STABLE PATTERNS IN SPATIALLY DISCRETE REACTION-DIFFUSION MODELS

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1 Introduction.

In modelling several natural phenomena, reaction-diffusion equations arise that have steady states of the form

$$\varepsilon^2 \Delta v - f(v) = 0\,, \tag{1}$$

where f is an odd function with $vf''(v) > 0$ for $v \neq 0$, $f'(0) < 0$ and $f(\pm v_s) = 0$ for some $v_s > 0$. For example,

$$v_t = \varepsilon^2 \Delta v - f(v)\,. \tag{2}$$

with reaction term of the form described above is variously known as an Allen-Cahn equation or a Ginzburg-Landau equation and models the kinetics of phase transitions. A typical reaction term is

$$f(v) = \lambda(v^3 - v)\,. \tag{3}$$

Such equations are known to form sharp transition layers between positive and negative regions quite rapidly, but these are metastable and eventually collapse, approaching an equilibrium that is constant in space [2,3,4,14,15].

In continuous space approximations of neural network models [5,6,8,9], similar equations arise of the form

$$v_t = \frac{\gamma}{G'(v)} \left[\varepsilon^2 \frac{\tau}{2} \Delta v + \tau_0 v - \frac{G(v)}{\gamma} \right]\,, \tag{4}$$

where $\tau, \tau_0, \varepsilon$ and γ are positive constants. Here, $G(v)$ is the inverse of a sigmoidal function, g. This g has range $(-1, 1)$ and describes the dependence of the firing rate of a neuron on its membrane potential. It is an increasing function and therefore so is G. Without loss of generality, we may take $G'(0) = 1$. These properties are summarized as

$$
\begin{gathered}
G(v) = -G(-v), \quad v \in (-1,1) \\
|G(v)| \to \infty \text{ as } |v| \to 1 \\
G'(v) > 0, \quad G'(0) = 1\,.
\end{gathered}
\tag{5}
$$

In the theory of [9], equation (4) is shown to be a good approximation to a class of neural networks for ε small and γ large. It also has equilibria of the form (1), taking

$$
f(v) = \frac{2}{\tau}\left(\frac{G(v)}{\gamma} - \tau_0 v\right)\,. \tag{6}
$$

If these equations are discretized in space via a central difference scheme with ε as the step size, systems of ordinary differential equations are obtained of the form:

$$
\dot{v}_i = \sum_{j\sim i} v_j - 2dv_i - f(v_i) \tag{7}
$$

for (2) or

$$
\dot{v}_i = \frac{\gamma}{G'(v_i)}\left[\frac{\tau}{2}\left(\sum_{j\sim i} v_j - 2dv_i\right) + \tau_0 v_i - \frac{G(v_i)}{\gamma}\right] \tag{8}
$$

for (4), where $\sum_{j\sim i}$ means the sum over nearest neighbours of i, and d is the number of spatial dimensions. In either case the equilibrium equations corresponding to (1) are

$$
\sum_{j\sim i} v_j - 2dv_i - f(v_i) = 0\,, \tag{9}
$$

or more simply

$$
\sum_{j\sim i} v_j = q(v_i)\,, \tag{10}
$$

with

$$q(v) = f(v) + 2dv\,. \tag{11}$$

Systems of ODE's of this form arise independently in other contexts. For example, the Ising model for spins of ferromagnetic paricles, lattice gasses, binary alloys, etc. with Glauber dynamics [10] and a mean field approximation, takes the form [7]

$$\dot{v}_i = -v_i + \tanh\left[K\sum_{j\sim i} v_j\right]\,, \tag{12}$$

where K is a parameter (inversely) related to temperature. Equilibria of this system can be written in the form

$$\sum_{j\sim i} v_j - \frac{1}{K}\tanh^{-1}(v_i) = 0\,, \tag{13}$$

which is again like (10) with

$$q(v_i) = \frac{1}{K}\tanh^{-1}(v_i)\,. \tag{14}$$

Discrete versions of the Allen-Cahn equation itself are used as models of binary alloys, for example [1]. In both of the above cases, the phenomena modelled are spatially discrete by nature.

It is known that reaction-diffusion equations of the form (2) in one spatial dimension and with natural boundary conditions (Neumann or periodic) have no stable steady states other than those which are constant in space (i.e. no stable patterns) [3,11,13]. This also holds for (4). In fact, both forms have the same energy functional (see below) as well as the same equilibria so their dynamics are essentially the same.

One might expect that the finite-difference approximations would demonstrate similar dynamics, especially when ε is small, or at least not be so dissimilar as to allow stable patterns. This expectation turns out not to be justified. This is somewhat surprising since it implies that high energy states can be stable. Intuitively, it results from the step size in the discretization being of the same order as the width of transition layers. Finer discretizations would be expected to follow the behaviour of the reaction-diffusion equations more closely. But the spatially discrete systems are of interest in their own right.

In this paper, we show how elementary methods may be used to demonstrate the existence of stable patterns for systems of ODE's of the forms (7), (8) or (12) above, for some functions $f(v)$. These stable patterns can exist on a whole range of scales (from the microscopic to the macroscopic) but are more likely to be stable at larger scales. We use periodic boundary conditions throughout, though we expect that the results will also hold for Neumann boundary conditions. We carry out the calculations in one spatial dimension for the most part, but indicate at the end how they can be extended to more than one dimension.

