# Graph Theory

## 1 Defining and representing graphs

A *graph* is an ordered pair $G = (V, E)$, where $V$ is a finite, non-empty set of objects called *vertices*, and $E$ is a (possibly empty) set of unordered pairs of distinct vertices (*i.e.*, 2-subsets of $V$) called *edges*.

- The set $V$ (or $V(G)$ to emphasize that it belongs to the graph $G$) is called the *vertex set* of $G$.
- The set $E$ (or $E(G)$ to emphasize as above) is called the *edge set* of $G$.
- If $e = \{u, v\} \in E(G)$, we say that vertices $u$ and $v$ are *adjacent* in $G$, and that $e$ *joins* $u$ and $v$. We'll also say that $u$ and $v$ are the *ends* of $e$.
- The edge $e$ is said to be *incident* with $u$ (and $v$), and vice versa.
- We write $uv$ (or $vu$) to denote the edge $\{u, v\}$, on the understanding that no order is implied.
- $E(G)$ is a set. This means that two vertices either are adjacent or are not adjacent. There is no possibility of more than one edge joining a pair of vertices.
- The elements of $E$ are 2-subsets of $V$. Thus a vertex can not be adjacent to itself.

Graphs are usually represented pictorially with a point (or dot) in the plane corresponding to each vertex and a line segment (or curve of some sort) joining the corresponding points for each pair of adjacent vertices. The picture tells you what the graph is, that is, it tells you what the vertices are, and what the edges are.

The *adjacency matrix* of a graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the $n \times n$ matrix $A$ whose $(i, j)$ entry, $A_{ij}$, is 1 if $v_i v_j \in E$ and 0 if $v_i v_j \notin E$.

- $A_{ij}$ is the truth value of the statement "$v_i$ and $v_j$ are adjacent".

- The matrix $A$ is symmetric.
- Since no vertex can be adjacent to itself, the diagonal entries of $A$ are zeros.

# 2 Equality and isomorphism of graphs

Two graphs are *equal* if they have the same vertex set and the same edge set.

Graphs $G$ and $H$ are *isomorphic* if there is a 1-1 correspondence
$f : V(G) \to V(H)$ such that $xy \in E(G) \Leftrightarrow f(x)f(y) \in E(H)$.

The function $f$ is called an *isomorphism.*

- Isomorphic graphs are identical in every respect other than the names given to the vertices.
- A way to prove two graphs are isomorphic is to relabel the vertices of one and obtain the other.
- A way to prove two graphs are not isomorphic, find a property that one graph has and the other one doesn't have.

# 3 Special graphs

A graph is *complete* if any two different vertices are adjacent. Since any two such graphs on the same number of vertices are isomorphic, we talk about *the complete graph on $n$ vertices* and denote it by $K_n$.

A graph $G$ is *bipartite* if there exist disjoint sets $X, Y \subseteq V$ such that $X \cup Y = V$ and every edge has one end in $X$ and the other in $Y$. The ordered pair $(X, Y)$ is called a *bipartition* of $G$.

A bipartite graph with bipartition $(X, Y)$ is a *complete bipartite graph* if every vertex in $X$ is adjacent to every vertex in $Y$. Since any two such graphs with $m = |X|$ and $n = |Y|$ are isomorphic, we talk about *the complete bipartite graph* and denote it by $K_{m,n}$ .

# 4 Vertex degrees

The *degree* of a vertex $v \in V$, denoted by $\deg(v)$ (or $\deg_G(v)$), is the number of vertices adjacent to $v$.

- For any vertex $v \in V$ we have $0 \leq \deg(v) \leq |V| - 1$.

**Proposition 4.1.** *Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} \deg(v) = 2|E|$.*

*Proof.* Every edge of $G$ is counted twice on the RHS. Since an edge $xy$ contributes one to $\deg(x)$ and one to $\deg(y)$, every edge is also counted twice on the LHS. Therefore LHS = RHS. $\square$

- By Proposition 4.1, the sum of the vertex degrees is even.
- By Proposition 4.1, the number of vertices of odd degree is even.
- Since every vertex of $K_n$ has degree $n - 1$, it has $n(n-1)/2 = \binom{n}{2}$ edges.

