COARSE AND EQUIVARIANT CO-ASSEMBLY MAPS

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Abstract. We study an equivariant co-assembly map that is dual to the usual Baum–Connes assembly map and closely related to coarse geometry, equivariant Kasparov theory, and the existence of dual Dirac morphisms. As applications, we prove the existence of dual Dirac morphisms for groups with suitable compactifications, that is, satisfying the Carlsson–Pedersen condition, and we study a K–theoretic counterpart to the proper Lipschitz cohomology of Connes, Gromov and Moscovici.

1. Introduction

This is a sequel to the articles [6, 7], which deal with a coarse co-assembly map that is dual to the usual coarse assembly map. Here we study an equivariant co-assembly map that is dual to the Baum–Connes assembly map for a group G.

A rather obvious choice for such a dual map is the map

$$p_{\mathcal{E}}G^*\colon \mathrm{KK}^G_*(\mathbb{C},\mathbb{C}) \to \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}) \tag{1}$$

induced by the projection $p_{\mathcal{E}}G\colon \mathcal{E}G \to \text{point}$. This map and its application to the Novikov conjecture go back to Kasparov ([10]). Nevertheless, (1) is not quite the map that we consider here. Our map is closely related to the coarse co-assembly map of [6]. It is an isomorphism if and only if the Dirac-dual-Dirac method applies to G. Hence there are many cases—groups with $\gamma \neq 1$—where our co-assembly map is an isomorphism and (1) is not.

Most of our results only work if the group G is (almost) totally disconnected and has a G–compact universal proper G–space $\mathcal{E}G$. We impose this assumption throughout the introduction.

First we briefly recall some of the main ideas of [6, 7]. The new ingredient in the coarse co-assembly map is the *reduced stable Higson corona* $\mathfrak{c}^{\mathrm{red}}(X)$ of a coarse space X. Its definition resembles that of the usual Higson corona, but its K–theory behaves much better. The coarse co-assembly map is a map

$$\mu\colon \mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathrm{red}}(X)\bigr) \to \mathrm{KX}^*(X), \tag{2}$$

where $\mathrm{KX}^*(X)$ is a coarse invariant of X that agrees with $\mathrm{K}^*(X)$ if X is uniformly contractible.

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If $|G|$ is the coarse space underlying a group G, then there is a commuting diagram

$$\begin{array}{ccc} \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(G)\bigr) & \xrightarrow{p_{\mathcal{E}}G^*} & \mathrm{RKK}^G_*\bigl(\mathcal{E}G; \mathbb{C}, C_0(G)\bigr) \\ \updownarrow\cong & & \updownarrow\cong \\ \mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) & \xrightarrow{\mu} & \mathrm{KX}^*(|G|). \end{array} \tag{3}$$

In this situation, $\mathrm{KX}^*(|G|) \cong \mathrm{K}^*(\mathcal{E}G)$ because $|G|$ is coarsely equivalent to $\mathcal{E}G$, which is uniformly contractible. The commuting diagram (3), coupled with the reformulation of the Baum Connes assembly map in [11], is the source of the relationship between the coarse co-assembly map and the Dirac-dual-Dirac method mentioned above.

If G is a torsion-free discrete group with finite classifying space BG, the coarse co-assembly map is an isomorphism if and only if the Dirac-dual-Dirac method applies to G. A similar result for groups with torsion is available, but this requires working equivariantly with respect to compact subgroups of G.

In this article, we work equivariantly with respect to the whole group G. The action of G on its underlying coarse space $|G|$ by isometries induces an action on $\mathfrak{c}^{\mathfrak{red}}(G)$. We consider a G–equivariant analogue

$$\mu\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) \to \mathrm{K}_*(C_0(\mathcal{E}G) \rtimes G) \tag{4}$$

of the coarse co-assembly map (2); here $\mathrm{K}^{\mathrm{top}}_*(G, A)$ denotes the domain of the Baum–Connes assembly map for G with coefficients A. We avoid $\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(X) \rtimes G)$ and $\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(X) \rtimes_{\mathrm{r}} G)$ because we can say nothing about these two groups. In contrast, the group $\mathrm{K}^{\mathrm{top}}_*\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr)$ is much more manageable. The only *analytical* difficulties in this group come from coarse geometry.

There is a commuting diagram similar to (3) that relates (4) to equivariant Kasparov theory. To formulate this, we need some results of [11]. There is a certain G–C^*-algebra P and a class $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ called *Dirac morphism* such that the Baum–Connes assembly map for G is equivalent to the map

$$\mathrm{K}_*\bigl((A \otimes \mathsf{P}) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G)$$

induced by Kasparov product with D. The Baum–Connes conjecture holds for G with coefficients in $\mathsf{P} \otimes A$ for any A. The Dirac morphism is a *weak equivalence*, that is, its image in $\mathrm{KK}^H(\mathsf{P}, \mathbb{C})$ is invertible for each compact subgroup H of G.

The existence of the Dirac morphism allows us to localise the (triangulated) category KK^G at the multiplicative system of weak equivalences. The functor from KK^G to its localisation turns out to be equivalent to the map

$$p_{\mathcal{E}}G^*\colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B).$$

One of the main results of this paper is a commuting diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) & \longrightarrow & \mathrm{K}^*(\mathcal{E}G) \\ \updownarrow\cong & & \updownarrow\cong \\ \mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) & \xrightarrow{p_{\mathcal{E}}G^*} & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}). \end{array} \tag{5}$$

In other words, the equivariant coarse co-assembly (4) is equivalent to the map

$$p_{\mathcal{E}}G^*\colon \mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}).$$

This map is our proposal for a dual to the Baum–Connes assembly map.

We should justify why we prefer the map (4) over (1). Both maps have isomorphic targets:

$$\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathsf{P}) \cong \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}) \cong \mathrm{K}_*(C_0(\mathcal{E}G)\rtimes G).$$

Even in the usual Baum–Connes assembly map, the analytical side involves a choice between full and reduced group C^*–algebras and crossed products. Even though the full group C^*–algebra has better functoriality properties and is sometimes preferred because it gives potentially finer invariants, the reduced one is used because its K–theory is closer to $\mathrm{K}^{\mathrm{top}}_*(G)$. In formulating a dual version of the assembly map, we are faced with a similar situation. Namely, the topological object that is dual to $\mathrm{K}^{\mathrm{top}}_*(G)$ is $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$. For the analytical side, we have some choices; we prefer $\mathrm{KK}^G_*(\mathbb{C},\mathsf{P})$ over $\mathrm{KK}^G_*(\mathbb{C},\mathbb{C})$ because the resulting co-assembly map is an isomorphism in more cases.

Of course, we must check that this choice is analytical enough to be useful for applications. The most important of these is the Novikov conjecture. Elements of $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$ yield maps $\mathrm{K}^{\mathrm{top}}_*(G)\to\mathbb{Z}$, which are analogous to higher signatures. In particular, (4) gives rise to such objects. The maps $\mathrm{K}^{\mathrm{top}}_*(G)\to\mathbb{Z}$ that come from a class in the range of (1) are known to yield homotopy invariants for manifolds because there is a pairing between $\mathrm{KK}^G_*(\mathbb{C},\mathbb{C})$ and $\mathrm{K}_*(C^*_{\max}G)$ (see [8]). But since (4) factors through (1), the former also produces homotopy invariants.

In particular, surjectivity of (4) implies the Novikov conjecture for G. More is true: since $\mathrm{KK}^G(\mathbb{C},\mathsf{P})$ is the home of a *dual-Dirac morphism*, (5) yields that G has a dual-Dirac morphism and hence a γ*–element* if and only if (4) is an isomorphism. This observation can be used to give an alternative proof of the main result of [7].

We call elements in the range of (4) *boundary classes*. These automatically form a graded ideal in the $\mathbb{Z}/2$-graded unital ring $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$. In contrast, the range of the unital ring homomorphism (1) need not be an ideal because it always contains the unit element of $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$. We describe two important constructions of boundary classes, which are related to compactifications and to the proper Lipschitz cohomology of G studied in [3, 4].

Let $\mathcal{E}G\subseteq Z$ be a G–equivariant compactification of $\mathcal{E}G$ that is compatible with the coarse structure in a suitable sense. Since there is a map

$$\mathrm{K}^{\mathrm{top}}_*\bigl(G, C(Z\setminus\mathcal{E}G)\bigr)\to \mathrm{K}^{\mathrm{top}}_*\bigl(G,\mathfrak{c}^{\mathfrak{red}}(\mathcal{E}G)\bigr),$$

we get boundary classes from the boundary $Z\setminus\mathcal{E}G$. This construction also shows that G has a dual-Dirac morphism if it satisfies the Carlsson–Pedersen condition. This improves upon a result of Nigel Higson ([9]), which shows split injectivity of the Baum–Connes assembly map with coefficients under the same assumptions.

