TWO APPLICATIONS OF TOPOLOGY TO MODEL THEORY

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ABSTRACT. By utilizing the topological concept of pseudocompactness, we simplify and improve a proof of Caicedo, Dueñez, and Iovino concerning Terence Tao's metastability. We also pinpoint the exact relationship between the Omitting Types Theorem and the Baire Category Theorem by developing a machine that turns topological spaces into abstract logics.

I. INTRODUCTION

The senior (third) author has often remarked that model theorists use topology, but mainly at a rather elementary level. The present work by current and former members of the Toronto Seminar applies more advanced general topology, first to simplify and improve a proof of Caicedo, Dueñez, and Iovino [CDI19] concerning Terence Tao's notion of metastability [Tao08], and second to produce and utilize a machine for converting topological spaces into abstract logics. This machine is then used to determine the exact relationship between the Omitting Types Theorem and the Baire Category Theorem. Morley's Categoricity Theorem has been said to be the beginning of modern model theory. Morley's original proof [Mor65] made extensive use of topology, especially the Cantor-Bendixson analysis of compact spaces. In the years since Morley's paper appeared there have been some uses of topology in model theory, but the topology has been fairly elementary, and in many cases combinatorial arguments have come to replace topological ones. We hope to encourage model theorists to consider the possible applications of more sophisticated topological methods in model theory. Our first application illustrates that some rather simple topology—albeit a topic likely not covered in the one graduate topology course an average model

Date: August 20, 2020.
2020 *Mathematics Subject Classification.* 03C15, 03C30, 03C45, 03C95, 03C98, 40A05, 54D99, 54E52, 54G20.
Key words and phrases. Metastability, countable compactness, pointwise convergence, uniform convergence, uniform metastability principle, pseudocompactness, omitting types, Baire category, type-space functor, Banach-Mazur game.
[1] Research supported by NSERC Grant A-7354.
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theorist might have taken—can easily simplify and improve model-theoretic arguments. Our second application introduces a machine which converts topological spaces into abstract logics, thus giving access to the vast field of strange topological spaces while searching for model-theoretic counterexamples.

Part 1. Pseudocompactness and the Uniform Metastability Principle

II. A Brief Introduction to Metastability

Definition 2.1. A *sampling of* ω is a family $\{ \eta_n : n < \omega \} \subseteq [\omega]^{<\omega}$ such that $\eta_n \subseteq \omega \setminus n$ for each $n < \omega$. Let $\mathcal{S}$ denote the set of all samplings of ω. Let (X, d) be a metric space. A sequence $\langle x_n : n < \omega \rangle$ is *metastable* if for each $\varepsilon > 0$ and each sampling η, there is an $m < \omega$ such that $(\forall i, j \in \eta_m)$ $(d(x_i, x_j) < \varepsilon)$.

It was proved by Tao [Tao08] that a sequence is metastable if and only if it is Cauchy. The relevant distinction occurs when one considers uniform metastability:

Definition 2.2. A family $A \subseteq X^\omega$, where (X, d) is a metric space, is *uniformly metastable* if there is a family $\{ E_{\varepsilon,\eta} : \varepsilon > 0,\ \eta \in \mathcal{S} \}$ such that whenever $\eta \in \mathcal{S}$ and $\varepsilon > 0$, each sequence in A is metastable witnessed by the same $m < E_{\varepsilon,\eta}$. A sequence of functions $\langle f_n : n < \omega \rangle$ in R^X is *uniformly metastable* if there is a family $\{ E_{\varepsilon,\eta} : \varepsilon > 0,\ \eta \in \mathcal{S} \}$ such that whenever $\eta \in \mathcal{S}$ and $\varepsilon > 0$, for each $x \in X$ the sequence $\langle f_n(x) : n < \omega \rangle$ is metastable witnessed by the same $m < E_{\varepsilon,\eta}$.

The following examples from an early version of [CDI19] show that uniform metastability is strictly in between uniform convergence and pointwise convergence:

- The family of all eventually 0 sequences in 2^ω is not uniformly metastable even though each sequence is trivially convergent. To see this, take the subfamily of all sequences with arbitrarily long initial segments with alternating 0's and 1's and $\eta_n = \{ n, n+1 \}$.
- The set of all monotonic sequences in 2^ω is uniformly metastable witnessed by $E_{\varepsilon,\eta} = \max \eta_0$. However, the convergence is not uniform.

It is a natural to ask when results regarding pointwise convergence of functions can be improved to uniform metastability in a way similar to that of Tao's Metastable Dominated Convergence Theorem [Tao08]. In [CDI19], a topological proof is given for the following fact: if X is countably compact, then on any closed subspace, there is no distinction between pointwise convergence and uniform metastability. The converse result is only proved in [CDI19] in a model theoretic setting using powerful machinery. We produced a topological proof of this converse result using the following fact: a countably compact space is a space with every closed subspace pseudocompact. The model theoretic result follows at once from this topological fact and a few basic remarks.

Definition 2.3. A topological space X is *pseudocompact* if every continuous real-valued function on X has bounded image.

There is a whole book devoted to pseudocompact spaces [HTMT18]. The following basic result can also be found in [Tka15]:

Proposition 2.4. A completely regular space X is pseudocompact if and only if every locally finite family of non-empty open sets (i.e. every point of X has a neighbourhood meeting at most finitely many members of the family) is finite.

Proof. Suppose X is pseudocompact and that there is an infinite locally finite family of non-empty open sets $\{ U_n : n < \omega \}$. Take $x_n \in U_n$ for each $n < \omega$. By complete regularity, take a continuous $f_n : X \to \mathbb{R}$ such that $f_n(x_n) = n$ and $f \restriction X \setminus U_n = 0$. Then $F = \sum_{n<\omega} f_n$ is continuous since $\{ U_n : n < \omega \}$ is locally finite: given $x \in X$, let $S_x = \{ n < \omega : x \in U_n \} \in [\omega]^\omega$; the continuity of F at x follows from $\bigcup_{n \in S_x} \overline{U}_n = \overline{\bigcup_{n \in S_x} U_n}$. Conversely, suppose X is not pseudocompact, then there is an unbounded continuous function $f : X \to \mathbb{R}$. Since f^2 is also continuous and unbounded, we can assume $f \geq 0$. Let $x_0 \in f[X]$; if $x_n \in f[X]$ has been constructed, take $x_{n+1} \in f[X]$ such that $f(x_{n+1}) > f(x_n) + 1$. If we denote the ball of center $f(x_n)$ and radius 1 by $B(f(x_n), 1)$, then $\mathcal{B} = \left\{ f^{-1}[B(f(x_n), 1)] : n < \omega \right\}$ is an infinite family of pairwise disjoint non-empty open sets. Now suppose $\mathcal{B}$ is not locally finite, then there is a point $x \in X$ such that every open neighbourhood of x contains elements with arbitrarily large images, contradicting the continuity of f.

□

Remark Notice that when pseudocompactness fails, one can get the infinite locally finite family of open sets to be pairwise disjoint. Also notice that complete regularity is unnecessary for the direction "every locally finite family of open sets is finite" implies pseudocompactness. However, regularity is required.