A graph is *$r$-regular* (or *regular of degree* $r$) if every vertex has degree $r$. A graph is regular if it is $r$-regular for some $r$.

- By Proposition 4.1, does not exist an $r$ regular graph on $n$ vertices if $r$ and $n$ are both odd.
- If $r$ or $n$ is even, then there exists an $r$-regular graph on $n$ vertices.

**Proposition 4.2.** *If a graph $G$ has at least two vertices, then it has two vertices of the same degree.*

*Proof.* Suppose the graph $G$ has $n > 1$ vertices, then $0 \leq \deg(v) \leq n - 1$ for any vertex $v$. But a vertex of degree 0 is not adjacent to any other vertex, while a vertex of degree $n-1$ is adjacent to every other vertex. Thus a graph with $n \geq 2$ vertices can not have both a vertex of degree 0 and a vertex of degree $n - 1$. It follows that either $0 \leq \deg(v) \leq n - 2$ for any vertex $v$, or $1 \leq \deg(v) \leq n - 1$ for any vertex $v$. In either case, there are $n - 1$ possible value for a collection of $n$ integers, so two of the integers must be equal. $\square$

A *degree sequence* of a graph $G$ with $n$ vertices is a sequence of length $n$ whose elements are the degrees of the vertices of $G$ (in some order).

- A graph could have many degree sequences.
- Non-isomorphic graphs might have the same degree sequence.

A sequence $d_1, d_2, \ldots, d_n$ of non-negative integers is called *graphical* if it is the degree sequence of some graph.

If a sequence of $n \geq 1$ non-negative integers is graphical, then

- the sum of the elements in the sequence is even;
- the largest element in the sequence must be at most $n-1$;
- if $n > 1$ then both $n-1$ and 0 can not both occur in the sequence.

# 5 Subgraphs

A graph $H$ is a *subgraph* of a graph $G = (V, E)$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A subgraph if a graph such that

- all of the vertices are vertices of $G$;
- all of the edges are edges of $G$;
- every edge of a subgraph joins two vertices belonging to the subgraph.

You can think of asubgraph of $G$ as being obtained from $G$ by first selecting some of the vertices of $G$ to belong to the subgraph, and then selecting some of the edges of $G$ joining vertices in this collection to be edges of the subgraph.

A *spanning* subgraph of a graph $G$ is a subgraph $H$ of $G$ such that $V(H) = V(G)$.

- The number of spanning subgraphs of a graph $G$ is $2^{|E(G)|}$.
- The number of subgraphs of the complete graph with vertex set $\{1, 2, \ldots, n\}$ is $2^{\binom{n}{2}}$. (Many of these are isomorphic.)
- The term *labelled* means the names of the vertices is important.
- There are $2^3 = 8$ labelled graphs on 3 vertices, but only 4 non-isomorphic (or, *unlabelled*) graphs on 3 vertices.

The *complement* of a graph $G$, denoted $\overline{G}$ is the graph with the same vertex set as $G$, and where distinct vertices $x$ and $y$ are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

- You can think of $\overline{G}$ as being obtained from the complete graph on $|V(G)|$ vertices by deleting the edges that belong to $G$.
- If $G$ and $H$ are isomorphic, then so are $\overline{G}$ and $\overline{H}$.
- $\overline{\overline{G}} = G$.
- A graph is *self-complementary* if it is isomorphic to its complement. Self complementary graphs exist if and only if the number of vertices is congruent to 0 or 1 modulo 4.