In one spatial dimension equations (2) and (4) become

$$v_t = \varepsilon^2 v_{xx} - f(v)\,. \tag{15}$$

and

$$v_t = \frac{\gamma}{G'(v)}\left[\varepsilon^2 \frac{\tau}{2} v_{xx} + \tau_0 v - \frac{G(v)}{\gamma}\right] \tag{16}$$

respectively. The corresponding systems of ODE's, (7) and (8), become

$$\dot{v}_m = v_{m+1} - 2v_m + v_{m-1} - f(v_m) \tag{17}$$

and

$$\begin{aligned} \dot{v}_m &= \frac{\gamma}{G'(v_m)} \left[\frac{\tau}{2} (v_{m+1} - 2v_m + v_{m-1}) + \tau_0 v_m - \frac{G(v_m)}{\gamma} \right] \\ &= \frac{\gamma}{G'(v_m)} \left[\frac{\tau}{2} (v_{m+1} + v_{m-1}) + (\tau_0 - \tau) v_m - \frac{G(v_m)}{\gamma} \right], \end{aligned} \tag{18}$$

respectively, and (12) becomes

$$\dot{v}_m = -v_m + \tanh \left[K \left(v_{m-1} + v_{m+1} \right) \right] . \tag{19}$$

The equilibrium equations for these systems of ODE's all become

$$v_{m+1} + v_{m-1} = q(v_m) = f(v_m) + 2v_m, \tag{20}$$

with appropriate choice of f or q, from (3), (6), (14).

2 Lyapunov functional.

We first note that equations (15) and (16) have the same Lyapunov functional, namely,

$$E[v] = \int \left(\frac{\varepsilon^2}{2} v_x^2 + F(v) \right) dx \,, \tag{21}$$

where

$$F'(v) = f(v)$$

or

$$F(v) = Q(v) - v^2$$

so that

$$Q'(v) = q(v) .$$

For (16),

$$Q'(v) = q(v) = \frac{2}{\tau}\left[\frac{G(v)}{\gamma} - (\tau_0 - \tau)v\right] , \tag{22}$$

so that (18) is

$$\dot{v}_m = \frac{\gamma}{G'(v_m)}\frac{\tau}{2}\left[v_{m+1} + v_{m-1} - q(v_m)\right] .$$

These are not difficult to prove, but we concentrate on the discrete space equations.

The spatial discretization, with $\mathbf{v} = (v_m)$, has the analogous Lyapunov functional

$$E[\mathbf{v}] = \sum_j Q(v_j) - \frac{1}{2}\sum_j v_j(v_{j+1} + v_{j-1}) . \tag{23}$$

Lemma 1 *$E[\mathbf{v}]$ in* (23) *is a Lyapunov functional for equation* (18) *with periodic boundary conditions.*

Proof To show that (23) is a Lyapunov functional, it is necessary to show that equilibria are critical points of the energy surface and that energy decreases with time. Note that equation (20) is the equilibrium equation for (18) with q given by (22). First,

$$\frac{d}{dv_j}E[\mathbf{v}] = q(v_j) - (v_{j-1} + v_{j+1}) ,$$

which is clearly zero at equilibria. Second,

$$\begin{aligned}
\dot{E}[\mathbf{v}] &= \sum_j q(v_j)\dot{v}_j - \frac{1}{2}\sum_j \left[v_j\dot{v}_{j+1} + v_{j+1}\dot{v}_j + v_j\dot{v}_{j-1} + v_{j-1}\dot{v}_j\right] \\
&= \sum_j q(v_j)\dot{v}_j - \frac{1}{2}\sum_j \left[(v_{j+1} + v_{j-1})\,\dot{v}_j\right] - \frac{1}{2}\sum_k v_{k+1}\dot{v}_k - \frac{1}{2}\sum_\ell v_{\ell-1}\dot{v}_\ell \\
&= \sum_j \left[q(v_j) - (v_{j+1} + v_{j-1})\right]\dot{v}_j \\
&= -\sum_j \frac{2G'(v_j)}{\gamma\tau}\dot{v}_j^2 \leq 0\,,
\end{aligned}$$

since τ , γ, and $G'(v_j) \geq 0$, and where we have used (22). □

$E[\mathbf{v}]$ is also an energy functional for (17) and (19), and the proofs are similar.

Thus, equilibria for the three equations are the same as are their stability properties. We can therefore focus our attention on equations (20) and (23) to obtain results for the discretizations of the neural network equation (16) or the Ginzburg-Landau (or Allen-Cahn) equation (15) or for the Ising model equations (19). We remark here that in the Ginzburg-Landau equation with f defined by (3), $|f(v)| \to \infty$ as $|v| \to \infty$. In the neural network theory we have f given by (6) which has the property that $|f(v)| \to \infty$ as $|v| \to 1$. However, as long as initially $v \in [-v_s, v_s]$, where v_s is the positive solution to $f(v) = 0$, the solutions remain in this interval, so the behaviour of f for $|v| > v_s$ is irrelevant. Thus, this $f(v)$ is also of the type studied by Carr and Pego, for example [2,3].

In the remainder of this paper we will use two examples of the general q. The form given by (22), which comes from the neural field model, we will refer to as q_1 and the corresponding f from (11) as f_1. Note that the Ising model equations (19) also have q in this form with $\tau = \tau_0 = 2$, $\gamma = K$ and $G = \tanh$. For comparison we also use the form given by (3) and (11), which is typical for the Ginzburg-Landau equation (15), and we will refer to this f as f_2, so that

$$q_2(v) = (2-\lambda)v + \lambda v^3\,. \tag{24}$$

It is sometimes convenient to work with $f(v)$ and sometimes with $q(v)$ and we shall use both as appropriate.