Although we have discussed only $\mathrm{KK}^G_*(\mathbb{C},\mathsf{P})$ so far, our main technical result is more general and can also be used to construct elements in Kasparov groups of the form $\mathrm{KK}^G_*\bigl(\mathbb{C},C_0(X)\bigr)$ for suitable G–spaces X. If X is a proper G–space, then we can use such classes to construct boundary classes in $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$. This provides a K–theoretic counterpart of the proper Lipschitz cohomology of G defined by Connes, Gromov, and Moscovici in [3]. Our approach clarifies the geometric parts of several constructions in [3]; thus we substantially simplify the proof of the homotopy invariance of Gelfand–Fuchs cohomology classes in [3].

2. Preliminaries

2.1. **Dirac-dual-Dirac method and Baum–Connes conjecture.** First, we recall the *Dirac-dual-Dirac method* of Kasparov and its reformulation in [11]. This is a technique for proving injectivity of the *Baum–Connes assembly map*

$$\mu\colon \mathrm{K}^{\mathrm{top}}_*(G,B)\to \mathrm{K}_*\bigl(C^*_r(G,B)\bigr), \tag{6}$$

where G is a locally compact group and B is a C^*–algebra with a strongly continuous action of G or, briefly, a *G–C^*-algebra*.

This method requires a proper G-C^*-algebra A and classes

$$d \in \mathrm{KK}^G(A, \mathbb{C}), \qquad \eta \in \mathrm{KK}^G(\mathbb{C}, A), \qquad \gamma := \eta \otimes_A d \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C}),$$

such that $p_{\mathcal{E}} G^*(\gamma) = 1_{\mathbb{C}}$ in $\mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. If these data exist, then the Baum–Connes assembly map (6) is injective for all B. If, in addition, $\gamma = 1_{\mathbb{C}}$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$, then the Baum–Connes assembly map is invertible for all B, so that G satisfies the Baum–Connes conjecture with arbitrary coefficients.

Let A and B be G–C^*-algebras. An element $f \in \mathrm{KK}^G(A, B)$ is called a *weak equivalence* in [11] if its image in $\mathrm{KK}^H(A, B)$ is invertible for each compact subgroup H of G.

The following theorem contains some of the main results of [11].

Theorem 1. *Let G be a locally compact group. Then there is a G–C^*-algebra P and a class $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ called* Dirac morphism *such that*

(a) *D is a weak equivalence;*
(b) *the Baum–Connes conjecture holds with coefficients in $A \otimes \mathsf{P}$ for any A;*
(c) *the assembly map* (6) *is equivalent to the map*

$$\mathsf{D}_*\colon \mathrm{K}_*(A \otimes \mathsf{P} \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G);$$

(d) *the Dirac-dual-Dirac method applies to G if and only if there is a class $\eta \in \mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ with $\eta \otimes_{\mathbb{C}} D = 1_A$, if and only if the map*

$$\mathsf{D}^*\colon \mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \to \mathrm{KK}^G_*(\mathsf{P}, \mathsf{P}), \qquad x \mapsto D \otimes x, \tag{7}$$

is an isomorphism.

Whereas [7] studies the invertibility of (7) by relating it to (2), here we are going to study the map (7) itself.

It is shown in [11] that the localisation of the category KK^G at the weak equivalences is isomorphic to the category $\mathrm{RKK}^G(\mathcal{E}G)$ whose morphism spaces are the groups $\mathrm{RKK}^G(\mathcal{E}G; A, B)$ as defined by Kasparov in [10]. This statement is equivalent to the existence of a Poincaré duality isomorphism

$$\mathrm{KK}^G_*(A \otimes \mathsf{P}, B) \cong \mathrm{RKK}^G_*(\mathcal{E}G; A, B) \tag{8}$$

for all G-C^*-algebras A and B (this notion of duality is analysed in [5]). The canonical functor from KK^G to the localisation becomes the obvious functor

$$p_{\mathcal{E}} G^*\colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B).$$

Since D is a weak equivalence, $p_{\mathcal{E}} G^*(\mathsf{D})$ is invertible. Hence the maps in the following commuting square are isomorphisms for all G–C^*-algebras A and B:

$$\begin{array}{ccc}
\mathrm{RKK}^G_*(\mathcal{E}G; A, B \otimes \mathsf{P}) & \xrightarrow[\cong]{\mathsf{D}_*} & \mathrm{RKK}^G_*(\mathcal{E}G; A, B) \\
\cong \big\downarrow \mathsf{D}^* & & \cong \big\downarrow \mathsf{D}^* \\
\mathrm{RKK}^G_*(\mathcal{E}G; A \otimes \mathsf{P}, B \otimes \mathsf{P}) & \xrightarrow[\cong]{\mathsf{D}_*} & \mathrm{RKK}^G_*(\mathcal{E}G; A \otimes \mathsf{P}, B).
\end{array}$$

Together with (8) this implies

$$\mathrm{KK}^G_*(A \otimes \mathsf{P}, B) \cong \mathrm{KK}^G_*(A \otimes \mathsf{P}, B \otimes \mathsf{P}).$$

In the following, it will be useful to turn the isomorphism

$$\mathrm{K}^{\mathrm{top}}_*(G, A) \cong \mathrm{K}_*\big((A \otimes \mathsf{P}) \rtimes_{\mathrm{r}} G\big)$$

in Theorem 1.(c) into a definition.

2.2. **Group actions on coarse spaces.** Let G be a locally compact group and let X be a right G–space and a coarse space. We always assume that G acts continuously and coarsely on X, that is, the set $\{(xg, yg) \mid g \in K, (x, y) \in E\}$ is an entourage for any compact subset K of G and any entourage E of X.

Definition 2. *We say that* G acts by translations *on* X *if* $\{(x, gx) \mid x \in X,\ g \in K\}$ *is an entourage for all compact subsets* $K \subseteq G$. *We say that* G *acts by* isometries *if every entourage of* X *is contained in a* G*–invariant entourage.*

Example 3. Let G be a locally compact group. Then G has a unique coarse structure for which the right translation action is isometric; the corresponding coarse space is denoted $|G|$. The generating entourages are of the form

$$\bigcup_{g \in G} Kg \times Kg = \{(xg, yg) \mid g \in G, x, y \in K\}$$

for compact subsets K of G. The *left* translation action is an action by translations for this coarse structure.

Example 4. More generally, any proper, G–compact G–space X carries a unique coarse structure for which G acts isometrically; its entourages are defined as in Example 3. With this coarse structure, the orbit map $G \to X$, $g \mapsto g \cdot x$, is a coarse equivalence for any choice of $x \in X$. If the G–compactness assumption is omitted, the result is a σ–coarse space. We always equip a proper G–space with this additional structure.

2.3. **The stable Higson corona.** We next recall the definition of the *stable Higson corona* of a coarse space X from [6, 7]. Let D be a C^*–algebra.

Let $\mathcal{M}(D \otimes \mathbb{K})$ be the multiplier algebra of $D \otimes \mathbb{K}$, and let $\bar{\mathfrak{B}}^{\mathrm{red}}(X, D)$ be the C^*–algebra of norm-continuous, bounded functions $f\colon X \to \mathcal{M}(D \otimes \mathbb{K})$ for which $f(x) - f(y) \in D \otimes \mathbb{K}$ for all $x, y \in X$. We also let

$$\mathfrak{B}^{\mathrm{red}}(X, D) := \bar{\mathfrak{B}}^{\mathrm{red}}(X, D)/C_0(X, D \otimes \mathbb{K}).$$

Definition 5. *A function* $f \in \bar{\mathfrak{B}}^{\mathrm{red}}(X, D)$ *has* vanishing variation *if the function* $E \ni (x, y) \mapsto \|f(x) - f(y)\|$ *vanishes at* ∞ *for any closed entourage* $E \subseteq X \times X$.

The *reduced stable Higson compactification* of X with coefficients D is the subalgebra $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D) \subseteq \bar{\mathfrak{B}}^{\mathrm{red}}(X, D)$ of vanishing variation functions. The quotient

$$\mathfrak{c}^{\mathrm{red}}(X, D) := \bar{\mathfrak{c}}^{\mathrm{red}}(X, D)/C_0(X, D \otimes \mathbb{K}) \subseteq \mathfrak{B}^{\mathrm{red}}(X, D)$$

is called *reduced stable Higson corona* of X. This defines a functor on the coarse category of coarse spaces: a coarse map $f\colon X \to X'$ induces a map $\mathfrak{c}^{\mathrm{red}}(X', D) \to \mathfrak{c}^{\mathrm{red}}(X, D)$, and two maps $X \to X'$ induce the same map $\mathfrak{c}^{\mathrm{red}}(X', D) \to \mathfrak{c}^{\mathrm{red}}(X, D)$ if they are close. Hence a coarse equivalence $X \to X'$ induces an isomorphism $\mathfrak{c}^{\mathrm{red}}(X', D) \cong \mathfrak{c}^{\mathrm{red}}(X, D)$.