*Notation*: If $G$ is a graph and $xy$ is an edge of $G$, we use $G - xy$ to denote the graph obtained from $G$ by deleting the edge $xy$. Formally, this graph has vertex set $V(G)$, and edge set $E(G) - \{xy\}$. Similarly, of $x \in V$ then $G - x$ is the graph obtained from $G$ be deleting $x$ and all edges incident with $x$. More generally, if $X \subseteq V$, then $G - X$ denotes the subgraphs of $G$ obtained by deleting the vertices in $X$ and all edges incident with them (so it has vertex set $V - X$ and edge set $E - Y$, where $Y = \{uv \in E : u \in X \text{ or } v \in X\}$) Similarly, if $Z \subseteq E$, then $G - Z$ denotes the subgraph of $G$ obtained by by deleting the edges belonging to $Z$.

# 6 Walk, trails, paths, circuits and cycles

A *walk* in a graph is an alternating sequence of vertices and edges, $v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, e_n, v_n$ such that $e_i = v_{i-1}v_i$ for $1 \leq i \leq n$.

- The integer $n$ is the *length* of the walk. (It is the number of edges in the walk.)
- Equivalently, a walk is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $v_{i-1}v_i \in E,\ 1 \leq i \leq n$.

A walk is

- a *trail* if no edge is repeated;
- a *path* if no vertex is repeated;
- *closed* if $v_0 = v_n$ (note that a path can't be closed).

- Every path is a trail, and every trail is a walk.
- There are walks which are not trails, and trails which are not paths.

**Proposition 6.1.** *If a graph $G$ has an walk that starts at $x$ and ends at $y$, then it has a path that starts at $x$ and ends at $y$.*

*Proof idea*: If a vertex is repeated, then part of the walk can be deleted so that a shorter walk from $x$ to $y$ is obtained. This a shortest walk from $x$ to $y$ must be a path.

*Proof.* Let $P : (x = x_0), x_1, x_2, \ldots, (x_k = y)$ be a shortest walk from $x$ to $y$. If it is not a path, then there is a repeated vertex. Therefore there exist subscripts $i$ and $j$ such that $0 \leq i < j \leq k$ such that $x_i = x_j$. But then $(x = x_0), x_1, x_2, \ldots, (x_i = x_j), x_{j+1}, \ldots, (x_k = y)$ is a shorter walk from $x$ to $y$, a contradiction. Therefore $P$ is a path. □

A *cycle* is a closed walk of length at least three in which the vertices are distinct except the first and last.

A closed trail is called a *circuit*.

- Circuits can have length 0, cycles can't.
- In a circuit no edge is repeated.

- Circuits can have repeated vertices other then the first and last, cycles can't.
- Every cycle is a circuit.
- There are many circuits that are not cycles.

Similar arguments as in the proof of Proposition 6.1 prove:

- If $G$ has a circuit of positive length containing the vertex $v$, then it has a cycle containing $v$.
- If $G$ has a closed walk of odd length, then it has a cycle of odd length.

It is not true that if a graph contains a closed walk (containing a vertex $v$) then it contains a cycle (containing a vertex $v$).

**Proposition 6.2.** *Let $G$ be a graph in which every vertex has degree at least $k$. Then, $G$ has a path of length at least $k$. Further, of $k \geq 2$, then $G$ has a cycle of length at least $k+1$.*

*Proof.* We prove the first statement. The proof of the second statement is similar. Let $P = v_0, v_1, \ldots, v_p$ be a longest path in $G$. If some vertex adjacent to $v_0$ does not belong to $P$, then there is a longer path than $P$. Hence, every vertex adjacent to $v_0$ belongs to $P$. Since $\deg(v_0) \geq k$ we must have $p \geq k$. Thus $G$ has a path of length at least $k$. □

# 7 Connectedness and components

A graph is *connected* if, for any two different vertices $u$ and $v$, there exists a walk from $u$ to $v$; otherwise it is *disconnected*. A (connected) *component* of a graph $G$ is a maximal connected subgraph of $G$.

- *maximal* is with respect to inclusion. (By contrast, "maximum" is with respect to size.)