The following proposition follows from a theorem and an exercise in [Eng89];

Proposition 2.5. A space is countably compact if and only if every closed subspace is pseudocompact.

Proof. If X is countably compact and there is a closed subspace $C \subseteq X$ that is not pseudocompact then, as in the proof of Proposition 2.1, C includes a closed discrete subspace and so does X, which contradicts countable compactness. Conversely, if X is not countably compact, it includes a discrete closed set $C = \{ x_n : n < \omega \}$. Letting $f(x_n) = n$, we obtain a continuous unbounded function on C. □

Now we present the connection between pseudocompactness and uniform metastability:

Proposition 2.6. Let X be a regular topological space. If every sequence of continuous real-valued functions $\langle f_n : n < \omega \rangle$ on X that converges pointwise is uniformly metastable, then X is pseudocompact.

Proof. Suppose X is not pseudocompact and let $\mathcal{B} = \{ U_n : n < \omega \}$ be a infinite locally finite family of pairwise disjoint non-empty open sets. For each $n < \omega$, take $x_n \in U_n$. Then consider the functions $f_n : X \to \mathbb{R}$ such that $f(x_n) = 1$ and $f \restriction X \setminus U_n = 0$. Then the function $F = \sum_{n<\omega} f_n$ is continuous since $\mathcal{B}$ is locally finite. Consider $g_n = \sum_{i \le n} f_i$. Then the sequence of continuous functions $\langle g_0, F, g_1, F, g_2, F, \dots \rangle$ converges pointwise to F but it is not uniformly metastable as it contains all eventually 1 sequences with arbitrarily long initial segments of alternating 0's and 1's. □

Definition 2.7. The *Topological Uniform Metastability Principle* holds for a topological space X if whenever a sequence of real-valued continuous functions converges pointwise on a closed subspace $C \subseteq X$, it is uniformly metastable on C.

The previous results allow us to characterize the equivalence between the topological uniform metastability principle and countable compactness.

Theorem 2.8. Let X be a completely regular space. Then X is countably compact if and only if the topological uniform metastability principle holds for X.

Proof. We reproduce the proof given in an early version of [CDI19] when X is countably compact: assume uniform metastability fails and let $\varepsilon > 0$ and $\eta \in \mathcal{S}$ be witnesses of this fact. Then, for each $n < \omega$, there is $x \in X$ such that for each $k < n$, $M_{x,k} = \max \{ |f_i(x) - f_j(x)| : i, j \in \eta_k \} \ge \varepsilon$. Then $x \in \bigcap_{k \le n} A_k$ where $A_k = \{ z \in X : M_{z,k} \ge \varepsilon \}$ is closed by the continuity of the f_n's. Thus $\{ A_k : k < \omega \}$ is centred and, by countable compactness, there is an $x \in \bigcap_{k<\omega} A_k$, which contradicts the convergence of $\langle f_n(x) : n < \omega \rangle$. Conversely, suppose X is not countably compact. Then, by Proposition 2.2, there is a closed subspace $C \subseteq X$ that is not pseudocompact and so, by Proposition 2.3, there is a sequence of continuous real-valued functions on X that converges pointwise on C but is not uniformly metastable on C. $\square$

Remark The current version of [CDI19] proves the previous equivalence assuming that X is regular and paracompact. We just showed that the paracompactness assumption can be replaced by assuming that X is completely regular. Also, [CDI19] points out that the analogue of the uniform metastability principle for nets, instead of sequences, is equivalent to X being compact.

III. The Uniform Metastability Principle

Logics for metric structures are properly presented in [Eag17] and [Cai17]. Given a logic for metric structures $\mathcal{L}$ and a signature τ, recall that the logic topology on the space of τ-structures $\mathrm{Str}(\tau)$ is determined by the basic closed sets $[\varphi] = \{\, \mathfrak{M} \in \mathrm{Str}(\tau) : \mathfrak{M} \models \varphi \,\}$. This topology is (up to the quotient by elementary equivalence) a special case of the more general framework described in Section V.2. We regard τ-sentences as continuous $[0,1]$-valued functions on the space of τ-structures in the natural way: $\mathfrak{M} \mapsto \varphi^{\mathfrak{M}}$. In this context, we now define the model theoretic analogue of metastability:

Definition 3.1. Let $\mathcal{L}$ be a logic for metric structures and τ a signature. Given a τ-theory T, we say that a sequence of τ-sentences $\langle \varphi_n : n < \omega \rangle$ *converges pointwise modulo* T if and only for for every model $\mathfrak{M}$ of T, the sequence $\langle {\varphi_n}^{\mathfrak{M}} : n < \omega \rangle$ converges. We say that the sequence is *uniformly metastable modulo* T if the family $\left\{\, \langle {\varphi_n}^{\mathfrak{M}} : n < \omega \rangle : \mathfrak{M} \models \varphi \,\right\}$ is uniformly metastable.

Definition 3.2. The Uniform Metastability Principle (UMP) for a logic $\mathcal{L}$ is the following statement: "if τ is a signature and T is an τ-theory, then every sequence of τ-sentences $\langle \varphi_n : n < \omega \rangle$ that converges pointwise modulo T is also uniformly metastable modulo T."

In an early version of [CDI19], it was proved that the UMP is equivalent to the logic being countably compact. This follows from the following two lemmas:

Lemma 3.3. The logic topology is completely regular.

Proof. Let $C \subseteq \mathrm{Str}(\tau)$ and $\mathfrak{M} \notin C$. Then there must be a formula φ such that $\varphi^{\mathfrak{M}} < 1$ and $(\forall \mathfrak{N} \in C)\ \varphi^{\mathfrak{N}} = 1$ (as otherwise $\mathfrak{M}$ would belong to C by the definition of the topology on $\mathrm{Str}(\tau)$). Then φ is the continuous function that separates C and $\mathfrak{M}$. $\square$

Lemma 3.4. The closed subspaces of the logic topology are completely determined by τ-theories, i.e. $C \subseteq \mathrm{Str}(\tau)$ is closed if and only if there is a τ-theory T such that C is the set of τ-structures that are models of T.

Proof. Suppose C is a closed set in $\mathrm{Str}(\tau)$; then C is the intersection of basic closed sets, say $C = \bigcap_{\alpha<\kappa}[\varphi_\alpha]$. Thus C is the set of τ-structures that are models of the theory $T = \{\, \varphi_\alpha : \alpha < \kappa \,\}$. Conversely, each model of a τ-theory T belongs to the intersection of all $[\varphi]$ where φ ranges over T. □

Definition 3.5. A *logic $\mathcal{L}$ is countably compact* if and only if given a signature τ, the space of τ-structures $\mathrm{Str}(\tau)$ is countably compact.

Putting all this together, we easily obtain the main result of the early version of [CDI19]:

Theorem 3.6. Let $\mathcal{L}$ be a logic for metric structures. The UMP holds if and only if $\mathcal{L}$ is countably compact.