For some technical purposes, we must allow unions $\mathscr{X} = \bigcup X_n$ of coarse spaces such that the embeddings $X_n \to X_{n+1}$ are coarse equivalences; such spaces are called σ*–coarse spaces*. The main example is the *Rips complex* $\mathscr{P}(X)$ of a coarse space X, which is used to define its coarse K–theory. More generally, if X is a proper but not G–compact G–space, then X may be endowed with the structure of a σ–coarse space. For coarse spaces of the form $|G|$ for a locally compact group G with a G–compact universal proper G–space $\mathcal{E}G$, we may use $\mathcal{E}G$ instead of $\mathscr{P}(X)$ because $\mathcal{E}G$ is coarsely equivalent to G and uniformly contractible. Therefore, we do not need σ–coarse spaces much; they only occur in Lemma 7.

It is straightforward to extend the definitions of $\bar{\mathfrak{c}}^{\mathrm{red}}(X, D)$ and $\mathfrak{c}^{\mathrm{red}}(X, D)$ to σ–coarse spaces (see [6, 7]). Since we do not use this generalisation much, we omit details on this.

Let H be a locally compact group that acts coarsely and properly on X. It is crucial for us to allow non-compact groups here, whereas [7] mainly needs equivariance for compact groups. Let D be an H–C^*-algebra, and let $\mathbb{K}_H := \mathbb{K}(\ell^2\mathbb{N}\otimes L^2H)$. Then H acts on $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D\otimes\mathbb{K}_H)$ by

$$(h\cdot f)(x) := h\cdot\bigl(f(xh)\bigr),$$

where we use the obvious action of H on $D\otimes\mathbb{K}_H$ and its multiplier algebra. The action of H on $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D\otimes\mathbb{K}_H)$ need not be continuous; we let $\bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D)$ be the subalgebra of H–continuous elements in $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D\otimes\mathbb{K}_H)$. We let $\bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D)$ be the subalgebra of vanishing variation functions in $\bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D)$. Both algebras contain $C_0(X, D\otimes\mathbb{K}_H)$ as an ideal. The corresponding quotients are denoted by $\mathfrak{B}^{\mathfrak{red}}_H(X, D)$ and $\mathfrak{c}^{\mathfrak{red}}_H(X, D)$. By construction, we have a natural morphism of extensions of H–C^*-algebras

$$\begin{array}{ccccc} C_0(X, D\otimes\mathbb{K}_H) & \rightarrowtail & \bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D) & \twoheadrightarrow & \mathfrak{c}^{\mathfrak{red}}_H(X, D) \\ \| & & \downarrow\subseteq & & \downarrow\subseteq \\ C_0(X, D\otimes\mathbb{K}_H) & \rightarrowtail & \bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D) & \twoheadrightarrow & \mathfrak{B}^{\mathfrak{red}}_H(X, D). \end{array} \tag{9}$$

Concerning the extension of this construction to σ–coarse spaces, we only mention one technical subtlety. We must extend the functor $\mathrm{K}^{\mathrm{top}}_*(H, \sqcup)$ from C^*–algebras to σ–H-C^*–algebras. Here we use the definition

$$\mathrm{K}^{\mathrm{top}}_*(H, A) \cong \mathrm{K}_*\bigl((A\otimes\mathsf{P})\rtimes_{\mathrm{r}} H\bigr), \tag{10}$$

where $\mathsf{D}\in\mathrm{KK}^H(\mathsf{P}, \mathbb{C})$ is a Dirac morphism for H. The more traditional definition as a colimit of $\mathrm{KK}^G_*(C_0(X), A)$, where $X\subseteq\mathcal{E}G$ is G–compact, yields a wrong result if A is a σ–H-C^*-algebra because colimits and limits do not commute.

Let H be a locally compact group, let X be a coarse space with an isometric, continuous, proper action of H, and let D be an H–C^*-algebra. The H–equivariant coarse K–theory $\mathrm{KX}^*_H(X, D)$ of X with coefficients in D is defined in [7] by

$$\mathrm{KX}^*_H(X, D) := \mathrm{K}^{\mathrm{top}}_*\bigl(H, C_0(\mathscr{P}(X), D)\bigr). \tag{11}$$

As observed in [7], we have $\mathrm{K}^{\mathrm{top}}_*\bigl(H, C_0(\mathscr{P}(X), D)\bigr)\cong\mathrm{K}_*(C_0(\mathscr{P}(X), D)\rtimes H)$ because H acts properly on $\mathscr{P}(X)$.

For most of our applications, X will be equivariantly uniformly contractible for all compact subgroups $K\subseteq H$, that is, the natural embedding $X\to\mathscr{P}(X)$ is a K–equivariant coarse homotopy equivalence. In such cases, we simply have

$$\mathrm{KX}^*_H(X, D)\cong\mathrm{K}^{\mathrm{top}}_*\bigl(H, C_0(X, D)\bigr). \tag{12}$$

In particular, this applies if X is an H–compact universal proper H–space (again, recall that the coarse structure is determined by requiring H to act isometrically).

The *H–equivariant coarse co-assembly map for X with coefficients in D* is a certain map

$$\mu^*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(H, \mathfrak{c}^{\mathfrak{red}}_H(X, D)\bigr)\to\mathrm{KX}^*_H(X, D)$$

defined in [7]. In the special case where we have (12), this is simply the boundary map for the extension $C_0(X, D\otimes\mathbb{K}_H)\rightarrowtail\bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D)\twoheadrightarrow\mathfrak{c}^{\mathfrak{red}}_H(X, D)$. We are implicitly using the fact that the functor $\mathrm{K}^{\mathrm{top}}_*(H, \sqcup)$ has long exact sequences for arbitrary extensions of H-C^*–algebras, which is proved in [7] using the isomorphism

$$\mathrm{K}^{\mathrm{top}}_*(H, B) := \mathrm{K}_*\bigl((B\otimes\mathsf{P})\rtimes_{\mathrm{r}} H\bigr)\cong\mathrm{K}_*\bigl((B\otimes_{\max}\mathsf{P})\rtimes H\bigr)$$

and exactness properties of maximal C^*–tensor products and full crossed products.

There is also an alternative picture of the co-assembly map as a forget-control map, provided X is uniformly contractible (see [7, §2.8]). We have the following equivariant version of this result:

Proposition 6. *Let G be a totally disconnected group with a G–compact universal proper G–space $\mathcal{E}G$. Then the G–equivariant coarse co-assembly map for G is equivalent to the map*

$$j_*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) \to \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr)$$

induced by the inclusion $j\colon \mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G, D) \to \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)$.

The equivalence of the two maps means that there is a natural commuting diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}_G(|G|, D)\bigr) & \xrightarrow{\mu^*} & \mathrm{KX}^G_*(|G|, D) \\ \updownarrow\cong & & \updownarrow\cong \\ \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) & \xrightarrow{j^*} & \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr). \end{array}$$

Recall that j is induced by the inclusion $\bar{\mathfrak{c}}^{\mathfrak{red}}_G(\mathcal{E}G, D) \to \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)$, which exactly forgets the vanishing variation condition. Hence j_* is a forget-control map.

