \right).$$

To see this, first note that if p is a non-trivial projection, then $\|p\| = 1$, $p = pp^*$, $\|1 - p\| = 1$ and $\|p - p^2\| = 0$, so the above sentence is not satisfied. Conversely, if X is connected and $f \in C(X)$ with $\|f\| = 1$ satisfies $\|1 - ff^*\| > 1/2$, then the minimum of the image of ff^* is less than $1/2$, while $\|ff^*\| = \|f\|^2 = 1$ implies that the maximum of the image of ff^* is greater than $1/2$. In particular, ff^* attains the value $1/2$, and so $\|ff^* - (ff^*)^2\| \geq 1/4$ as required.

We let T_{cmpt} denote the theory whose models are exactly the commutative C*-algebras, and T_{conn} denote the theory whose models are exactly the commutative C*-algebras without non-trivial projections. As we have seen above, both of these theories are universally axiomatizable.

4.1.1 Connections to prior work

We view the model-theoretic treatment of commutative unital C*-algebras as an indirect model theory of compacta, via the Gelfand-Naimark theorem. The earliest work in model-theoretic topology used discrete structures as replacements for topological spaces, so that the methods of first-order model theory could be applied (see [64]). Another approach, introduced by Bankston, develops a model theory for compacta by directly "dualizing" model-theoretic notions. Since we will make use of results which were stated and proved in this framework, we include here a description of how Bankston's terminology translates to ours. The central notion of Bankston's model theory of compacta is the following *ultracoproduct* of compact spaces.

Definition 4.1.4 ([6]). Let $(X_i)_{i \in I}$ be compacta, and let $\mathcal{U}$ be an ultrafilter on the index set I. Let $Y = \bigcup_{i \in I}(X_i \times \{i\})$ be the topological disjoint union of the X_i, so Y is a locally compact space. Let $q : Y \to I$ be the projection onto the second coordinate, and extend q to the Stone-Čech compactification of Y, $q : \beta Y \to \beta I$. The *ultracoproduct* $\sum_{i \in I} X_i$ is then defined to be

$$\sum_{i \in I} X_i = q^{-1}(\{\mathcal{U}\}).$$

When $X_i = X$ for all i, we speak of the *ultracopower*, and denote it by $X_{\mathcal{U}}$.

The most important fact for us is that this ultracoproduct construction exactly corresponds to the (metric) ultraproduct of the corresponding C*-algebras.

Fact 4.1.5 ([57]). *For any compacta $(X_i)_{i \in I}$, and any ultrafilter $\mathcal{U}$ on I,*

$$C\left(\sum_{\mathcal{U}} X_i\right) \cong \prod_{\mathcal{U}} C(X_i).$$

In ordinary model theory there is a canonical diagonal embedding from a structure to its ultrapower. In the topological setting, there is the *co-diagonal surjection* $\nabla_{\mathcal{U},X} : X_{\mathcal{U}} \to X$, defined to be the restriction to $X_{\mathcal{U}}$ of the Stone-Čech lifting of the projection $X \times I \to X$. From this description it is clear that $\nabla_{\mathcal{U},X}$ is a continuous surjection, and the dual map from $C(X)$ to $C(X)^{\mathcal{U}}$ is the diagonal embedding.

Definition 4.1.6 ([6])**.** Compacta X and Y are *co-elementarily equivalent* if there is an ultrafilter $\mathcal{U}$ on some index set I such that $X_\mathcal{U}$ is homeomorphic to $Y_\mathcal{U}$.

Fact 4.1.5, together with the Keisler-Shelah theorem for metric structures, immediately implies the following.

Fact 4.1.7. *Compacta X and Y are co-elementarily equivalent if and only if $C(X)$ and $C(Y)$ are elementarily equivalent as metric structures.*

A number of interesting topological properties are known to be preserved by co-elementary equivalence – see Section 4.2 below.

Definition 4.1.8 ([6], [8])**.** Let X and Y be compacta. A function $f : X \to Y$ is *co-elementary* if there is an ultrafilter $\mathcal{U}$, and a homeomorphism $H : X_\mathcal{U} \to Y_\mathcal{U}$, such that $\nabla_{\mathcal{U},Y} \circ H = f \circ \nabla_{\mathcal{U},X}$.

The function f is *co-existential* if there is a compactum Z, a co-elementary map $g : Z \to Y$, and a continuous surjection $h : Z \to X$, such that $g = f \circ h$.

Co-elementary and co-existential maps are clearly continuous surjections. It is also clear from the definition that the co-diagonal map $\nabla_{\mathcal{U},X} : X_\mathcal{U} \to X$ is co-elementary. The well-known characterization of elementary and existential maps which inspired the above definition shows:

Fact 4.1.9. *A map $f : X \to Y$ is co-elementary (resp. co-existential) if and only if the dual map $\tilde{f} : C(Y) \to C(X)$ is elementary (resp. existential).*

Recall that a metric structure M is a *existentially closed* model of a theory T if $M \models T$ and whenever $N \models T$ every embedding of M into N is existential (i.e., preserves the values of $\exists$-formulas). The corresponding notion for compacta is a *co-existentially closed* member of a class $\mathcal{K}$ of compacta.

Definition 4.1.10 ([9])**.** Let $\mathcal{K}$ be a class of compacta. A space $X \in \mathcal{K}$ is a *co-existentially closed member of $\mathcal{K}$* if whenever $Y \in \mathcal{K}$ and $f : Y \to X$ is a continuous surjection, then f is a co-existential map. Particularly, when $\mathcal{K}$ is the class of all compacta we say X is a *co-existentially closed compactum*, and when $\mathcal{K}$ is the class of continua we say X is a *co-existentially closed continuum*.

The expected translation to C*-algebras follows from Fact 4.1.9

Fact 4.1.11. *A space X is a co-existentially closed compactum (resp. continuum) if and only if $C(X)$ is an existentially closed model of T_{cmpt} (resp. T_{conn}).*

As mentioned above, another frequently used technique in the model theory of compacta is to use discrete structures as stand-ins for the compact spaces, so that the usual methods of first-order model theory can be applied. In the special case where the spaces are 0-dimensional this approach works perfectly, thanks to Stone duality, with Boolean algebras used as the discrete counterpart to the 0-dimensional spaces. There is no such duality between general compact spaces and discrete structures in any finite language (see [3]), but it has nevertheless been fruitful to consider lattice bases of closed sets. The relevant result for our purposes is the following.

Fact 4.1.12 ([6])**.** *Let X and Y be compacta, and let L and L' be lattices which are bases for the closed sets of X and Y, respectively. If L and L' are elementarily equivalent as lattices, then X and Y are co-elementarily equivalent.*

In [5] Bankston shows that the converse to the above fact is false – co-elementarily equivalent spaces can have non-elementarily equivalent lattice bases of closed sets.

As an example of how lattices can be used in the study of compacta, in [35] Dow and Hart use lattices and a saturating ultrafilter to show that under $\mathrm{cov}(\text{meagre}) = 2^{<\mathfrak{c}} = \mathfrak{c}$ there exists a surjectively universal compactum and a surjectively universal continuum of weight $2^{\aleph_0}$. We can obtain the same result by applying Theorem 2.4.4 to take an ultrapower of $C(2^{\mathbb{N}})$ (respectively, $C([0,1])$) by a metric saturating ultrafilter, and then applying Gelfand-Naimark duality; Fact 4.1.5 shows that this method obtains the same spaces as in [35], namely the ultracopowers $(2^{\mathbb{N}})_{\mathcal{U}}$ and $([0,1])_{\mathcal{U}}$, respectively.

4.2 Elementary properties

In this section we describe some of the properties of the space X which are reflected in the first-order continuous theory of $C(X)$, as well as properties of elements which can be detected from their types. In the case where X is 0-dimensional we give a full description of all possible complete theories of $C(X)$.

Many interesting topological properties have been shown to be preserved by co-elementary equivalence.