Part 2. Omitting Types and the Baire Category Theorem

IV. Definitions

The fundamental topological notion we will consider in this part is that of a *type-space functor*, which is an abstraction of the spaces of complete types from first-order logic. A single type-space functor can be thought of as capturing the topological content of the model theory of a single signature. We then describe how to combine various type-space functors to produce a *topological logic*, which amounts to a topological description of an abstract model-theoretic logic.

IV.1. Type-space functors.

To simplify notation, whenever κ is a cardinal and $A \in [\kappa]^n$, we write $A = \{a_0 < \ldots < a_{n-1}\}$ to mean that $A = \{a_0, \ldots, a_{n-1}\}$ and $a_0 < \ldots < a_{n-1}$.

Definition 4.1. A *type-space functor* S takes each $n \in \omega$ to a topological space S_n, and each $f : n \to m$ to a continuous open map $Sf : S_m \to S_n$, satisfying the following conditions. Here $i_k : k \to k+1$ is the inclusion, and $d_m : m+1 \to m+2$ is $d(j) = j$ for $j < m$ and $d(m) = m+1$.

(1) For all $f : n \to m$ and $g : m \to k$, $S(g \circ f) = (Sf) \circ (Sg)$.
(2) If $\iota_n : n \to n$ is the identity function then $S\iota_n : S_n \to S_n$ is the identity function.
(3) For each $m \in \omega$, $p \in S_m$, $q \in (Si_m)^{-1}(\{p\})$, and non-empty open $U \subseteq (Si_m)^{-1}(\{p\}$, let $WAP_S(m, p, q, U)$ be the statement that there is an $r \in S_{m+2}$ such that $(Si_{m+1})(r) = q$ and $(Sd_m)(r) \in U$. We require that $WAP_S(m, q, p, U)$ holds for all such m, p, q, U.

Our definition is based on the one in Knight [Kni07], with some of the simplifications introduced in [Kni10]. We differ from Knight in that we require each map Sf to be open and we only require a weak version of the amalgamation property. The basic example of a type-space functor is when each S_n is the set of complete n-types of some first-order theory; see Section V for other examples.

Definition 4.2. Let S be a type-space functor. Define S_ω to be the inverse limit of the spaces S_n, using each $S\iota_{n,m}$ as a bonding map for $n < m$. Concretely,

$$S_\omega = \{(a_n)_{n<\omega} \in \prod_{n<\omega} S_n : \text{ for all } n < m,\, a_n = (S\iota_{n,m})(a_m)\},$$

with the subspace topology.

For each map $f : n \to \omega$ we have a map $Sf : S_\omega \to S_n$. To define this map, let m be large enough so that the image of f is included in m. Then define $f' : n \to m$ to be $f'(i) = f(i)$ for all $i < n$. Finally, define $Sf : S_\omega \to S_n$ by $(Sf)((a_j)_{j<\omega}) = (Sf')(a_m)$.

In order to view a type-space functor as having model-theoretic content, we need a notion of a *model*, which we take from [Kni07, Definition 2.9].

Definition 4.3. Let S be a type-space functor, and let κ be a cardinal. A *model of size* κ for S is a function M, whose domain is $[\kappa]^{<\omega}$, satisfying the list of properties below for all $A = \{a_0 < \ldots < a_{n-1}\} \in [\kappa]^n$.

(1) $M(A) \in S_n$.
(2) If $B = \{b_0 < \ldots < b_{m-1}\} \in [\kappa]^m$, $A \subseteq B$, and $g : n \to m$ is the function satisfying $a_i = b_{g(i)}$ for all i, then $M(A) = (Sg)(M(B))$.
(3) If $U \subseteq (S\iota_{n,m})^{-1}(M(A))$ is open, then there is a $B = \{b_0 < \ldots < b_{m-1}\} \in [\kappa]^m$ with $A \subseteq B$, and a permutation g of m satisfying $a_i = b_{g(i)}$ for all $i < n$, such that $(Sg)(M(B)) \in U$.

Interpreting these conditions in the context of traditional model theory may help illuminate their meaning. In that context, condition (1) simply says that if $M \models T$ and $(a_1, \ldots, a_n) \in M^n$, then $\mathrm{tp}^M(a_1, \ldots, a_n)$ is a complete n-type of T. Condition (2) corresponds to the fact that if $k < n$, then $\mathrm{tp}^M(a_1, \ldots, a_k)$ consists of those formulas in $\mathrm{tp}^M(a_1, \ldots, a_n)$ that only use the variables $x_1, \ldots, x_k$. Condition (3) is an analogue of the fact that structures are closed under existential quantification. In classical model theory condition (3) expresses that if p is a complete n-type, and q is a complete m-type with $n < m$ and $p \subseteq q$, and if $(a_1, \ldots, a_n)$ realizes p in a model M, then for any formula $\varphi(x_1, \ldots, x_m) \in q$ we can find $a_{n+1}, \ldots, a_m \in M$ such that $M \models \varphi(a_1, \ldots, a_m)$.

Definition 4.4. Let S be a type-space functor, let M be a model for S of size κ, and let $(a_0, \ldots, a_{n-1})$ be a tuple of length n from κ. Let $A = \{a_0, \ldots, a_{n-1}\} = \{c_0 < \ldots < c_{k-1}\}$. Let $g : n \to k$ be the function such that $c_{g(i)} = a_i$ for all $i < n$. Then we define

$$M \models p(a_0, \ldots, a_{n-1}) \iff p = (Sg)(M(A)).$$

In this case we also say that $(a_0, \ldots, a_{n-1})$ *realizes* p in M. If there is no tuple $(a_0, \ldots, a_{n-1})$ realizing p in M then we say M *omits* p.

If $A \subseteq S_n$, we write $M \models A(a_0, \ldots, a_{n-1})$ to mean $M \models p(a_0, \ldots, a_{n-1})$ for some $p \in A$.

IV.2. **Topological logics.** In abstract model theory one is interested in a wide variety of logics, such as logics with infinitely long formulas, or logics with non-classical quantifiers . Lindström [Lin69] was the first to give axioms unifying the various extended logics that had been studied, which provided a fruitful and very general setting for studying non-classical model theory (see [BF85] for an extensive survey of this area). In the same paper, Lindström proved his well-known result that first-order logic is maximal amongst compact logics satisfying the downward Löwenheim-Skolem theorem. Our type-space functors can be used to produce logics satisfying Lindström's definition, which we now state (following [Vää12]).

The structures under consideration in abstract model theory are the same as those in classical model theory. For our purposes it is harmless to assume that our signatures are relational. We recall the definition of (relational) signatures and structures from model theory:

Definition 4.5. A *signature* is a set of *relation symbols*, each with an associated *arity*. If τ is a signature, a τ*-structure* $\mathcal{M}$ is a non-empty set M, together with, for each n-ary relation symbol $R \in \tau$, a set $R^{\mathcal{M}} \subseteq M^n$. For each signature τ, the class of τ-structures is denoted by Str_τ.

Where abstract model theory differs from classical model theory is in allowing a very general definition of "sentence" and a similarly general notion of "satisfaction" between structures and sentences.