Proof. We may replace $|G|$ by $\mathcal{E}G$ because $\mathcal{E}G$ is coarsely equivalent to $|G|$. The coarse K–theory of $\mathcal{E}G$ agrees with the usual K–theory of $\mathcal{E}G$ (see [7]). A slight elaboration of the proof of [7, Lemma 15] shows that

$$\mathrm{K}^H_*\bigl(\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) \cong \mathrm{KK}^H_*\bigl(\mathbb{C}, \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr)$$

vanishes for all compact subgroups H of G. This yields $\mathrm{K}^{\mathrm{top}}_*\bigl(G, \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) = 0$ by a result of [2]. Now the assertion follows from the Five Lemma and the naturality of the K–theory long exact sequence for (9) as in [7]. □

3. Classes in Kasparov theory from the stable Higson corona

In this section, we show how to construct classes in equivariant KK-theory from the K–theory of the stable Higson corona. The following lemma is our main technical device:

Lemma 7. *Let G and H be locally compact groups and let X be a coarse space equipped with commuting actions of G and H. Suppose that G acts by translations and that H acts properly and by isometries. Let A and D be H–C^*-algebras, equipped with the trivial G–action. We abbreviate*

$$B_X := C_0(X, D \otimes \mathbb{K}_H \otimes_{\max} A) \rtimes H, \qquad E_X := (\bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D) \otimes_{\max} A) \rtimes H$$

and similarly for $\mathscr{P}(X)$ instead of X. There are extensions $B_X \rightarrowtail E_X \twoheadrightarrow E_X/B_X$ and $B_{\mathscr{P}(X)} \rightarrowtail E_{\mathscr{P}(X)} \twoheadrightarrow E_{\mathscr{P}(X)}/B_{\mathscr{P}(X)}$ with

$$E_{\mathscr{P}(X)}/B_{\mathscr{P}(X)} \cong E_X/B_X \cong (\mathfrak{c}^{\mathfrak{red}}_H(X, D) \otimes_{\max} A) \rtimes H,$$

and a natural commuting diagram

$$\begin{array}{ccc} \mathrm{K}_{*+1}(E_X/B_X) & \xrightarrow{\partial} & \mathrm{K}_*(B_{\mathscr{P}(X)}) \\ \downarrow\psi & & \downarrow\phi \\ \mathrm{KK}^G_*(\mathbb{C}, B_X) & \xrightarrow{p_\mathcal{E} G^*} & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, B_X). \end{array} \tag{13}$$

Proof. The quotients E_X/B_X and $E_{\mathscr{P}(X)}/B_{\mathscr{P}(X)}$ are as asserted and agree because $X \to \mathscr{P}(X)$ is a coarse equivalence and because maximal tensor products and full crossed products are exact functors in complete generality, unlike spatial tensor products and reduced crossed products. We let ∂ be the K–theory boundary map for the extension $B_{\mathscr{P}(X)} \rightarrowtail E_{\mathscr{P}(X)} \twoheadrightarrow E_X/B_X$.

Since we have a natural map $\mathfrak{c}_H^{\mathfrak{re}\mathfrak{d}}(X,D)\otimes_{\max}A\to\mathfrak{c}_H^{\mathfrak{re}\mathfrak{d}}(X,D\otimes_{\max}A)$, we may replace the pair (D,A) by $(D\otimes_{\max}A,\mathbb{C})$ and omit A if convenient. Stabilising D by $\mathbb{K}_H$, we can further eliminate the stabilisations.

First we lift the K–theory boundary map for the extension $B_X\rightarrowtail E_X\twoheadrightarrow E_X/B_X$ to a map $\psi\colon \mathrm{K}_{*+1}(E_X/B_X)\to\mathrm{KK}_*^G(\mathbb{C},B_X)$. The G–equivariance of the resulting Kasparov cycles follows from the assumption that G acts on X by translations.

We have to distinguish between the cases $*=0$ and $*=1$. We only write down the construction for $*=0$. Since the algebra E_X/B_X is matrix-stable, $\mathrm{K}_1(E_X/B_X)$ is the homotopy group of unitaries in E_X/B_X without further stabilisation. A cycle for $\mathrm{KK}_0^G(\mathbb{C},B_X)$ is given by two G–equivariant Hilbert modules $\mathcal{E}_\pm$ over B_X and a G–continuous adjointable operator $F\colon\mathcal{E}_+\to\mathcal{E}_-$ for which $1-FF^*$, $1-F^*F$ and $gF-F$ for $g\in G$ are compact; we take $\mathcal{E}_\pm=B_X$ and let $F\in E_X\subseteq\mathcal{M}(B_X)$ be a lifting for a unitary $u\in E_X/B_X$. Since G acts on X by translations, the induced action on $\bar{\mathfrak{c}}^{\mathfrak{re}\mathfrak{d}}(X,D)$ and hence on E_X/B_X is trivial. Hence u is a G–invariant unitary in E_X/B_X. For the lifting F, this means that

$$1-FF^*,\ 1-F^*F,\ gF-F\in B_X.$$

Hence F defines a cycle for $\mathrm{KK}_0^G(\mathbb{C},B_X)$. We get a well-defined map $[u]\mapsto[F]$ from $\mathrm{K}_1(E_X/B_X)$ to $\mathrm{KK}_0^G(\mathbb{C},B_X)$ because homotopic unitaries yield operator homotopic Kasparov cycles.

Next we have to factor the map $p_{\mathcal{E}}G^*\circ\psi$ in (13) through $\mathrm{K}_0(B_{\mathscr{P}(X)})$. The main ingredient is a certain continuous map $\bar{c}\colon\mathcal{E}G\times X\to\mathscr{P}(X)$. We use the same description of $\mathscr{P}(X)$ as in [7] as the space of positive measures on X with $1/2<\mu(X)\le1$; this is a σ–coarse space in a natural way, we write it as $\mathscr{P}(X)=\bigcup\mathrm{P}_d(X)$.

There is a function $c\colon\mathcal{E}G\to\mathbb{R}_+$ for which $\int_{\mathcal{E}}Gc(\mu g)\,\mathrm{d}g=1$ for all $\mu\in\mathcal{E}G$ and $\operatorname{supp}c\cap Y$ is compact for G–compact $Y\subseteq\mathcal{E}G$. If $\mu\in\mathcal{E}G$, $x\in X$, then the condition

$$\langle\bar{c}(\mu,x),\alpha\rangle:=\int_G c(\mu g)\alpha(g^{-1}x)\,\mathrm{d}g$$

for $\alpha\in C_0(X)$ defines a probability measure on X. Since such measures are contained in $\mathscr{P}(X)$, $\bar{c}$ defines a map $\bar{c}\colon\mathcal{E}G\times X\to\mathscr{P}(X)$. This map is continuous and satisfies $\bar{c}(\mu g,g^{-1}xh)=\bar{c}(\mu,x)h$ for all $g\in G$, $\mu\in\mathcal{E}G$, $x\in X$, $h\in H$.

For a C^*–algebra Z, let $C(\mathcal{E}G,Z)$ be the σ–C^*-algebra of all continuous functions $f\colon\mathcal{E}G\to Z$ without any growth restriction. Thus $C(\mathcal{E}G,Z)=\varprojlim C(K,Z)$, where K runs through the directed set of compact subsets of $\mathcal{E}G$.

We claim that $(\bar{c}^*f)(\mu)(x):=f\bigl(\bar{c}(\mu,x)\bigr)$ for $f\in C_0(\mathscr{P}(X),D)$ defines a continuous $*$–homomorphism

$$\bar{c}^*\colon C_0(\mathscr{P}(X),D)\to C\bigl(\mathcal{E}G,C_0(X,D)\bigr).$$

If $K\subseteq\mathcal{E}G$ is compact, then there is a compact subset $L\subseteq G$ such that $c(\mu\cdot g)=0$ for $\mu\in K$ and $g\notin L$. Hence $\bar{c}(\mu,x)$ is supported in $L^{-1}x$ for $\mu\in K$. Since G acts on X by translations, such measures are contained in a filtration level $\mathrm{P}_d(X)$. Hence $\bar{c}^*(f)$ restricts to a C_0-function $K\times X\to D$ for all $f\in C_0(\mathscr{P}(X),D)$. This proves the claim. Since $\bar{c}$ is H–equivariant and G–invariant, we get an induced map

$$B_{\mathscr{P}(X)}=C_0(\mathscr{P}(X),D)\rtimes H\to\bigl(C\bigl(\mathcal{E}G,C_0(X,D)\bigr)\rtimes H\bigr)^G=C(\mathcal{E}G,B_X)^G,$$

where $Z^G\subseteq Z$ denotes the subalgebra of G–invariant elements. We obtain an induced $*$–homomorphism between the stable multiplier algebras as well.