Theorem 4.2.1 ([6], [7], [11]). *Let X and Y be compacta with $X \equiv Y$. Then:*

1. *X is a continuum if and only if Y is continuum,*

2. *if X and Y are locally connected and metrizable, the X is an arc if and only if Y is an arc and X is a simple closed curve if and only if Y is a simple closed curve,*

3. *X is a decomposable (resp. indecomposable, hereditarily indecomposable) continuum if and only if Y is a decomposable (resp. indecomposable, hereditarily indecomposabl) continuum,*

4. *X and Y have the same number of isolated points as counted in $\mathbb{N} \cup \{\infty\}$*

5. *X and Y have the same Lebesgue covering dimension.*

Among properties *not* preserved by co-elementary equivalence we clearly have metrizability and weight. More surprisingly, hereditary decomposability is not preserved by co-elementary equivalence [11, Proposition 4.3]

We will later need to know what information is captured by the type of an element of $C(X)$. For quantifier-free types the situation is particularly simple.

Lemma 4.2.2. *Let X be any compactum. For any $f, g \in C(X)$, $\mathrm{qftp}(f) = \mathrm{qftp}(g)$ if and only if $f[X] = g[X]$.*

Proof. If $f[X] = g[X]$, then the C*-algebras generated by $(1, f)$ and $(1, g)$ are isomorphic by a map sending f to g [32, Corollary I.3.2], from which it follows that $\mathrm{qftp}(f) = \mathrm{qftp}(g)$.

Conversely, suppose that $\mathrm{qftp}(f) = \mathrm{qftp}(g)$. Note that the function $|f| : X \to \mathbb{C}$ is quantifier-free definable from f, since $|f|$ is the unique function satisfying $|f|^2 = ff^*$. Let $N \in \mathbb{N}$ be sufficiently large

so that $N > |f(x)|, |g(x)|$ for all $x \in X$. Then for any $\lambda \in \mathbb{C}$ with $|\lambda| < N$, we have:

$$
\begin{aligned}
\lambda \in f[X] &\iff \text{ there is } x \in X \text{ such that } \big| N1 - |f(x) - \lambda 1| \big| = N \\
&\iff \big\| N1 - |f - \lambda 1| \big\| = N \\
&\iff \big\| N1 - |g - \lambda 1| \big\| = N \qquad\qquad\qquad \text{since } \mathrm{qftp}(f) = \mathrm{qftp}(g) \\
&\iff \lambda \in g[X]
\end{aligned}
$$

$\square$

Lemma 4.2.3. *There is a formula $\varphi(x)$ such that for any compactum X, $C(X) \models \varphi(f)$ if and only if f is a minimal projection.*

Proof. Let $\mathcal{P}$ denote the set of projections in $C(X)$. The set $\mathcal{P}$ is definable (in the sense of [13] – see the remarks following Lemma 2.1 in [26]), and hence we can relativize quantifiers to $\mathcal{P}$. The necessary formula is

$$
\big(\|x - x^*\| = \|x - x^2\| = 0 \big) \wedge \big(\|x\| = 1 \big) \wedge \big(\|1 - x\| = 1 \big) \wedge \left(\sup_{q \in \mathcal{P}} \big(\|q - p\| = 1 \big) \vee \|q\| = 0 \right).
$$

$\square$

4.2.1 Complete theories for 0-dimensional X

As we saw in the previous section, the covering dimension of a compactum X can be detected from the continuous first-order theory of $C(X)$. When X is 0-dimensional we have two algebras associated to X, namely the Boolean algebra $CL(X)$ of clopen subsets of X and the C*-algebra $C(X)$. By the Stone and Gelfand-Naimark dualities, respectively, each of these algebras determine X up to homeomorphism. The connection between the discrete first-order theory of the Boolean algebra and the continuous metric theory of the C*-algebra is similarly complete. The results of this subsection appear in [40].

Theorem 4.2.4. *Let X and Y be 0-dimensional compacta. Then $C(X) \equiv C(Y)$ if and only if $CL(X) \equiv CL(Y)$.*

Proof. This follows from Fact 4.1.7 and results of [6], but we give a direct proof that avoids co-elementary equivalence. Let $\mathcal{U}$ be an ultrafilter on some set I, and let M be either a metric or discrete structure. For each $a \in \mathcal{U}$, let $M_a = \prod_{i \in a} M$. Then the ultrapower $M^{\mathcal{U}}$ is the colimit of the collection of M_a's along the partial order category $(\mathcal{U}, \subseteq)^{\mathrm{op}}$. Let F be the functor from Boolean algebras to commutative unital C*-algebras given by $F(A) = C(\mathrm{St}(A))$, where $\mathrm{St}(A)$ is the Stone space of A. This functor is an equivalence of categories when the codomain is restricted to algebras of the form $C(X)$ for X 0-dimensional. Equivalences of categories preserve products and colimits, so the above discussion shows that for any Boolean algebra A, $F(A^{\mathcal{U}}) \cong F(A)^{\mathcal{U}}$. That is, for any compactum X, $C(X)^{\mathcal{U}} \cong C(\mathrm{St}(CL(X)^{\mathcal{U}}))$. The proof is then complete by applying the Keisler-Shelah theorem. $\square$

It is interesting to note that the above result fails when $C(X)$ is considered only as a discrete ring (see [4, Section 2]).

Corollary 4.2.5. *There are exactly $\aleph_0$ distinct complete theories of C*-algebras of the form $C(X)$ for X a 0-dimensional compactum.*

Proof. There are exactly $\aleph_0$ distinct complete theories of Boolean algebras; see [28, Theorem 5.5.10] for a description of these theories. $\square$

Corollary 4.2.6. *If X and Y are infinite, compact, 0-dimensional spaces both with the same finite number of isolated points or both having a dense set of isolated points, then $C(X) \equiv C(Y)$.*

In particular, let α be any infinite ordinal. Then $C(\alpha + 1) \equiv C(\beta\omega)$. Moreover, if α is a countable limit, $C(2^{\mathbb{N}}) \equiv C(\beta\omega \setminus \omega) \equiv C(\beta\alpha \setminus \alpha)$.

Proof. Given X, Y as in the hypothesis, again by [28, Theorem 5.5.10] we have that $CL(X) \equiv CL(Y)$. $\square$

The results above give a complete description of the theories of $C(X)$ when X is compact and 0-dimensional. Every compact space is the perfect image of a 0-dimensional compactum, and perfect maps are known to preserve many interesting topological properties, so it might be hoped that the general model theory of algebras $C(X)$ could be reduced to the case where X is 0-dimensional, at least in some cases. We do not presently know of any applications of this sort, but we point out that if $f : Y \to X$ is a perfect map then the dual map $f^* : C(X) \to C(Y)$ is an embedding, but not an elementary embedding, since the property of Y being 0-dimensional can be expressed as a sentence of first-order continuous logic.

4.3 Quantifier elimination

We consider the question of which complete theories of commutative unital C*-algebras have quantifier elimination. The main result is:

Theorem 4.3.1. *The complete theories of commutative unital C*-algebras which have quantifier elimination are exactly the theories of $\mathbb{C}, \mathbb{C}^2$, and $C(2^{\mathbb{N}})$.*

This result was originally obtained in several stages with a variety of co-authors. Our presentation here is a simplification and synthesis of arguments found in [38], [39], and [40]. Before beginning the proof we point out that in the non-commutative setting we show in [38] that only one further theory of C*-algebras with quantifier elimination exists, namely the theory of $M_2(\mathbb{C})$ (the algebra of 2×2 matrices over $\mathbb{C}$). As this thesis is focusing on the case of commutative C*-algebras, we will not prove this statement about non-commutative algebras, which uses techniques significantly different from the primarily topological ones we will need in the commutative case.

4.3.1 Algebras with quantifier elimination

We begin with the positive results. Two finite-dimensional algebras were shown to have quantifier elimination in [38].