Definition 4.6. An *abstract logic* is a pair $L = (S, \models)$, where S is a set, and $\models$ is a relation between structures and elements of S, satisfying the following closure properties:

(Isomorphisms): If $\mathcal{M}, \mathcal{N}$ are structures and $\mathcal{M} \cong \mathcal{N}$, then for any $\varphi \in S$, $\mathcal{M} \models \varphi$ if and only if $\mathcal{N} \models \varphi$.

(Renaming): Suppose that τ and τ' are signatures, $\pi : \tau \to \tau'$ is a bijection that respects arity, and $\hat{\pi} : \mathrm{Str}_\tau \to \mathrm{Str}_{\tau'}$ is the natural extension of τ to the class of τ-structures. Then for any $\varphi \in S$ there is a $\varphi' \in S$ such that for every τ-structure $\mathcal{M}$, $\mathcal{M} \models \varphi$ if and only if $\hat{\pi}(\mathcal{M}) \models \varphi'$.

(Free expansions): Suppose that τ and τ' are signatures with $\tau \subseteq \tau'$, and $\varphi \in S$. Then there is a $\varphi' \in S$ such that for any τ'-structure $\mathcal{M}$, $\mathcal{M} \models \varphi'$ if and only if $\mathcal{M}|\tau \models \varphi$.

(Negation): For any $\varphi \in S$ there is a $\neg\varphi \in S$ such that for all $\mathcal{M}$, $\mathcal{M} \models \neg\varphi$ if and only if $\mathcal{M} \not\models \varphi$.

(Conjunction): For any $\varphi, \psi \in S$ there is a $\varphi \wedge \psi \in S$ such that for all $\mathcal{M}$, $\mathcal{M} \models \varphi \wedge \psi$ if and only if $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$.

(Existential quantification): Suppose that τ is a signature, and c is a constant symbol in τ. For every $\varphi \in S$ there is a $\varphi' \in S$ such that for any $\tau \setminus \{c\}$-structure $\mathcal{M}$, $\mathcal{M} \models \varphi'$ if and only if there is some $c^{\mathcal{M}} \in \mathcal{M}$ such that $(\mathcal{M}, c^{\mathcal{M}}) \models \varphi$.

In order to use type-space functors to build abstract logics, we must put together several type-space functors in a suitably compatible way.

Definition 4.7. A *topological logic* consists of, for each signature τ, a type-space functor $\mathcal{S}^\tau$, together with a function C_τ with domain τ, such that for each n-ary relation symbol $R \in \tau$, $C_\tau(R)$ is a closed subset of $\mathcal{S}^\tau_n$. For each $n < \omega$ and each signature τ, we define $B_{\tau,n}$ to be the collection of $C_\tau(R)$'s, where R is an n-ary relation symbol, together with the collection of preimages of such sets under the maps from $\mathcal{S}^\tau_n$ to $\mathcal{S}^\tau_m$ induced by the inclusion maps $i : m \to n$ when $m < n$. We also impose the following requirements:

- For each signature τ and each $n < \omega$, the space $\mathcal{S}^\tau_n$ is 0-dimensional.
- For each n, the collection $B_{\tau,n}$ is a base of closed sets for $\mathcal{S}^\tau_n$.
- If τ, τ' are signatures, and $\pi : \tau \to \tau'$ is a renaming, then π induces an isomorphism of the type-space functors $\mathcal{S}^\tau$ and $\mathcal{S}^{\tau'}$.

Our notion of topological logic is, in fact, the topological content of abstract logics.

Theorem 4.8. Each topological logic determines an abstract logic.

Proof. Let $\mathcal{S}$ be a topological logic. We define an abstract logic $L = (S, \models)$ as follows.

First, we define S by defining that the elements of S are exactly the sets that are closed in some $\mathcal{S}^\tau_n$.

Suppose that $\mathcal{M}$ is a τ-structure. Without loss of generality, we may assume that the universe of $\mathcal{M}$ is a cardinal κ. Define a function f with domain $[\kappa]^{<\omega}$ by setting $f(a_0, \ldots, a_{n-1})$ to be the unique $p \in \mathcal{S}^\tau_n$ such that for all $X \in B_{\tau,n}$, $p \in X$ if and only if $(a_0, \ldots, a_{n-1})$ is an element of $\mathcal{M}$'s interpretation of the relation symbol from which X was obtained.

We now define the satisfaction relation $\mathcal{M} \models \varphi(a_0, \ldots, a_{n-1})$, where $\varphi \in S$ and $a_0, \ldots, a_{n-1} \in \kappa$. First, if $\varphi \not\subseteq \mathcal{S}_n^\tau$, then we declare $\mathcal{M} \not\models \varphi(a_0, \ldots, a_{n-1})$. So suppose that φ is a closed subset of $\mathcal{S}_n^\tau$. Then we declare $\mathcal{M} \models \varphi(a_0, \ldots, a_{n-1})$ if and only if $f(a_0, \ldots, a_{n-1}) \in \varphi$.

Our set of sentences is closed under negation because each $\mathcal{S}_n^\tau$ is 0-dimensional. It is closed under conjunctions because the closed sets of any space are closed under finite intersections. Closure under existential quantification follows from the fact that for any signature τ, $\mathcal{S}_n^{\tau\cup\{c\}} \cong \mathcal{S}_{n+1}^\tau$, and the requirement that in any type-space functor the maps Sg are continuous and open. Closure under renaming is guaranteed by the third point in the definition of a topological logic. □

The converse of Theorem 4.8 is also true. Since we will not make use of this fact, we omit the proof. The proof is very similar to the method of producing a type-space functor from a first-order theory, described in Section V.1 below.

Theorem 4.9. Each abstract logic determines a topological logic.

We also note that the above results can be proved without the requirement that the spaces involved are all 0-dimensional, provided that the spaces are regular and that we work with a suitably adapted notion of "abstract logic" for metric structures.

The discussion above explains our earlier claim that a single type-space functor should be thought of as the topological representation of the model theory of an abstract logic in one signature. In light of this, and because the model-theoretic results we will be interested in are concerned only with single signatures, we will focus on individual type-space functors instead of topological logics.

V. Examples

V.1. **First-order type-space functors.** The basic example of a type-space functor arises from a first-order theory, and indeed the properties of type-space functors are designed to abstract the properties of this example.

Definition 5.1. Let T be a first-order theory. The *type-space functor of* T, $S(T)$, consists of the following data. For each $n < \omega$, let S_n be the set of all complete n-types of T, considered with the logic topology (that is, the topology generated by basic closed sets of the form $[\varphi] = \{p \in S_n : \varphi \in p\}$ for each n-ary formula φ). To each $f : n \to m$ associate the map $Sf : S_m \to S_n$ defined by $(Sf)(p) = \{\varphi(x_0, \ldots, x_{n-1}) : \varphi(x_{f(0)}, \ldots, x_{f(n-1)}) \in p\}$.

We say that a type-space functor is a *first-order type space functor* if it is the type-space functor of some first-order theory T.

The following proposition follows directly from the definitions, together with basic facts from first-order model theory.

Proposition 5.2. For every first-order theory T, the type-space functor of T is a type-space functor. The space S_ω is homeomorphic to the logic topology on the set of ω-types of T.

