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Noncommutative Poincaré duality for boundary actions of hyperbolic groups

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Abstract. For a large class of word hyperbolic groups Γ the cross product C^*-algebras $C(\partial\Gamma) \rtimes \Gamma$, where $\partial\Gamma$ denotes the Gromov boundary of Γ satisfy Poincaré duality in K-theory. This class strictly contains fundamental groups of compact, negatively curved manifolds. We discuss the general notion of Poincaré duality for C^*-algebras, construct the fundamental classes for the aforementioned algebras, and prove that KK-products with these classes induce inverse isomorphisms. The Baum-Connes Conjecture for amenable groupoids is used in a crucial way.

1. Introduction

It is well known that if M^n is a compact n-dimensional spin^c-manifold, the C^*-algebra $C(M^n)$ of continuous functions on M^n exhibits Poincaré duality in K-theory. Specifically, if $[D] \in K_n(M)$ is the K-homology class of the Dirac operator on M, then cap product with $[D]$ induces an isomorphism $K^*(M^n) \xrightarrow{\cong} K_{*+n}(M^n)$. It is natural to ask whether there are noncommutative C^*-algebras exhibiting the same phenomenon. In [7] A. Connes introduced the appropriate formalism for this question, defining the analog for C^*-algebras of Spanier-Whitehead duality for finite complexes. Two C^*-algebras A and B shall be said to be dual if there exists a class Δ in the K-homology of $A \otimes B$, and a class $\hat{\Delta}$ in the K-theory of $A \otimes B$ such that $\hat{\Delta} \otimes_B \Delta = 1_A$ and $\hat{\Delta} \otimes_A \Delta = 1_B$. If A and B are dual, cap product with Δ induces an isomorphism $K_*(A) \to K^*(B)$. A special case is where $B = A^{\mathrm{op}}$, which we term Poincaré duality, while a C^*-algebra satisfying Poincaré duality we shall call in this paper a Poincaré duality algebra. Known *commutative* examples of Poincaré duality algebras are given by continuous functions on spaces homotopy equivalent to one of the aforementioned M^n above; it is unknown to the author whether there are other commutative examples. The first nontrivial example of a noncommutative Poincaré duality algebra was given by Connes (see [6]) in the form of the irrational rotation algebra A_θ. In this paper we shall prove that if Γ is a hyperbolic group satisfying a certain mild symmetry property, and $\partial\Gamma$ is its Gromov boundary, then the cross product $C(\partial\Gamma) \rtimes \Gamma$ is a Poincaré duality algebra.

Examples of pairs of algebras A and B dual in the above sense were given by Kaminker and Putnam (see [19]); the pairs were O_M and O_{M^t} respectively, where for a square 0–1 valued matrix M, O_M refers to the corresponding Cuntz-Krieger algebra. Their result is a special case of a more general one, in which the stable and unstable Ruelle algebras R^{s} and R^{u} associated to a hyperbolic dynamical system are shown to be dual (see [20]).

A particular example of a hyperbolic dynamical system is provided by an Anosov diffeomorphism of a compact manifold; thus the duality discovered by Kaminker and Putnam holds for these. An obvious question is whether or not the same duality holds for Anosov *flows*. The principal example of such a flow is given by geodesic flow on a compact, negatively curved Riemannian manifold M. The algebras R^{s} and R^{u} can in this case be regarded as foliation algebras as follows. Define two equivalence relations on SM by respectively $v \sim_{\mathrm{s}} w$ if $\limsup_{t\to\infty} d_{SM}(g_t v, g_t w) = 0$, and $v \sim_{\mathrm{u}} w$ if $\limsup_{t\to-\infty} d_{SM}(g_t v, g_t w) = 0$. Define weak versions of these equivalence relations by respectively $v \sim_{\mathrm{ws}} w$ if $g_t(v) \sim_{\mathrm{s}} w$ for some t, and similarly for $v \sim_{\mathrm{wu}} w$. The equivalence classes of these latter two relations make up two codimension-1 foliations $\mathscr{F}^{\mathrm{ws}}$ and $\mathscr{F}^{\mathrm{wu}}$ of SM. We can then form (see e.g. [6]) the corresponding foliation algebras $C_r^*(\mathscr{F}^{\mathrm{ws}})$ and $C_r^*(\mathscr{F}^{\mathrm{wu}})$. The work of Kaminker and Putnam then suggested that $C_r^*(\mathscr{F}^{\mathrm{ws}})$ should be dual in the aforementioned sense to $C_r^*(\mathscr{F}^{\mathrm{wu}})$.

Now it is easy to see that the unit tangent sphere at a point of M acts as a transversal to both foliations. We may therefore reduce the two holonomy groupoids to this transversal and so obtain equivalent groupoids, which are now r-discrete. Finally, it is easy to see that these groupoids are in fact equivalent, and can be each identified with the transformation groupoid $\partial\tilde{M} \rtimes \Gamma$, where $\Gamma = \pi_1(M)$ and the boundary $\partial\tilde{M}$ is that associated to the Gromov hyperbolic metric space $\tilde{M}$, acted apon by Γ by an extension of the action of Γ by deck transformations on $\tilde{M}$ to an action by homeomorphisms of $\partial\tilde{M}$. Since M is compact and negatively curved, the group Γ is of course hyperbolic in the sense of Gromov, and $\partial\tilde{M}$ can be equivariantly identified with $\partial\Gamma$. Consequently, if $C_r^*(\mathscr{F}^{\mathrm{ws}})$ is to be dual to $C_r^*(\mathscr{F}^{\mathrm{ws}})$, we expect that the strongly Morita equivalent algebra $C(\partial\Gamma) \rtimes \Gamma$ will be then dual to itself, or, equivalently, to its opposite algebra. In other words, we can reformulate the question of duality for the foliation algebras purely geometric-group theoretically as follows: is $C(\partial\Gamma) \rtimes \Gamma$ a Poincaré duality algebra when $\Gamma = \pi_1(M)$, for a compact, negatively curved manifold M?

It is not difficult to see that the answer to this question is yes. Consider first the case where M has constant negative curvature. For then, if say $n = 2$ for simplicity, we may take Γ to be a uniform lattice in $G = \mathrm{PSL}_2(\mathbb{R})$, and then for P equal to the parabolic subgroup of upper triangular matrices of determinant 1, we may identify SM with G/Γ and $\partial\Gamma$ with G/P. Since the groupoids $G/P \rtimes \Gamma$ and $G/\Gamma \rtimes P$ are equivalent, and since by two applications of the Thom Isomorphism, $C(G/\Gamma) \rtimes P$ is KK-equivalent to $C(G/\Gamma) \cong C(SM)$, we see $C(\partial\Gamma) \rtimes \Gamma$ is KK-equivalent to $C(SM)$. Since SM is a spin$^{\mathrm{c}}$ manifold, $C(SM)$ has Poincaré duality in K-theory, and therefore so does $C(\partial\Gamma) \rtimes \Gamma$.

Similar arguments can be used for the higher dimensional cases of constant negative curvature. On the other hand, if the curvature is variable, it seems to be necessary to use the infinite dimensional techniques of Higson, Kasparov and Tu ([30]). One then argues as follows. The Baum-Connes conjecture for the amenable groupoid $\partial\Gamma \rtimes \Gamma$ tells us that $C(\partial\Gamma) \rtimes \Gamma$ is KK-equivalent to $C_0(\partial\Gamma \times \underline{E}\Gamma) \rtimes \Gamma \cong C_0(S\tilde{M}) \rtimes \Gamma$ which in turn is strongly

Morita equivalent to $C(SM)$. Again, as SM is a compact spin^c manifold, $C(SM)$ has Poincaré duality, and we are done.

These arguments do not however provide a concrete description of the fundamental class Δ, which is desirable at least from the point of view of noncommutative geometry (whose basic data are cycles, not merely classes). To find such a concrete description was in fact the starting point of our investigation. We wished, moreover, to describe such a cycle, purely in terms of the action of Γ on its Gromov boundary and without reference to spin^c manifolds, Dirac operators, and so on. That such a description exists was suggested by the following example, of quite a different type from the above.

Let $\Gamma = \mathbb{F}_2$. Then Γ is a hyperbolic group, with boundary a Cantor set. It is easy to check (see e.g. [29]) that $C(\partial\Gamma) \rtimes \Gamma$ is in fact isomorphic to a Cuntz-Krieger algebra O_M with matrix M symmetric. By the results of Kaminker and Putnam, we conclude for reasons having apparently nothing to do with topology (but instead with the combinatorics of subshifts of finite type) that $C(\partial\Gamma) \rtimes \Gamma$ is a Poincaré duality algebra. For in this case $O_M \cong O_{M^t}$. Similar calculations verify that $C(\partial\Gamma) \rtimes \Gamma$ is a Poincaré duality algebra when Γ is a free product of cyclic groups.

Motivated by the latter calculations, we will in this paper approach the problem from a different point of view, which will turn out to be quite fruitful, yielding a Poincaré duality result for a very wide class of hyperbolic groups, where neither the argument above in the case of $\Gamma = \pi_1(M)$ nor that of Kaminker and Putnam appear (as far as we know) to apply.