3 Stability of flat equilibria.

We wish to establish that there exist stable patterns for these dynamical systems; i.e. that there are stable equilibria that are not constant in space. We examine equations that have equilibria given by (20) and energy functional given by (23). In these equations we will allow $f(v)$ to be any smooth function satisfying the conditions

$$\begin{aligned} &f(-v) = -f(v),\ \text{i.e. } f \text{ odd, so } f(0)=0\,,\\ &vf''(v) > 0 \text{ for } v \neq 0\,. \end{aligned} \tag{25}$$

From (11) it is clear that the same conditions must also apply to $q(v)$. These conditions hold for $f(v)$ given by (6) and $q(v)$ by (22) if G satisfies conditions (5). They hold for the example in (3) or (24) if $\lambda > 0$. In addition, after this section we will require that

$$\exists v_s > 0 \text{ such that } f(\pm v_s) = 0\,. \tag{26}$$

Then, $f'(0) < 0$ and f is of the form considered by Carr and Pego [2,3]. For convenience, we continue to assume periodic boundary conditions.

First, we examine the stability of the trivial equilibrium, $v \equiv 0$.

Proposition 3 *The trivial solution to* (20), $v \equiv 0$, *is asymptotically stable for* $q'(0) \geq 2$ *and unstable for* $q'(0) < 2$.

Proof The result is easy to see if we rewrite the energy functional (23) in the form

$$E[\mathbf{v}] = \sum_j Q(v_j) - \sum_j v_j^2 + \frac{1}{4} \sum_j \left[(v_{j+1} - v_j)^2 + (v_{j-1} - v_j)^2 \right] \tag{27}$$

which can be shown to be equivalent under periodic boundary conditions. Now it is clear that the last term, which gives a contribution from interactions, can only increase as any constant equilibrium is perturbed and the first two terms contain no interactions and so may be treated separately for each v_j. Letting $F(v) = Q(v) - v^2$, we have that $F'(0) = 0$ by (25) and $F''(0) = q'(0) - 2$. Thus, if $q'(0) > 2$, then $F''(0) > 0$ and $F(v) > 0$ for v near 0, so the 0 equilibrium is asymptotically stable. If $q'(0) = 2$, then $F''(0) = 0$ and $F'''(0) = q''(0) = 0$, since q is an odd function, but $F^{(4)}(0) = q'''(0) > 0$, since $q''(v) < 0$ for $v < 0$ and $q''(v) > 0$ for $v > 0$. Thus, the 0 equilibrium is still asymptotically stable for $q'(0) = 2$. If $q'(0) < 2$, then $F''(0) < 0$ and $F(v) < 0$ for v near 0. A perturbation that is constant in space will decrease the energy, so the 0 equilibrium is unstable. □

The proposition is illustrated by our two examples:

Example 1 For $q_1(v)$ from (22), the condition $q'(0) < 2$ for instability of the trivial equilibrium becomes $\tau_0 > \frac{1}{\gamma}$. (Recall that $G'(v) \geq G'(0) = 1$). □

Example 2 For $q_2(v)$ given by (24), the trivial solution is unstable if $\lambda > 0$. □

We are really only interested in the case $f'(0) < 0$ ($q'(0) < 2$), so that the constant space solutions ($v_j = v_s$ for all j or $v_j = -v_s$ for all j) exist and the zero solution is unstable. These constant-space solutions are always stable.

Proposition 4 *The two solutions $v_j = v_s$ for all j, and $v_j = -v_s$ for all j, where v_s is the positive solution to $f(v) = 0$ when f satisfies conditions (25) and (26), are asymptotically stable.*

Proof As before, use the energy functional in the form (27). Again the last term cannot decrease when a constant equilibrium is perturbed and the first two terms may be treated separately for each v_j. With $F(v) = Q(v) - v^2$, we have

$$F'(\pm v_s) = q(\pm v_s) \mp 2v_s = f(\pm v_s) = 0,$$

and

$$F''(\pm v_s) = f'(\pm v_s) > 0\,.$$

Therefore, the energy is greater for perturbations of this equilibrium and it is asymptotically stable. □

4 A stable equilibrium of period 6.

Now we demonstrate the existence, under certain conditions on $q(v)$, of stable equilibria that are not constant in space. Consider a period 6 equilibrium, $\mathbf{v}^*$, of the form

$$v_1 = v_4 = 0; \quad v_2 = v_3 = -v_5 = -v_6 = B > 0, \tag{28}$$

where B is a constant.