The model theory also agrees with classical model theory in this case. Suppose that T is a first-order theory, and S is the corresponding type-space functor. Suppose also that $\mathcal{M} \models T$ is enumerated as $\{m_\alpha : \alpha < \kappa\}$. Define M on $[\kappa]^{<\omega}$ by $M(\{i_0 < \ldots < i_{n-1}\}) = \mathrm{tp}^M(m_0, \ldots, m_{n-1})$. It is then routine to verify that M is a model (in the sense of Definition 4.3), and that for any $p \in S_n$ and any $i_0, \ldots, i_{n-1} \in \kappa$,

$$M \models p(i_0, \ldots, i_{n-1}) \iff \mathcal{M} \models p(m_{i_0}, \ldots, m_{i_{n-1}}).$$

Conversely, Knight [Kni07, Proposition 2.10] showed that every model of S arises in this way from a (classical) model of T.

V.2. **Type-space functors from other logics.** The process described above for first-order theories can be readily adapted to spaces of types arising from more general logics such as $L_{\omega_1,\omega}$, or indeed any abstract logic. In a partial converse to this process, Morley [Mor74] in effect showed that if a type-space functor has each S_n a 0-dimensional Polish space, and if a stronger amalgamation condition holds, then S arises from a theory in a countable fragment of $L_{\omega_1,\omega}$ in the manner described above, and moreover the theory obtained is essentially unique. Ben Yaacov [Ben05] showed that without the 0-dimensionality assumption it is still often possible to give a syntactic presentation of a type-space functor, but the associated logic is that of metric structures (see also [BYBHU08]). If one forms the type-space functor arising from a signature τ of continuous first-order logic for metric structures, then S_0 is precisely the quotient space of $\mathrm{Str}(\tau)$ by the elementary equivalence relation (which is the same as the topological indistinguishability relation).

V.3. **Type-space functors generated by a space.** Our second class of examples of type-space functors gives examples that do not come from theories in classical logics. Examples of this kind are the ones that we will use to produce counterexamples.

Definition 5.3. Let X be any topological space. The *type-space functor of* X, denoted S_X, consists of the following data. For each $n < \omega$, define $S_n = X^n$, and to each $f : n \to m$, associate the map $(Sf) : X^m \to X^n$ defined by $(Sf)(x_0, \ldots, x_{m-1}) = (x_{f(0)}, \ldots, x_{f(n-1)})$.

The following proposition follows directly from the definitions.

Proposition 5.4. For any topological space X, the type-space functor of X is a type-space functor. The space S_ω is homeomorphic to the (Tychonoff) product topology on X^ω.

VI. Omitting Types

Proofs of Omitting Types Theorems using the Baire Category Theorem have been given for a variety of logics; for some examples, see [Mor74], [Poi00], [CI14], [Eag14]. In this section we describe the relationship between Baire category properties and omitting types for type-space functors. Throughout this section, S denotes a type-space functor.

Starting from the type-space functor S we will be focusing on a certain subspace $S_{\mathcal{W}}$ of S_ω. The motivation for the following definition is that we are defining an analogue of the space of ω-types of the form $\mathrm{tp}(a_0, a_1, \ldots)$, where $(a_0, a_1, \ldots)$ enumerates a countable model of a theory. In fact, we will see in Lemma 6.4 that there is a correspondence between elements of the following space, and the models defined in Section V.3 above.

Definition 6.1. Suppose that $\sigma \in S_\omega$. For $A \in [\omega]^n$, let $f_A : n \to \omega$ be the map sending i to the ith element of A (in increasing order). Then we define

$$M_\sigma(A) = (Sf_A)(\sigma).$$

In the opposite direction, given a countable model M, for each $n < \omega$ define $\sigma_n = M(\{0, 1, \ldots, n-1\})$, and let σ_M be the equivalence class of $(\sigma_0, \sigma_1, \ldots)$ in S_ω.

Lemma 6.2. For each $\sigma \in S_\omega$, the map M_σ satisfies conditions (1) and (2) of Definition 4.3.

Proof. Condition (1) is clear from the definition. For (2), suppose that $A = \{a_0 < \ldots < a_{n-1}\} \in [\omega]^n$, $B = \{b_0 < \ldots < b_{m-1}\} \in [\omega]^m$, $A \subseteq B$, and $g : n \to m$ satisfies $a_i = b_{g(i)}$ for all i. Then for each i,

$$f_B \circ g(i) = f_B(g(i)) = b_{g(i)} = a_i = f_A(i).$$

Therefore

$$(Sg)(M_\sigma(B)) = (Sg)((Sf_B)(\sigma)) = (S(f_B \circ g))(\sigma) = (Sf_A)(\sigma) = M_\sigma(A).$$

□

In general we cannot expect M_σ to be a model (that is, to satisfy condition (3) of Definition 4.3), just as we cannot expect an arbitrary ω-type of a first-order theory to specify a witness to every existential formula it implies. We define $S_{\mathcal{W}}$ to be the set of those $\sigma \in S_\omega$ for which M_σ is a model. Formally:

Definition 6.3. For $\sigma \in S_\omega$, we put $\sigma \in S_{\mathcal{W}}$ if and only if for every $n < \omega$, every $A \in [\omega]^n$, every $m \geq n$, and every open $U \subseteq S_{\iota_{n,m}}^{-1}(\{M_\sigma(A)\})$, there is a $B \in [\omega]^m$ and a permutation g of m such that $B \supseteq A$, and $(Sg)(M_\sigma)(B)) \in U$.

Note that in this definition the set U could equivalently be required to come from a fixed base for the topology of S_ω.

Lemma 6.4. The map $\sigma \mapsto M_\sigma$ is a one-to-one correspondence between $S_{\mathcal{W}}$ and the set of countable models of S, with inverse $M \mapsto \sigma_M$.

Proof. Given $\sigma \in S_{\mathcal{W}}$, it is clear that M_σ satisfies condition (1) of Definition 4.3. For condition (2), suppose that $A = \{a_0 < \ldots < a_{n-1}\} \in [\omega]^n$, $B = \{b_0 < \ldots < b_{m-1}\} \in [\omega]^m$, $A \subseteq B$, and $g : n \to m$ satisfies $a_i = b_{g(i)}$ for all i. Then for each i,

$$f_B \circ g(i) = f_B(g(i)) = b_{g(i)} = a_i = f_A(i).$$

Therefore

$$(Sg)(M_\sigma(B)) = (Sg)((Sf_B)(\sigma)) = (S(f_B \circ g))(\sigma) = (Sf_A)(\sigma) = M_\sigma(A).$$

The definition of $S_{\mathcal{W}}$ exactly ensures that condition (3) is satisfied, so M_σ is a model. It is straightforward to check that for any $\sigma \in S_{\mathcal{W}}$ we have $\sigma = \sigma_{M_\sigma}$, and for any model M we have $M = M_{\sigma_M}$. $\square$

In light of Lemma 6.4, we will sometimes identify a countable model M with the sequence σ_M.

We define several omitting types properties that S may have. Another omitting types property, involving topological games, will appear in Section VI.1.