Let then Γ be an arbitrary hyperbolic group and $A = C(\partial\Gamma) \rtimes \Gamma$ the corresponding cross product. Our method is as follows. We construct a canonical extension of $A \otimes A^{\mathrm{op}}$ by the compact operators based on simple considerations of the action of the group Γ on its compactification $\overline{\Gamma}$. Specifically, associated to the compactification, there are two extensions of the algebra $C(\partial\Gamma) \rtimes \Gamma$ by the compact operators, one corresponding, roughly, to the action of Γ on $l^2\Gamma$ by left translation and the action of $C(\overline{\Gamma})$ by multiplication operators, and the other to the action of Γ by right translation and the action of $C(\overline{\Gamma})$ by multiplication operators twisted by inversion on the group. Each extension yields a map $C(\partial\Gamma) \rtimes \Gamma$ into the Calkin algebra, and these two maps into the Calkin algebra commute as a consequence of the compactification being, in the language of [15], 'good,' which simply means that metric balls of uniform size become small in the topology of the compactification near the boundary. Using this asymptotic commutativity, we obtain a single map from $A \otimes A^{\mathrm{op}}$ into the Calkin algebra; i.e. an extension of $A \otimes A^{\mathrm{op}}$ by the compact operators. We define Δ to be the corresponding KK-class.

We will then set about proving that the class $\Delta \in KK^1(A \otimes A^{\mathrm{op}}, \mathbb{C})$ induces Poincaré duality, provided Γ is torsion-free and a certain condition regarding geodesics is met. The latter can be stated as: the boundary has a continuous self map with no fixed points; it is needed for a selection argument in the latter stages of the proof. This technical condition is of course satisfied by groups whose boundaries are spheres or Cantor sets; it is unknown to the author whether there are any groups whose boundaries do not satisfy it. In our argument we will still make use of the Baum-Connes conjecture for the groupoid $\partial\Gamma \rtimes \Gamma$, but this time not to produce a class which *a priori* we know induces Poincaré duality, as in the discussion of $\Gamma = \pi_1(M)$ above, but to show that our class Δ does.

The first step in proving that product with Δ does indeed induce a Poincaré duality isomorphism, is to construct an inverse, or dual element $\hat{\Delta} \in KK^{-1}(\mathbb{C}, A \otimes A^{\mathrm{op}})$. We do this using a construction of Gromov, which produces a sort of geodesic flow for an arbitrary hyperbolic group. We then show that $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta = 1_A$. A calculation in [9] showed that in the case of the free group $\mathbb{F}_2$, the cycle corresponding to the product $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta$ was a compact perturbation of the "γ-element" cycle constructed by Julg and Vallette in [18], parameterised by the points of $\partial\Gamma$. In other words in this case the statement $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta = 1$ was equivalent to the statement $\gamma_{\partial\mathbb{F}_2 \rtimes \mathbb{F}_2} = 1_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2}$ where $\gamma_{\partial\mathbb{F}_2 \rtimes \mathbb{F}_2}$ is the γ-element for this transformation groupoid, and so roughly equivalent to the statement that the Baum-Connes map for the groupoid is an isomorphism. The latter has been verified by Tu ([30]) for general hyperbolic groups, and we are able to resolve the general case in a somewhat analogous way.

The organization of the paper is as follows. In Section 2 we provide a summary of the basic facts from *KK*-theory which we will need. In Section 3 we set up the formalism of *K*-theoretic Poincaré duality. In Section 4 we construct the fundamental class Δ, which as mentioned exists for every hyperbolic group, with or without torsion, and with or without a fixed-point-free map on the boundary. We then construct the dual element $\hat{\Delta}$ using an analog for hyperbolic groups of geodesic flow on a negatively curved manifold. In Section 5 we begin the process of verifying the fundamental equation of Poincaré duality: $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta = 1_A$, where $A = C(\partial\Gamma) \rtimes \Gamma$.

Given the class $\gamma_A = \hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta \in KK(A, A)$, we wish to show it is 1_A. We first describe a cycle for $KK(A, A)$ representing the class γ_A. We then make use of this calculation to show that γ_A lies in the range of the descent map

$$\lambda : RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}) \to KK(A, A).$$

We reduced to showing that its preimage, $\gamma_{\partial\Gamma}$, is $1_{\partial\Gamma} \in RKK(\partial\Gamma; \mathbb{C}, \mathbb{C})$, since the descent map has the property that $\lambda(1_{\partial\Gamma}) = 1_A$. The Baum-Connes conjecture for the amenable groupoid $\partial\Gamma \rtimes \Gamma$ implies that there is an isomorphism $RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}) \cong RKK_\Gamma(\partial\Gamma \times \underline{E}\Gamma; \mathbb{C}, \mathbb{C})$, where $\underline{E}\Gamma$ is the classifying space for proper actions of Γ, and so it will suffice to show that the image of $\gamma_{\partial\Gamma}$ under this isomorphism is $1_{\partial\Gamma \times \underline{E}\Gamma}$. This calculation, which though not difficult is slightly involved, is performed in Sections 6 and 7. It is at this point that we require the hypothesis that the boundary of Γ possesses a fixed-point-free map.

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2. *KK*-theoretic preliminaries

Kasparov's *KK*-theory, along with some of its elaborations, will be used extensively in this paper. *KK* can be understood categorically ([14]). From this latter point of view, there is a category **KK** whose objects are separable, nuclear C^*-algebras and whose morphisms $A \to B$ are the elements of $KK(A, B)$. There is a functor from the category of C^*-algebras to the category **KK**. There is a composition, or intersection product operation $KK(A, D) \times KK(D, B) \to KK(A, B)$ which we denote by $(\alpha, \beta) \mapsto \alpha \otimes_D \beta$. If D is a C^*-

algebra, there is a natural map $KK(A,B) \to KK(A \otimes D, B \otimes D)$, $\alpha \mapsto \alpha \otimes 1_D$, and similarly a map $KK(A,B) \to KK(D \otimes A, D \otimes B)$. The above three operations imply the existence of a mixed cup-cap product

$$KK(A_1, B_1 \otimes D) \times KK(D \otimes B_2, A_2) \to KK(A_1 \otimes B_2, B_1 \otimes A_2)$$

which is denoted $(\alpha, \beta) \mapsto \alpha \otimes_D \beta$, and defined by $\alpha \otimes_D \beta = (\alpha \otimes 1_{B_2}) \otimes_{B_1 \otimes D \otimes B_2} (1_{B_1} \otimes \beta)$. There are higher KK groups $KK^i(A,B)$ for all $i \in \mathbb{Z}$, defined by $KK^i(A,B) = KK(A, B \otimes C_i)$ where C_i is the ith complex Clifford algebra, and one of the features of the theory is that the intersection product is graded commutative. If $A_1, \dots, A_n$ are C^*-algebras, let σ_{ij} denote the map

$$A_1 \otimes \cdots A_i \otimes \cdots A_j \otimes \cdots \otimes A_n \to A_1 \otimes \cdots A_j \otimes \cdots A_i \otimes \cdots \otimes A_n$$

obtained by flipping the two factors. Then by graded commutativity we mean:

Lemma 1. *If $\alpha \in KK^i(A_1, B_1)$ and $\beta \in KK^j(A_2, B_2)$, then*

$$\alpha \otimes_{\mathbb{C}} \beta = (-1)^{ij} (\sigma_{12})_* \sigma_{12}^* (\beta \otimes \alpha) \in KK(A_1 \otimes A_2, B_1 \otimes B_2).$$

Let Λ be a discrete group. Then as well as the category **KK** there is the category $\mathbf{KK}_\Lambda$, whose objects are $\Lambda - C^*$-algebras and whose morphisms are the elements of $KK_\Lambda(A,B)$. We can think of these as equivariant morphisms. There is a descent map

$$\lambda : KK_\Lambda(A,B) \to KK(A \rtimes \Lambda, B \rtimes \Lambda)$$

producing from an equivariant morphism a nonequivariant one. There is a map backwards if A and B happen both to be trivial $\Lambda - C^*$-algebras in the sense that every $\gamma \in \Lambda$ acts as the identity automorphism. The descent map is natural: that is, $\lambda(\alpha \otimes_D \beta) = \lambda(\alpha) \otimes_{D \rtimes \Lambda} \lambda(\beta)$. The group $KK_\Lambda(A,A)$ is a ring with the intersection product, and there is an identity in this ring, denoted 1_A, and it satisfies $\lambda(1_A) = 1_{A \rtimes \Lambda}$.

Finally, let X be a locally compact Λ space. Then there is another category, denoted $\mathcal{RKK}_\Lambda$, this time whose objects are $\Lambda - C(X)$-algebras (see [21]) and whose morphisms are the elements of $\mathcal{RKK}_\Lambda(X; A, B)$. In the case of $A = C_0(X) \otimes A_0$ and $B = C_0(X) \otimes B_0$, with A_0 and B_0 $\Lambda - C^*$-algebras, we denote, following Kasparov, the group $\mathcal{RKK}_\Lambda(X; A, B)$ by $RKK_\Lambda(X; A_0, B_0)$. The intersection product

$$\mathcal{RKK}_\Lambda(X; A, D) \times \mathcal{RKK}_\Lambda(X; D, B) \to \mathcal{RKK}_\Lambda(X; A, B)$$

is denoted $(\alpha, \beta) \mapsto \alpha \otimes_{X,D} \beta$, and similarly for RKK_Λ. Note also that $RKK_\Lambda(X; A, A)$ has a unit, which is denoted $1_{X,A}$, and if $A = \mathbb{C}$ we denote this unit simply by 1_X. Finally, if Z is any space, there is a natural map

$$p_Z^* : RKK_\Lambda(X; A, B) \to \mathcal{RKK}_\Lambda(X \times Z; A, B).$$

This map is natural with respect to intersection products and thus is a ring homomorphism when $A = B$. Under certain special circumstances it is an isomorphism (see Theorem 54).