- A graph is connected if and only if it has only one component, and disconnected if and only if it has at least two components.
- Because of Proposition 6.1, the term "walk" in the definition of connected could be (and often is) replaced by "path".

# 8 Eulerian graphs

An *Euler trail* (or *Eulerian trail*) in a graph $G$ is a trail that includes all of the vertices and edges of $G$. A closed Euler trail is called an *Euler tour* or *Eulerian circuit*. A graph is *Eulerian* if it has an Eulerian circuit.

**Theorem 8.1.** *A graph $G$ has an Eulerian circuit if and only if it is connected and every vertex has even degree.*

*Proof.* It is clear that a graph with an Eulerian circuit is connected because the circuit contains every vertex of $G$. Each time an Eulerian circuit passes through a vertex, it uses exactly two edges incident with that vertex. The first and last edge of the trail each account for one edge incident with the same vertex. Thus the tour accounts for an even number of edges incident with each vertex. Since it contains all of the vertices and edges of $G$, it follows that the degree of each vertex of $G$ is even.

The proof of the converse is by induction on the number, $m$, of edges of $G$. The statement is true if $m = 0$ because the only possibility is that $G \cong K_1$. Suppose the statement is true for all connected graphs with fewer than $m$ edges in which every vertex has even degree. Let $G$ be a connected graph with $m$ edges in which every vertex has even degree.

Let $C$ be a longest circuit in $G$. By Proposition 6.2, the length of $C$ is positive. We claim that $C$ is an Eulerian circuit. Suppose not.

Let $E(C)$ denote the edge set of $C$. Since $C$ uses an even number of edges (possibly 0) incident with each vertex of $G$, every vertex of $G - E(C)$ has even degree. By the induction hypothesis, every component of $G - E(C)$ has an Eulerian circuit. Since $C$ is not an Eulerian circuit, there is a component of $G - E(C)$ in which the Eulerian circuit has positive length, call it $H$.

Since $G$ is connected and $G - E(C)$ is not connected, some vertex $x \in V(H)$ belongs to $C$. But then the circuit obtained by following $C$ up to $x$, then the Eulerian circuit of $H$, then the remainder of $C$ is longer than $C$, a contradiction. Therefore $C$ is an Eulerian circuit.

The result now follows by induction. □

- Theorem 8.1 gives an easy criteria for proving that a graph has/doesn't have an Eulerian circuit. If it does, describe the tour, and if it doesn't then show that it (the graph) is disconnected or has a vertex of odd degree.
- The proof also gives an algorithm for constructing the trail when it exists. Start at any vertex of odd degree, and construct a maximal trail. If it contains all of the edges in the graph, then you're done. Otherwise, delete all of the edges in your trail. In each component of the resulting graph, all vertices will have even degree. The graph is necessarily disconnected because the trail contains all edges incident with its last vertex (otherwise it would not be maximal!). There will be a last vertex $w$ on the trail which belongs to a component that has edges. This component has an Eulerian circuit starting and ending at $w$, and this "little tour" can be "added in" to your trail to give a longer trail. The little tour can be be found using the same procedure (or, assumed to exist by induction). Repeating this process for each extended trail eventually results in an Eulerian circuit of the graph.

**Corollary 8.2.** *A graph has an Euler trail if and only if it is connected and has 0 or 2 vertices of odd degree.*

*Proof.* If there are no vertices of odd degree, the result follows from Theorem 8.1. Suppose there are 2 vertices of odd degree; call them $u$ and $v$.

Construct a new graph $G'$ by adding a new vertex, $x$, and joining it to $u$ and to $v$. Then every vertex of $G'$ has ever degree, so $G'$ has an Eulerian circuir. Without loss of generality the circuits starts at $x$, and the first vertex after $x$ is $u$. Since $x$ has degree 2, the penultimate vertex in the circuit (the next to last vertex in the circuit) is $v$. Since $G$ is obtained from $G'$ by deleting $x$, it follows that $G$ has an Euler trail that starts at $u$ and ends at $v$. □

- From the counting argument in the proof of Theorem 8.1, if $G$ is connected and has exactly 2 vertices of odd degree, then an Euler trail must start at one of them and end at the other.