Proposition 4.3.2. *The theories of $\mathbb{C}$ and $\mathbb{C}^2$ have quantifier elimination.*

Proof. We show the proof for $\mathbb{C}^2$, the proof for $\mathbb{C}$ being similar and easier. Since $\mathbb{C}^2$ is finite-dimensional, it is the unique (up to isomorphism) model of its first-order theory. We use the quantifier elimination test of Proposition 2.3.2. After identifying isomorphic algebras, the diagram takes the form
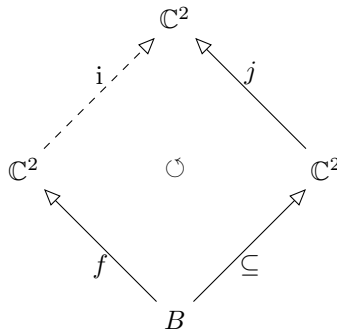
Here B is a subalgebra of $\mathbb{C}^2$, so is either isomorphic to $\mathbb{C}$ or $\mathbb{C}^2$. Again identifying B with one of these two algebras, elementary linear algebra shows that f and the inclusion map are conjugate by an automorphism of $\mathbb{C}^2$. Conjugating j by the same automorphism produces the required embedding i. $\square$

To show quantifier elimination for $C(2^{\mathbb{N}})$, we give a detailed version of the proof from [39], which also shows that $\text{Th}(C(2^{\mathbb{N}}))$ is the model completion of T_{cmpt}. A direct proof of quantifier elimination, due to Farah and Hart, is presented in [40]. The results about model companions and completions that we use here are described in Section 2.3.

Proposition 4.3.3. *The theory of $C(2^{\mathbb{N}})$ is the model completion of T_{cmpt}, and hence has quantifier elimination.*

Proof. We first show that the theory of $C(2^{\mathbb{N}})$ is the model companion of T_{cmpt}. In [8] Bankston shows that the co-existentially closed compacta are exactly the 0-dimensional compacta without isolated points. By the translations described in Section 4.1.1, together with Theorem 4.2.4, this is equivalent to saying that the existentially closed models of T_{cmpt} are exactly the algebras elementarily equivalent to $C(2^{\mathbb{N}})$. It follows that $\text{Th}(C(2^{\mathbb{N}}))$ is the model companion of T_{conn}.

Next, we see that T_{cmpt} is the model *completion* of T_{conn}. It suffices to show that T_{cmpt} has the amalgamation property. Phrased in terms of compacta, we must show that whenever X, Y, and Z are compacta and $f : X \to Z$ and $g : Y \to Z$ are continuous surjections, there is a compactum W and continuous surjections $r : W \to X$ and $s : W \to Y$ such that $f \circ r = g \circ s$. It is not difficult to see that we can take $W = \{\, (x,y) \in X \times Y : f(x) = g(y) \,\}$, and let r and s be the projections onto the first and second coordinates, respectively.

Finally, $\text{Th}(C(2^{\mathbb{N}}))$ has quantifier elimination, because it is the model completion of the universally axiomatizable theory T_{cmpt} (see Proposition 2.3.12). $\square$

The fact that the algebra $C(2^{\mathbb{N}})$ is one of few C*-algebras to possess desirable model-theoretic properties can be explained in several ways. We have just shown that its theory is the model completion of T_{cmpt}. One can also construct $C(2^{\mathbb{N}})$ as the Fraïssé limit of the class of finite-dimensional commutative C*-algebras (in an appropriate context for understanding Fraïssé limits of C*-algebras, as developed in [37]).

4.3.2 Topological consequences of quantifier elimination

Towards showing that our current list of theories of commutative C*-algebras with quantifier elimination is complete, we show some consequences for X of $\text{Th}(C(X))$ having quantifier elimination. The reduction to the connected case that we prove in this subsection appeared in [38].

Lemma 4.3.4. *Suppose that* $\mathrm{Th}(C(X))$ *has quantifier elimination. If* X *has a non-trivial connected clopen subset then every non-trivial clopen subset of* X *is connected.*

Proof. Immediate from quantifier elimination, Lemma 4.2.2, and Lemma 4.2.3. □

Lemma 4.3.5. *If* $\mathrm{Th}(C(X))$ *has quantifier elimination and* X *is* 0*-dimensional with at least three points then* X *has no isolated points.*

Proof. If $x \in X$ is isolated then by Lemma 4.3.4 $X \setminus \{x\}$ is connected. Since X is 0-dimensional and compact it is totally disconnected, so $X \setminus \{x\}$ is a singleton. □

Proposition 4.3.6. *If* $\mathrm{Th}(C(X))$ *has quantifier elimination and* X *has at least three points then either* X *is connected or* $C(X) \equiv C(2^{\mathbb{N}})$.

Proof. Suppose that X has at least three points, is not connected, and $\mathrm{Th}(C(X))$ has quantifier elimination. As finite compacta are discrete, Lemma 4.3.5 implies that X is infinite. Without loss of generality we may assume that $C(X)$ is separable, so that X is metrizable.

Let $\emptyset \neq C \subseteq X$ be clopen, and suppose for a contradiction that C is connected. Then by Lemma 4.3.4 $X \setminus C$ is also connected. Let $f, g \in C(X)$ be such that $f[C] = [0,1]$, $f[X \setminus C] = \{0\}$, $g[C] = [0, 1/2]$, and $g[X \setminus C] = [1/2, 1]$. It follows from Lemma 4.2.2 and quantifier elimination that $\mathrm{tp}(f) = \mathrm{tp}(g)$. On the other hand, there is a projection q such that $fq = q$, but for any projection q we have $\|gq - q\| \geq \frac{1}{2}$, giving the desired contradiction.

Now by repeatedly splitting each clopen subset of X into two clopen subsets we obtain a continuous surjection $X \to 2^{\mathbb{N}}$, and hence an embedding $C(2^{\mathbb{N}}) \to C(X)$. On the other hand, $2^{\mathbb{N}}$ is a surjectively universal compact metric space, so there is a surjection of $2^{\mathbb{N}}$ onto X, and hence an embedding of $C(X) \to C(2^{\mathbb{N}})$. We therefore have a chain of embeddings:

$$C(2^{\mathbb{N}}) \to C(X) \to C(2^{\mathbb{N}}) \to C(X) \to \cdots$$

Let A be the union of this chain. Then A is the union of a chain of structures isomorphic to $C(X)$, and by the assumption of quantifier elimination this chain is elementary, so $A \equiv C(X)$. The algebra A is also the union of a chain of algebras isomorphic to $C(2^{\mathbb{N}})$, and this chain is also elementary by Proposition 4.3.3, so $A \equiv C(2^{\mathbb{N}})$. Therefore $C(X) \equiv C(2^{\mathbb{N}})$. □

4.3.3 Quantifier elimination and continua

In light of the work done above, the remainder of the proof of Theorem 4.3.1 concerns the model theory of algebras $C(X)$, where X is a continuum. We will be interested only in non-degenerate continua, that is, continua with more than one point, so for brevity we say "continuum" for "non-degenerate continuum". We call a theory a *theory of continua* if all of its models are of the form $C(X)$ for X a continuum. Recalling that $C(X)$ is finite-dimensional if and only if X is finite, we see that a theory of continua is any consistent extension of T_{conn} whose models are infinite-dimensional. Importantly, $C(X)$ is infinite-dimensional if and only if the unit ball of $C(X)$ is not totally bounded, and this can be expressed by a $\forall\exists$-theory. The material of this subsection appears in [39]. The next proposition should be compared with the situation for T_{cmpt}, Proposition 4.3.3.

Proposition 4.3.7. T_{conn} *does not have a model completion.*

Proof. If T_{conn} has a model completion then the class of models of T_{conn} has the amalgamation property. As in Proposition 4.3.3, we state the amalgamation property in terms of topological spaces, rather than C*-algebras. In terms of continua, the amalgamation property states that whenever X, Y, and Z are continua and $f : X \to Z$ and $g : Y \to Z$ are continuous surjections, there is a continuum W and continuous surjections $r : W \to X$ and $s : W \to Y$ such that $f \circ r = g \circ s$. We give an example, pointed out to us by Logan Hoehn, to show that the class of continua does not enjoy this property.