An element of $\mathrm{K}_0\bigl(B_{\mathscr{P}(X)}\bigr)$ is represented by a self-adjoint bounded multiplier $F\in\mathcal{M}(B_{\mathscr{P}(X)}\otimes\mathbb{K})$ such that $1-FF^*$ and $1-F^*F$ belong to $B_{\mathscr{P}(X)}\otimes\mathbb{K}$. Now $\tilde{F}:=\bar{c}^*(F)$ is a G–invariant bounded multiplier of $C(\mathcal{E}G,B_X\otimes\mathbb{K})$ and hence a G–invariant multiplier of $C_0(\mathcal{E}G,B_X\otimes\mathbb{K})$, such that $\alpha\cdot(1-\tilde{F}\tilde{F}^*)$ and $\alpha\cdot(1-\tilde{F}^*\tilde{F})$

belong to $C_0(\mathcal{E}G, B_X \otimes \mathbb{K})$ for all $\alpha \in C_0(\mathcal{E}G)$. This says exactly that $\tilde{F}$ is a cycle for $\mathrm{RKK}_0^G(\mathcal{E}G; \mathbb{C}, B_X)$. This construction provides the natural map

$$\phi\colon \mathrm{K}_0(B_{\mathscr{P}(X)}) \to \mathrm{RKK}_0^G(\mathcal{E}G; \mathbb{C}, B_X).$$

Finally, a routine computation, which we omit, shows that the two images of a unitary $u \in E_X/B_X$ differ by a compact perturbation. Hence the diagram (13) commutes. □

We are mainly interested in the case where A is the source P of a Dirac morphism for H. Then $\mathrm{K}_{*+1}(E_X/B_X) = \mathrm{K}_*^{\mathrm{top}}\bigl(H, \mathfrak{c}_H^{\mathfrak{red}}(X, D)\bigr)$, and the top row in (13) is the H–equivariant coarse co-assembly map for X with coefficients D. Since we assume H to act properly on X, we have a KK^G-equivalence $B_X \sim C_0(X, D) \rtimes H$, and similarly for $\mathscr{P}(X)$. Hence we now get a commuting square

$$\begin{array}{ccc} \mathrm{K}_{*+1}^{\mathrm{top}}\bigl(H, \mathfrak{c}_H^{\mathfrak{red}}(X, D)\bigr) & \xrightarrow{\quad\partial\quad} & \mathrm{KX}_H^*(X, D) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} \\ \mathrm{KK}_*^G(\mathbb{C}, C_0(X, D) \rtimes H) & \xrightarrow{p_{\mathcal{E}} G^*} & \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, C_0(X, D) \rtimes H). \end{array} \tag{14}$$

If, in addition, $D = \mathbb{C}$ and the action of H on X is free, then we can further simplify this to

$$\begin{array}{ccc} \mathrm{K}_{*+1}^{\mathrm{top}}\bigl(H, \mathfrak{c}_H^{\mathfrak{red}}(X)\bigr) & \xrightarrow{\quad\partial\quad} & \mathrm{KX}_H^*(X) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} \\ \mathrm{KK}_*^G\bigl(\mathbb{C}, C_0(X/H)\bigr) & \xrightarrow{p_{\mathcal{E}} G^*} & \mathrm{RKK}_*^G\bigl(\mathcal{E}G; \mathbb{C}, C_0(X/H)\bigr). \end{array} \tag{15}$$

We may also specialise the space X to $|G|$, with G acting by multiplication on the left, and with $H \subseteq G$ a compact subgroup acting on $|G|$ by right multiplication. This is the special case of (13) that is used in [7]. The following applications will require other choices of X.

3.1. **Applications to Lipschitz classes.** Now we use Lemma 7 to construct interesting elements in $\mathrm{KK}_*^G\bigl(\mathbb{C}, C_0(X)\bigr)$ for a G–space X. This is related to the method of Lipschitz maps developed by Connes, Gromov and Moscovici in [3].

3.1.1. *Pulled-back coarse structures.* Let X be a G–space, let Y be a coarse space and let $\alpha\colon X \to Y$ be a proper continuous map. We pull back the coarse structure on Y to a coarse structure on X, letting $E \subseteq X \times X$ be an entourage if and only if $\alpha_*(E) \subseteq Y \times Y$ is one. Since α is proper and continuous, this coarse structure is compatible with the topology on X. For this coarse structure, G *acts by translations* if and only if α satisfies the following *displacement condition* used in [3]: for any compact subset $K \subseteq G$, the set

$$\bigl\{\bigl(\alpha(gx), \alpha(x)\bigr) \in Y \times Y \bigm| x \in X,\ g \in K\bigr\}$$

is an entourage of Y. The map α becomes a coarse map. Hence we obtain a commuting diagram

$$\begin{array}{ccccc} \mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}(Y)\bigr) & \xrightarrow{\alpha^*} & \mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}(X)\bigr) & \xrightarrow{\psi} & \mathrm{KK}_*^G\bigl(\mathbb{C}, C_0(X)\bigr) \\ \downarrow{\scriptstyle\partial^Y} & & \downarrow{\scriptstyle\partial^X} & & \downarrow{\scriptstyle p_{\mathcal{E}} G^*} \\ \mathrm{KX}^*(Y) & \xrightarrow{\alpha^*} & \mathrm{KX}^*(X) & \xrightarrow{\phi} & \mathrm{RKK}_*^G\bigl(\mathcal{E}G; \mathbb{C}, C_0(X)\bigr). \end{array}$$

with ψ and ϕ as in Lemma 7.

The constructions of [3, §I.10] only use $Y = \mathbb{R}^N$ with the Euclidean coarse structure. The coarse co–assembly map is an isomorphism for $\mathbb{R}^N$ because $\mathbb{R}^N$ is scalable. Moreover, $\mathbb{R}^N$ is uniformly contractible and has bounded geometry. Hence we obtain canonical isomorphisms

$$\mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\bigr) \cong \mathrm{KX}^*(\mathbb{R}^N) \cong \mathrm{K}^*(\mathbb{R}^N).$$

In particular, $\mathrm{K}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\bigr) \cong \mathbb{Z}$ with generator $[\partial\mathbb{R}^N]$ in $\mathrm{K}_{1-N}\bigl(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\bigr)$. This class is nothing but the usual dual-Dirac morphism for the locally compact group $\mathbb{R}^N$. As a result, any map $\alpha\colon X \to \mathbb{R}^N$ that satisfies the displacement condition above induces

$$[\alpha] := \psi\bigl(\alpha^*[\partial\mathbb{R}^N]\bigr) \in \mathrm{KK}^G_{-N}\bigl(\mathbb{C}, C_0(X)\bigr).$$

The commutative diagram (13) computes $p_{\mathcal{E}}G^*[\alpha] \in \mathrm{RKK}^G_{-N}\bigl(\mathcal{E}G; \mathbb{C}, C_0(X)\bigr)$ in purely topological terms.

3.1.2. *Principal bundles over coarse spaces.* As in [3], we may replace a fixed map $X \to \mathbb{R}^N$ by a section of a vector bundle over X. But we need this bundle to have a G–equivariant spin structure. To encode this, we consider a G–equivariant $\mathrm{Spin}(N)$–principal bundle $\pi\colon E \to B$ together with actions of G on E and B such that π is G–equivariant and the action on E commutes with the action of $H := \mathrm{Spin}(N)$. Let $T := E \times_{\mathrm{Spin}(N)} \mathbb{R}^N$ be the associated vector bundle over B. It carries a G–invariant Euclidean metric and spin structure. As is well-known, sections $\alpha\colon B \to T$ correspond bijectively to $\mathrm{Spin}(N)$-equivariant maps $\alpha'\colon E \to \mathbb{R}^N$; here a section α corresponds to the map $\alpha'\colon E \to \mathbb{R}^N$ that sends $y \in E$ to the coordinates of $\alpha\pi(y)$ in the orthogonal frame described by y. Since the group $\mathrm{Spin}(N)$ is compact, the map α' is proper if and only if $b \mapsto \|\alpha(b)\|$ is a proper function on B.

As in §3.1.1, a $\mathrm{Spin}(N)$–equivariant proper continuous map $\alpha'\colon E \to Y$ for a coarse space Y allows us to pull back the coarse structure of Y to E; then $\mathrm{Spin}(N)$ acts by isometries. The group G acts by translations if and only if α' satisfies the displacement condition from §3.1.1. If $Y = \mathbb{R}^N$, we can rewrite this in terms of $\alpha\colon B \to T$: we need

$$\sup\bigl\{\|g\alpha(g^{-1}b) - \alpha(b)\| \bigm| b \in B,\ g \in K\bigr\}$$

to be bounded for all compact subsets $K \subseteq G$.