Definition 6.5. (1) S has the *classical omitting types property* if for every non-empty closed $T \subseteq S_0$, and every sequence $(E_j)_{j<\omega}$ such that E_j is meagre in $(S\iota_{0,j})^{-1}(T)$, there exists a model $M \models T$ such that M omits every E_j.

(2) S has the *strong omitting types property* if for every non-empty closed $C \subseteq S_{\mathcal{W}}$, and every meagre $E \subseteq C$, there is a model in C omitting E.

Proposition 6.6. The strong omitting types property implies the classical omitting types property.

Proof. Fix a non-empty closed $T \subseteq S_0$. To simplify notation, for $\alpha \leq \omega$, let $A_\alpha = (S\iota_{0,\alpha})^{-1}(T)$. For each $j < \omega$ let $E_j \subseteq A_j$ be meagre. For each $j < \omega$, and each $\mathbf{i} \in \omega^j$, let $f_{j,\mathbf{i}} : j \to \omega$ be defined by $f_{j,\mathbf{i}}(k) = \mathbf{i}_k$, where $\mathbf{i}_k$ is the kth element of $\mathbf{i}$ in increasing order. Next, for each j and $\mathbf{i}$, define

$$C_{j,\mathbf{i}} = (Sf_{j,\mathbf{i}})^{-1}(E_j).$$

Then each $\mathcal{C}_{j,\mathbf{i}}$ is meagre in $S_{\mathcal{W}} \cap A_\omega$ because $Sf_{j,\mathbf{i}}$ is continuous, open, and surjective. Finally, define

$$F = \bigcup_{j<\omega} \bigcup_{\mathbf{i}\in\omega^j} C_{j,\mathbf{i}}.$$

Then F is meagre in $S_{\mathcal{W}} \cap A_\omega$. By the strong omitting types property we can find a model M such that M (or, more precisely, σ_M) is in $(S_{\mathcal{W}} \cap A_\omega) \setminus F$. For such an M we have $(S\iota_{0,\omega})(M) \in T$, so $M \models T$.

To see that M omits each E_j, suppose that $A \in \omega^j$. Write $A = \{a_0, \ldots, a_{j-1}\} = \{c_0 < \ldots < c_{k-1}\}$, and let $g : j \to k$ be such that $c_{g(i)} = a_i$ for each $i < j$. According to Definition 4.4, to show that M omits E_j we must show that in this situation $(Sg)(M(A)) \notin E_j$. Unwinding Definition 6.1, we obtain

$$(Sg)(M(A)) = (Sg)(M_{\sigma_M}(A)) = (Sg)(Sf_A)(\sigma_M) = S(f_A \circ g)(\sigma_M).$$

In the above calculation $f_A : k \to \omega$ sends i to c_i, so we have $f_A \circ g(i) = c_{g(i)} = a_i$. Letting $\mathbf{i} = (a_0, a_1, \ldots, a_{j-1})$ we therefore have $f_A \circ g = f_{j,\mathbf{i}}$. Combining the above calculations, and using that we chose M so that $(Sf_{j,\mathbf{i}})(\sigma_M) \notin E_j$, we get

$$(Sg)(M(A)) = (Sf_{j,\mathbf{i}})(\sigma_m) \notin E_j.$$

□

Our omitting types properties conclude that certain countable models exist, but there are type-space functors with no countable models at all. In order to conclude omitting types properties from topological facts about the type-space functor S we must also assume that the collection of countable models for S is sufficiently rich. For type-space functors coming from countable theories this richness is provided by the downward Löwenheim-Skolem theorem. In general, we make the following definition.

Definition 6.7. Let S be a type-space functor. We say that S has *enough countable models* if $S_{\mathcal{W}}$ is dense in S_ω.

Lemma 6.8. *Let S be a type-space functor. If S is the functor associated to a countable first-order theory, or if S is generated by a separable topological space, then S has enough countable models.*

Proof. Suppose first that T is a countable first-order theory. Then a basic open set $O \subseteq S_\omega$ is the set of all ω-types of T containing some particular formula φ. If $O \neq \emptyset$ then there is a model $\mathcal{M} \models T$ containing a tuple $\vec{a}$ satisfying φ, and by Löwenheim-Skolem we may assume $\mathcal{M}$ is countable. If σ is the type of an enumeration of $\mathcal{M}$ in the appropriate order (so that the elements of $\vec{a}$ have the same indices as the variables appearing in φ), then $\sigma \in O \cap S_{\mathcal{W}}$.

Now suppose that X is a space and $S = S_X$. Let $O \subseteq X^\omega$ be a basic open set. Let $D \subseteq X$ be a countable dense set, and let $\sigma \in O$ be such that every element of D is listed in σ infinitely many times. We show that $\sigma \in \mathcal{W}$. We are given $n < \omega$, $m \geq n$, $A = \{a_0 < \ldots < a_{n-1}\} \in [\omega]^n$, and a non-empty basic open set $U \subseteq (S^{-1}_{\iota_{n,m}})(M_\sigma(A))$. Unraveling the definitions, this means that there are open sets $V_n, V_{n+1}, \ldots, V_{m-1} \subseteq X$ such that elements in U are exactly those sequences of the form $(x_0, \ldots, x_{m-1})$ where $x_i = \sigma(a_i)$ for $i < n$ and $x_i \in V_i$ for $n \leq i < m$. Choose $B = \{b_0 < \ldots < b_{m-1}\}$ such that $b_i = a_i$ for $i < n$, and such that $\sigma(b_i) \in V_i$ for $n \leq i < m$ (this is possible by our choice of σ). Then $M_\sigma(B) = (\sigma(b_0), \ldots, \sigma(b_{m-1})) \in U$ (so also $(Sg)(M_\sigma(B)) \in U$ where $g : m \to m$ is the identity function). Therefore $\sigma \in S_{\mathcal{W}}$, and hence $S_{\mathcal{W}} \cap O \neq \emptyset$. □

The topological content of the omitting types theorem for first-order logic is captured by the following:

Theorem 6.9. Let S be a type-space functor with enough countable models. If every closed subspace of $S_{\mathcal{W}}$ is non-meagre in itself then S has the classical omitting types property.

Proof. The proof is nearly identical to the proof of Proposition 6.6. As in that proof, we fix $T \subseteq S_0$ closed, and for $\alpha \leq \omega$ let $A_\alpha = (S_{\iota_{0,\alpha}})^{-1}(T)$. For each $j < \omega$, let $E_j \subseteq A_j$ be meagre. For each $j < \omega$ and $\mathbf{i} \in \omega^j$, define $f_{j,\mathbf{i}} : j \to \omega$ by $f_{j,\mathbf{i}}(k) = \mathbf{i}_k$, and define $C_{j,\mathbf{i}} = (Sf_{j,\mathbf{i}})^{-1}(E_j)$; then each $C_{j,\mathbf{i}}$ is meagre in $S_{\mathcal{W}} \cap A_\omega$ (here we use that S has enough countable models, which was not necessary in Proposition 6.6). Define

$$F = \bigcup_{j<\omega} \bigcup_{\mathbf{i}\in\omega^j} C_{j,\mathbf{i}}.$$

Then F is meagre in $S_{\mathcal{W}} \cap A_\omega$, and since $S_{\mathcal{W}} \cap A_\omega$ is non-meagre in itself by hypothesis, we can find $M \in (S_{\mathcal{W}} \cap A_\omega) \setminus F$. This M satisfies T and omits each E_j. □

To characterize the strong omitting types property topologically we will need some terminology. A topological space is *completely Baire* if every closed subspace is Baire, and is *completely non-meagre* if every closed subspace is non-meagre in itself.