Let $X = Y = [0,1]$, and let Z be the circle $\mathbb{S}^1$, which, for convenience, we view as the subset of $\mathbb{C}$ consisting of complex numbers $e^{i\theta}$. Let $f : X \to Z$ be $f(x) = e^{2\pi i x}$ and $g : Y \to Z$ be $g(y) = e^{\pi i(2y+1)}$. Suppose that W, r, s complete the amalgamation, in the above sense. Let $A = r^{-1}([0,1/2]) \cap s^{-1}([1/2,1])$ and $B = r^{-1}([1/2,1]) \cap s^{-1}([0,1/2])$, both of which are closed in W. It is easy to see that $A \cup B = W$. Now suppose that $w \in A \cap B$. Then $r(w) = s(w) = 1/2$, so $f(r(w)) = e^{\pi i} \neq e^{2\pi i} = g(s(w))$, contradicting the assumption that $f \circ r = g \circ s$. Therefore $A \cap B = \emptyset$, and so W is disconnected, yielding a contradiction. $\square$

We are now in a position to complete the proof of Theorem 4.3.1, by showing that no theory of continua can have quantifier elimination. We will need the following important fact about theories of continua.

Lemma 4.3.8. *If X and Y are any two continua, then $\mathrm{Th}_\forall(C(X)) = \mathrm{Th}_\forall(C(Y))$.*

Proof. By the Downward Löwenheim-Skolem theorem (Theorem 2.2.5) we may assume that $C(X)$ and $C(Y)$ are separable, that is, that X and Y are metrizable. In [60, Proposition 3.1] Hart shows that there exists a metrizable continuum Z such that:

- Y is a continuous image of Z, and

- X and Z have lattice bases of closed sets which are elementarily equivalent in the discrete first-order language of lattices.

(1) implies that $C(Z) \subseteq C(Y)$. By Facts 4.1.7 and 4.1.12 condition (2) implies that $C(X) \equiv C(Z)$. We therefore have $C(X) \equiv C(Z) \subseteq C(Y)$, so $\mathrm{Th}_\forall(C(Y)) \subseteq \mathrm{Th}_\forall(C(X))$. Interchanging the roles of X and Y we obtain the conclusion. $\square$

Corollary 4.3.9. *No theory of continua has quantifier elimination.*

Proof. If a theory of continua had quantifier elimination then, by Lemma 4.3.8, that theory would be the model completion of T_{conn}, contradicting Proposition 4.3.7. $\square$

The combination of Propositions 4.3.2, 4.3.3, and 4.3.6, and Corollary 4.3.9 completes the proof of Theorem 4.3.1. Before turning to the question of whether there are any theories of continua which are model complete, we pause to observe that Lemma 4.3.8 implies that every theory of continua has the maximum possible number of separable models.

Proposition 4.3.10. *Every theory of continua has $2^{\aleph_0}$ non-isomorphic separable models.*

Proof. Let T be a theory of continua. Let $\mathcal{C}$ be a family of non-homeomorphic metrizable continua such that each $X \in \mathcal{C}$ has $C(X) \models T$, and every complete separable model of T is isomorphic to $C(X)$ for some $X \in \mathcal{C}$. We first observe that every metrizable continuum is the continuous image of a continuum in $\mathcal{C}$. Let X be any metrizable continuum, and let Y be a metrizable continuum such that $C(Y) \models T$. By Lemma 4.3.8 there is a continuum Z such that $C(Y) \equiv C(Z)$, and $C(X)$ embeds in $C(Z)$; it follows from

the Downward Löwenheim-Skolem theorem that $C(Z)$ can be chosen to be separable, so Z is metrizable. This completes the claim.

It is shown in [83, Section 20] that there is a family $\mathcal{K}$ of metrizable continua such that $|\mathcal{K}| = 2^{\aleph_0}$ and no metrizable continuum maps onto uncountably many members of $\mathcal{K}$. In particular, the continua in $\mathcal{C}$ can together map onto at most $|\mathcal{C}| \cdot \aleph_0$ of the continua in $\mathcal{K}$. As we have just shown that every continuum in $\mathcal{K}$ is the image of a continuum in $\mathcal{C}$, we must have $|\mathcal{C}| = |\mathcal{K}| = 2^{\aleph_0}$. □

4.3.4 Model completeness and the pseudoarc

Having shown that no theory of continua has quantifier elimination, it is natural to ask about the weaker property of model completeness. To conclude our discussion of topics related to quantifier elimination, we prove that the only possible theory of continua which is model complete is $\mathrm{Th}(C(\mathbb{P}))$, where $\mathbb{P}$ is the pseudoarc. The material of this section appears in [39].

Lemma 4.3.11. *There is at most one model complete theory of continua; if there is one, it is the model companion of T_{conn}.*

Proof. Immediate from Lemma 4.3.8. □

In fact, we can obtain more information from Lemma 4.3.8, though we will not make further use of the following observation.

Lemma 4.3.12. *There is at most one complete $\forall\exists$-axiomatizable theory of continua.*

Proof. Suppose that T and T' are complete $\forall\exists$-axiomatizable theories of continua. Given $A_0 \models T$, by Lemma 4.3.8 there is $A_1 \models T'$ such that A_0 embeds in A_1. Then again by Lemma 4.3.8 there is $A_2 \models T$ such that A_1 embeds in A_2. Continuing in this way we produce a chain $A_0 \subseteq A_1 \subseteq \cdots$, with even indices models of T and odd indices models of T'. The union A of this chain is the union of the chain of odd-indexed models, and so is the union of a chain of models of the $\forall\exists$-axiomatiable theory T. By the standard preservation result it follows that $A \models T$. Similarly we have $A \models T'$. □

If T_{conn} has a model companion then that model companion is the theory of the existentially closed models of T_{conn}, by Proposition 2.3.11. We therefore wish to know which models of T_{conn} are existentially closed, or, equivalently, which continua are co-existentially closed. The most natural candidate is the pseudoarc, and indeed in [10] Bankston asked if the pseudoarc is a co-existentially closed continuum. We answer this question in the affirmative.

The pseudoarc is a continuum originally discovered independently by Knaster [71] and Moïse [86]. Rather than give a construction of this space here, we instead describe two properties which characterize the pseudoarc up to homeomorphism.

Definition 4.3.13. A continuum is *chainable* if every open cover can be refined to a finite open cover $(U_n)_{n<N}$ such that $U_n \cap U_m \neq \emptyset$ if and only if $|n - m| \leq 1$.

A continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua, and *hereditarily indecomposable* if every subcontinuum is indecomposable.

The abstract characterization of the pseudoarc, due to Bing [17], is that the *pseudoarc* is the unique non-degenerate, hereditarily indecomposable, chainable continuum. It is one-dimensional, and is homeomorphically embedded in $\mathbb{R}^2$. For more on the pseudoarc, see [79]. Bankston [11, Corollary 4.13] showed

that a co-existentially closed continuum is necessarily one-dimensional and hereditarily indecomposable, motivating his question of whether the pseudoarc is co-existentially closed.