If the displacement condition holds, then we are in the situation of Lemma 7 with $H = \mathrm{Spin}(N)$ and $X = E$. Since H acts freely on E, $C_0(E) \rtimes H$ is G–equivariantly Morita–Rieffel equivalent to $C_0(B)$. We obtain canonical maps

$$\begin{aligned}\mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr) &\xrightarrow{(\alpha')^*} \mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(E)\bigr)\\ &\xrightarrow{\psi} \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(E) \rtimes \mathrm{Spin}(N)\bigr) \cong \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(B)\bigr).\end{aligned}$$

The $\mathrm{Spin}(N)$–equivariant coarse co-assembly map for $\mathbb{R}^N$ is an isomorphism by [7] because the group $\mathbb{R}^N \rtimes \mathrm{Spin}(N)$ has a dual-Dirac morphism. Using also the uniform contractibility of $\mathbb{R}^N$ and $\mathrm{Spin}(N)$-equivariant Bott periodicity, we get

$$\mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr) \cong \mathrm{KX}^*_{\mathrm{Spin}(N)}(\mathbb{R}^N) \cong \mathrm{K}^*_{\mathrm{Spin}(N)}(\mathbb{R}^N) \cong \mathrm{K}^{*+N}_{\mathrm{Spin}(N)}(\mathrm{point}).$$

The class of the trivial representation in $\mathrm{Rep}(\mathrm{Spin}N) \cong \mathrm{K}_0^{\mathrm{Spin}(N)}(\mathbb{C})$ is mapped to the usual dual-Dirac morphism $[\partial\mathbb{R}^N] \in \mathrm{K}^{\mathrm{Spin}(N)}_{1-N}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr)$ for $\mathbb{R}^N$. As a result, any proper section $\alpha\colon B \to T$ satisfying the displacement condition induces

$$[\alpha] := \psi\bigl(\alpha^*[\partial\mathbb{R}^N]\bigr) \in \mathrm{KK}^G_{-N}\bigl(\mathbb{C}, C_0(B)\bigr).$$

Again, the commutative diagram (13) computes $p_{\mathcal{E}}G^*[\alpha] \in \mathrm{RKK}^G_{-N}\bigl(\mathcal{E}G; \mathbb{C}, C_0(X)\bigr)$ in purely topological terms.

3.1.3. *Coarse structures on jet bundles.* Let M be an oriented compact manifold and let $\mathrm{Diff}^+(M)$ be the infinite-dimensional Lie group of orientation-preserving diffeomorphisms of M. Let G be a locally compact group that acts on M by a continuous group homomorphism $G \to \mathrm{Diff}^+(M)$. The *Gelfand–Fuchs cohomology* of M is part of the group cohomology of $\mathrm{Diff}^+(M)$ and by functoriality maps to the group cohomology of G. It is shown in [3] that the range of Gelfand–Fuchs cohomology in the cohomology of G yields homotopy-invariant higher signatures.

This argument has two parts; one is geometric and concerns the construction of a class in $\mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(X)\bigr)$ for a suitable space X; the other uses cyclic homology to construct linear functionals on $\mathrm{K}_*(C_0(X) \rtimes_{\mathrm{r}} G)$ associated to Gelfand–Fuchs cohomology classes. We can simplify the first step; the second has nothing to do with coarse geometry.

Let $\pi^k\colon \mathrm{J}^k_+(M) \to M$ be the *oriented k–jet bundle* over M. That is, a point in $\mathrm{J}^k_+(M)$ is the kth order Taylor series at 0 of an orientation-preserving diffeomorphism from a neighbourhood of $0 \in \mathbb{R}^n$ into M. This is a principal H–bundle over M, where H is a connected Lie group whose Lie algebra $\mathfrak{h}$ is the space of polynomial maps $p\colon \mathbb{R}^n \to \mathbb{R}^n$ of order k with $p(0) = 0$, with an appropriate Lie algebra structure. The maximal compact subgroup $K \subseteq H$ is isomorphic to $\mathrm{SO}(n)$, acting by isometries on $\mathbb{R}^n$. It acts on $\mathfrak{h}$ by conjugation.

Since our construction is natural, the action of G on M lifts to an action on $\mathrm{J}^k_+(M)$ that commutes with the H–action. We let H act on the right and G on the left. Define $X_k := \mathrm{J}^k_+(M)/K$. This is the bundle space of a fibration over M with fibres H/K. Gelfand–Fuchs cohomology can be computed using a chain complex of $\mathrm{Diff}^+(M)$-invariant differential forms on X_k for $k \to \infty$. Using this description, Connes, Gromov, and Moscovici associate to a Gelfand–Fuchs cohomology class a functional $\mathrm{K}_*(C_0(X_k) \rtimes_{\mathrm{r}} G) \to \mathbb{C}$ for sufficiently high k in [3].

Since $\mathrm{J}^k_+(M)/H \cong M$ is compact, there is a unique coarse structure on $\mathrm{J}^k_+(M)$ for which H acts isometrically (see §2.2). With this coarse structure, $\mathrm{J}^k_+(M)$ is coarsely equivalent to H. The compactness of $\mathrm{J}^k_+(M)/H \cong M$ also implies easily that G acts by translations. We have a Morita–Rieffel equivalence $C_0(X_k) \sim C_0(\mathrm{J}^k_+M) \rtimes K$ because K acts freely on $\mathrm{J}^k_+(M)$. We want to study the map

$$\mathrm{K}_{*+1}(\mathfrak{c}_K^{\mathfrak{red}}(\mathrm{J}^k_+M) \rtimes K) \xrightarrow{\psi} \mathrm{KK}^G_*(\mathbb{C}, C_0(\mathrm{J}^k_+M) \rtimes K) \cong \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(X_k)\bigr)$$

produced by Lemma 7.

Since H is almost connected, it has a dual-Dirac morphism by [10]; hence the K–equivariant coarse co-assembly map for H is an isomorphism by the main result of [7]. Moreover, H/K is a model for $\mathcal{E}G$ by [1]. We get

$$\mathrm{K}_{*+1}(\mathfrak{c}_K^{\mathfrak{red}}(\mathrm{J}^k_+M) \rtimes K) \cong \mathrm{K}_{*+1}(\mathfrak{c}_K^{\mathfrak{red}}(|H|) \rtimes K) \cong \mathrm{KX}^*_K(|H|) \cong \mathrm{K}^*_K(H/K).$$

Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H and K. There is a K–equivariant homeomorphism $\mathfrak{h}/\mathfrak{k} \cong H/K$, where K acts on $\mathfrak{h}/\mathfrak{k}$ by conjugation. Now we need to know whether there is a K–equivariant spin structure on $\mathfrak{h}/\mathfrak{k}$. One can check that this is the case if $k \equiv 0, 1 \bmod 4$. Since we can choose k as large as we like, we can always assume that this is the case. The spin structure allows us to use Bott periodicity to identify $\mathrm{K}^*_K(H/K) \cong \mathrm{K}^H_{*-N}(\mathbb{C})$, which is the representation ring of K in degree $-N$, where $N = \dim \mathfrak{h}/\mathfrak{k}$. Using our construction, the trivial representation of K yields a canonical element in $\mathrm{KK}^G_{-N}\bigl(\mathbb{C}, C_0(X_k)\bigr)$.

This construction is much shorter than the corresponding one in [3] because we use Kasparov's result about dual-Dirac morphisms for almost connected groups.

Much of the corresponding argument in [3] is concerned with proving a variation on this result of Kasparov.

4. Computation of $\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P})$

So far, we have merely used the diagram (13) to construct certain elements in $\mathrm{KK}^G_*(\mathbb{C}, B)$. Now we show that this construction yields an isomorphic description of $\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P})$. This assertion requires G to be a totally disconnected group with a G–compact universal proper G–space. We assume this throughout this section.

Lemma 8. *In the situation of Lemma 7, suppose that $X = |G|$ with G acting by left translations and that $H \subseteq G$ is a compact subgroup acting on X by right translations; here $|G|$ carries the coarse structure of Example 3. Then the maps ψ and ϕ are isomorphisms.*

Proof. We reduce this assertion to results of [7]. The C^*–algebras $\mathfrak{c}_H^{\mathfrak{red}}(|G|, D) \rtimes H$ and $\mathfrak{c}_H^{\mathfrak{red}}(|G|, D)^H$ are strongly Morita equivalent, whence have isomorphic K–theory. It is shown in [7] that

$$\mathrm{K}_{*+1}\bigl(\mathfrak{c}_H^{\mathfrak{red}}(|G|, D)^H\bigr) \cong \mathrm{KK}^G_*(\mathbb{C}, \mathrm{Ind}_H^G D). \tag{16}$$

Finally, $\mathrm{Ind}_H^G(D) = C_0(G, D)^H$ is G–equivariantly Morita–Rieffel equivalent to $C_0(G, D) \rtimes H$. Hence we get

$$\begin{aligned}\mathrm{K}_{*+1}(\mathfrak{c}_H^{\mathfrak{red}}(|G|, D) \rtimes H) \cong \mathrm{K}_{*+1}\bigl(\mathfrak{c}_H^{\mathfrak{red}}(|G|, D)^H\bigr) &\cong \mathrm{KK}^G_*\bigl(\mathbb{C}, \mathrm{Ind}_H^G(D)\bigr)\\ &\cong \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(G, D) \rtimes H\bigr).\end{aligned} \tag{17}$$