Throughout this paper we will let $B(\mathscr{E})$ denote bounded operators on a Hilbert module $\mathscr{E}$, $K(\mathscr{E})$ compact operators, and $Q(\mathscr{E})$ the Calkin algebra $B(\mathscr{E})/K(\mathscr{E})$. The projection map $B(\mathscr{E}) \to Q(\mathscr{E})$, which will be invoked frequently, will always be denoted by π.

Following Kasparov ([21]), if $\mathscr{E}$ is a Hilbert B-module and A acts on $\mathscr{E}$ by a homomorphism $A \to B(\mathscr{E})$, we will refer to $\mathscr{E}$ as a Hilbert (A, B)-bimodule.

Because all the algebras in this paper are ungraded—or alternatively, have trivial grading—we can make certain simplifications in the definitions of the KK groups (see [4]). With such ungraded A and B, cycles for $KK(A, B)$ are given simply by pairs $(\mathscr{E}, F)$ where $\mathscr{E}$ is an (A, B)-bimodule, F commutes modulo compact operators with the action of A, and $a(F^*F - 1)$ and $a(FF^* - 1)$ are compact for every $a \in A$.

Cycles for $KK^1(A, B)$ are given by pairs $(\mathscr{E}, P)$ for which P is as before an operator on the (A, B)-bimodule $\mathscr{E}$ as above, and where P satisfies the three conditions $[a, P]$, $a(P^2 - P)$, and $a(P - P^*)$ are compact for all $a \in A$. Such pairs are equivalently given by *extensions*, i.e. homomorphisms $A \mapsto Q(\mathscr{E})$. For by the Stinespring construction, under our nuclearity assumptions, for each such homomorphism τ there exists a Hilbert (A, B)-module $\tilde{\mathscr{E}}$, an isometry $U : \mathscr{E} \to \tilde{\mathscr{E}}$, and an operator P on $\tilde{\mathscr{E}}$ such that $a(P^2 - P)$, $[a, P]$, and $a(P - P^*)$ are compact for all $a \in A$, and $\pi(U^*PaPU) = \tau(a)$ for all $a \in A$.

Recall that $KK^{-1}\big(\mathbb{C}, C^*(\mathbb{R})\big) \cong \mathbb{Z}$ and is generated by the class $[\hat{d}_{\mathbb{R}}]$ of the Dirac operator on $\mathbb{R}$, viewed as an unbounded self-adjoint multiplier of $C^*(\mathbb{R})$. The class $[\hat{d}_{\mathbb{R}}]$ allows us to identify, for any C^*-algebras A and B, the groups $KK^1\big(C^*(\mathbb{R}) \otimes A, B\big)$, and $KK(A, B)$, by the map $KK^1\big(C^*(\mathbb{R}) \otimes A, B\big) \to KK(A, B)$, $x \mapsto [\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} x$. We shall need to compute this map at the level of cycles in several simple cases.

Let ψ be the function in $C^*(\mathbb{R})$ whose Fourier transform is $\dfrac{-2i}{z + i}$. It has the property that $\psi + 1$ is unitary in $C^*(\mathbb{R})^+$.

Lemma 2. *Let A be a C^*-algebra, φ a homomorphism $C^*(\mathbb{R}) \to A$, and suppose $\tau : A \to Q(H)$ is a homomorphism to the Calkin algebra. Let $[\tau]$ denote the class in $KK^1(A, \mathbb{C})$ corresponding to τ. Then the class $[\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} \varphi^*([\tau]) \in KK(\mathbb{C}, \mathbb{C})$ is represented by the cycle $(H, T + 1)$, where T is any operator on H such that $\pi(T) = \tau\big(\varphi(\psi)\big)$.*

We will also need the following simple lemma.

Corollary 3. *Define a class $[\tau] \in KK^1\big(C^*(\mathbb{R}), \mathbb{C}\big)$ by means of the homomorphism $\tau : C^*(\mathbb{R}) \to Q\big(L^2(\mathbb{R})\big)$,*

$$f \mapsto \pi\big(\chi \cdot \lambda(f)\big),$$

where λ is the left regular representation of $C^*(\mathbb{R})$ and χ is the characteristic function of the left half-line. Then $[\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} [\tau] = [1_{\mathbb{C}}] \in KK(\mathbb{C}, \mathbb{C})$.

Proof. This follows from Lemma 2 and a calculation; one checks simply that $\chi \cdot \psi$ as an operator on $L^2(\mathbb{R})$ has index 1. One can do this by solving a simple differential equation. (See [10].) □

Note 4. Remark that the function χ above can be replaced by any function on $\mathbb{R}$ which is 1 at $-\infty$ and 0 at $+\infty$. For any such function gives the same extension.

Next, let A_1 and A_2 be $\Lambda - C^*$-algebras, where Λ is a discrete group. An action of Λ on an (A_1, A_2)-bimodule $\mathscr{E}$ will always refer to an action of Λ as complex linear maps compatible with the inner product in the sense that $\langle \gamma\xi, \gamma\eta \rangle_{A_2} = \gamma(\langle \xi, \eta \rangle_{A_2})$, and compatible with the bimodule structure in the sense that $\gamma(a\xi b) = \gamma(a)\gamma(\xi)\gamma(b)$. Such $\mathscr{E}$ will be referred to as a $\Lambda - (A_1, A_2)$-bimodule. If we wish to possibly waive the part of the last requirement that states that $\gamma(a\xi) = \gamma(a)\gamma(\xi)$, whilst maintaining the requirement that $\gamma(\xi b) = \gamma(\xi)\gamma(b)$, we will simply call $\mathscr{E}$ a $\Lambda - A_2$-module. Thus, such a module satisfies $\gamma(\xi b) = \gamma(\xi)\gamma(b)$, but the homomorphism $A_1 \to \mathscr{B}(\mathscr{E})$ may not necessarily be Λ-equivariant.

Cycles for $KK_\Lambda(A_1, A_2)$ are then given by pairs $(\mathscr{E}, F)$ where $\mathscr{E}$ is a $\Lambda - (A_1, A_2)$-bimodule, and where $F \in B(\mathscr{E})$ with $a(F^*F - 1)$ and $a(FF^* - 1)$ compact for all $a \in A_1$, and $\gamma(F) - F$ compact for all $\gamma \in \Lambda$. Cycles for $KK^1_\Lambda(A_1, A_2)$ are given by pairs $(\mathscr{E}, P)$ where $\mathscr{E}$ is a $\Lambda - (A_1, A_2)$-bimodule and P is an operator with $a(P^2 - P)$, $a(P - P^*)$, and $[a, P]$ compact for all $a \in A_1$, and $\gamma(P) - P$ compact for all $\gamma \in \Lambda$.

A minor technical issue which in general we do not know how to resolve concerns the question of whether or not an equivariant map $A_1 \to Q(\mathscr{E})$, where $\mathscr{E}$ is a $\Lambda - A_2$-module, produces an element of $KK^1_\Lambda(A_1, A_2)$. If Λ is the trivial group this is of course the Stinespring construction, given our standing assumption that all C^*-algebras (with the obvious exceptions of Calkin algebras and so on) are nuclear. In the general case, an equivariant homomorphism $A_1 \to Q(\mathscr{E})$ yields a homomorphism $A_1 \rtimes \Lambda \to Q(\mathscr{E} \rtimes \Lambda)$ where $\mathscr{E} \rtimes \Lambda$ is as in [21], being a certain $(A_1 \rtimes \Lambda, A_2 \rtimes \Lambda)$-bimodule (this is part of the definition of the descent map) and so an element of $KK^1(A_1 \rtimes \Lambda, A_2 \rtimes \Lambda)$ as long as not merely A_1 and A_2 are nuclear, but also $A_1 \rtimes \Lambda$ and $A_2 \rtimes \Lambda$ are nuclear. But such an element may not necessarily come under descent from an element of $KK^1_\Lambda(A_1, A_2)$. To avoid this issue, we make the following definition.

Definition 5. Let Λ be a discrete group, let A_1 and A_2 be $\Lambda - C^*$-algebras and let $\mathscr{E}$ be a $\Lambda - A_2$-module. Let $\tau : A_1 \to Q(\mathscr{E})$ be a Λ-equivariant homomorphism. We say τ is dilatable if there is a $\Lambda - (A_1, A_2)$-bimodule $\tilde{\mathscr{E}}$, an operator P on $\tilde{\mathscr{E}}$ such that $[a, P]$, $a(P^2 - P)$, $a(P^* - P)$ and $\gamma(P) - P$ are compact for all $a \in A_1$, $\gamma \in \Lambda$, and if there exists an isometry $U : \mathscr{E} \to \tilde{\mathscr{E}}$, such that $\pi(U^*PaPU) = \tau(a) \in Q(\mathscr{E})$ for all $a \in A_1$.

As mentioned above, if Λ is the trivial group then every homomorphism $A_1 \to Q(\mathscr{E})$ is dilatable. The same is clearly true of finite Λ. In general, with the hypothesis of dilatibility, we do clearly have the following:

Lemma 6. *If A_1, A_2, $\mathscr{E}$, Λ and τ as above, and if τ is dilatable, then τ defines a class $[\tau]$ in $KK^1_\Lambda(A_1, A_2)$ by the pair $(\tilde{\mathscr{E}}, P)$.*

We next pass to a case where to calculate the Kasparov product of two elements one of which is given by a dilatable homomorphism, we do not need to explicitly involve the dilation. We will use this technical lemma several times, sometimes with Λ the trivial group. In the latter case, the lemma gives a method of avoiding explicit construction of a completely positive section.