Proposition 5 *An equilibrium of the form* (28) *exists if and only if*

$$q'(0) < 1, \ i.e. \ f'(0) < -1. \tag{29}$$

Proof An equilibrium must satisfy (20) at every point. Here, we always have $v_3+v_5 = q(v_4) = q(0) = 0$, by (25), so the equilibrium exists exactly when it is possible to find a B such that $B = v_2+v_4 = q(v_3) = q(B)$ (the other cases are taken care of by the symmetries in (28) and in q). That is, the equilibrium exists when $q'(0) < 1$, by (25) applied to q. Also, $B = q(B)$ is equivalent to $B = 2B + f(B)$ or $f(B) = -B$, which has a positive solution when $f'(0) < -1$. □

Example 1 (continued) Condition (29) implies $\tau_0 - \frac{\tau}{2} > \frac{1}{\gamma}$ for $q_1(v)$ as in (22), which is not possible if $2\tau_0 \leq \tau$ since $\gamma > 0$, so the condition for existence of this period 6 equilibrium is in this case $2\tau_0 > \tau$ and $\gamma > \frac{1}{\tau_0 - \tau/2}$. □

Example 2 (continued) Condition (29) implies $\lambda > 1$ for $q_2(v)$ as in (24). □

We can determine the stability of this equilibrium from the energy functional (23) by means of the matrix of second derivatives $\frac{\partial^2 E}{\partial v_i \partial v_j}$. The equilibrium $\mathbf{v}^*$ is asymptotically stable if and only if this matrix evaluated at $\mathbf{v}^*$ is positive definite, since this ensures that the energy is greater in a neighbourhood of the equilibrium (for Lyapunov's stability theorem, see, e.g. [12]). This matrix is

$$P = \frac{\partial^2 E}{\partial v_i \partial v_j}[\mathbf{v}^*] = \begin{bmatrix} q'(-B) & -1 & 0 & 0 & 0 & -1 \\ -1 & q'(0) & -1 & 0 & 0 & 0 \\ 0 & -1 & q'(B) & -1 & 0 & 0 \\ 0 & 0 & -1 & q'(B) & -1 & 0 \\ 0 & 0 & 0 & -1 & q'(0) & -1 \\ -1 & 0 & 0 & 0 & -1 & q'(-B) \end{bmatrix}. \tag{30}$$

Proposition 6 *P is positive definite if and only if*

$$q'(0) > 0 \tag{31}$$

and

$$q'(\pm B) - \frac{2}{q'(0)} - 1 > 0. \tag{32}$$

Proof Let $\mathbf{x}$ be an arbitrary vector in $\mathbf{R}^6$. Then

$$\mathbf{x}^t P\mathbf{x} = q'(-B)\left[x_1^2 + x_6^2\right] + q'(B)\left[x_3^2 + x_4^2\right] + q'(0)\left[x_2^2 + x_5^2\right]$$

$$- 2\left[x_6 x_1 + x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_6\right].$$

Now if $q'(0) < 0$ then we can take $x_1 = x_3 = x_4 = x_6 = 0$ and $x_2 = x_5 \neq 0$ to make $\mathbf{x}^t P\mathbf{x} < 0$ so P is not positive definite. If $q'(0) = 0$ then we may take $x_1 = x_3 = x_4 = x_6 = 1$, say, and $x_2 = x_5 > \frac{q'(B)-1}{2}$ so that

$$\mathbf{x}^t P\mathbf{x} = 2q'(-B) + 2q'(B) - 2\left[4x_2 + 2\right] = 4\left[q'(B) - 1 - 2x_2\right] < 0$$

and again, P is not positive definite. If $q'(0) > 0$, we can rewrite $\mathbf{x}^t P\mathbf{x}$ as

$$\begin{aligned}\mathbf{x}^t P\mathbf{x} = {} & q'(0)\left[x_2 - \frac{1}{q'(0)}\left(x_1 + x_3\right)\right]^2 + q'(0)\left[x_5 - \frac{1}{q'(0)}\left(x_4 + x_6\right)\right]^2 \\ & + \left(x_3 - x_4\right)^2 + \left(x_1 - x_6\right)^2 + \frac{1}{q'(0)}\left[\left(x_1 - x_3\right)^2 + \left(x_4 - x_6\right)^2\right] \\ & + \left[q'(-B) - \frac{2}{q'(0)} - 1\right]\left[x_1^2 + x_6^2\right] + \left[q'(B) - \frac{2}{q'(0)} - 1\right]\left[x_3^2 + x_4^2\right]\end{aligned} \tag{33}$$

to see that P is positive definite when $q'(\pm B) - \frac{2}{q'(0)} - 1 > 0$. If this quantity is ≤ 0, we can take $x_1 = x_3 = x_4 = x_6$ and $x_2 = x_5 = \frac{2}{q'(0)} x_1$ to make $\mathbf{x}^t P\mathbf{x} \leq 0$ for $\mathbf{x} \neq 0$, so P is not positive definite. $\square$

Note that exactly the same approach works for multiple periods (where there are extra degrees of freedom in the perturbation vector $\mathbf{x}$). The vectors are in $\mathbf{R}^{6m}$ for m periods and $\mathbf{x}^t P\mathbf{x}$ will contain more terms of the same form so that the conditions for stability do not change.