It is a routine exercise to verify that this composition agrees with the map ψ in (13). Similar considerations apply to the map ϕ. □

We now set $X = |G|$, and let $G = H$. The actions of G on $|G|$ on the left and right are by translations and isometries, respectively. Lemma 7 yields a map

$$\Psi_*\colon \mathrm{K}_{*+1}\bigl((\mathfrak{c}^{\mathfrak{red}}(|G|, D) \otimes_{\max} A) \rtimes G\bigr) \to \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(|G|, D \otimes_{\max} A) \rtimes G\bigr). \tag{18}$$

for all A, D, where we use the G-equivariant Morita–Rieffel equivalence between $C_0(|G|, D) \rtimes G$ and D. It fits into a commuting diagram

$$\begin{array}{ccc}\mathrm{K}_{*+1}\bigl((\mathfrak{c}_G^{\mathfrak{red}}(|G|, D) \otimes_{\max} A) \rtimes G\bigr) & \xrightarrow{\partial} & \mathrm{K}_*(C_0(\mathcal{E}G, D \otimes_{\max} A) \rtimes G)\\ \downarrow{\scriptstyle \Psi_*^{D,A}} & & \downarrow{\scriptstyle \cong}\\ \mathrm{KK}^G_*(\mathbb{C}, D \otimes_{\max} A) & \xrightarrow{p_{\mathcal{E}} G^*} & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, D \otimes_{\max} A).\end{array}$$

Lemma 9. *The class of G-C^*-algebras A for which $\Psi_*^{D,A}$ is an isomorphism for all D is triangulated and thick and contains all G-C^*–algebras of the form $C_0(G/H)$ for compact open subgroups H of G.*

Proof. The fact that this category of algebras is triangulated and thick means that it is closed under suspensions, extensions, and direct summands. These formal properties are easy to check.

Since $H \subseteq G$ is open, there is no difference between H–continuity and G–continuity. Hence

$$\begin{aligned}\bigl(\mathfrak{c}_G^{\mathfrak{red}}(|G|, D) \otimes_{\max} C_0(G/H)\bigr) \rtimes G \cong \bigl(\mathfrak{c}_H^{\mathfrak{red}}(|G|, D) \otimes_{\max} C_0(G/H)\bigr) \rtimes G\\ \sim \bigl(\mathfrak{c}_H^{\mathfrak{red}}(|G|, D) \rtimes H,\end{aligned}$$

where $\sim$ means Morita–Rieffel equivalence. Similar simplifications can be made in other corners of the square. Hence the diagram for $A = C_0(G/H)$ and G acting on

the right is equivalent to a corresponding diagram for trivial A and H acting on the right. The latter case is contained in Lemma 8. □

Theorem 10. *Let G be an almost totally disconnected group with G–compact $\mathcal{E}G$. Then for every $B \in \mathrm{KK}^G$, the map*

$$\Psi_*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|, B)\bigr) \to \mathrm{KK}^G_*(\mathbb{C}, B \otimes \mathsf{P})$$

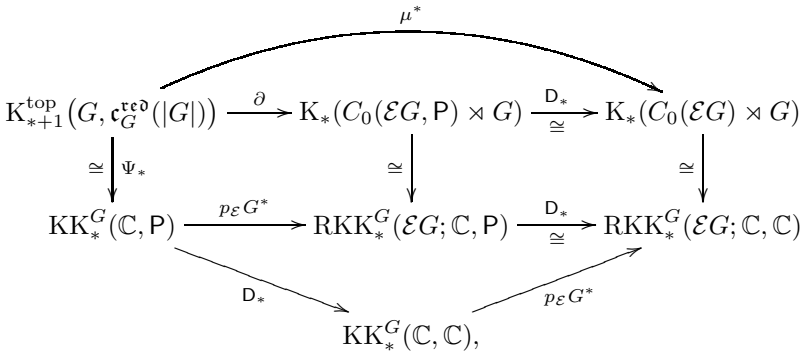
is an isomorphism and the diagram

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|, B)\bigr) & \xrightarrow{\mu^*} & \mathrm{KX}^*_G(|G|, B) \\ \cong\downarrow \Psi_* & & \cong\downarrow \\ \mathrm{KK}^G_*(\mathbb{C}, B \otimes \mathsf{P}) & \xrightarrow{p_{\mathcal{E}} G^*} & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, B \otimes \mathsf{P}) \end{array}$$

commutes. In particular, $\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr)$ is naturally isomorphic to $\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P})$.

Proof. It is shown in [7] that for such groups G, the algebra P belongs to the thick triangulated subcategory of KK^G that is generated by $C_0(G/H)$ for compact subgroups H of G. Hence the assertion follows from Lemma 9 and our definition of $\mathrm{K}^{\mathrm{top}}$. □

Corollary 11. *Let $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ be a Dirac morphism for G. Then the following diagram commutes*



where Ψ_* is as in Theorem 10 *and μ^* is the G–equivariant coarse co-assembly map for $|G|$, and the indicated maps are isomorphisms.*

Proof. This follows from Theorem 10 and the general properties of the Dirac morphism discussed in §2.1. □

Definition 12. *We call $a \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$*

(a) *a* boundary class *if it lies in the range of*

$$\mu^*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C});$$

(b) properly factorisable *if $a = p_{\mathcal{E}} G^*(b \otimes_A c)$ for some proper G-C^*-algebra A and some $b \in \mathrm{KK}^G_*(\mathbb{C}, A)$, $c \in \mathrm{KK}^G_*(A, \mathbb{C})$;*

(c) proper Lipschitz *if $a = p_{\mathcal{E}} G^*(b \otimes_{C_0(X)} c)$, where $b \in \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(X)\bigr)$ is constructed as in §3.1.1 and §3.1.2 and $c \in \mathrm{KK}^G_*(C_0(X), \mathbb{C})$ is arbitrary.*

Proposition 13. *Let G be a totally disconnected group with G–compact $\mathcal{E}G$.*

(a) *A class $a \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ is properly factorisable if and only if it is a boundary class.*

(b) *Proper Lipschitz classes are boundary classes.*

(c) *The boundary classes form an ideal in the ring $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$.*

(d) *The class $1_{\mathcal{E}}G$ is a boundary class if and only if G has a dual-Dirac morphism; in this case, the G–equivariant coarse co-assembly map μ^* is an isomorphism.*

(e) *Any boundary class lies in the range of*

$$p_{\mathcal{E}}G^*\colon \mathrm{KK}^G_*(\mathbb{C},\mathbb{C}) \to \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$$

and hence yields homotopy invariants for manifolds.

Proof. By Corollary 11, the equivariant coarse co-assembly map

$$\mu^*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G,\mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) \to \mathrm{K}_*\bigl(C_0(\mathcal{E}G\rtimes G)\bigr) \cong \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$$

is equivalent to the map

$$\mathrm{KK}^G_*(\mathbb{C},\mathsf{P}) \xrightarrow{p_{\mathcal{E}}G^*} \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathsf{P}) \cong \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}).$$

If we combine this with the isomorphism $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}) \cong \mathrm{KK}^G_*(\mathsf{P},\mathsf{P})$, the resulting map

$$\mathrm{KK}^G_*(\mathbb{C},\mathsf{P}) \to \mathrm{KK}^G_*(\mathsf{P},\mathsf{P})$$

is simply the product (on the left) with $\mathsf{D}\in\mathrm{KK}^G(\mathsf{P},\mathbb{C})$; this map is known to be an isomorphism if and only if it is surjective, if and only if 1_{P} is in its range, if and only if the H–equivariant coarse co-assembly map

$$\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}_H(|G|)\rtimes H) \to \mathrm{KX}^*_H(|G|)$$

is an isomorphism for all compact subgroups H of G by [7].

