Mathematical Induction

Principle of Mathematical Induction: Let $P(n)$ be a statement involving the integer n. IF the statement is true when $n = 1$, and whenever the statement is true for $n = k$, then it is also true for $n = k + 1$, THEN the statement is true for all integers $n \geq 1$.

There is nothing special about the integer 1 in the statement above. It can be replaced (in both places it occurs) by any other integer, and the Principle still works.

Example 1. Prove that for all $n \geq 5$, $2^n > n^2$.

BASIS: When $n = 5$, $2^n = 2^5 = 32$ and $n^2 = 5^2 = 25$. As $32 > 25$ the statement is true for $n = 5$.

INDUCTION HYPOTHESIS: Suppose $2^k > k^2$, for some $k \geq 5$.

INDUCTION STEP: We want to show $2^{k+1} > (k+1)^2$. Consider $(k+1)^2 = k^2 + 2k + 1 < k^2 + 2k + k$ (as $k \geq 5 > 1$) $= k^2 + 3k < k^2 + k(k)$ (as $k \geq 5 > 3$) $= 2k^2 < 2(2^k)$ (by the induction hypothesis) $= 2^{k+1}$, which is what we wanted.

Therefore, by the Principle of Mathematical Induction, for all $n \geq 5$, $2^n > n^2$.

Here's my view of PMI: First you have a proof showing the statement you want to prove is true for $n = 1$ (say). Usually this amounts to no more than just checking some easy cases. Then, you supply an argument that shows that IF the statement is true for $n = k$, THEN it must also be true for $n = k+1$. Now, you know the statement is true for $n = 1$, so your argument shows it is true for $n = 2$. But now you know it is true for $n = 2$, so your argument shows it is true for $n = 3$. But now you know it is true for $n = 3$, so your argument shows it is true for $n = 4$. Clearly this process can be repeated as many times as you want, so the only reasonable conclusion is that the statement is true for every integer greater than or equal to 1.

When you make your induction hypothesis, you assume that the statement you want to prove is true for $n = k$. But think about the above paragraph: By the time you know the statement is true for $n = k$, you've already proved that it is true for all numbers between 1 and k inclusive. So, why not just make this stronger hypothesis instead? There is no reason not to. This version of PMI is sometimes called Strong Induction, because it appears that you are making a stronger induction hypothesis (i.e., you are assuming more). However, in fact, the two principles are equivalent, and anything that can be proved with one can be proved with the other; its just a matter of how much writing you have to do. Using strong induction sometimes makes the proofs a lot shorter.

Principle of Strong Mathematical Induction: Let $P(n)$ be a statement involving the integer n. IF the statement is true when $n = 1$, and whenever the statement is true for $n = 1, 2, \ldots k$, then it is also true for $n = k + 1$, THEN the statement is true for all integers $n \geq 1$.

There is nothing special about the integer 1 in the statement above. It can be replaced (in all three places it occurs) by any other integer, and the Principle still works.

Example 2. The Fibonacci numbers $f_0, f_1, f_2, \ldots$ are defined by $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. Prove that for all $n \geq 0$, $f_n \leq (7/4)^n$.

BASIS: When $n = 0$ we have $f_n = f_0 = 1$ and $(7/4)^n = (7/4)^0 = 1$ As $1 \leq 1$, the statement is true when $n = 1$. When $n = 1$ we have $f_n = f_1 = 1$ and $(7/4)^n = (7/4)^1 = (7/4)$. As $1 \leq (7/4)$ the statement is true when $n = 1$.

(*Comment: Notice that I used 2 cases in the basis here, whereas the principle seems to only require one case. The reason for this is that in the induction step I will want to take f_{k+1} and replace it by $f_k + f_{k-1}$. I can only do this when $k+1$ is at least 2, which means the case $k+1 = 1$ needs to be treated separately. I have made a choice to do this as part of the basis. It could equally well have been done as part of the induction step.*)

INDUCTION HYPOTHESIS: Assume that $f_i \leq (7/4)^i$ for $i = 1, 2, \ldots, k$. (Because of the basis, we can assume $k \geq 1$.)