Lemma 7. *Let A_1, A_2 be $\Lambda - C^*$-algebras and $\mathscr{E}$ be a $\Lambda - A_2$-module. Let $[h]$ be a class in $KK^1_\Lambda(C^*(\mathbb{R}) \otimes A_1, A_2)$ given by a Λ-equivariant dilatable homomorphism $h : C^*(\mathbb{R}) \otimes A_1 \to Q(\mathscr{E})$ of the form $x \otimes a_1 \mapsto h'(x)h''(a_1)$, where h' and h'' are Λ-equivariant homomorphisms. Suppose that the homomorphism h'' lifts to a Λ-equivariant homomorphism $\tilde{h}'' : A_1 \to B(\mathscr{E})$. Then the class $[\hat{d}_\mathbb{R}] \otimes_{C^*(\mathbb{R})} [h] \in KK_\Lambda(A_1, A_2)$ is represented by the following cycle. The module is $\mathscr{E}$ with its original $\Lambda - A_2$-module structure and the left A_1-module structure given by the homomorphism $\tilde{h}''$. The operator is given by $F + 1$ where F is any operator on $\mathscr{E}$ such that $\pi(F) = h'(\psi)$.*

Remark 8. Similar lemmas can be formulated and proved for the RKK_Λ category, but we leave it to the reader to formulate them.

3. Formalism of noncommutative Poincaré duality

Let us begin with a lemma. See [19] for a similar discussion.

Lemma 9. *Let A and B be C^*-algebras and let Δ and $\hat{\Delta}$ be two elements in $KK^i(A \otimes B, \mathbb{C})$ and $KK^{-i}(\mathbb{C}, A \otimes B)$ respectively. Define a map $\hat{\Delta}_j : K^j(B) \mapsto K_{j-i}(A)$ by $\hat{\Delta}_j(x) = \hat{\Delta} \otimes_B x$. Define a map $\Delta_j : K_j(A) \mapsto K^{j+i}(B)$ by $\Delta_j(y) = y \otimes_A \Delta$. Define also two classes in respectively $KK(A, A)$ and $KK(B, B)$ by $\gamma_A = (\hat{\Delta} \otimes 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \sigma^*_{12}(\Delta))$, and $\gamma_B = ((\sigma_{12})_*(\hat{\Delta}) \otimes 1_B) \otimes_{B \otimes A \otimes B} (1_B \otimes \Delta)$. Then we have:*

$$\Delta_{j-i}(\hat{\Delta}_j(x)) = (-1)^{ij} \gamma_B \otimes_B x, \quad x \in K^j(B);$$

and

$$\hat{\Delta}_{j+i}(\Delta_j(y)) = (-1)^{ij} y \otimes_A \gamma_A, \quad y \in K_j(A).$$

Proof. We verify the first equation; the second follows similarly. Let $x \in K^j(B)$. Then it follows from the definition that

$$\Delta_{j-i}(\hat{\Delta}_j(x)) = (\hat{\Delta} \otimes 1_B) \otimes_{A \otimes B \otimes B} (1_A \otimes x \otimes 1_B) \otimes_{A \otimes B} \Delta.$$

By functoriality of the intersection product we may write this

$$((\sigma_{12})_*(\hat{\Delta}) \otimes 1_B) \otimes_{B \otimes A \otimes B} \sigma^*_{12}(1_A \otimes x \otimes 1_B) \otimes_{A \otimes B} \Delta.$$

On the other hand, again by definition, we have

$$\gamma_B \otimes_B x = ((\sigma_{12})_*(\hat{\Delta}) \otimes 1_B) \otimes_{B \otimes A \otimes B} (1_B \otimes \Delta) \otimes_B x.$$

So we are reduced to proving that $(1_A \otimes x \otimes 1_B) \otimes_{A \otimes B} \Delta = (-1)^{ij}(1_B \otimes \Delta) \otimes_B x$. But this follows immediately from Lemma 1. □

In view of this theorem, we will take as the definition of duality between two C^*-algebras the following (compare [7], p. 588):

Definition 10. Two separable, unital, and nuclear C^*-algebras A and B are *dual* with a dimension shift of i if there exist $\Delta \in KK^i(A \otimes B, \mathbb{C})$, $\hat{\Delta} \in KK^{-i}(\mathbb{C}, A \otimes B)$ such that

$$\hat{\Delta} \otimes_B \Delta = 1_A$$

and

$$\hat{\Delta} \otimes_A \Delta = (-1)^i 1_B.$$

We will call such a pair $(\hat{\Delta}, \Delta)$ a duality pair.

Theorem 11. *If A and B are dual in the sense of Definition* 10, *then the maps $\hat{\Delta}_*$ and Δ_* defined in Lemma* 9 *induce inverse isomorphisms up to the signs specified there $K_j(A) \cong K^{j+i}(B)$ and $K^j(B) \cong K_{j-i}(A)$.*

For the next piece of terminology recall that for a C^*-algebra A, A^{op} denotes the opposite algebra of A.

Definition 12. A separable, nuclear C^*-algebra A is a Poincaré duality algebra if A and A^{op} are dual in the sense of Definition 10. We will refer to Δ as the fundamental class of A, and $(\hat{\Delta}, \Delta)$ as a Poincaré duality pair.

4. The main theorem

Let Γ be a hyperbolic group. We shall assume here and throughout this paper that Γ is torsion-free. To Γ we can add a boundary $\partial\Gamma$ which compactifies the group Γ understood as a metric space. Thus, $\overline{\Gamma} = \Gamma \cup \partial\Gamma$ can be given the structure of a compact metrizable space in which Γ is contained as a dense, open subset. For details see [11]. The group Γ acts by homeomorphisms on $\partial\Gamma$ and this action is topologically amenable in the sense of [2] (see the appendix of [2] for a proof of this fact). Therefore, to each hyperbolic group we can associate an amenable r-discrete amenable groupoid $\partial\Gamma \rtimes \Gamma$ and then a groupoid C^*-algebra $C(\partial\Gamma) \rtimes \Gamma$ which for the rest of this paper we shall denote by A. The C^*-algebra A is separable, simple, nuclear and purely infinite (see [29] or [1]).

Our goal is to show that for a large subclass of hyperbolic groups Γ, A is a Poincaré duality algebra in the sense of Definition 12. Let us first state certain simple facts we shall require.

Note 13. When we are thinking of elements of Γ as simply points in the metric space Γ, we shall use the notation x, y, etc. In particular, x_0 will always refer to the identity of the group, viewed as a natural basepoint. Also, for any $R \geqq 0$ and any $x \in \Gamma$, $B_R(x)$ denotes the ball of radius R (with respect to the word metric) centered at x.

For convenience we will also fix a metric $d_{\overline{\Gamma}}$ on $\overline{\Gamma}$ compatible with the topology. The following lemma then follows from the definition of the topology on $\overline{\Gamma}$ (see [11]).

Lemma 14. *If $\varepsilon > 0$, there exists $R \geqq 0$ such that if $a, b \in \overline{\Gamma}$ and $d_{\overline{\Gamma}}(a, b) \geqq \varepsilon$, then every geodesic from a to b passes through $B_R(x_0)$. Conversely, if $R \geqq 0$, there exists $\varepsilon > 0$ such that if every geodesic between a and b passes through $B_R(x_0)$, then $d_{\overline{\Gamma}}(a, b) \geqq \varepsilon$.*

We will also require the following. Recall that $(x|y)$ denotes the Gromov product of $x, y \in \Gamma$ (see [12] or [11]). For the proof of this lemma see for example [28].

Lemma 15. *If f is a bounded function on Γ, then f extends to a continuous function on $\overline{\Gamma}$ if and only if for all $\varepsilon > 0$ there exists $R \geqq 0$ such that if $(x|y) > R$, then $|f(x) - f(y)| < \varepsilon$.*

We shall need an explicit description of the classifying space for proper actions of Γ. This is given by the Rips construction.

Definition 16. The Rips complex for Γ of parameter N, $P_N(\Gamma)$, is the simplicial complex whose vertices are the points of Γ, and whose k-simplices are the sets of cardinality k of diameter less than or equal to N.

Let $\bar{P}_N(\Gamma)$ denote the realization of the Rips complex. It can be viewed as the collection of finitely supported probability measures on Γ whose support has diameter $\leqq N$. This point of view will be useful later on the proof when some linear interpolation will be needed from Γ to $\bar{P}_N(\Gamma)$. Note that Γ is embedded naturally in $\bar{P}_N(\Gamma)$. Clearly $\bar{P}_N(\Gamma)$ carries a free, simplicial, isometric, proper, co-compact action of Γ.

A proof of the following may be found in [24].

Lemma 17. *For large enough N, $\bar{P}_N(\Gamma)$ is the classifying space $\underline{E}\Gamma$ for proper actions of Γ.*

Note 18. From this point onwards, we fix N sufficiently large as in the above lemma, and denote the realization of the Rips complex with parameter N simply by $\underline{E}\Gamma$. We will also fix a simplicial metric $d_{\underline{E}\Gamma}$ on $\underline{E}\Gamma$, so that Γ is quasi-isometrically embedded in $\underline{E}\Gamma$ as the vertices of the complex. Then $\underline{E}\Gamma$ is of course a hyperbolic space in its own right, and is quasi-isometric to Γ.