Hurewicz [Hur28] proved that a metrizable space is completely Baire if and only if the space does not include a closed copy of the space $\mathbb{Q}$ of rational numbers. Since $\mathbb{Q}$ is meagre in itself, it follows immediately that a metrizable space is completely Baire if and only if it is completely non-meagre. For this latter claim much weaker assumptions than metrizability are sufficient. The one we will use is the following.

Definition 6.10. A topological space is *quasi-regular* if each open set includes the closure of an open set. A space is *completely quasi-regular* if each closed subspace is quasi-regular.

Quasi-regularity is commonly required to prove results about Baire spaces (see e.g. [Oxt57]).

Lemma 6.11. A completely quasi-regular space is completely Baire if and only if it is completely nonmeagre.

Proof. That completely Baire implies completely nonmeagre is immediate. For the other direction, let F be a closed subspace of a completely nonmeagre, completely quasi-regular space X. Let $\{U_n\}_{n<\omega}$ be a collection of dense open subspaces of F. If $\bigcap_{n<\omega} U_n$ were not dense in F, then there would be a $V \subseteq F$, V open in F, such that $V \cap \bigcap_{n<\omega} U_n = \emptyset$. Let W be open in V with $\overline{W} \subseteq V$. Then $\overline{W} \cap \bigcap_{n<\omega} U_n \neq \emptyset$, because $\overline{W}$ is nonmeagre. This contradicts $V \cap \bigcap_{n<\omega} U_n = \emptyset$. □

We note that regularity of type spaces can serve as a kind of weak negation. For example, in continuous first-order logic for metric structures one does not have a classical negation, but the connective $1 - x$ acts as an approximate negation, and closure under that connective is also the essential ingredient in the proof that the type spaces in continuous logic are regular. See [Cai95] for more about the role of topological separation axioms in abstract model theory. In our context we are assuming even less than regularity, though it is not clear exactly how to translate quasi-regularity into logical terms, owing to the difficulty of computing closures in the type spaces of traditional logics.

Theorem 6.12. Let S be a type-space functor with enough countable models, and such that $S_{\mathcal{W}}$ is quasi-regular. Then the following are equivalent:

(1) S has the strong omitting types property.
(2) $S_{\mathcal{W}}$ is completely non-meagre.
(3) $S_{\mathcal{W}}$ is completely Baire.

Proof. (1) → (2): Suppose that $C' \subseteq S_{\mathcal{W}}$ is meagre in itself, and let C be a closed subset of S_ω such that $C' = S_{\mathcal{W}} \cap C$. Let E_n be nowhere dense in C', such that $C' = \bigcup_{n<\omega} E_n$. Then each E_n remains nowhere dense in S_ω, so C' is meagre in S_ω. The closed set C and the meagre set C' contradict the statement of the strong omitting types property, because any model in C is in $C \cap S_{\mathcal{W}}$, and therefore does not omit C'.

(2) → (3): Apply Lemma 6.11.

(3) $\to$ (1): Suppose that every closed subspace of S is Baire, let $C \subseteq S_\omega$ be closed, and let $E \subseteq C$ be meagre. Let $C' = C \cap S_{\mathcal{W}}$, so C' is closed in $S_{\mathcal{W}}$. Let $E' = E \cap S_{\mathcal{W}}$. By the assumption that S has enough countable models, $S_{\mathcal{W}}$ is dense in S_ω, so E' is meagre in C'. Since $S_{\mathcal{W}}$ is completely Baire, C' is Baire, and hence $C' \setminus E' \neq \emptyset$. Any element of $C' \setminus E'$ corresponds to a model of C omitting E' (just as in the proof of Proposition 6.6). □

It is usually easier to understand the topology of S_ω than the topology of $S_{\mathcal{W}}$. In concrete situations it is therefore useful to have information about how $S_{\mathcal{W}}$ sits as a subspace of S_ω. Recall that the *weight* of a topological space X is the minimum cardinality of a base for the topology of X. The following lemma is very useful, and is also immediate from the definition of $S_{\mathcal{W}}$.

Lemma 6.13. Let S be a type-space functor, and for each $n < \omega$ let $w(S_n)$ be the weight of S_n. Then $S_{\mathcal{W}}$ is the intersection of $\sum_n w(S_n)$-many open subsets of S_ω.

Example 6.14. Many *Omitting Types Theorems* in the literature can be easily derived from Theorem 6.9, after translating our topological statement into model-theoretic terminology. We present here just a few examples. The spaces in the following examples are *Čech-complete*; a space X is *Čech-complete* if it is a G_δ in some (equivalently, every) compactification of X. Completely metrizable spaces are Čech-complete, as are locally compact Hausdorff spaces, and every Čech-complete space is Baire.

(1) Let T be a first-order theory in a countable language, and let S be the associated type-space functor (as described in Section V.1). In this context each S_n is a second countable space, so $S_{\mathcal{W}}$ is a dense G_δ in S_ω by Lemmas 6.13 and 6.8. By the compactness theorem, S_ω is a compact space, and is therefore Čech-complete. Čech-completeness is inherited by dense G_δ subspaces, and by closed subspaces, so it follows that every closed subspace of $S_{\mathcal{W}}$ is Čech-complete, and hence Baire. The Omitting Types Theorem for first-order logic then follows from Theorem 6.9, together with the observation that a type $p \in S_n(T)$ is principal if and only if p is an isolated point of $S_n(T)$ (see [Mar02, Section 4.2]).

(2) The above discussion also works more generally, if T is a theory in a countable fragment of $L_{\omega_1,\omega}$. In this case S_ω is not compact, but it is Polish, and so is still Čech-complete. We obtain the Omitting Types Theorem for countable fragments of $L_{\omega_1,\omega}$, originally due to Keisler [Kei71]. This proof of omitting types for countable fragments of $L_{\omega_1,\omega}$ is fundamentally the same as the one given by Morley [Mor74].

(3) Similarly, if T is a theory in a countable fragment of the logic $L_{\omega_1,\omega}$ for metric structures, then S_ω is Čech-complete. Translating Theorem 6.9 into model-theoretic terminology gives the Omitting Types Theorem for (not necessarily complete) metric structures from [Eag14].

Not every omitting types theorem from the literature is a direct consequence of the topological version presented here. Notably, the omitting types theorem for continuous logic [BYBHU08], which requires that the models omitting the given types be based on *complete* metric spaces, does not directly follow from our results; see [FM18] for a discussion of the subtleties that arise in omitting types in complete metric structures. We note also that our topological approach to obtaining omitting types theorems bears some resemblance to Keisler's [Kei73], which develops both omitting types and set-theoretic forcing as a result of a more general notion of forcing that is closely related to Baire category.