INDUCTION STEP: We want to show that $f_{k+1} \le (7/4)^{k+1}$. Consider $f_{k+1} = f_k + f_{k-1}$ (We can do this as $k+1$ is at least 2; see the comment following the basis) $< (7/4)^k + (7/4)^{k-1}$ (by the Induction Hypothesis; notice how the stronger hypothesis comes in handy here.) $= (7/4)^{k-1}(7/4 + 1) = (7/4)^{k-1}(11/4) = (7/4)^{k-1}(44/16) \le (7/4)^{k-1}(49/16) = (7/4)^{k-1}(7/4)^2 = (7/4)^{k+1}$, which is what we wanted.

Therefore, by Strong Induction, for all $n \ge 0$, $f_n \le (7/4)^n$.

Example 3. Prove that any positive integer greater than or equal to 8 can be written as a sum of 3's and 5's. (For example, $8 = 5+3, 9 = 3+3+3, 10 = 5+5, 11 = 3+3+5$, etc.). The statement we want to prove is "any integer $n \ge 8$ can be written as a sum of 3's and 5's".

BASIS: Since $8 = 5+3$, $9 = 3+3+3$, and $10 = 5+5$, each of 8, 9, and 10 can be written as a sum of 3's and 5's. (The reason for using 3 cases in the basis will become apparent below.)

INDUCTION HYPOTHESIS: Assume each of $8, 9, 10, \ldots k$ can be written as a sum of 3's and 5's. Because of the basis we can assume $k \ge 10$.

INDUCTION STEP: We want to show that $k+1$ can be written as a sum of 3's and 5's. Since $k \ge 10$, $k+1-3 \ge 8$, so by the induction hypothesis, $k+1-3$ can be written as a sum of 3's and 5's (*Comment: this is why we used 3 cases in the basis.*). But then adding three to this sum gives $k+1$ as a sum of 3's and 5's, which is what we wanted.

Therefore, by Strong Induction, any integer $n \ge 8$ can be written as a sum of 3's and 5's.

To illustrate the induction step, suppose $k+1 = 11$. Then $k+1-3 = 11-3 = 8 = 5+3$, so $k+1 = 8+3 = 5+3+3$, a sum of 3's and 5's.

Problems

1. Consider the sequence $a_0, a_1, a_2, \ldots$ of integers defined by $a_0 = 10$ and $a_n = 2a_{n-1}$, $n \ge 1$. Prove that $a_n = 2^n 10$ for all $n \ge 0$.

2. Guess and prove a formula for $1 - 2 + 3 - 4 + \ldots + (-1)^{n-1}n$ (i.e., one that works for any $n \ge 1$).

3. Prove that for all $n \ge 1$, $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1}n^2 = (-1)^n n(n+1)/2$.

4. Prove that for all $n \ge 1$, $1(2) + 2(3) + 3(4) + \ldots + n(n+1) = n(n+1)(n+2)/3$.

5. Prove that for all $n \ge 1$, $1^3 + 2^3 + 3^3 + \ldots n^3 = n^2(n+1)^2/4$.

6. Prove that for all $n \ge 1$, $\frac{1}{1(2)} + \frac{1}{2(3)} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

7. Prove that for all $n \ge 1$, $1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1$.

8. Prove by induction that if n distinct dice are rolled, then the number of outcomes where the sum of the faces is an even integer equals the number of outcomes where the sum of the faces is an odd integer.

9. Prove by induction that the number of binary sequences of length n with an even number of ones equals the number of binary sequences of length n with an odd number of ones.

10. Prove by induction that for any integer $n \ge 1$, $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

11. Let f_n denote the n-th Fibonacci number. Prove that for all $n \ge 1$, $f_n \ge (3/2)^{n-1}$.

12. Prove that any integer greater than or equal to 35 can be written as a sum of 5's and 6's.