- The proof of Corollary 8.2, together with the proof of Theorem 8.1, tells you how to find and Euler train an a connected graph with exactly 2 vertices of odd dergee.

# 9 Hamiltonian graphs

A *Hamilton cycle* (or *Hamiltonian cycle*) in a graph $G$ s a cycle that contains every vertex of $G$. A graph that has a hamilton cycle is said to be *Hamiltonian.*

A *Hamilton path* (or *Hamiltonian path*) in a graph i$G$ s a path that contains every vertex of $G$.

- A graph with a Hamiltonian cycle is connected.

- A Hamilton cycle of a contains every vertex of a graph exactly once, whereas an Eulerian circuit contains every edge exactly once (and every vertex at least once).

- There are graphs with both a Hamilton cycle and an Eulerian circuit, graphs with neither, and graphs with one but not the other.

- There is no known theorem that characterizes the graphs that have a Hamilton cycle (or a Hamilton path).

Let $n \geq 0$ be an integer. The *n-cube* is the graph, $Q_n$, with vertex set equal to the set of binary sequences of length $n$, and in which vertices (sequences) $s_1$ and $s_2$ are adjacent if and only if they differ in exactly one place.

- $|V(Q_n)| = 2^n$.

- $Q_n$ is $n$-regular.

- $|E(Q_n)| = n2^{n-1}$.

- $Q_n$ is bipartite.

For $n > 0$, let $V_0$ be the set of vertices of $Q_n$ in which the first entry is 0, and let $V_1$ be the set in which the first entry is 1. Then $V = V_0 \cup V_1$. The subgraph with vertex set $V_0$ and all edges with both ends in $V_0$ is isomorphic to $Q_{n-1}$. The same holds for $V_1$. Thus, $Q_n$ can be constructed from 2 copies of $Q_{n-1}$ by prefixing the vertices of one copy with 0, prefixing the vertices of the other copy with 1, and then joining $0s$ to $1s$ for each vertex $s$ of $Q_{n-1}$.

**Theorem 9.1.** *For all $n \geq 2$, the graph $Q_n$ is Hamiltonian.*

*Proof idea: use the recursive structure of $Q_n$ to piece together Hamilton cycles in the two smaller cubes from which $Q_n$ is constructed.*

*Proof.* By induction on $n$. The graph $Q_2$ itself a cycle of length 4, so it is Hamiltonian. Suppose that $Q_n$ is Hamiltonian for some $n \geq 2$. Let $s_1, s_2, \ldots, s_{2^n}, s_1$ be a Hamilton cycle in $Q_n$.

Now consider $Q_{n+1}$. By the induction hypothesis and the recursive construction of $Q_n$, the sequence

$$0s_1, 0s_2, \ldots, 0s_{2^n}, 1s_{2^n}, 1s_{2^n-1}, \ldots 1s_1, 0s_1$$

is a Hamilton cycle. The result now follows by induction. □

# 10 Trees

A *tree* is a connected graph that has no cycles (*i.e.*, a connected acyclic graph).

By Proposition 6.2, a connected graph with no cycles can not have all vertices of degree at least 2. Therefore there is a vertex of degree 1.

A *leaf* of a tree is a vertex of degree 1.

The following is proved by considering a longest path, and then using the same idea as in the proof of Proposition 6.2, together with the fact that there are no cycles, to show that its ends must have degree 1.

**Proposition 10.1.** *A tree with at least 2 vertices has at least 2 vertices of degree 1.*

Leaves are useful because, if $\ell$ is a leaf, then it can not appear in a path between two vertices $u$ and $v$ in $V - \{x\}$. Thus, if $T$ is a tree, and $x$ is a leaf, then $T - x$ (the subgraph obtained by deleting $x$ and the edge incident with it) is also a tree.