VI.1. **A game version of omitting types.** The Banach-Mazur game on a topological space X is a game played between two players, called EMPTY and NONEMPTY, as follows. The players alternate choosing open sets $O_0 \supseteq O_1 \supseteq \cdots$, with EMPTY choosing first. The player NONEMPTY wins if $\bigcap_{n<\omega} O_n \neq \emptyset$, otherwise EMPTY wins. The connection between the Banach-Mazur game and Baire spaces is the following well-known result.

Theorem 6.15 (see e.g. [Oxt57])**.** A topological space X is a Baire space if and only if EMPTY does not have a winning strategy in the Banach-Mazur game.

There are examples of spaces X for which the Banach-Mazur game is not determined [Oxt57], so asserting that NONEMPTY has a winning strategy is strictly stronger than asserting that EMPTY does not have one. This stronger property was introduced by Choquet [Cho69] who called it **weakly α-favourable**. Weak α-favourability was further investigated by H. E. White [Whi75], who, among other results, proved it was preserved by topological products — even box products, unlike the usual Baire Category Theorem [Fle78].

In light of Theorems 6.9 and 6.12 it is natural to ask how the Omitting Types Theorem is strengthened by using weakly α-favourable spaces instead of Baire spaces. By analogy to the case of first-order logic, we will refer to a closed subset of S_ω as a *partial ω-type*. It is then convenient to state the Banach-Mazur game in dual form.

Definition 6.16. Let S be a type-space functor, and let $C \subseteq S_\omega$ be a partial ω-type. The *omitting types game* on C is played by two players, OMIT and REALIZE, as follows. The players alternate picking partial ω-types

$F_0 \subseteq F_1 \subseteq \cdots$, with REALIZE playing first, and with each F_i omissible in a model realizing C. The player OMIT wins if $\bigcup_{n<\omega} F_n$ is omissible in a model realizing C, otherwise REALIZE wins.

We say that S has the *game omitting types property* if OMIT has a winning strategy in the omitting types game on C, for every C.

We call a space X *completely weakly α-favourable* if every closed subspace of X is weakly α-favourable. The definition of the omitting types game immediately gives the following statement, analogous to Theorem 6.12.

Theorem 6.17. Let S be a type-space functor with enough countable models. The following are equivalent:

(1) S has the game omitting types property.
(2) $S_{\mathcal{W}}$ is completely weakly α-favourable.

We immediately obtain the following game version of the omitting types theorem for countable fragments of $L_{\omega_1,\omega}$, which to the best of our knowledge has not been explicitly stated elsewhere.

Theorem 6.18. Let T be a theory in a countable fragment of $L_{\omega_1,\omega}$. Two players OMIT and REALIZE play the following game: REALIZE plays first, and the players alternate picking a sequence of partial ω-types $\Sigma_0 \supseteq \Sigma_1 \supseteq \ldots$ (the inclusions being as sets of formulas), such that each Σ_i is omissible in a model of T. Player OMIT has a strategy to ensure that $\bigcap_{n<\omega} \Sigma_i$ is omissible in a model of T.

Proof. In the type-space functor S of T the space $S_{\mathcal{W}}$ is Polish (see [Mor74]), and therefore completely weakly α-favourable. It follows that S has the game omitting types property. The statement of the game omitting types property, together with the definition of the logic topology, give the desired conclusion. □

In many cases of interest it is possible to deduce the game omitting types property from the topology of S_ω, rather than $S_{\mathcal{W}}$.

Theorem 6.19. Let S be a type-space functor with enough countable models, such that each S_n is separable and metrizable. If S_ω is completely weakly α-favourable then S satisfies the game omitting types property.

Proof. In this context S_ω is, by definition, a subspace of a product of separable metrizable spaces, and hence is itself separable and metrizable. Moreover, $S_{\mathcal{W}}$ is a dense G_δ in S_ω by Lemma 6.13 and the definition of "enough countable models". By Theorem 6.17 it suffices to prove the purely topological claim that if X is a separable metrizable completely weakly α-favourable space and Y is a dense G_δ in X, then Y is completely weakly α-favourable.

Let Z be a closed subspace of Y, and let $\overline{Z}$ be the closure of Z in X. Since Y is metrizable and Z is closed in Y, Z is a G_δ in Y. Y itself is a G_δ in X, so Z is a G_δ in X, and hence Z is a G_δ in $\overline{Z}$. On the other hand, $\overline{Z}$ is weakly α-favourable by hypothesis, and of course Z is dense in $\overline{Z}$. White [Whi75] proved that dense G_δ subspaces of weakly α-favourable regular spaces are weakly α-favourable, so Z is weakly α-favourable as required. □

VII. Distinguishing the omitting types properties

Our original motivation for this paper was to determine whether or not the Omitting Types Theorem is equivalent to the Baire Category Theorem. We are now prepared to address this question. It suffices to find a space X satisfying the Baire Category Theorem while the type-space functor it generates does not satisfy OTT. A Baire X such that X^ω is not Baire, and hence has no dense G_δ Baire subspaces will suffice, e.g. the Baire X with X^2 not Baire of Fleissner and Kunen [Fle78] will do the trick. A more nuanced example is due to Aarts and Lutzer [AL73]. They construct a completely Baire separable metric space with a dense completely metrizable subspace such that X^2 is not completely Baire. X is actually weakly α-favorable, so X^ω is as well, so X^ω is Baire, but not completely Baire.

We end by noting that the game version of omitting types is genuinely stronger than the strong version.

Lemma 7.1. Suppose that X is a separable metrizable space X such that X^ω is completely Baire, but X^ω does not include a dense completely metrizable subspace. Then the type-space functor $S(X)$ has the strong omitting types property but does not have the game omitting types property.

Proof. By Proposition 5.4 $S(X)_\omega = X^\omega$, and by Lemmas 6.8 and 6.13 $S(X)_\mathcal{W}$ is a dense G_δ in X^ω. Medini and Zdomskyy [MZ15] proved that every dense G_δ subspace of a completely Baire space is completely Baire, so our assumption that X^ω is completely Baire implies that $S(X)_\mathcal{W}$ is completely Baire, and hence by Theorem 6.12 $S(X)$ has the strong omitting types property.

Since X is a separable metrizable space so is X^ω, and hence also $S(X)_\mathcal{W}$. Telgársky [Tel87] proved that a separable metrizable space is weakly α-favourable if and only if it has a dense completely metrizable subspace. Therefore if $S(X)$ had the game omitting types property, then $S(X)_\mathcal{W}$ would have a completely metrizable dense subspace, and hence X^ω would also have such a subspace, contrary to our hypothesis. So $S(X)$ does not have the game omitting types property. □

A space satisfying the hypotheses of Lemma 7.1, and hence giving rise to a type-space functor which satisfies the strong omitting types property but not the game version, was constructed in [TZ19], in response to an earlier version of this manuscript which had shown that existence of such an example is consistent with ZFC.

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