Combinatorial Arguments

A *combinatorial argument*, or *combinatorial proof*, is an argument that involves counting. We have already seen this type of argument, for example in the section on Stirling numbers of the second kind.

Remember that we could define $\binom{n}{k}$ *combinatorially* (meaning as an expression that counts the number of objects of some kind) as the number of ways of selecting k distinct objects from n distinct objects without regard for order. Then, since $\binom{n}{k}$ counts selections, it must be that $\binom{n}{k} = 0$ if $k < 0$, or $k < n$, or $n < 0$. (Note: the previous sentence is a combinatorial argument.)

Example (easy combinatorial argument). We show that $\binom{n}{k} = \binom{n}{n-k}$: The LHS counts the number of ways to select k people from a group of n people to receive a candy. The RHS counts the same thing by counting the number of ways to select the $n-k$ people who will not receive a candy.

Pascal's Identity. For integers n and k, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof.

The LHS counts the number of ways to select k out of n children to have their face painted. The RHS counts the same thing according to two cases: either a specific child of the n, say Gary, is in the group selected, or he is not selected. In the first case the remaining $k-1$ children in the group must be selected from the remaining $n-1$ children. The number of ways to do this is $\binom{n-1}{k-1}$. In the second case all k children in the group must be selected from the remaining $n-1$ children. The number of ways to do this is $\binom{n-1}{k}$. By the Rule of Sum, the number of selections is $\binom{n-1}{k-1} + \binom{n-1}{k}$. Hence, the result. ∎

Advice (using combinatorial arguments to prove identities). You need to argue that LHS and RHS are just different expressions for the same number. This is often done by counting something and obtaining the LHS, and then counting it in a different way and obtaining the RHS. If one side involves a summation, take that as a suggestion to proceed by organising the counting into cases and using the Rule of Sum. A good way to break into cases is by considering what happens with some special object (Gary, in the above example). If products are involved somewhere, then the counting likely involves a sequence of steps and the Rule of Product. Also, there is no "right" thing to count, so you might as well pick something to make it entertaining.

Exercises. Let $P(n, k)$ be the number of k-permutations of n distinct objects. That is, $P(n, k)$ equals the number of ways to arrange k of n distinct objects in a line.

(a) Give a combinatorial argument to show that $P(n, k) = \binom{n}{k}k!$.

(b) By counting directly, show that for $0 \leq k \leq n$ $P(n, k) = \frac{n!}{(n-k)!}$. Use this result and part (a) to show that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \leq k \leq n$.

(c) Give a combinatorial argument to show that $P(n, k) = P(n-1, k) + kP(n-1, k-1)$.

We now prove the Binomial Theorem using a combinatorial argument. It can also be proved by other methods, for example by induction, but the combinatorial argument explains where the coefficients come from.

Binomial Theorem. If $n \geq 0$ is an integer, then $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Proof.

Consider $(x+y)^n = (x+y)(x+y)\cdots(x+y)$ multiplied out and not simplified. Each summand is a product of n symbols, one from each factor $(x+y)$. In each summand, some k symbols (where $0 \leq k \leq n$) equal x and $(n-k)$ symbols equal y. For each of the possible values of k, the number of such summands equals the number of ways of choosing which of the n factors $(x+y)$ contribute an x to the summand, which is $\binom{n}{k}$. The result follows from the Rule of Sum. ∎

It is because they appear in the Binomial Theorem that the numbers $\binom{n}{k}$ are called *binomial coefficients*. If you think of the rows of Pascal's Triangle as being numbered $0, 1, \ldots$ and the entries in each row as also being numbered $0, 1, \ldots$, then the entry number k in row n of the triangle is the coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$. By the Binomial Theorem this equals $\binom{n}{k}$. Pascal's Identity says that the binomial coefficients satisfy the same identity used to construct the triangle.

Definition ($\binom{n}{n_1, n_2, \ldots, n_t}$). If $n_1, n_2, \ldots, n_t$ are integers and $n = n_1 + n_2 + \cdots + n_t$, then the symbol $\binom{n}{n_1, n_2, \ldots, n_t}$ is defined to equal the number of ways to select n objects from a collection of objects of t different types, such that n_i objects of type i are selected, $1 \leq i \leq t$.

It is not difficult to show by direct counting that

$$\binom{n}{n_1, n_2, \ldots, n_t} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{t-1}}{n_t} = \frac{n!}{n_1!n_2!\ldots n_t!}.$$

The symbol is often called a *multinomial coefficient* because of its appearance in the Multinomial Theorem below. Notice that when $t = 2$ the multinomial coefficient $\binom{n}{k, n-k}$ is an alternate notation for $\binom{n}{k}$. When $t = 2$ the Multinomial Theorem is just the Binomial Theorem expressed using different notation.

Multinomial Theorem. For integers $n \geq 0$ and $t \geq 1$,

$$(x_1 + x_2 + \cdots x_t)^n = \sum \binom{n}{n_1, n_2, \ldots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

where the sum is over all ways of writing n as a sum of non-negative integers $n_1, n_2, \ldots, n_t$.

Proof.

Exercise. (Hint; Consider the LHS multiplied out and not simplified, and count how many terms have n_1 symbols x_1, n_2 symbols x_2, etc.) ∎

We now give examples of combinatorial arguments. These demonstrate counting the number of ways to form committees, the number of ways to select subsets of sets, the number bit strings of certain types, the number of lattice paths, and the number of pairs of objects. (Almost anything you can think of has a chance of working.)