**Theorem 10.2.** *Suppose that $T$ has $n \geq 1$ vertices. The following statements are equivalent:*

1. *$T$ is a tree.*
2. *$T$ is a connected graph with $|E| = n - 1$.*
3. *There is a unique path between any two vertices of $T$.*

*Proof.* (1) $\Rightarrow$ (2). Since $T$ is a tree, it is connected. We show that $|E| = n-1$ by induction on $n$. A tree with 1 vertex has no edges, so the statement is true when $n = 1$. Suppose that every tree with $k$ vertices has exactly $k - 1$ edges, for some $k \geq 1$.

Let $T$ be a tree with $k + 1$ vertices. By Proposition 10.1, the tree $T$ has a leaf, $\ell$. Since $T - \ell$ is a tree with $k$ vertices, by the induction hypothesis it has exactly $k - 1$ edges. Since $T$ has 1 more edge than $T - \ell$, it has $k$ edges. The result now follows by induction.

(2) $\Rightarrow$ (1). The proof is by induction on $n$. A graph with 1 vertex and no edges is a tree. Suppose that every connected graph with $k$ vertices and $k-1$ edges is a tree, for some $k \geq 1$.

Let $T$ be a connected graph with $k + 1$ vertices and $k$ edges. Since the sum of vertex degrees is $2((k + 1) - 1) < 2(k + 1$, the graph $T$ has a vertex $x$ of degree less than 2. Since $T$ is connected, the vertex $x$ must have degree 1. The graph $T - x$ is connected and has $k - 1$ edges. Hence, by the induction hypothesis, it is a tree.

But then $T$ is also a tree. It is connected by hypothesis. A vertex of degree 1 can not be in a cycle, so it is acyclic (since $T - x$ is acyclic, any cycle of $T$ must contain $x$). The result now follows by induction.

(1) $\Rightarrow$ (3). Since $T$ is connected, there is a path between any two vertices. It remains to show uniqueness. Suppose that $P : (x = v_0), v_1, \ldots, (v_k = y)$ and $Q : (x = w_0), w_1, \ldots, (w_\ell = y)$ are two different paths from $x$ to $y$. Since $v_0 = w_0$ and the paths are different, there exists a smallest subscript, $i$, such that $v_{i+1} \neq w_{i+1}$. Similarly, since $w_\ell = v_k$, there exists a smallest subscript $j > i + 1$ such that $w_j$ belongs to $P$; say $w_j = v_t$. Note that $t > i$ since $P$ is a path. Then $(v_i = w_i), w_{i+1}, \ldots, (w_j = v_t), v_{t-1}, \ldots, v_i$ is a cycle in $T$, a contradiction. Thus any two vertices of $T$ are joined by a unique path.

(3) $\Rightarrow$ (1). If any two vertices of $T$ are joined by a unique path, then $T$ is connected. Since there are two different paths between any two vertices belonging to a cycle, it follows that $T$ is acyclic. Therefore, $T$ is a tree. □

# 11 Planar graphs

A graph is *planar* if it can be drawn in the plane so that edges meet only at their ends. A *plane embedding* of a planar is a drawing of the graph in the plane so that edges meet only at their ends. The term *plane graph* is used to refer to a planar embedding of a planar graph.

- a graph is planar of it can be drawn in the plane so no edges cross.
- A graph is planar if it has a plane embedding. Planar graphs also have drawings in which edges cross.

A plane graph divides the plane into a number of (topologically) connected regions called *faces*.

- If the graph is finite, one of the faces is infinite. It is sometimes called the *exterior face*.
- Every edge is incident with (that is, touches) one or two faces. An edge in a cycle is incident with 2 faces.