We now pass to the construction of the fundamental class $\Delta \in KK^1(A \otimes A^{\mathrm{op}}, \mathbb{C})$, which will arise naturally as an *extension*, or equivalently as a homomorphism $A \otimes A^{\mathrm{op}} \to Q(H)$ for some Hilbert space H. This map $A \otimes A^{\mathrm{op}} \to Q(H)$ will be given by two commuting maps $A \to Q(H)$ and $A^{\mathrm{op}} \to Q(H)$, which we shall denote by λ and λ^{op} respectively.

Passing to the description of λ, let us put $H = l^2(\Gamma)$. This notation will be retained throughout the rest of this paper. Let e_x, $x \in \Gamma$ denote the standard basis element of H corresponding to point mass at x. For $\gamma \in \Gamma$ let u_γ denote the unitary in $B(H)$ given by left translation by γ, i.e. $u_\gamma(e_x) = e_{\gamma x}$. Let $\lambda(\gamma)$ denote the image of u_γ in the Calkin algebra. Let f be a function in $C(\partial\Gamma)$, apply the Tietze extension theorem to extend f to a continuous function $\tilde{f}$ on $\overline{\Gamma}$, and let $\lambda(f)$ denote the image in $Q(H)$ of the operator on H given by multiplication by $\tilde{f}$, in other words the operator $e_x \mapsto \tilde{f}(x)e_x$. Remark that though the map $\gamma \to u_\gamma$, $f \to \tilde{f}$ is not well-defined into $B(H)$, it *is* well-defined into $Q(H)$, since any two extensions of a function f differ by a function vanishing at ∞ and thus by a compact operator on $l^2(\Gamma)$. The following lemma is a trivial calculation:

Lemma 19. *The assignment $\gamma \mapsto \lambda(\gamma)$, $f \mapsto \lambda(f)$, defines a covariant pair for the C^*-dynamical system $\big(C(\partial\Gamma), \Gamma\big)$, and so a homomorphism $A \to Q(H)$.*

Next, define a map $\lambda^{\mathrm{op}} : A^{\mathrm{op}} \to Q(H)$ as follows. First, let v_γ, for $\gamma \in \Gamma$, denote the unitary operator of *right* translation by $\gamma : v_\gamma(e_x) = e_{x\gamma}$. Let $\lambda^{\mathrm{op}}(\gamma)$ denote the image of this unitary operator in the Calkin algebra. If now $f \in C(\partial\Gamma)$, let $\tilde{f}$ denote an extension of f to a continuous function on $\overline{\Gamma}$ as before, and let $\lambda^{\mathrm{op}}(f)$ denote the image in the Calkin algebra of the multiplication operator given by multiplication by the function $x \mapsto \tilde{f}(x^{-1})$. These two maps are similarly well-defined into the Calkin algebra, and we have easily:

Lemma 20. *The assignment $\gamma \mapsto \lambda^{\mathrm{op}}(\gamma)$, $f \mapsto \lambda^{\mathrm{op}}(f)$, defines a covariant pair with respect to the opposite action of Γ on $C(\partial\Gamma)$ and hence a homomorphism $A^{\mathrm{op}} \to Q(H)$.*

We next show the two homomorphisms λ and λ^{op} commute as maps into the Calkin algebra. This follows from the following.

Lemma 21. *Let $\tilde{f}$ be a function on Γ, viewed as a multiplication operator on H, and let $\gamma \in \Gamma$.*

(1) *If $x \mapsto \tilde{f}(x)$ is continuous on $\overline{\Gamma}$, then $[v_\gamma, \tilde{f}]$ is a compact operator.*

(2) *If $x \mapsto \tilde{f}(x^{-1})$ is continuous on $\overline{\Gamma}$, then $[u_\gamma, \tilde{f}]$ is a compact operator.*

Proof. Let $\tilde{f}$ be as in (1). Choose $\varepsilon > 0$. Remark if $x, \gamma \in \Gamma$ we have $(x, x\gamma) \geqq |x| - |\gamma|$. From this and Lemma 15 we see: there exists $R \geqq 0$ such that

$$|x| > R \Rightarrow |\tilde{f}(x) - \tilde{f}(x\gamma)| < \varepsilon.$$

In other words, the function $\tilde{f}(x) - \tilde{f}(x\gamma)$ vanishes at infinity. It follows immediately that $v_\gamma \tilde{f} v_{\gamma^{-1}} - \tilde{f}$ is compact; for this operator is precisely multiplication by this function. Hence $(v_\gamma \tilde{f} v_{\gamma^{-1}} - \tilde{f}) v_\gamma = [v_\gamma, \tilde{f}]$ is also a compact operator. (2) follows from (1) by conjugating by the unitary $H \to H$ induced from inversion on the group. □

Remark 22. The above lemma can be restated in a slightly more general way. Having fixed a left-invariant metric on Γ, as we have done, *right* translation by a fixed $\gamma \in \Gamma$ gives an operator of finite propagation; on the other hand any operator of finite propagation commutes modulo compacts with multiplication by a function in $C(\overline{\Gamma})$ by the same proof as that of Lemma 21.

Definition 23. Let Γ be any hyperbolic group and $\partial\Gamma$ its Gromov boundary. Let H denote $l^2(\Gamma)$. We define the fundamental class of the C^*-algebra $A = C(\partial\Gamma) \rtimes \Gamma$ to be the class Δ in $KK^1(A \otimes A^{\mathrm{op}}, \mathbb{C})$ corresponding to the homomorphism $A \otimes A^{\mathrm{op}} \to Q(H)$ induced by the two commuting homomorphisms λ and λ^{op}.

Remark 24. Let Γ be a discrete, not necessarily hyperbolic group acting cocompactly and properly on a nonpositively curved space X, and let ∂X denote the visibility boundary of X. The visibility boundary compactifies the group Γ and all of the above constructions extend to this situation. We thus obtain a map $C(\partial X) \rtimes \Gamma \otimes_{\max} \left(C(\partial X) \rtimes \Gamma\right)^{\mathrm{op}} \to Q(H)$ in the same way. However, as the Γ-action on ∂X is no longer amenable, it is no longer necessarily the case that such a map defines a KK^1 element.

Remark 25. If J denotes the conjugate linear operator $H \to H$ sending the element $\sum_\gamma a_\gamma e_\gamma \in C_c(\Gamma)$ to the element $\sum_\gamma \bar{a}_\gamma e_{\gamma^{-1}}$, then the equation $J\lambda(a^*)J^{-1} = \lambda^{\mathrm{op}}(a)$ holds for any $a \in A$. This is the content of Connes' reality axiom (see [6]), except that the relation holds in the Calkin algebra rather than in $B(H)$. In fact, it is easy to see that all our constructions are compatible with the various real structures on the algebras, Hilbert spaces, and so on, concerned, and that the cycle Δ in actually gives a KR-homology class. Similarly we shall see that $\hat{\Delta}$ gives a KR class, and that the duality we are going to prove holds in the real as well as the complex setting.

We now proceed to the element $\hat{\Delta}$, to construct which we shall use an idea of Gromov and subsequent work by Champetier and Matheus. Theorem 27 was first stated by Gromov (see [12], p. 222), with a sketch of a proof; details were added by the latter two authors in respectively [5] and [23]. As the latter authors' work does not seem to be very well known, we provide here a brief discussion of it.

Let us denote by $\partial^2\Gamma$ the space $\{(a,b) \in \partial\Gamma \rtimes \partial\Gamma \mid a \neq b\}$. Let $\widetilde{G\Gamma}$ denote the collection of geodesics in $\underline{E}\Gamma$. Note that $\widetilde{G\Gamma}$ has a natural metric with respect to which it is quasi-isometric to $\underline{E}\Gamma$ and hence to Γ. Furthermore $\widetilde{G\Gamma}$ carries commuting free and proper actions of $\mathbb{R}$ and Γ, and the action of Γ is co-compact. It is not in general true that a pair (a,b) of distinct boundary points of Γ is connected by a unique element up to re-parameterization of $\widetilde{G\Gamma}$. In other words, it is not quite true that $\widetilde{G\Gamma}/\mathbb{R} \cong \partial^2\Gamma$, which is what we would like. This may be remedied as follows.

One defines an equivalence relation $\sim$ on $\widetilde{G\Gamma}$ such that $G\Gamma = \widetilde{G\Gamma}/\sim$ is Hausdorff and in fact with the Hausdorff metric on equivalence classes is a metric space quasi-isometric to $\widetilde{G\Gamma}$ with the quotient map $q : \widetilde{G\Gamma} \to G\Gamma$ providing the quasi-isometry. The relation $\sim$ is Γ-equivariant, and Γ thus acts on $G\Gamma$ and q is a Γ-invariant map. The relation $\sim$ is not quite compatible with the action of $\mathbb{R}$ on $\widetilde{G\Gamma}$, but it is possible to define a new $\mathbb{R}$ action on $G\Gamma$ commuting with the Γ-action and with the following property: if $(a,b) \in \partial^2\Gamma$, the $\mathbb{R}$ orbits of all the geodesics in $\widetilde{G\Gamma}$ from a to b are collapsed by the quotient map to a single orbit of the new action of $\mathbb{R}$ on $G\Gamma$. This enables us to identify $G\Gamma/\mathbb{R}$ with $\partial^2\Gamma$.