Following the terminology of [45] (see also [47]), we say that a class $\mathcal{K}$ of separable models of T_{conn} is *definable by a uniform sequence of universal types* if there are *existential* formulas $(\varphi_{m,n}(\vec{x}_m))_{m,n\in\mathbb{N}}$ such that a model $C(X)$ of T_{conn} belongs to $\mathcal{K}$ if and only if, for all $m \in \mathbb{N}$, we have

$$C(X) \models \sup_{\vec{x}_m} \inf_n \varphi_{m,n}(\vec{x}_m).$$

Theorem 4.3.14. *The class of models $C(X)$ of T_{conn} with X chainable is uniformly definable by a sequence of universal types.*

Proof. Consider the following quantifier-free formulae:

- $\varphi^k(\vec{f}) = |\|\sum f_i^* f_i\| - \|(\|\sum f_i^* f_i\| - \sum f_i^* f_i)\||$

- $\psi_1^{l,m}(\vec{g}, w) := d((g_1^* g_1 + \cdots + g_m^* g_m) - \frac{1}{l} \cdot 1, w^* w)$

- $\psi_2^m(\vec{g}) := \max_{|i-j|\geq 2} \|g_i g_j\|$

- $\psi_3^{k,m}(\vec{f}, \vec{g}, \vec{h}) := \max_j \min_i d(f_i^* f_i - g_j^* g_j, h_{ij}^* h_{ij})$

Let $\sigma_k(\vec{f})$ denote

$$\inf_{(m,l)} \inf_{\vec{g}} \inf_{\vec{h}} \inf_{w} \min(\varphi^k, \max(\psi_1^{l,m}, \psi_2^m, \psi_3^{k,m})).$$

We show that a metrizable continuum X is chainable if and only if, for all k, we have

$$\left(\sup_{\vec{f}} \sigma_k(\vec{f})\right)^{C(X)} = 0.$$

We start with the "if" implication: pick a finite open cover (U_i) of X and take a sequence (f_i) from $C(X)$ with $\|f_i\| = 1$ and such that $U_i = \{x \in X \ : \ f_i(x) \neq 0\}$. (This is possible for, in a metrizable compact space, every closed set is the zeroset of a continuous function.) It follows that $\sum f_i^* f_i$ is invertible, so $\varphi^k(\vec{f}) \neq 0$, and hence there is m and functions $g_1, \ldots, g_m$ with $\sum g_i^* g_i$ invertible. Setting $V_i = \{x \in X \mid g_i(x) \neq 0\}$, we have the required refinement.

For the converse, note that $\varphi^k(\vec{f}) = 0$ if and only if 0 is in the spectrum of $\sum f_i^* f_i$ if and only if $U_i := \{x \in X \mid f_i(x) \neq 0\}$ is not an open cover for X. Given the refinement provided by chainability of X, the construction of the appropriate g_i's is immediate from the aforementioned fact that in a metrizable compact space every closed set is a zero set. $\square$

By a *condition* we mean a finite set of expressions of the form $\varphi(\vec{x}) < r$ where $\varphi(\vec{x})$ is a quantifier-free formula and $r \in \mathbb{R}^+$. If A is a C*-algebra and $\vec{a}$ is a tuple from A, we say that $\vec{a}$ *satisfies* the condition $p(\vec{x})$ if $\varphi(\vec{a})^A < r$ for all expressions $\varphi(\vec{x}) < r$ belonging to $p(\vec{x})$.

Fact 4.3.15 ([54, Appendix A])**.** *Suppose that $\mathcal{K}$ is a class of separable models of T_{conn} that is uniformly definable by a sequence of universal types as witnessed by the formulas $(\varphi_{m,n}(\vec{x}_m))$. Further suppose that, for every $\epsilon > 0$, every $m \in \mathbb{N}$ and every satisfiable condition $p(\vec{x})$ there is a model $C(X)$ of T_{conn} and $\vec{f} \in C(X)$ that satisfies $p(\vec{x})$ and for which $\inf_n \varphi_{m,n}(\vec{f})^{C(X)} < \epsilon$. Then there is a separable existentially closed model of T_{conn} that belongs to the class $\mathcal{K}$.*

Corollary 4.3.16. *There is a separable existentially closed model of T_{conn} that is chainable, which is thus necessarily isomorphic to $C(\mathbb{P})$.*

Proof. Suppose that $p(\vec{x})$ is a condition that is satisfied in $C(X)$ for some continuum X. By embedding $C(X)$ into $C(Y)$ with $C(Y) \equiv C(\mathbb{P})$, we see that $p(\vec{x})$ is satisfied in $C(Y)$ and hence in $C(\mathbb{P})$. In particular, for any $m, k \geq 1$, we have $\vec{f} \in C(\mathbb{P})$ such that $\vec{f}$ satisfies $p(\vec{x})$ and $\sigma_k(\vec{f}) < \frac{1}{m}$. Thus, we can apply Fact 4.3.15 and Theorem 4.3.14 to obtain a separable existentially closed model of T_{conn} that is chainable. $\qquad\square$

Remark 4.3.17. It is known that $\mathbb{P}$ is generic in the descriptive set-theoretic sense, that is, in the space of subcontinua of $[0,1]^{\mathbb{N}}$, the set of those continua homeomorphic to $\mathbb{P}$ is a dense G_δ set. One can view Corollary 4.3.16 as the statement that the pseudoarc is also model-theoretically generic, as it arises as the generic model constructed using model-theoretic forcing in the proof of Fact 4.3.15.

Corollary 4.3.18. *The following are equivalent:*

1. *$\mathrm{Th}(C(\mathbb{P}))$ is model complete,*

2. *there is a nondegenerate continuum X such that $\mathrm{Th}(C(X))$ is model complete,*

3. *T_{conn} has a model companion.*

If these equivalent conditions hold, then $\mathrm{Th}(C(\mathbb{P}))$ is the model companion of T_{conn}, and so $\mathrm{Th}(C(\mathbb{P}))$ is the unique theory of continua which is model complete. If the conditions do not hold, then there is no model complete theory of continua.

Proof. The implication (1) implies (2) is obvious, and (2) implies (3) is Lemma 4.3.11.

Suppose that T^* is the model companion of T_{conn}. Since T_{conn} is universally axiomatizable T^* is the theory of existentially closed models of T_{conn}, so by Theorem 4.3.16 $C(\mathbb{P}) \models T^*$. Suppose that X is another continuum such that $C(X) \models T^*$. By Lemma 4.3.8 there is a continuum X' such that $C(X) \equiv C(X')$ and $C(\mathbb{P})$ embeds in $C(X')$; since $C(X')$ and $C(\mathbb{P})$ are models of the model complete theory T^*, we see that the embedding of $C(\mathbb{P})$ into $C(X')$ is elementary, and in particular $C(\mathbb{P}) \equiv C(X') \equiv C(X)$. Therefore $T^* = \mathrm{Th}(C(\mathbb{P}))$, and $\mathrm{Th}(C(\mathbb{P}))$ is model complete.

The final assertions in the statement follow immediately from Lemma 4.3.11 $\qquad\square$

The main issue left unresolved in this section is whether or not the equivalent conditions of Corollary 4.3.18 hold.

Question 4.3.19. Is $\mathrm{Th}(C(\mathbb{P}))$ model complete?

A useful first step to answering this question would be to understand other models of $\mathrm{Th}(C(\mathbb{P}))$. Many such separable models exist, by Proposition 4.3.10.

Question 4.3.20. Give a concrete description of the metrizable continua X not homeomorphic to $\mathbb{P}$ such that $C(X) \equiv C(\mathbb{P})$.

Answering the question above would also be a useful step towards determining whether or not $\mathrm{Th}(C(\mathbb{P}))$ has a prime model. Since $C(\mathbb{P})$ is the only model of its theory which is chainable, and we saw above that for separable models of T_{conn} chainability can be described as omitting a certain family of types, we have:

Proposition 4.3.21. *If $\mathrm{Th}(C(\mathbb{P}))$ has a prime model, then that prime model is $C(\mathbb{P})$.*

Question 4.3.22. Does $\mathrm{Th}(C(\mathbb{P}))$ have a prime model?

4.4 Saturation

In this final section of the chapter we concern ourselves with how the $\aleph_1$-saturation of an algebra $C(X)$ relates to the topology of the compactum X. We will also consider two weakenings of $\aleph_1$-saturation. The contents of this section are from [40].