Thus the period 6 equilibrium exists and is asymptotically stable exactly when $0 < q'(0) < 1$ and $q'(\pm B) - \frac{2}{q'(0)} - 1 > 0$. In terms of f, conditions (31) and (32) are

$$1 + f'(\pm B) > \frac{2}{2 + f'(0)}\,, \tag{34}$$

$$f'(0) > -2\,. \tag{35}$$

Example 1 (continued) For equation (18), we have seen that the period 6 equilibrium exists only when $\tau_0 - \frac{\tau}{2} > \frac{1}{\gamma}$ and stability requires at least the equivalent of (31), which is $\tau_0 - \tau < \frac{1}{\gamma}$. Thus, a necessary condition for the existence of a stable period 6 equilibrium for large γ (i.e. as $\gamma \to \infty$) is

$$\frac{\tau}{2} < \tau_0 \le \tau\,.$$

If we consider the case $\tau_0 = \tau = 2$, as for the Ising model, (31) is automatically satisfied and (32) becomes

$$\frac{G'(B)}{\gamma} - 1 > 2\gamma\,. \tag{36}$$

Since, by definition, $q_1(B) = B$, we have (with $\tau_0 = \tau = 2$) $f_1(B) = -B$ or $\frac{G(B)}{\gamma} = B$ so that $\gamma = \frac{G(B)}{B}$ and B goes from 0 to 1 as γ goes from 1 to ∞. Hence, with these parameters, we may express (36) as

$$\frac{BG'(B)}{G(B)} - 1 > \frac{2G(B)}{B}\,.$$

If $G(v) = \tanh^{-1} v$, again as for the Ising model, this condition becomes

$$\frac{1}{1-B^2} > \frac{\tanh^{-1} B}{B} + 2\left(\frac{\tanh^{-1} B}{B}\right)^2 ,$$

and it can be shown by asymptotic analysis [7] that this is true for large enough B (or γ). A numerical calculation shows that it is true for $\gamma > 1.8576$. $\square$

Example 2 (continued) For $q_2(v)$ given by (24), condition (34) and (35) becomes

$$1 + \lambda\left(3B^2 - 1\right) > \frac{2}{2-\lambda}, \quad \lambda < 2 .$$

In this case, we can find B in terms of λ as follows:

$$q_2(B) = B \Rightarrow f_2(B) = -B \Rightarrow \lambda\left(B^3 - B\right) = -B \Rightarrow B^3 = \left(1 - \frac{1}{\lambda}\right) B$$
$$\Rightarrow B^2 = 1 - \frac{1}{\lambda} .$$

Using this, our stability condition becomes

$$2\lambda - 2 > \frac{2}{2-\lambda} ,$$

where the denominator on the right is positive since $\lambda < 2$. Thus,

$$(2-\lambda)(\lambda - 1) > 1$$

or

$$\lambda^2 - 3\lambda + 3 < 0 .$$

But this quadratic inequality is not satisfied for any real λ, showing that the period 6 equilibrium is never stable for f_2. $\square$

5 Large scale stable patterns.

The existence of a stable period 6 equilibrium for some $q(v)$ is of limited interest in itself, since we want to consider ε to be small, and as $\varepsilon \to 0$, the grid shrinks and the period 6 equilibrium oscillates at very high frequency (and thus, in a sense, consists only of transition layers). However, there can also exist stable equilibria of arbitrarily large period, for appropriate $q(v)$. This is demonstrated by the following series of propositions.

Proposition 7 *If $0 < q'(0) < 2$ and q satisfies* (25) *then there exists a periodic solution to* (20) *of period N for all even $N \geq 6$,*

$$N > \frac{2\pi}{\cos^{-1}\left(\frac{q'(0)}{2}\right)} . \tag{37}$$

Proof Let $N \geq 6$ be even. Let $\phi(n) = \sin\left(\frac{2\pi n}{N}\right)$. We carry out the proof by an iteration. Define an initial vector (of period N) as

$$v_n^{(0)} = \delta\phi(n) \text{ for } n = 0, 1, \ldots, N-1, \tag{38}$$

where the constant $\delta > 0$ is to be chosen.

Now iterate according to

$$v_n^{(m+1)} = q^{-1}\left(v_{n+1}^{(m)} + v_{n-1}^{(m)}\right) , \tag{39}$$

where we have used the fact that $q'(0) > 0$ to ensure that q is strictly increasing so that q^{-1} exists and is also increasing. If $v_n^{(m)} \geq v_n^{(m-1)}$ for all n, then

$$v_n^{(m+1)} \geq q^{-1}\left(v_{n+1}^{(m-1)} + v_{n-1}^{(m-1)}\right) = v_n^{(m)}$$

for all n. So an initially increasing sequence must continue to increase. Similarly, an initially decreasing sequence must continue to decrease. Now, for the initial vector in (38), the points $v_n^{(m)}$ where $n = \frac{kN}{2}$, i.e. multiples of $N/2$, will be zero and will remain zero due to the symmetries in the vector and the iteration. Thus, we need only show that in the positive parts of the initial vector, each point increases on the first iteration (and in the negative parts each point decreases) to get a monotone increasing (monotone decreasing) sequence of points for each n. That is, we need

$$v_n^{(0)} = \delta\phi(n) \leq q^{-1}\left(\delta\left(\phi(n-1)+\phi(n+1)\right)\right) = q^{-1}\left(v_{n-1}^{(0)} + v_{n+1}^{(0)}\right) = v_n^{(1)}.$$

I.e.