We remark that this identification may be seen in another way. If r is a point of $G\Gamma$, the curve $t \mapsto g_t(r)$, where g_t denotes the $\mathbb{R}$-action on $G\Gamma$, is a quasi-geodesic in $G\Gamma$. If under the identification $G\Gamma/\mathbb{R} \cong \partial^2\Gamma$ the $\mathbb{R}$-orbit of r corresponds to $(a,b) \in \partial^2\Gamma$, then it is also the case that $\lim_{t\to-\infty} g_t(r) = a$ and $\lim_{t\to+\infty} g_t(r) = b$, where the limits are taken in the Gromov hyperbolic metric space $G\Gamma$.

We will only need some of the details of this construction in the proof of Lemma 30. Apart from this lemma, we will only need the properties of $G\Gamma$ stated in Theorem 27 below.

Remark 26. We choose this moment to note that the only Γ-invariant homeomorphism $\partial\Gamma \to \partial\Gamma$ is the identity homeomorphism. For, as is well known, the action of Γ on $\partial\Gamma$ is strongly proximal. If ϕ is a Γ-invariant homeomorphism of $\partial\Gamma$, by amenability of $\mathbb{Z}$, ϕ leaves invariant some probability measure μ. But then for all $\gamma \in \Gamma$, $\phi_*\gamma_*(\mu) = \gamma_*\mu$. Choose $a \in \partial\Gamma$. By strong proximality we can choose a sequence of $\gamma \in \Gamma$ such that $\gamma_*(\mu) \to \delta_a$ where δ_a denotes point mass at a, and the convergence is wk*. It follows ϕ fixes a. Since a was arbitrary, ϕ is the identity map.

Theorem 27. *There exists a proper metric space $G\Gamma$ on which Γ acts, for which*:

(1) *$G\Gamma$ has the structure of a locally trivial principal $\mathbb{R}$-bundle over $\partial^2\Gamma$.*

(2) *Γ acts on $G\Gamma$ freely, properly and co-compactly, and its action commutes with the $\mathbb{R}$ action.*

(3) *There is a continuous involution $G\Gamma \to G\Gamma$ denoted $r \mapsto \hat{r}$, which commutes with the Γ action, and satisfies $g_t(\hat{r}) = \widehat{g_{-t}r}$ for all t, where g_t denotes the $\mathbb{R}$ action.*

Note 28. Elements of the space $G\Gamma$ should be thought of as geodesics in $\underline{E}\Gamma$, and so we shall call them *pseudogeodesics*. The $\mathbb{R}$-orbit of a pseudogeodesic is determined by a pair of distinct boundary points (a, b). We will call such a pseudogeodesic a "pseudogeodesic from a to b." In such a case, we denote by $r(-\infty)$ the point a, and by $r(+\infty)$ the point b. As per the discussion prior to Remark 26, actually the curve $t \mapsto g_t(r)$ is a quasi-geodesic in $G\Gamma$ viewed as a hyperbolic metric space quasi-isometric to Γ, and $a = \lim_{t\to-\infty} g_t(r)$, and $b = \lim_{t\to\infty} g_t(r)$, so this notation is actually quite suitable.

Remark 29. If Γ acts properly, isometrically and co-compactly on a $CAT(-\varepsilon)$ space X for $\varepsilon > 0$ we may take for our purposes the space $G\Gamma$ to be the space of actual (parameterized) geodesics in X, rendering the lemma superfluous. For convexity in $CAT(-\varepsilon)$ spaces implies that any two distinct boundary points are joined by a unique geodesic.

We will also need the following lemma.

Lemma 30. *Let $G\Gamma$ be as in Theorem 27. Then there exists a proper Γ-equivariant map $G\Gamma \to \underline{E}\Gamma$, denoted $r \mapsto r(0)$ and satisfying*

$$\lim_{t\to\infty} g_t(r)(0) = r(+\infty) \quad \textit{and} \quad \lim_{t\to-\infty} g_t(r)(0) = r(-\infty),$$

where the limits are taken in the Gromov compactification $\overline{\underline{E}\Gamma}$ of the hyperbolic metric space $\underline{E}\Gamma$.

Proof. Fixing a point of $G\Gamma$, the orbit map $\Gamma \to G\Gamma$ is a quasi-isometry which therefore induces a Γ-invariant homeomorphism $\partial\Gamma \to \partial G\Gamma$. We may thus identify these two spaces, and the identification is independent of the point chosen, since any two such identifications differ by a Γ-invariant homeomorphism $\partial\Gamma \to \partial\Gamma$, and the only such is the identity by Remark 26.

On the other hand, by the universal property of $\underline{E}\Gamma$ (see [8]), there exists a proper, continuous Γ-equivariant map $\alpha : G\Gamma \to \underline{E}\Gamma$. Such a map is necessarily a quasi-isometry, since the actions of Γ on $G\Gamma$ and $\underline{E}\Gamma$ are co-compact. Hence α extends to a Γ-invariant homeomorphism $\alpha : \partial\Gamma = \partial G\Gamma \to \partial\Gamma$. Since it is Γ-invariant, it must be the identity map, again by Remark 26.

Now if r is a pseudogeodesic from a to b where a and b are points of $\partial\Gamma$ viewed by our identification as points of $\partial G\Gamma$, then $t \mapsto g_t(r)$ is a quasi-geodesic in $G\Gamma$ and we have $\lim_{t\to-\infty} g_t(r) = a$ and $\lim_{t\to+\infty} g_t(r) = b$. Since α is a quasi-isometry, $t \mapsto \alpha(g_t(r))$ is a quasi-geodesic in $\underline{E}\Gamma$, and we have $\lim_{t\to-\infty} \alpha(g_t r) = a$ and $\lim_{t\to+\infty} \alpha(g_t r) = b$ since α extends to the identity map on the boundary, and we are done.

Note from this point onward we shall drop the notation $r \mapsto \alpha(r)$, replacing it with $r \mapsto r(0)$ as in the statement of the theorem. □

Remark 31. Let M be a compact spin$^{\mathrm{c}}$ manifold, so that $C(M)$ is a Poincaré duality algebra in the sense of Definition 12. The fundamental class Δ is obtained by pushing forward the class of the Dirac operator on M by the diagonal map $M \to M \times M$ to a class in $K_*(M \times M) \cong K^*\big(C(M) \otimes C(M)\big)$. Let U be a tubular neighborhood of the diagonal in $M \times M$. There is an inclusion of C^*-algebras $C_0(U) \to C(M) \otimes C(M)$, and the dual element $\hat{\Delta}$ is constructed by pushing forward by this inclusion the Thom class in $K^*(U) \cong K_*\big(C_0(U)\big)$ to an element of $K^*(M \times M) \cong K_*\big(C(M) \otimes C(M)\big)$. In our situation, which is vaguely analogous, there is an inclusion of C^*-algebras

$$C_0(\partial^2\Gamma) \rtimes \Gamma \to A \otimes A,$$

and the algebra on the left hand side is strongly Morita equivalent to a cross product by $\mathbb{R}$, and thus has a Thom class, namely the generator of the flow, which may similarly be pushed forward to a class in $K_1(A \otimes A)$ and then to a class in $K_1(A \otimes A^{\mathrm{op}})$ using the isomorphism $A \cong A^{\mathrm{op}}$. This is how we shall construct $\hat{\Delta}$.

Note 32. For the following we will denote by $(a, b) \mapsto r_{a,b}$ a continuous selection of pseudogeodesic from a to b. Such a continuous (but not Γ-equivariant) selection exists by Theorem 27.1 and by paracompactness of $\partial^2\Gamma$ (see [12]).

Define a right $C_0(\partial^2\Gamma) \rtimes \Gamma$-valued inner product on the linear space $C_c(G\Gamma)$ by the formula:

$$\langle \xi, \eta \rangle_{C_0(\partial^2\Gamma)\rtimes\Gamma}\big((a,b),\gamma\big) = \int_{\mathbb{R}} \bar{\xi}\big(g_t(r_{a,b})\big)\eta\big(g_t\gamma^{-1}(r_{a,b})\big)\,dt.$$

Define a right $C_0(\partial^2\Gamma) \rtimes \Gamma$-module structure on $C_c(G\Gamma)$ by

$$(\xi \cdot f)(r) = \xi(r) f\big(r(-\infty), r(+\infty)\big), \quad f \in C_0(\partial^2\Gamma), \quad \text{and} \quad (\xi \cdot \gamma)(r) = \xi(\gamma r), \quad \text{for } \gamma \in \Gamma.$$

Note this right module structure is compatible with the inner product.

Definition 33. Let E denote the completion of $C_c(G\Gamma)$ to a right Hilbert $C_0(\partial^2\Gamma) \rtimes \Gamma$-module with respect to the above inner product.

Definition 34. Define a left action of $C^*(\mathbb{R})$ on E by the unitary representation $t \mapsto U_t$, where $(U_t\xi)(r) = \xi\big(g_{-t}(r)\big)$.

Remark 35. It follows from the definition that the finite rank operators on E as a $C_0(\partial^2\Gamma) \rtimes \Gamma$-module are linear combinations of the operators

$$K\xi(r) = \sum_{\gamma \in \Gamma} \zeta(\gamma^{-1}r) \int_{\mathbb{R}} \overline{\eta(g_t r)}\xi(\gamma^{-1}g_t r)\,dt,$$

where ζ and η are elements of E, which fact we will use in the proof (which we have extracted from [27]) of the following lemma.