Definition 4.4.1. Let Φ be a set of formulas in the language of C*-algebras. We say that a C*-algebra A is Φ-$\aleph_1$-*saturated* if A realizes all Φ-types over countable parameter sets.

Of particular interest are the cases where Φ is the set of all quantifier-free formulas, in which case we speak of *quantifier-free $\aleph_1$-saturation*, and the case where Φ is the set of formulas of the form $\|P(\vec{x})\|$ where P is a degree-1 $*$-polynomial, in which case we speak of *degree-1 $\aleph_1$-saturation*. As usual, when Φ contains all formulas we speak simply of $\aleph_1$-*saturation*.

Clearly $\aleph_1$ saturation implies quantifier-free $\aleph_1$-saturation, which in turn implies degree-1 $\aleph_1$-saturation. Degree-1 $\aleph_1$-saturation was introduced by Farah and Hart in [44], where it was shown that a variety of important properties of (generally non-commutative) C*-algebras follow from this weak version of saturation. It may be expected that there is a hierarchy of saturation properties between degree-1 $\aleph_1$-saturation and quantifier-free $\aleph_1$-saturation, but it was shown in [44] that degree-2 $\aleph_1$-saturation is already equivalent to quantifier-free $\aleph_1$-saturation.

When X is finite the algebra $C(X)$ is finite-dimensional, and hence has compact unit ball. It follows that in this case $C(X)$ is the only model of its continuous first-order theory, and hence is $\aleph_1$-saturated (and, in fact, κ-saturated for every κ). We will therefore only be interested in algebras of the form $C(X)$ for X an infinite compactum. It will be useful to know when saturation can be witnessed by particular kinds of functions.

Lemma 4.4.2 ([44, Lemma 2.1]). *In a degree-1 $\aleph_1$-saturated C*-algebra, if a type can be finitely approximately satisfied by self-adjoint elements then it can be realized by self-adjoint elements, and similarly with "self-adjoint" replaced by "positive".*

The first limiting conditions for the weakest degree of saturation are given by the following Proposition. Recall that a compactum X is *sub-Stonean* if pairs of disjoint open σ-compact sets have disjoint closures, and is *Rickart* if open σ-compact sets have open closures. Rickart spaces are sub-Stonean: If X is Rickart, and A and B are disjoint open σ-compact sets, then $A \subseteq (X \setminus B)^\circ = X \setminus \overline{B}$, and this latter set is closed because $\overline{B}$ is open, so $\overline{A} \subseteq X \setminus \overline{B}$. The converse is false, with $\beta\mathbb{N} \setminus \mathbb{N}$ being an example of a space which is sub-Stonean but not Rickart. For more about both sub-Stonean and Rickart spaces, see [56].

Proposition 4.4.3. *Let X be an infinite compactum, and suppose that X satisfies one of the following conditions:*

1. *X has the countable chain condition;*

2. *X is separable;*

3. *X is metrizable;*

4. *X is homeomorphic to a product of two infinite compacta;*

5. *X is not sub-Stonean;*

   6. X is Rickart.

Then $C(X)$ is not countably degree-1 saturated.

Proof. We have 3 implies 2 which implies 1, so let us begin by showing that 1 implies that $C(X)$ is not degree-1 $\aleph_1$-saturated. Assume X has the countable chain condition and $C(X)$ is degree-1 $\aleph_1$-saturated. The countable chain condition for X translates to the statement that in $C(X)$ every family of positive elements of norm 1 whose pairwise products are 0 must be countable. Using Zorn's lemma, find a subset Z of the positive elements of the unit ball of $C(X)$ which is maximal (under inclusion) with respect to the property that if $f, g \in Z$ and $f \neq g$, then $fg = 0$. By hypothesis, the set Z is countable; list it as $Z = \{f_n\}_{n \in \mathbb{N}}$.

   For each $n \in \mathbb{N}$, define $P_n(x) = f_n x$, and let $K_n = 0$. Let $P_{-1}(x) = x$, and $K_{-1} = 1$. The type $\{\, \|P_n(x)\| = K_n : n \geq -1 \,\}$ is finitely satisfiable. Indeed, by definition of Z, for any $m \in \mathbb{N}$ and any $0 \leq n \leq m$ we have $\|P_n(f_{m+1})\| = \|f_n f_{m+1}\| = 0$, and $\|f_{m+1}\| = 1$. By degree-1 $\aleph_1$-saturation and Lemma 4.4.2 there is a positive element b in the unit ball of $C(X)$ such that $\|P_n(b)\| = 0$ for all $n \in \mathbb{N}$. This contradicts the maximality of Z.

   If $C(X)$ is degree-1 $\aleph_1$-saturated then by [44, Proposition 2.7] $C(X)$ is a SAW*-algebra; all we need to know of this property is that the main result of [51] states that such algebras cannot be written as the tensor product of two infinite-dimensional C*-algebras. For commutative C*-algebras we have $C(Y) \otimes C(Z) \cong C(Y \times Z)$, from which the claim for 4 follows.

   In the following two cases it will be useful to observe that for self-adjoint $a, b \in C(X)$, if $\|a - b - 1\| \leq 1$ then $b \leq a$. Indeed, for each $x \in X$ we have either $0 \leq |a(x) - b(x) - 1| = a(x) - b(x) - 1$, in which case $1 \leq a(x) - b(x)$, so $b(x) \leq a(x)$, or else $|a(x) - b(x) - 1| = 1 + b(x) - a(x) \leq 1$, in which case again $b(x) \leq a(x)$. The consequence of this observation is that we can use expressions of the form $b \leq a$ for self-adjoint elements in degree-1 types.

   We now consider 5. It is shown in [101, Theorem 4.6] that X is sub-Stonean if and only if $C(X)$ has the property that whenever $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences of self-adjoint elements in $C(X)$ such that $x_n \leq x_{n+1} \leq \cdots \leq y_{n+1} \leq y_n$ for all n, then there is a self-adjoint $z \in C(X)$ such that $x_n \leq z \leq y_n$ for all n. Suppose we are given such sequences in a degree-1 $\aleph_1$-saturated algebra $C(X)$, and by renormalizing if necessary, assume that $\|y_1\| = 1$. Let $\Sigma(z)$ be the type over $\{x_n, y_n : n \in \mathbb{N}\}$ consisting of all formulas expressing $x_n \leq z$ and $z \leq y_n$. Any finite subset of Σ is satisfied in the unit ball of $C(X)$ by a sufficiently large x_n, which is self-adjoint, so by degree-1 $\aleph_1$-saturation and Lemma 4.4.2, Σ is realized in the unit ball of $C(X)$ by a self-adjoint z, and hence X is sub-Stonean. A more general version of this argument appears in [44, Proposition 2.6].

   It remains to consider 6. Let X be Rickart. The Rickart condition can be rephrased as saying that any bounded increasing monotone sequence of self-adjoint functions in $C(X)$ has a least upper bound in $C(X)$ (see [56, Theorem 2.1]).

   Consider a sequence $(a_n)_{n \in \mathbb{N}} \subseteq C(X)$ of positive elements of norm 1 whose pairwise products are 0, and let $b_n = \sum_{i \leq n} a_i$. Then $(b_n)_{n \in \mathbb{N}}$ is a bounded increasing sequence of positive functions, so it has a least upper bound b. Since $\|b_n\| = 1$ for all n, we also have $\|b\| = 1$. The type consisting of $\|x\| = 1$, $1 \leq \|b - x\| \leq 2$, $\|b - x - 1\| = 1$, and $x \geq b_n$ for $n \in \mathbb{N}$ is consistent, with partial solution b_{n+1} for the first three formulas and the remaining ones up to n. If $C(X)$ is degree-1 $\aleph_1$-saturated then this type has a positive solution y, but in that case $y \geq b_n$ for all $n \in \mathbb{N}$, yet $b > y$, showing that X is not Rickart. $\quad\square$

   Note that the preceding proof shows that the existence of a particular increasing bounded sequence

that is not norm-convergent but does have a least upper bound (a condition apparently weaker than being Rickart) is sufficient to prove that $C(X)$ does not have degree-1 $\aleph_1$-saturation. We also note that for any countably incomplete ultrafilter $\mathcal{U}$ we have that $C(X_{\mathcal{U}}) \cong C(X)^{\mathcal{U}}$ is $\aleph_1$-saturated, so Proposition 4.4.3 generalizes [6, Proposition 2.3.1], which states that $X_{\mathcal{U}}$ does not have the countable chain condition.