**Theorem 11.1** (Euler's formula)**.** *Let $G$ be a connected plane graph with $v$ vertices, $e$ edges and $f$ faces. Then*

$$v - e + f = 2.$$

- By Euler's formula, any two plane embeddings of a planar graph have the same number of faces.

Euler's formula can be used to show $K_{3,3}$ and $K_5$ are not planar. We will see later (Kuratowski's theorem) that there is a sense in which any non-planar graph "contains" one of them as a subgraph.

**Proposition 11.2.** *The complete bipartite graph $K_{3,3}$ is not planar.*

*Proof idea: Use Euler's formula to show that it has the wrong number of edges.*

*Proof.* Suppose $K_{3,3}$ is planar. Since it has 6 vertices and 9 edges, by Euler's formula a plane embedding of $K_{3,3}$ has 5 faces.

Since $K_{3,3}$ is bipartite, it contains no cycle of length 3 (as a 3-cycle is not bipartite). Thus the boundary of each face contains at least 4 edges. Since every edge is incident with at most 2 faces, we have $4f \leq 2e$. With $f = 5$ and $e = 9$, this inequality is $20 < 18$, a contradiction. $\square$

The same proof idea can be used to show the following.

**Proposition 11.3.** *The complete graph $K_5$ is not planar.*

*Proof.* Suppose $K_5$ is planar. Since it has 5 vertices and 10 edges, by Euler's formula a plane embedding of $K_5$ has 7 faces.

The boundary of each face contains at least 3 edges. Since every edge is incident with at most 2 faces, we have $3f \leq 2e$. With $f = 7$ and $e = 10$, this inequality is $21 < 20$, a contradiction. $\square$

And the same idea again gives an upper bound on the number of edges in a planar graph.

**Theorem 11.4.** *If $G$ is a connected planar graph with $v$ vertices and $e$ edges, then $v \leq 3e - 6$.*

*Proof.* Since a connected graph with at most 3 vertices has at most 3 edges, the theorem is true for all connected planar graphs with at most 3 vertices. Assume $v \geq 4$. If $e < 3$ then the desired inequality holds, so assume $e \geq 3$.

The boundary of each face contains at least 3 edges. Since every edge is incident with at most 2 faces, we have $3f \leq 2e$.

By Euler's formula, $2 = v - e + f \leq v - e + 2e/3 = v - e/3$, so that $6 \leq 3v - e$. Rearranging gives $e \leq 3v - 6$. ☐

**Corollary 11.5.** *Every planar graph has a vertex of degree at most 5.*

*Proof.* Let $G$ be a planar graph. We can assume $G$ is connected, otherwise the argument can be applied to a component. If every vertex has degree at least 6, then

$$6|V| \leq \sum_{v \in V} \deg(v) = 2|E| \leq 2(3|V| - 6) = 6|V| - 12,$$

a contradiction. ☐

An edge $xy$ of a graph $G$ is said to be *subdivided* if it is deleted and replaced by a path of length 2: $x, v_{xy}, y$, where $v_{xy}$ is a new vertex. A graph $H$ is a *subdivision* of a graph $G$ if a graph isomorphic to $H$ can be obtained from $G$ by a subdividing edges (possible none).

**Theorem 11.6** (Kuratowski's Theorem, 1930)**.** *A graph is planar if and only if it has no subgraph which is a subdivision of $K_{3,3}$ or $K_5$.*

Kuratowski's Theorem provides a method for certifying that a graph is, or is not planar. To show that a graph is planar, give a plane embedding. To show that a graph is not planar, give a subgraph which is a subdivision of $K_{3,3}$ or $K_5$.

# 12 Colouring

A *k-colouring* of a graph $G$ is an assignment of $k$ colours $\{1, 2, \ldots, k\}$ to the vertices of $G$ so that adjacent vertices are assigned different colours. A graph

$G$ is called *$k$-colourable* if there exists a $k$-colouring of $G$. The *chromatic number* of a graph $G$, denoted $\chi(G)$, is the smallest $k$ such that $G$ is $k$-colourable. When $G$ is clear in the context, we write $\chi$ instead of $\chi(G)$.