$$q\left[\delta\phi(n)\right] \leq \delta\left(\phi(n-1)+\phi(n+1)\right) .$$

There exists a δ such that this relation is satisfied as long as the slope of the function on the right hand side, considered as a function of δ, is greater than the slope of the function on the left at $\delta = 0$. That is, the inequality can be satisfied if

$$q'(0)\phi(n) < \phi(n-1) + \phi(n+1)\,.$$

Now expanding the sine functions gives

$$\phi(n-1) + \phi(n+1) = 2\phi(n)\cos\left(\frac{2\pi}{N}\right) ,$$

so the condition becomes

$$q'(0) < 2\cos\left(\frac{2\pi}{N}\right) ,$$

or equivalently,

$$N > \frac{2\pi}{\cos^{-1}\left(\frac{q'(0)}{2}\right)} .$$

If this is satisfied, we get a monotone increasing sequence $v_n^{(m)}$ for each n, $0 < n < \frac{N}{2}$. Recall that $q(v_s) = 2v_s$, where v_s is the positive solution to $f(v) = 0$ (see (25) and (11)). So $v_s = q^{-1}(2v_s)$. Since q and therefore q^{-1} are increasing functions, we have by the iteration scheme (39) that $\left|v_n^{(m+1)}\right| < v_s$ as long as $\left|v_{n-1}^{(m)}\right| < v_s$ and $\left|v_{n+1}^{(m)}\right| < v_s$. Thus, if we take δ small enough so that $\left|v_n^{(0)}\right| < v_s$ for all n, then $\left|v_n^{(m)}\right| < v_s$ for all n and m. Then each monotone increasing sequence must converge (to v_n, say). For $\frac{N}{2} < n < N$, each sequence is monotone decreasing and converges to $v_n = -v_{N-n}$. The resulting vector $\mathbf{v}^*$ satisfies the equilibrium equation (20) at every point and is therefore a solution. □

We will require some properties of these equilibria. First, it is evident by the method of construction that these equilibria have symmetries. In particular,

$$v_0 = v_{\frac{N}{2}} = v_N = 0,$$

$$v_n = -v_{\frac{N}{2}+n},$$

and

$$v_n = v_{\frac{N}{2}-n}.$$

Now define the second difference,

$$\Delta^2 v_i \equiv v_{i-1} + v_{i+1} - 2v_i .$$

From equation (20), it is clear that for any equilibrium,

$$\Delta^2 v_i = q(v_i) - 2v_i = f(v_i),$$

which is negative for $0 < v_i < v_s$ and positive for $-v_s < v_i < 0$. Thus, an equilibrium must be concave down where it is positive and concave up where it is negative. This concavity also implies that the equilibria found in Proposition 7 increase to a maximum and decrease to zero again from v_0 to $v_{\frac{N}{2}}$. Thus,

$$v_n \geq v_1$$

for $0 < n < \frac{N}{2}$ and in particular

$$v_2 > v_1$$

for even $N \geq 8$.

Furthermore, we know that f has a unique minimum on $(0, v_s)$, say at $v = \eta$, by (25) and since $f'(0) < 0$ (i.e. $q'(0) < 2$). For $v_i \geq \eta$, $f(v_i)$ is an increasing function of v_i. Thus, for an equilibrium, $v_i > v_j \geq \eta$ implies $f(v_i) > f(v_j)$ and therefore

$$\Delta^2 v_i > \Delta^2 v_j .$$

Proposition 8 *The solution to* (20) *of period* $N \geq 8$ *given by Proposition 7 exists and is stable if*

$$0 < q'(0) < 2\cos\left(\frac{2\pi}{N}\right) \tag{40}$$

and

$$q'(v_1^*) > \frac{2}{q'(0)} + 1 . \tag{41}$$

Proof As for the period 6 equilibrium, stability is demonstrated by showing that the matrix of second partial derivatives, $P = \frac{\partial^2 E}{\partial v_i \partial v_j}\left[\mathbf{v}^*\right]$, where $\mathbf{v}^*$ is the equilibrium, is positive definite. The matrix will be similar to that in (30), having $q'(v_i^*)$ on the diagonal in the i^{th} position and -1 in adjacent positions. Letting $\mathbf{x}$ be an arbitrary vector in $\mathbf{R}^N$, we have

$$\mathbf{x}^t P\mathbf{x} = \sum_{i=0}^{N-1} q'(v_i^*)x_i^2 - 2\sum_{i=0}^{N-1} x_i x_{i+1}\,. \tag{42}$$

In order to see when this must be positive, we need to express it as a sum of squares with positive coefficients, as in (33). The interaction terms in the last sum can be handled by including terms like $(x_i - x_{i+1})^2$ for each adjacent pair, and then the extra $2x_i^2$ for each point will have to be subtracted from $q'(v_i^*)x_i^2$. However, this will not work for the points where $v_i^* = 0$, i.e. when $i = \frac{kN}{2}$, for some integer k, since the equilibrium only exists for $0 < q'(0) < 2\cos\left(\frac{2\pi}{N}\right) < 2$, so that $(q'(0) - 2)\,x_i^2 < 0$. Thus, we handle the points $i = \frac{kN}{2}$ and their interactions with adjacent points separately to get the equivalent expression

$$\begin{aligned}\mathbf{x}^t P\mathbf{x} = &\sum_{i=\frac{kN}{2}}\sum_{j\sim i}\frac{q'(0)}{2}\left[x_i - \frac{2}{q'(0)}x_j\right]^2 + \frac{1}{2}\sum_{i\neq\frac{kN}{2}}\sum_{\substack{j\sim i\\ j\neq\frac{kN}{2}}}[x_i - x_j]^2\\ &+\sum_{i=\frac{kN}{2}}\sum_{j\sim i}\left[q'(v_j^*) - \frac{2}{q'(0)} - 1\right]x_j^2 + \sum_{|v_i^*|>|v_1^*|}[q'(v_i^*) - 2]\,x_i^2\,,\end{aligned} \tag{43}$$

where the sums over $j \sim i$ mean sums over the immediate neighbours of i. Thus, the second sum above is over all adjacent pairs where neither is a zero point and the last sum is over all points aside from the zero points and those adjacent to them. Now by (25), $vq''(v) > 0$, so that $q'(v)$ increases with $|v|$ and so (43) is positive if (41) is satisfied and $q'(v_2^*) > 2$. This last condition follows from (40) and (41), however. Note that $q'(v_2^*) > q'(v_1^*)$ since $|v_2^*| > |v_1^*|$, which is a property of the equilibria from the discussion following Proposition