For any G–C^*-algebra B, the $\mathbb{Z}/2$–graded group $\mathrm{K}^{\mathrm{top}}_*(G,B)\cong \mathrm{K}_*\bigl((B\otimes\mathsf{P})\rtimes_{\mathrm{r}}G\bigr)$ is a graded module over the $\mathbb{Z}/2$–graded ring $\mathrm{KK}^G_*(\mathsf{P},\mathsf{P})\cong\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$ in a canonical way; the isomorphism between these two groups is a ring isomorphism because it is the composite of the two ring isomorphisms

$$\mathrm{KK}^G_*(\mathsf{P},\mathsf{P}) \xrightarrow[\cong]{p_{\mathcal{E}}G^*} \mathrm{RKK}^G_*(\mathcal{E}G;\mathsf{P},\mathsf{P}) \xleftarrow[\cong]{{}_{\sqcup}\otimes\mathsf{P}} \mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}).$$

Hence we get module structures on $\mathrm{K}^{\mathrm{top}}_*\bigl(G,\mathfrak{c}^{\mathfrak{red}}(|G|)\bigr)$ and

$$\mathrm{K}^{\mathrm{top}}_*\bigl(G,C_0(\mathcal{E}G,\mathbb{K})\bigr)\cong \mathrm{K}_*(C_0(\mathcal{E}G)\rtimes G)\cong\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C}).$$

The latter isomorphism is a module isomorphism; thus $\mathrm{K}^{\mathrm{top}}_*\bigl(G,C_0(\mathcal{E}G,\mathbb{K})\bigr)$ is a free module of rank 1. The equivariant co-assembly map is natural in the formal sense, so that it is an $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$-module homomorphism. Hence its range is a submodule, that is, an ideal in $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$ (since this ring is graded commutative, there is no difference between one- and two-sided graded ideals). This also yields that μ^* is surjective if and only if it is bijective, if and only if the unit class $1_{\mathcal{E}}G$ belongs to its range; we already know this from [7].

If A is a proper G-C^*–algebra, then $\mathrm{id}_A\otimes\mathsf{D}\in\mathrm{KK}^G(A\otimes\mathsf{P},A)$ is invertible ([11]). If $b\in\mathrm{KK}^G_*(\mathbb{C},A)$ and $c\in\mathrm{KK}^G_*(A,\mathbb{C})$, then we can write the Kasparov product $b\otimes_A c$ as

$$\mathbb{C}\xrightarrow{b}A\xleftarrow[\cong]{\mathrm{id}_A\otimes\mathsf{D}}A\otimes\mathsf{P}\xrightarrow{c\otimes\mathrm{id}_{\mathsf{P}}}\mathsf{P}\xrightarrow{\mathsf{D}}\mathbb{C},$$

where the arrows are morphisms in the category KK^G. Therefore, $b\otimes_A c$ factors through D and hence is a boundary class by Theorem 10. □

5. Dual-Dirac morphisms and the Carlsson–Pedersen condition

Now we construct boundary classes in $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$ from more classical boundaries. We suppose again that $\mathcal{E}G$ is G–compact, so that $\mathcal{E}G$ is a coarse space.

A *metrisable compactification* of $\mathcal{E}G$ is a metrisable compact space Z with a homeomorphism between $\mathcal{E}G$ and a dense open subset of Z. It is called *coarse* if all scalar-valued functions on Z have vanishing variation; this implies the corresponding assertion for operator-valued functions because $C(Z,D)\cong C(Z)\otimes D$. Equivalently, the embedding $\mathcal{E}G\to Z$ factors through the Higson compactification of $\mathcal{E}G$. A compactification is called *G–equivariant* if Z is a G–space and the embedding $\mathcal{E}G\to Z$ is G–equivariant. An equivariant compactification is called *strongly contractible* if it is H–equivariantly contractible for all compact subgroups H of G. The *Carlsson–Pedersen condition* requires that there should be a G–compact model for $\mathcal{E}G$ that has a coarse, strongly contractible, and G–equivariant compactification.

Typical examples of such compactifications are the Gromov boundary for a hyperbolic group (viewed as a compactification of the Rips complex), or the visibility boundary of a CAT(0) space on which G acts properly, isometrically, and cocompactly.

Theorem 14. *Let G be a locally compact group with a G–compact model for $\mathcal{E}G$ and let $\mathcal{E}G\subseteq Z$ be a coarse, strongly contractible, G–equivariant compactification. Then G has a dual-Dirac morphism.*

Proof. We use the C^*–algebra $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)$ as defined in §2.3. Since Z is coarse, there is an embedding $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)\subseteq\bar{\mathfrak{c}}^{\mathfrak{red}}_G(\mathcal{E}G)$. Let $\partial Z:=Z\setminus\mathcal{E}G$ be the boundary of the compactification. Identifying

$$\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z)\cong\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)/C_0(\mathcal{E}G,\mathbb{K}_G),$$

we get a morphism of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\mathcal{E}G,\mathbb{K}_G) & \longrightarrow & \bar{\mathfrak{c}}^{\mathfrak{red}}_G(\mathcal{E}G) & \longrightarrow & \mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G) & \longrightarrow & 0\\
 & & \| & & \uparrow{\scriptstyle\subseteq} & & \uparrow{\scriptstyle\subseteq} & & \\
0 & \longrightarrow & C_0(\mathcal{E}G,\mathbb{K}_G) & \longrightarrow & \bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z) & \longrightarrow & \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z) & \longrightarrow & 0.
\end{array}$$

Let H be a compact subgroup. Since Z is compact, we have $\bar{\mathfrak{B}}_G(Z)=C(Z,\mathbb{K})$. Since Z is H–equivariantly contractible by hypothesis, $\bar{\mathfrak{B}}_G(Z)$ is H–equivariantly homotopy equivalent to $\mathbb{C}$. Hence $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)$ has vanishing H–equivariant K–theory. This implies $\mathrm{K}^{\mathrm{top}}_*\bigl(G,\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)\bigr)=0$ by [2], so that the connecting map

$$\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G,\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z)\bigr)\to\mathrm{K}^{\mathrm{top}}_*\bigl(G,C_0(\mathcal{E}G)\bigr)\cong\mathrm{K}_*(C_0(\mathcal{E}G)\rtimes G) \tag{19}$$

is an isomorphism. This in turn implies that the connecting map

$$\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G,\mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G)\bigr)\to\mathrm{K}^{\mathrm{top}}_*\bigl(G,C_0(\mathcal{E}G)\bigr)$$

is surjective. Thus we can lift $1\in\mathrm{RKK}^G_0(\mathcal{E}G;\mathbb{C},\mathbb{C})\cong\mathrm{K}_0(C_0(\mathcal{E}G)\rtimes G)$ to

$$\alpha\in\mathrm{K}^{\mathrm{top}}_1\bigl(G,\mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G)\bigr)\cong\mathrm{K}^{\mathrm{top}}_1\bigl(G,\mathfrak{c}^{\mathfrak{red}}_G(|G|)\bigr).$$

Then $\Psi_*(\alpha)\in\mathrm{KK}^G_0(\mathbb{C},\mathsf{P})$ is the desired dual-Dirac morphism. □

The group $\mathrm{K}^{\mathrm{top}}_*\bigl(G,\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z)\bigr)$ that appears in the above argument is a *reduced* topological G–equivariant K–theory for ∂Z and hence differs from $\mathrm{K}^{\mathrm{top}}_*\bigl(G,C(\partial Z)\bigr)$. The relationship between these two groups is analysed in [5].

References

[1] Herbert Abels, *Parallelizability of proper actions, global K-slices and maximal compact subgroups*, Math. Ann. **212** (1974/75), 1–19. MR **0375264**

[2] Jérôme Chabert, Siegfried Echterhoff, and Hervé Oyono-Oyono, *Going-down functors, the Künneth formula, and the Baum–Connes conjecture*, Geom. Funct. Anal. **14** (2004), no. 3, 491–528. MR **2100669**

[3] Alain Connes, Mikhail Gromov, and Henri Moscovici, *Group cohomology with Lipschitz control and higher signatures*, Geom. Funct. Anal. **3** (1993), no. 1, 1–78. MR **1204787**

[4] Alexander N. Dranishnikov, *Lipschitz cohomology, Novikov's conjecture, and expanders*, Tr. Mat. Inst. Steklova **247** (2004), no. Geom. Topol. i Teor. Mnozh., 59–73 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. (2004), no. 4 (247), 50–63. MR **2168163**

[5] Heath Emerson and Ralf Meyer, *Euler characteristics and Gysin sequences for group actions on boundaries*, Math. Ann. **334** (2006), no. 4, 853–904. MR **2209260**

[6] ———, *Dualizing the coarse assembly map*, J. Inst. Math. Jussieu **5** (2006), no. 2, 161–186. MR **2225040**

[7] ———, *A descent principle for the Dirac–dual-Dirac method*, Topology **46** (2007), no. 2, 185–209. MR **2313071**

[8] Steven C. Ferry, Andrew Ranicki, and Jonathan Rosenberg, *A history and survey of the Novikov conjecture*, (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, Cambridge, 1995, pp. 7–66. MR **1388295**

[9] Nigel Higson, *Bivariant K-theory and the Novikov conjecture*, Geom. Funct. Anal. **10** (2000), no. 3, 563–581. MR **1779613**

[10] Gennadi G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201. MR **918241**

[11] Ralf Meyer and Ryszard Nest, *The Baum–Connes conjecture via localisation of categories*, Topology **45** (2006), no. 2, 209–259. MR **2193334**

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