- Every graph with $n$ vertices is $n$-colourable: assign a different colour to every vertex. Hence, there is a smallest $k$ such that $G$ is $k$-colourable.
- There is no requirement in the definition that all colours be used. Thus, a graph which is $k$-colourable is $t$-colourable for every $t \geq k$.
- Vertices of the same colour are not adjacent.
- A graph has $\chi = 1$ if and only if it has no edges.

By definition, a graph is 2-colourable if and only if the colours 1 and 2 can be assigned to its vertices so that every edge has one end coloured 1 and the other end coloured 2. Thus, if $X$ is the set of vertices of colour 1, and $Y$ is the set of vertices coloured 2, then $(X, Y)$ is a bipartition of $G$. Conversely, if $G$ is bipartite with bipartition $(X, Y)$, then assigning colour 1 to all vertices in $X$ and colour 2 to all vertices in $Y$ is a 2-colouring of $G$.

- A graph is 2-colourable if and only if it is bipartite.
- A graph has chromatic number 2 if and only if it is bipartite and has at least 1 edge.

**Theorem 12.1.** *A graph $G$ is bipartite if and only if it has no cycles of odd length.*

*Proof.* Suppose $G$ is bipartite with bipartition $(X, Y)$. Since the vertices of any cycle are alternately in $X$ and $Y$, every cycle has even length.

Suppose $G$ has no cycles of odd length. It suffices to prove the statement for connected graphs. Let $v \in V$. Let $X$ be the set of all vertices joined to $v$ by a path of even length, and let $Y$ be the set of all vertices joined to $v$ by a path of odd length.

We claim that $(X, Y)$ is a bipartition of $G$. Since $G$ is connected, $X \cup Y = V$. Suppose $X \cap Y \neq \emptyset$, and let $w \in X \cap Y$. Then there is a path from $v$ to $w$ of even length, and a path from $v$ to $w$ of odd length. Thus, $G$ has a closed walk of odd length, and therefore has a cycle of odd length, a contradiction. Hence $G$ is bipartite. □

- To show a graph is bipartite, describe the bipartition. To show a graph is not bipartite, describe an odd cycle.
- Every complete bipartite graph $K_{m,n}$ has chromatic number 2, but the vertex degrees can all be much larger than 2.
- A 5-cycle has $\chi = 2$ but no subgraph isomorphic to $K_3$. There are graphs with arbitrarily high chromatic number and no subgraph isomorphic to $K_3$.

**Proposition 12.2.** *Let $\Delta(G)$ be the largest degree of a vertex of the graph $G$. Then $\chi(G) \leq 1 + \Delta(G)$.*

*Proof.* By induction on $|V(G)$. The statement is true for all graphs with one vertex as $\chi = 1 \leq 0 + 1$. Suppose that $\chi(G) \leq 1 + \Delta(G)$ for all graphs $G$ with $k$ vertices, for some $k \geq 1$.

Let $G$ be a graph with $k + 1$ vertices, and let $x \in V(G)$. Then the graph $G - x$ has $k$ vertices, and $\Delta(G - x) \leq \Delta(G)$. By the induction hypothesis, $\chi(G - x) \leq 1 + \Delta(G - x) \leq 1 + \Delta(G)$.

Consider a $(1 + \Delta(G))$-colouring of $G - x$. Since $x$ has degree at most $\Delta(G)$, at most $\Delta(G)$ colours appear on vertices adjacent to $x$. Thus, one of the $1 + \Delta(G)$ colours available does not appear on a vertex adjacent to $x$. Assigning that colour to $x$ gives a $(1 + \Delta(G))$-colouring of $G$.

The result now follows by induction. □

- It is known that complete graphs and odd cycles are the only graphs, $G$, with $\chi(G) = 1 + \Delta(G)$. This result is known as *Brooks' Theorem.*