7. So, using (40) again,

$$q'(v_2^*) > q'(v_1^*) > \frac{1}{\cos\left(\frac{2\pi}{N}\right)} + 1 \geq 2\,.$$

Thus, conditions (40) and (41) give the result. □

Note that multiple periods can again be handled in exactly the same way.

Example 1 (continued) For $q_1(v)$ given by (22) with $\tau = \tau_0 = 2$, conditions (40) and (41) become

$$0 < \frac{1}{\gamma} < 2\cos\left(\frac{2\pi}{N}\right), \tag{44}$$

$$\frac{G'(v_1^*)}{\gamma} - 1 > 2\gamma\,. \tag{45}$$

The first of these (44) is always true with $\gamma > 1$, say (for $N \geq 6$). □

In order to establish the existence of these stable patterns for this example, it is necessary to show that (45) can also be satisfied. We do this with the help of some earlier results from [7]. First, we look at the difference between two equilibria, i.e. two solutions to (20). Recall that f has a minimum at $v = \eta$.

Lemma 9 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be solutions of* (20) *and let* $\mathbf{z} = \mathbf{u} - \mathbf{v}$*. Suppose* $u_m, v_m \geq \eta$ *and* $z_m \geq 0$ *for some* m*, then* $z_{m+1} \geq 2z_m - z_{m-1}$*.*

Proof From the properties of equilibria discussed after Proposition 7, $z_m \geq 0$ implies that $\Delta^2 u_m > \Delta^2 v_m$. That is, $u_{m+1} + u_{m-1} - 2u_m \geq v_{m+1} + v_{m-1} - 2v_m$, which when rearranged, gives the desired result. □

Lemma 10 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be solutions of* (20) *and let* $\mathbf{z} = \mathbf{u} - \mathbf{v}$*. Suppose* $u_1, \ldots, u_m \geq$

η, $v_1, \ldots, v_m \geq \eta$ and $z_1, \ldots, z_m \geq 0$. Then

$$z_{m+1} \geq (m+1)z_1 - mz_0 \, .$$

Proof Let k be an integer such that $1 \leq k \leq m-1$. Note that from Lemma 9, we have $-z_m \geq -\frac{1}{2}z_{m+1} - \frac{1}{2}z_{m-1}$. Using this,

$$(k+1)z_m - kz_{m-1} \geq (k+1)z_m - \frac{k}{2}z_m - \frac{k}{2}z_{m-2} = \left(\frac{k}{2}+1\right)z_m - \frac{k}{2}z_{m-2}$$

and then using Lemma 9 again on the first term,

$$(k+1)z_m - kz_{m-1} \geq \left(\frac{k}{2}+1\right)(2z_{m-1} - z_{m-2}) - \frac{k}{2}z_{m-2} = (k+2)z_{m-1} - (k+1)z_{m-2}.$$

Now apply this result $m-1$ times to $z_{m+1} \geq 2z_m - z_{m-1}$, with $k = 1, 2, \ldots, m-1$, in turn to get the result. □

Corollary 11 *If $z_1 \geq 0$ and $z_0 = 0$ in Lemma 10, then $z_2, \ldots, z_m \geq 0$ is automatic, since $z_{k+1} \geq (k+1)z_1$ for each k, $1 \leq k \leq m-1$. Therefore, the result holds for z_{m+1}, i.e.*

$$z_{m+1} \geq (m+1)z_1 \, .$$

□

Now suppose that we have two solutions to (20), one of even period M, call it $\mathbf{u}$, and one of larger even period, say $N > M$, call it $\mathbf{v}$. Suppose also that $u_0 = v_0 = 0$. We claim that $v_1 > u_1$ and therefore, that v_1 is an increasing function of N, at least when $u_1, v_1 \geq \eta$.

Proposition 12 *Let $\mathbf{u}$ and $\mathbf{v}$ be solutions to (20) of even period M and N respectively, with $6 \leq M < N$. Let $u_0 = v_0 = 0$. Suppose $u_1, v_1 \geq \eta$. Then $v_1 > u_1$.*

Proof Since $u_1 \geq \eta$ and $\mathbf{u}$ has even period M, we have $u_i \geq \eta$ for $1 \leq i < \frac{M}{2}$ (this is a property of the equilibria from the discussion following Proposition 7). Similarly, $v_i \geq \eta$ for $1 \leq i < \frac{N}{2}$. Now suppose that $u_1 \geq v_1$, so that $z_1 = u_1 - v_1 \geq 0$. Then we can apply Corollary 11 with $m = \frac{M}{2} - 1$, to show that $z_{M/2} \geq \frac{M}{2} z_1 \geq 0$. However, $u_{M/2} = 0$ and $v_{M/2} > 0$ so $z_{M/2} < 0$ and we have a contradiction. Thus $u_1 < v_1$ and since this is true for arbitrary even periods $M, N \geq 6$, v_1 is an increasing function of the period, N. □

This result can be applied to stability of periodic equilibria as follows. Since $q'(v)$ is an increasing function for positive v (25), $q'(v_1^*)$ is an increasing function of the period N by Proposition 12. Thus, if for some N, condition (41) is satisfied, then it will also be satisfied for all larger N. Also, increasing N increases the upper bound on $q'(0)$ in condition (40). So if, for a particular q, the existence of a stable equilibrium of even period N can be established, then the equilibria of larger even period also exist and are stable.