Vandermonde's Identity. $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

Proof.

The LHS counts the number of ways to choose a committee of r people from a group of m men and n women. The RHS counts the same thing according to cases depending on the number of men on the committee, which can range from 0 to r. If there are t men, then there must be $r - t$ women. Since in such a case there are $\binom{m}{t}$ ways to select the men and $\binom{n}{r-t}$ ways to select the women, the number of such committees is $\binom{m}{t}\binom{n}{r-t}$. The result now follows from the Rule of Sum. ∎

A good time to try a committee selection argument is when proving an identity involving binonial coeffocients and the "top" of one side of the equality involves a sum like $m + n$. This suggests one might try counting selections of people from a group of m men and n women. (Note: $2n = n + n$.)

Corollary. $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

Exercise. Use a committee selection argument to show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Example (counting bit strings). Prove that $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$.

The LHS counts the number of bit strings of length $n + 1$ with $r + 1$ ones. The RHS counts the same thing according to cases depending on the position of the rightmost one. It must be in one of the positions $r + 1, r + 2, \ldots, n + 1$. If it is in position t, then r of the $t - 1$ positions to its left hold a 1. Thus, the number of bit strings where the rightmost 1 is in position t is $\binom{t-1}{r}$. After letting $k = t - 1$, the result follows from the Rule of Sum. ∎

Exercise. Use an argument involving counting bit strings to give a combinatorial proof that $2^n - 1 = \sum_{k=0}^{n-1} 2^k$.

Example (counting lattice paths). Lattice paths are paths in the plane from $(0, 0)$ to (x, y) where at each step you must move either one unit up or one unit right. The main idea is that you choose the steps on which you move up (say). We prove the same identity as above, $\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$, using this method.

The LHS counts the number of such paths from $(0, 0)$ to $(n - r, r + 1)$. The RHS counts the same thing according to cases depending on the last time you move right. Each path involves $n + 1$ moves, of which $n - r$ are up, and $r + 1$ are right. Since you move right $r + 1$ times, the last move right occurs on one of moves $r + 1, r + 2, \ldots, n + 1$, and all subsequent moves are up. The number of paths in which the last move right occurs on move $n + 1$ equals the number of paths from $(0, 0)$ to $(n - r, r)$, which is $\binom{n}{r}$. The number where the last move right occurs on move n equals the number of paths from $(0, 0)$ to $(n - r - 1, r)$, which is $\binom{n-1}{r}$. The number where the last move right occurs on move $n - 1$ equals the number of paths from $(0, 0)$ to $(n - r - 2, r)$, which is $\binom{n-2}{r}$ and so on, until finally, the number where the last move right occurs on move $r + 1$ equals the number of paths from $(0, 0)$ to $(0, r)$, which is $\binom{r}{r}$. The result now follows from the Rule of Sum.

Exercise. Use an argument involving counting lattice paths to give a combinatorial proof that $2^n - 1 = \sum_{k=0}^{n-1} 2^k$.

Example (counting pairs). We prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

The LHS counts the number of pairs (x, S), where S is a k-subset of $\{1, 2, \ldots, n\}$ and $x \in S$. There are $\binom{n}{k}$ choices for k, and for each of these there are k choices for x. The RHS counts the same thing in a different order. First choose x - there are n choices - and then choose the other $k-1$ elements of S from the remaining $n-1$ elements. of $\{1, 2, \ldots, n\}$.

Exercise. Show that $\binom{n}{k}\binom{k}{t} = \binom{n}{t}\binom{n-t}{k-t}$.

Example (counting pairs). We prove that $\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$.

We claim that both sides count the number of pairs (X, Y) of disjoint subsets of $\{1, 2, \ldots, n\}$ such that $|X \cup Y| = m$.

On the one hand there are $\binom{n}{m}$ ways to choose the set $X \cup Y$, and for each of these there are 2^m ways to select a subset of these m elements to form the subset X (the remaining elements will comprise the subset Y). Thus, by the Rule of Product, the number of pairs is $2^m\binom{n}{m}$, which is the RHS.

On the other hand, the set X must have some number k of elements, where $0 \leq k \leq m$. For each such k there are $\binom{n}{k}$ ways to select the elements of X, and then $\binom{n-k}{m-k}$ ways to select the elements of Y from the $n-k$ elements of $\{1, 2, \ldots, n\}$ that are not in X. Thus, for each possible k, the number of pairs (X, Y) with $|X| = k$ and satisfying the given conditions is $\binom{n}{k}\binom{n-k}{m-k} = 2^m\binom{n}{m}$. By the Rule of Sum, the total number of pairs is $\sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{m-k}$, which is the LHS.

Exercise. Use an argument involving counting pairs (x, S), where S is a subset of $\{1, 2, \ldots, n\}$ and $x \in S$ to show that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Exercise. We start a combinatorial proof that $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n-1)2^{n-2} + n2^{n-1}$. The exercise is to finish it.

Each term on the LHS counts the number of ways to choose a subset S of k elements of $\{1, 2, \ldots, n\}$ and then a sequence of two not necessarily distinct elements from S. Thus, the LHS counts the number of ways to select a triple (S, x, y), where S is a subset of $\{1, 2, \ldots, n\}$ and x and y are not necessarily distinct elements from S. The RHS counts the same thing by ...