Lemma 36. *Every element of* $C^*(\mathbb{R})$ *acts on* E *as a compact operator. Therefore* E *defines a class* $[E] \in KK\big(C^*(\mathbb{R}), C_0(\partial^2\Gamma) \rtimes \Gamma\big)$.

Proof (see [27]). As GX/Γ is compact, we may find a compact fundamental domain F for the Γ action on GX. Choose $\varepsilon > 0$. Then we may choose open sets U_i of GX such that $F \subset \bigcup U_i$, and such that for all i, $U_i \cap g_t(U_i) = \emptyset$ for all $|t| \geqq \varepsilon$. Choose then (see [27]) functions $\zeta_{i,\varepsilon} \in C_c(GX)$ such that $\zeta_{i,\varepsilon} \in C_c(U_i)$, and such that

$$(*) \qquad \sum_{\gamma \in \Gamma} \zeta_{i,\varepsilon}(\gamma^{-1}r) \int_{\mathbb{R}} \zeta_{i,\varepsilon}(\gamma^{-1}g_t r)\, dt = 1$$

for all $r \in GX$. Define then operators K_ε on E by

$$K_\varepsilon \xi(r) = \sum_i \sum_{\gamma \in \Gamma} \zeta_{i,\varepsilon}(\gamma^{-1}r) \int_{\mathbb{R}} \zeta_{i,\varepsilon}(g_t r) \xi(g_t \gamma^{-1} r)\, dt.$$

From Remark 35, each K_ε is a compact operator, and from condition $(*)$ above and the fact that each $\zeta_{i,\varepsilon}(r)\zeta_{i,\varepsilon}(g_t r) = 0$ if $|t| \geqq \varepsilon$ and $r \in GX$, it can easily be seen that for $\varphi \in C^*(\mathbb{R})$,

$$\varphi \cdot K_\varepsilon \to \varphi$$

in operator norm, as $\varepsilon \to 0$. Since each $\varphi \cdot K_\varepsilon$ is compact, so is φ. □

Definition 37. Let the class $[D] \in KK^{-1}\big(\mathbb{C}, C_0(\partial^2\Gamma) \rtimes \Gamma\big)$ be defined by

$$[D] = [\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} [E],$$

where $[E]$ denotes the class in $KK\big(C^*(\mathbb{R}), C_0(\partial^2\Gamma) \rtimes \Gamma)\big)$ of the cycle $(E, 0)$.

Remark 38. It will be useful for later to note the following. By the Stabilization Theorem ([21]) we may embed E as a direct summand of a trivial Hilbert $C_0(\partial^2\Gamma) \rtimes \Gamma$-module $C_0(\partial^2\Gamma) \rtimes \Gamma \otimes V$, where V is any separable Hilbert space. Then the left action $C^*(\mathbb{R}) \to B(E)$ of $C^*(\mathbb{R})$ on E may be composed with the embedding, yielding a homomorphism $v : C^*(\mathbb{R}) \to K\big(C_0(\partial^2\Gamma) \rtimes \Gamma \otimes V\big) \cong C_0(\partial^2\Gamma) \rtimes \Gamma \otimes K(V)$. $[D]$ then becomes $v_*([\hat{d}_{\mathbb{R}}])$. Note also that since any two choices of v are related by a unitary equivalence, this construction is not dependent on the choice of embedding $E \to C_0(\partial^2\Gamma) \rtimes \Gamma \otimes V$.

We next note the following trivial

Lemma 39. *The* C^**-algebra* $A = C(\partial\Gamma) \rtimes \Gamma$ *is isomorphic to its opposite algebra.*

Proof. Define a map $j : A \to A^{\mathrm{op}}$ by the covariant pair $j(f) = f$ and $j(\gamma) = \gamma^{-1}$. Then j induces the required isomorphism. □

For what follows, observe that there is a canonical inclusion $C_0(\partial^2\Gamma) \rtimes \Gamma \to A \otimes A$ given by the composition $C_0(\partial^2\Gamma) \rtimes \Gamma \to C(\partial\Gamma \times \partial\Gamma) \rtimes \Gamma \cong C(\partial\Gamma) \otimes C(\partial\Gamma) \rtimes \Gamma \to C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma = A \otimes A$. We shall denote this inclusion by i.

Definition 40. We define the element $\hat{\Delta} \in KK^{-1}(\mathbb{C}, A \otimes A^{\mathrm{op}})$ to be

$$\hat{\Delta} = (1_A \otimes j)_* i_*([D]) \in KK^{-1}(\mathbb{C}, A \otimes A^{\mathrm{op}}).$$

We are finally in a position to state our main theorem.

Theorem 41. *Let Γ be a torsion-free hyperbolic group and $\partial\Gamma$ its Gromov boundary. Assume that $\partial\Gamma$ has a self-map with no fixed points. Let A denote the cross product $C(\partial\Gamma) \rtimes \Gamma$. Let Δ and $\hat{\Delta}$ be the classes constructed in respectively Definitions* 23 *and* 40. *Then A is a Poincaré duality algebra in the sense of Definition* 12 *and $(\hat{\Delta}, \Delta)$ is a Poincaré duality pair.*

The rest of this paper is devoted to the proof of Theorem 41.

5. Various reductions

Let Γ be a torsion-free hyperbolic group as in the previous section, A the cross product $C(\partial\Gamma) \rtimes \Gamma$, and $\Delta \in KK^1(A \otimes A^{\mathrm{op}}, \mathbb{C})$ and $\hat{\Delta} \in KK^{-1}(\mathbb{C}, A \otimes A^{\mathrm{op}})$ the KK-classes specified in respectively Definition 23 and Definition 40. To prove Theorem 41 we must verify that

$$\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta = 1_A$$

and

$$\hat{\Delta} \otimes_A \Delta = -1_{A^{\mathrm{op}}}.$$

Set $\gamma_A = \hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta$ and $\gamma_{A^{\mathrm{op}}} = \hat{\Delta} \otimes_A \Delta$. Using the map j of Lemma 39 we may identify $KK(A^{\mathrm{op}}, A^{\mathrm{op}})$ with $KK(A, A)$. We will first prove that with this identification, γ_A and $\gamma_{A^{\mathrm{op}}}$ are the same up to sign, which implies we will only need to compute one of the above products.

Let

$$\Delta_0 = (1_A \otimes j)^*(\Delta) \in KK^1(A \otimes A, \mathbb{C}) \quad \text{and} \quad \hat{\Delta}_0 = (1_A \otimes j^{-1})_*(\hat{\Delta}) \in KK^{-1}(\mathbb{C}, A \otimes A).$$

Recall that $\sigma_{12} : A \otimes A \to A \otimes A$ denotes the flip. We first note:

Lemma 42. *The classes Δ_0 and $\hat{\Delta}_0$ satisfy $\sigma_{12}^*(\Delta_0) = \Delta_0$; and $(\sigma_{12})_*(\hat{\Delta}_0) = -\hat{\Delta}_0$.*

Proof. Beginning with $\hat{\Delta}_0$, note $\hat{\Delta}_0 = i_*([D])$. Hence it suffices to show

$$(\sigma_{12} \circ i)_*([D]) = -i_*([D]).$$

Recall $[D]$ is given by $[\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} [E]$. Hence $(\sigma_{12} \circ i)_*([D]) = [\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} (\sigma_{12} \circ i)_*[E]$. Let $u : C^*(\mathbb{R}) \to C^*(\mathbb{R})$ denote the homomorphism corresponding to $t \mapsto -t$. Based on a simple index calculation we see $u_*([\hat{d}_{\mathbb{R}}]) = -[\hat{d}_{\mathbb{R}}]$. Furthermore we have

$$(\sigma_{12} \circ i)_*([E]) = u^* i_*([E]).$$

Hence $(\sigma_{12} \circ i)_*([D]) = u_*([\hat{d}_{\mathbb{R}}]) \otimes_{C^*(\mathbb{R})} i_*([E]) = -[\hat{d}_{\mathbb{R}}] \otimes_{C^*(\mathbb{R})} i_*([E]) = -i_*([D])$, and we are done.

The class Δ_0 is represented by the map $A \otimes A \to Q(H)$, $a \otimes b \mapsto \lambda(a)\rho(b)$, where λ is as before, and $\rho(b) = I\lambda(a)I$, with I the unitary $H \to H$ induced from inversion on the group. Applying the flip σ_{12}^* to Δ_0 results in the map $A \otimes A \to Q(H)$ given by $a \otimes b \mapsto \rho(b)\lambda(a)$. Since this is conjugate, via I, to Δ, the class of these two extensions is the same: $\sigma_{12}^*(\Delta_0) = \Delta_0$. □

Corollary 43. *We have*: $(j_*^{-1})j^*(\gamma_{A^{\mathrm{op}}}) = -\gamma_A$. *Hence if* $\gamma_A = 1$ *then* $\gamma_{A^{\mathrm{op}}} = -1_{A^{\mathrm{op}}}$.