Example 1 (continued) For q_1 as given by equation (22), with $\tau = \tau_0 = 2$, conditions (40) and (41) become (44) and (45) but these are true for period $N = 6$ as shown in the previous section. Thus the equilibria for all even periods $N \geq 6$ exist and are stable for this q.

Example 2 (continued) For q_2 given by (24), there was no stable period 6 equilibrium so it would be necessary to find one of larger period to get the large scale stable patterns in this case. Numerical experiment indicates that no matter how large the period, the conditions (40) and (41) cannot both be satisfied for this q. However, note that Proposition 7 gives only a sufficient condition for existence of stable patterns. Only for the period 6 case did we have a necessary *and* sufficient condition.

6 Patterns in d-dimensions.

The above results can be extended to two or more dimensions without much difficulty. There is, of course, a larger choice of patterns that can be examined for stability. For example, we can obtain a d-dimensional analogy to Proposition 7, for $0 < q'(0) < 2d$ either by starting with an initial function

$$\phi(\mathbf{n}) = \prod_{r=1}^{d} \sin\left(\frac{2\pi n_r}{N}\right) \tag{46}$$

for $\mathbf{n}$ a grid point with coordinates n_r, which in 2 dimensions will produce a checkerboard pattern of positive and negative square regions, or with

$$\phi(\mathbf{n}) = \sin\left(\frac{2\pi \sum_{r=1}^{d} n_r}{N}\right), \tag{47}$$

which in 2 dimensions will produce a pattern of diagonal ridges and valleys. In either case the condition for existence of this equilibrium is

$$q'(0) < 2d\cos\left(\frac{2\pi}{N}\right) \text{ or } N > \frac{2\pi}{\cos^{-1}\left(\frac{q'(0)}{2d}\right)}.$$

In the case of (47), the condition for stability of the equilibrium (denoted $\mathbf{v}^*$) is that the following be positive for all non-zero $\mathbf{x} \in \mathbf{R}^{(N^d)}$.

$$\begin{aligned}
\mathbf{x}^t P\mathbf{x} &= \sum_i q'(0)x_i^2 - 2\sum_i \sum_{j\sim i} x_i x_j \\
&= \sum_{v_i^*=0} \sum_{j\sim i} \frac{q'(0)}{2d}\left[x_i - \frac{2d}{q'(0)}x_j\right]^2 + \frac{1}{2}\sum_{v_i^*\neq 0} \sum_{\substack{j\sim i \\ v_j^*\neq 0}} [x_i - x_j]^2 \\
&\quad + \sum_{|v_i^*|=|v_1^*|} \left[q'(v_i^*) - \frac{2d}{q'(0)} - d\right] x_i^2 + \sum_{|v_i^*|>|v_1^*|} [q'(v_i^*) - 2d]\, x_i^2.
\end{aligned}$$

This is analogous to (33) and (43). So a sufficient condition for stability is that the equilibrium exists and

$$q'(v_1^*) > \frac{2d}{q'(0)} + d\,,$$

$$q'(v_2^*) > 2d\,.$$

Again, for large N we expect these conditions to be satisfied for some q. Other types of patterns are, of course, possible.

7 Discussion.

The reaction-diffusion equations we started with are approximations in continuous space to systems of ordinary differential equations, some of which have large scale stable patterns as we have shown. The error in the $\varepsilon^2 v_{xx}$ term is $O(\varepsilon^4)$. This suggests that for small ε solutions to the reaction-diffusion equations approximate solutions to the systems of ODEs very closely for some time. (This type of approximation is made more rigorous for the neural network equation in [9]). So we expect that solutions to the PDEs that start near a stable pattern of the corresponding system of ODEs should stay near it for some time though such a pattern cannot be stable for the reaction-diffusion equation. This is of course suggestive of the metastability that occurs in the analysis of the Ginzburg-Landau or Allen-Cahn type of equation as studied in [2,3,4,14,15]. Random initial conditions even for the discrete space equations may, however, lead to metastable patterns of transition layers since the stable patterns seem to depend on equal spacing of layers.

Furthermore, the stable patterns that exist for the reaction term of the Ising model equations and of the neural network equations are of arbitrarily high energy. We may take as many multiples of a period N equilibrium as we like and the equilibrium is still stable. For m multiples of the period,

$$E[\mathbf{v}] = m\left(\sum_{j=1}^{N} Q(v_j) - \frac{1}{2}\sum_{j=1}^{N} v_j(v_{j+1}+v_{j-1})\right),$$

which can be made arbitrarily large by making m large. Thus, despite the known lack of stable patterns for the reaction-diffusion equations, stable patterns of arbitrarily high energy can exist for their spatial discretizations, even when the diffusion coefficient is very small. Of course, if the step size of the discretization were much smaller than ε, which is the scale of the transition layer widths, these stable patterns should be lost, but the spatially discrete systems studied here model phenomena of interest independent of the continuous space equations.

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