The remaining results of this section compare the saturation of $C(X)$ with the saturation of the clopen algebra $CL(X)$, where the latter is viewed as a discrete structure in the signature of Boolean algebras.

Theorem 4.4.4. *Let X be a 0-dimensional compactum. Then*

$$C(X) \text{ is } \aleph_1\text{-saturated} \Rightarrow CL(X) \text{ is } \aleph_1\text{-saturated}$$

and

$$CL(X) \text{ is } \aleph_1\text{-saturated} \Rightarrow C(X) \text{ is quantifier-free } \aleph_1\text{-saturated}.$$

Theorem 4.4.5. *Let X be a 0-dimensional compactum, and assume further that X has a finite number of isolated points. If $C(X)$ is countably degree-1 saturated, then $CL(X)$ is countably saturated. Moreover, if X has no isolated points, then degree-1 $\aleph_1$-saturation and countable saturation coincide for $C(X)$.*

4.4.1 Proof of Theorem 4.4.4

One direction of Theorem 4.4.4 is straightforward, namely showing that if $C(X)$ is $\aleph_1$-saturated then so is $CL(X)$. Indeed, we have that $CL(X)$ is isomorphic (as a Boolean algebra) to the algebra of projections in $C(X)$, and the set of projections in $C(X)$ is definable. Using the same translation from discrete to continuous logic as appeared in the proof of Theorem 2.4.4, and relativizing all quantifiers to the projections, we see that saturation of $C(X)$ implies saturation of $CL(X)$.

The other direction will require more effort. To start, we will to need the following Proposition, relating elements of $C(X)$ to certain collections of clopen sets:

Proposition 4.4.6. *Let X be a 0-dimensional compactum, and let $f \in C(X)$ have norm at most 1. Then there exists a countable collection of clopen sets $\tilde{Y}_f = \{Y_{n,f} : n \in \mathbb{N}\}$ which completely determines f, in the sense that for each $x \in X$, the value $f(x)$ is completely determined by $\{n : x \in Y_{n,f}\}$.*

Proof. Let $\mathbb{C}_{m,1} = \{\frac{j_1 + \sqrt{-1}j_2}{m} : j_1, j_2 \in \mathbb{Z} \wedge \|j_1 + \sqrt{-1}j_2\| \leq m\}$.

For every $y \in \mathbb{C}_{m,1}$ consider $X_{y,f} = f^{-1}(B_{1/m}(y))$. We have that each $X_{y,f}$ is a σ-compact open subset of X, so is a countable union of clopen sets $X_{y,f,1}, \ldots, X_{y,f,n}, \ldots \in CL(X)$. Note that $\bigcup_{y \in \mathbb{C}_{m,1}} \bigcup_{n \in \mathbb{N}} X_{y,f,n} = X$. Let $\tilde{X}_{m,f} = \{X_{y,f,n}\}_{(y,n) \in \mathbb{C}_{m,1} \times \mathbb{N}} \subseteq CL(X)$.

We claim that $\tilde{X}_f = \bigcup_m \tilde{X}_{m,f}$ describes f completely. Fix $x \in X$. For every $m \in \mathbb{N}$ we can find a (not necessarily unique) pair $(y, n) \in \mathbb{C}_{m,1}$ such that $x \in X_{y,f,n}$. Note that, for any $m, n_1, n_2 \in \mathbb{N}$ and $y \neq z$, we have that $X_{y,f,n_1} \cap X_{z,f,n_2} \neq \emptyset$ implies $|y - z| \leq \sqrt{2}/m$. In particular, for every $m \in \mathbb{N}$ and $x \in X$ we have

$$2 \leq |\{y \in \mathbb{C}_{m,1} : \exists n(x \in X_{y,f,n})\}| \leq 4.$$

Let $A_{x,m} = \{y \in \mathbb{C}_{m,1} : \exists n(x \in X_{y,f,n})\}$ and choose $a_{x,m} \in A_{x,m}$ to have minimal absolute value. Then $f(x) = \lim_m a_{x,m}$ so the collection $\tilde{X}_f$ completely describes f in the desired sense. $\square$

The above proposition will be the key technical ingredient in proving the second implication in Theorem 4.4.4. We will proceed by first obtaining the desired result under the Continuum Hypothesis, and then showing how to eliminate the set-theoretic assumption.

Lemma 4.4.7. *Assume the Continuum Hypothesis. Let B be an $\aleph_1$-saturated Boolean algebra of cardinality $2^{\aleph_0} = \aleph_1$. Then $C(S(B))$ is $\aleph_1$-saturated.*

Proof. Let $B' \preceq B$ be countable, and let $\mathcal{U}$ be a non-principal ultrafilter on ω. By the uniqueness of $\aleph_1$-saturated models of size $\aleph_1$, and the continuum hypothesis, we have $(B')^{\mathcal{U}} \cong B$. We therefore have $C(S(B)) \cong C(S(B'))^{\mathcal{U}}$, and hence $C(S(B))$ is $\aleph_1$-saturated. $\qquad\square$

Theorem 4.4.8. *Assume the Continuum Hypothesis. Let X be a 0-dimensional compactum. If $CL(X)$ is $\aleph_1$-saturated as a Boolean algebra, then $C(X)$ is quantifier-free $\aleph_1$-saturated.*

Proof. Let $\Sigma = \{ \|P_n\| = r_n : n < \omega \}$ be a collection of conditions, where each P_n is a 2-degree $*$-polynomial in $x_0, \ldots, x_n$, such that there is a collection $F = \{f_{n,i}\}_{n \leq i}$ in the unit ball of $C(X)$ with the property that for all i we have $r_n - \frac{1}{i} < \|P_n(f_{0,i}, \ldots, f_{n,i})\| < r_n + \frac{1}{i}$ for all $n \leq i$.

For any n, we have that P_n has finitely many coefficients. Consider G the set of all coefficients of every P_n, and L the set of all possible 2-degree $*$-polynomials in $F \cup G$. Note that for any $n \leq i$ we have that $P_n(f_{0,i}, \ldots, f_{n,i}) \in L$ and that L is countable. For any element $f \in L$ consider a countable collection $\tilde{X}_f$ of clopen sets describing f, as in Proposition 4.4.6.

Since $CL(X)$ is $\aleph_1$-saturated, and $2^{\aleph_0} = \aleph_1$, we can find an $\aleph_1$-saturated Boolean algebra $B \subseteq CL(X)$ such that $\emptyset, X \in B$, for all $f \in L$ we have $\tilde{X}_f \subseteq B$, and $|B| = \aleph_1$. Then B is isomorphic to an ultrapower, and hence $C(S(B))$ is also isomorphic to an ultrapower, and hence is $\aleph_1$-saturated. By Stone and Gelfand-Naimark dualities we have that $C(S(B)) \subseteq C(X)$. Our choice of B ensures that $C(S(B))$ has all of the functions necessary to describe the type Σ, and by $\aleph_1$-saturation Σ is realized in $C(S(B))$. The assumption that Σ is a quantifier-free type implies that the realization of Σ in $C(S(B))$ is also a realization in $C(X)$. $\qquad\square$

To remove the Continuum Hypothesis from Theorem 4.4.8 we will show that the result is preserved by σ-closed forcing. The proof is based on the general absoluteness result for satisfaction of $\mathcal{L}_{\omega_1,\omega}$ formulas, Proposition 3.1.9.

Proposition 4.4.9. *Let $\mathbb{P}$ be a σ-closed notion of forcing. Let M be a metric structure, and let Φ be a set of (finitary) formulas. Then M is $\aleph_1$-Φ-saturated in V if and only if M is $\aleph_1$-Φ-saturated in the forcing extension $V[G]$.*

Proof. First, observe that since $\mathbb{P}$ is σ-closed, forcing with $\mathbb{P}$ does not introduce any new countable set. In particular, the set of types which must be realized for M to be countably Φ-saturated are the same in V and in $V[G]$.

Let $\Sigma(\vec{x})$ be a set of instances of formulas from Φ with parameters from a countable set $A \subseteq M$. Add new constants to the language for each $a \in A$, so that we may view Σ as a type without parameters. Define

$$\varphi(\vec{x}) = \inf\{\psi(\vec{x}) : \psi \in \Sigma\}.$$

Note that $\varphi^M(\vec{a}) = 0$ if and only if $\vec{a}$ satisfies Σ in M. This φ is an $\mathcal{L}_{\omega_1,\omega}$ formula, so by Proposition 3.1.9, for any $\vec{a}$ from M we have that $\varphi^M(\vec{a}) = 0$ in V if and only if $\varphi^M(\vec{a}) = 0$ in $V[G]$. As the same finite tuples $\vec{a}$ from M exist in V and in $V[G]$, this completes the proof. $\qquad\square$

Finally, we return to the proof of Theorem 4.4.4. All that remains is to show:

Lemma 4.4.10. *The Continuum Hypothesis can be removed from the hypothesis of Theorem 4.4.8.*

Proof. Let X be a 0-dimensional compactum such that $CL(X)$ is $\aleph_1$-saturated, and suppose that the Continuum Hypothesis fails. Let $\mathbb{P}$ be a σ-closed forcing which collapses $2^{\aleph_0}$ to $\aleph_1$ (see [78, Chapter 7, §6]). Let $A = C(X)$ and $B = CL(X)$. Observe that since $\mathbb{P}$ is σ-closed we have that A remains a complete metric space in $V[G]$, and by Lemma 3.1.9 A still satisfies the axioms asserting that $A = C(Y)$ for a compact 0-dimensional space Y. Also by Lemma 3.1.9 we have that B remains a Boolean algebra, and the set of projections in A in both V and $V[G]$ is B. We note that it may not be true in $V[G]$ that $X = S(B)$, or even that X is compact (see [33]), but this causes no problems because it follows from the above that $A = C(S(B))$ in $V[G]$. By Proposition 4.4.9 B remains $\aleph_1$-saturated in $V[G]$. Since $V[G]$ satisfies the Continuum Hypothesis we can apply Theorem 4.4.8 to conclude that A is quantifier-free $\aleph_1$-saturated in $V[G]$, and hence also in V by Proposition 4.4.9. $\qquad\square$

With the continuum hypothesis removed from Theorem 4.4.8, we have completed the proof of Theorem 4.4.4. It would be desirable to improve this result to say that if $CL(X)$ is $\aleph_1$-saturated then $C(X)$ is also $\aleph_1$-saturated. We note that if we could choose $C(S(B)) \preceq C(X)$ in the proof of Theorem 4.4.4 then the same proof would give the improved conclusion.

4.4.2 Proof of Theorem 4.4.5

We now turn to the proof of Theorem 4.4.5.

Proposition 4.4.11. *If X is a 0-dimensional compactum with finitely many isolated points such that $C(X)$ is degree-1 $\aleph_1$-saturated, then the Boolean algebra $CL(X)$ is $\aleph_1$-saturated.*

Proof. Assume first that X has no isolated points. In this case we get that $CL(X)$ is atomless, so it is enough to see that $CL(X)$ has the property that for two countable directed sets $Y, Z \subseteq CL(X)$ with $Y < Z$, there is $c \in CL(X)$ such that $Y < c < Z$ (see [85, Theorem 2.7]).

Assume for the moment that both Y and Z are infinite. Passing to a cofinal increasing sequence in Z and a cofinal decreasing sequence in Y, we can suppose that $Z = \{U_n\}_{n \in \mathbb{N}}$ and $Y = \{V_n\}_{n \in \mathbb{N}}$, where

$$U_1 \subsetneq \ldots \subsetneq U_n \subsetneq U_{n+1} \subsetneq \ldots \subsetneq V_{n+1} \subsetneq V_n \subsetneq \ldots \subsetneq V_1.$$

If $\bigcup_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n$ then $\bigcup_{n \in \mathbb{N}} U_n$ is a clopen set, so by the remark following the proof of Lemma 4.4.3, we have a contradiction to the degree-1 $\aleph_1$-saturation of $C(X)$.

For each $n \in \mathbb{N}$, let $p_n = \chi_{U_n}$ and $q_n = \chi_{V_n}$, where χ_A denotes the characteristic function of the set A. Then

$$p_1 < \ldots < p_n < p_{n+1} < \ldots < q_{n+1} < q_n < \ldots < q_1$$

and by degree-1 $\aleph_1$-saturation there is a positive r such that $p_n < r < q_n$ for every n. In particular $A = \{x \in X : r(x) = 0\}$ and $C = \{x \in X : r(x) = 1\}$ are two disjoint closed sets such that $\overline{\bigcup_{n \in \mathbb{N}} U_n} \subseteq C$ and $\overline{X \setminus \bigcap_{n \in \mathbb{N}} V_n} \subseteq A$. We want to find a clopen set D such that $A \subseteq D \subseteq X \setminus C$. For each $x \in A$ pick W_x a clopen neighborhood contained in $X \setminus C$. Then $A \subseteq \bigcup_{x \in A} W_x$. By compactness we can cover A with finitely many of these sets, say $A \subseteq \bigcup_{i \leq n} W_{x_i} \subseteq X \setminus C$, so $D = \bigcup_{i \leq n} W_{x_i}$ is the desired clopen set.

Essentially the same argument works when either Y or Z is finite. We need only change some of the inequalities from $<$ with $\leq$, noting that a finite directed set has always a maximum and a minimum.

If X has a finite number of isolated points, write $X = Y \cup Z$, where Y has no isolated points and Z is finite. Then $C(X) = C(Y) \oplus C(Z)$ and $CL(X) = CL(Y) \oplus CL(Z)$. The above proof shows that $CL(Y)$ is $\aleph_1$-saturated, and $CL(Z)$ is saturated because it is finite, so $CL(X)$ is again $\aleph_1$-saturated. $\quad\square$

Recall that when X has no isolated points then $C(X) \equiv C(2^{\mathbb{N}})$ (Theorem 4.2.4), and that the theory of $C(2^{\mathbb{N}})$ has quantifier elimination (Proposition 4.3.3). The proof of Theorem 4.4.5 is therefore complete by combining Theorem 4.4.4 and Proposition 4.4.11.

Remark 4.4.12. Throughout this chapter we have focused either on the case where the space X is 0-dimensional, or where the space is connected. Rather little is known about the model theory of C*-algebras associated to other spaces.

We have also focused entirely on the case of compact spaces. When X is locally compact but not compact we can consider the non-unital C*-algebra $C_0(X)$ consisting of continuous complex-valued functions on X that vanish at infinity. A version of the Gelfand-Naimark theorem shows that every non-unital commutative C*-algebra is of this form. The main reason that we have not considered this more general setting is that even when the X_i's are only locally compact their ultracoproduct is compact. More precisely, when $(X_i)_{i \in I}$ is a sequence of locally compact spaces and $\mathcal{U}$ is an ultrafilter on I, the ultracoproduct $\sum_{\mathcal{U}} X_i$ is homeomorphic to the ultracoproduct $\sum_{\mathcal{U}} \beta X_i$ of the Stone-Čech compactifications of the X_i's – see [4, Lemma 3.1]. Nevertheless, it remains possible that future model-theoretic study of $C_0(X)$ may yield interesting information about locally compact spaces.

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