Proof. One checks first that:

$$(1) \qquad (j^{-1})_*(1_{A^{\mathrm{op}}} \otimes \Delta) = (j^{-1} \otimes 1_A \otimes j^{-1})^*(1_A \otimes \Delta_0),$$

$$(2) \qquad j^*(j^{-1} \otimes 1_A \otimes j^{-1})_*\big((\sigma_{12})_*(\hat{\Delta}) \otimes 1_{A^{\mathrm{op}}}\big) = (\sigma_{12})_*(\hat{\Delta}_0) \otimes 1_A,$$

$$(3) \qquad \gamma_{A^{\mathrm{op}}} = \big((\sigma_{12})_*(\hat{\Delta}) \otimes 1_{A^{\mathrm{op}}}\big) \otimes_{A^{\mathrm{op}} \otimes A \otimes A^{\mathrm{op}}} (1_{A^{\mathrm{op}}} \otimes \Delta).$$

Hence, using (3), then (1), and then functoriality of the intersection product, we have

$$(4) \qquad (j^{-1})_*j^*(\gamma_{A^{\mathrm{op}}}) = j^*(j^{-1} \otimes 1_A \otimes j^{-1})_*\big((\sigma_{12})_*(\hat{\Delta}) \otimes 1_{A^{\mathrm{op}}}\big) \otimes_{A \otimes A \otimes A} (1_A \otimes \Delta_0).$$

Using (2) we have

$$(5) \qquad (j^{-1})_*j^*(\gamma_{A^{\mathrm{op}}}) = \big((\sigma_{12})_*(\hat{\Delta}_0) \otimes 1_A\big) \otimes (1_A \otimes \Delta_0).$$

On the other hand,

$$(6) \qquad \gamma_A = (\hat{\Delta}_0 \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta_0),$$

and now, comparing (5) and (6) we are done by Lemma 42. □

We are therefore reduced in the proof of Theorem 41 to proving $\gamma_A = 1_A$, where, as stated above, γ_A is the class $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta$.

Note 44. Recall that if $\mathscr{E}$ is a Hilbert A-module, we are denoting by $B(\mathscr{E})$ the bounded operators on $\mathscr{E}$, $K(\mathscr{E})$ the compact operators, and $Q(\mathscr{E})$ the quotient $B(\mathscr{E})/K(\mathscr{E})$. With $\mathscr{E} = A \otimes H$ the standard Hilbert A-module, we have natural maps $A \otimes B(H) \to B(A \otimes H)$, $A \otimes K(H) \to K(A \otimes H)$ and $A \otimes Q(H) \to Q(A \otimes H)$. We will sometimes suppress these maps, writing for instance an element of $B(A \otimes H)$ in the form $a \otimes T$, for $a \in A$ and $T \in B(H)$.

Remark 45. For what follows it will be useful to note that any function f on $\partial\Gamma \times \Gamma$ continuous in the $\partial\Gamma$-variable may be regarded via the formula $f(a)(e_x) = f(a, x)e_x$ as an element of $C\big(\partial\Gamma, B(H)\big) \cong C(\partial\Gamma) \otimes B(H)$ whence (see note above), as an element of $B(A \otimes H)$, and then, by application of the quotient map, an element of $Q(A \otimes H)$.

For further convenience, let us denote the C^*-algebra $C_0(\partial^2\Gamma) \rtimes \Gamma$ by B.

Now, from Equation (6) in the proof of Corollary 43, from $\hat{\Delta}_0 = i_*([D])$, and by functoriality of the intersection product, we have

$$\gamma_A = ([D] \otimes 1_A) \otimes_{B \otimes A} (i \otimes 1_A)^*(1_A \otimes \sigma_{12}^* \Delta_0).$$

We will begin by examining the term $(i \otimes 1_A)^*(1_A \otimes \sigma_{12}^* \Delta_0) \in KK^1(B \otimes A, A)$.

Define a covariant pair for the dynamical system $\big(C_0(\partial^2\Gamma), \Gamma\big)$ as follows. If F is a function on $\partial^2\Gamma$ and $\tilde{F}$ denotes an extension of F to a continuous function on $\partial\Gamma \times \overline{\Gamma}$, let $\tau(F)$ be the element of $Q(A \otimes H)$ corresponding (see Remark 45) to the function $\tau(F)(a, x) = \tilde{F}\big(x^{-1}(a), x^{-1}\big)$ on $\partial\Gamma \times \Gamma$. This is independent of the extension $\tilde{F}$ of F. For $\gamma \in \Gamma$, set $\tau(\gamma) = 1 \otimes \lambda^{\mathrm{op}}(\gamma^{-1}) \in Q(A \otimes H)$. It is easy to check that these two assignments define a covariant pair.

Definition 46. Let $\tau : B \to Q(A \otimes H)$ be the homomorphism corresponding to the above covariant pair.

For $\gamma \in \Gamma$ recall that u_γ denotes left translation by γ. Define a covariant pair for the dynamical system $\big(C(\partial\Gamma), \Gamma\big)$ by $\varphi(f) = f \otimes 1 \in B(A \otimes H)$, and $\varphi(\gamma) = \gamma \otimes u_\gamma \in B(A \otimes H)$.

Definition 47. Let $\varphi : A \to B(A \otimes H)$ denote the homomorphism corresponding to the above covariant pair.

The following proposition, though depending only on a simple property of hyperbolic groups, is central to the proof that $\gamma_A = 1_A$. It represents a sort of untwisting of the product $\hat{\Delta} \otimes_{A^{\mathrm{op}}} \Delta$.

Proposition 48. *The class* $(i \otimes 1_A)^*(1_A \otimes \sigma_{12}^* \Delta_0) \in KK^1(B \otimes A, A)$ *is represented by the homomorphism* $\imath : B \otimes A \to Q(A \otimes H)$,

$$\imath(b \otimes a) = \tau(b)\pi\big(\varphi(a)\big),$$

where φ, τ *are as in Definitions* 46 *and* 47.

We will require the following:

Lemma 49. *Let* $F \in C_c(\partial^2\Gamma \times \partial\Gamma)$, *and* $\tilde{F}$ *an extension of* F *to a continuous function on* $\partial\Gamma \times \overline{\Gamma} \times \overline{\Gamma}$. *Then the two functions on* $\partial\Gamma \times \Gamma$

$$(a, x) \mapsto \tilde{F}\big(x^{-1}(a), x^{-1}, x\big)$$

and

$$(a, x) \mapsto \tilde{F}\big(x^{-1}(a), x^{-1}, a\big)$$

are the same modulo $C_0(\partial\Gamma \times \Gamma)$.

Proof. Let F be as in the statement of the lemma. Then for some $\varepsilon > 0$, F is supported on the set of $(a, b, c) \in \partial\Gamma \times \partial\Gamma \times \Gamma$ such that $d_{\overline{\Gamma}}(a, b) \geqq \varepsilon$. Therefore F can be extended to a function $\tilde{F}$ supported on those $(a, b, c) \in \partial\Gamma \times \overline{\Gamma} \times \overline{\Gamma}$ for which $d_{\overline{\Gamma}}(a, b) \geqq \varepsilon$.

Let R correspond to ε as in Lemma 14. It suffices to show that for $a \in \partial\Gamma$ fixed and x_n a sequence in Γ converging to a boundary point $b \in \partial\Gamma$, the sequence

$$\tilde{F}\big(x_n^{-1}(a), x_n^{-1}, x_n\big) - \tilde{F}\big(x_n^{-1}(a), x_n^{-1}, a\big)$$

converges to 0 as $n \to \infty$. Since if $d_{\overline{\Gamma}}(x_n^{-1}a, x_n^{-1}) < \varepsilon$, then both $\tilde{F}\big(x_n^{-1}(a), x_n^{-1}, x_n\big) = 0$ and $\tilde{F}\big(x_n^{-1}(a), x_n^{-1}, a\big) = 0$, we may assume after extracting a subsequence if necessary, that $d_{\overline{\Gamma}}\big(x_n^{-1}(a), x_n^{-1}\big) \geqq \varepsilon$ for all n. Then by choice of R, $d\big(x_0, [x_n^{-1}, x_n^{-1}a)\big) = d\big(x_n, [e, a)\big) \leqq R$ for all n, where $[e, a)$ denotes any geodesic ray from e to a. Hence $x_n \to a$, and the result follows from continuity of $\tilde{F}$ in the third variable. $\square$

Proof of Proposition 48. Consider the class $(i \otimes 1_A)^*(1_A \otimes \sigma_{12}^*\Delta_0)$. It is represented by the homomorphism $B \otimes A \to Q(A \otimes H)$

$$a_1 \otimes a_2 \otimes a_3 \mapsto a_1 \otimes \rho(a_2)\lambda(a_3),$$

where we have suppressed the inclusion $i : B \to A \otimes A$ so that in the above formula $a_1 \otimes a_2$ is regarded as an element of B. Here $\rho(a) = \lambda^{\mathrm{op}}\big(j(a)\big)$ as in the proof of Lemma 42. Define a unitary map of Hilbert modules $U : A \otimes H \to A \otimes H$ by the formula $U(a \otimes e_x) = x \cdot a \otimes e_x$. Let Ad_U denote the inner automorphism of $Q(A \otimes H)$ given by $\pi(T) \mapsto \pi(UTU^*)$ and let $\imath'$ denote the homomorphism $B \otimes A \to Q(A \otimes H)$

$$\imath'(a_1 \otimes a_2 \otimes a_3) = \mathrm{Ad}_U\big(a_1 \otimes \rho(a_2)\lambda(a_3)\big).$$

We claim that $\imath' = \imath$. It is a simple matter to check that $\imath|_{B \otimes C_r^*(\Gamma)} = \imath'|_{B \otimes C_r^*(\Gamma)}$, where $B \otimes C_r^*(\Gamma)$ is viewed as a sub-algebra of $B \otimes A$, and that for $b \in B$ and $f \in C(\partial\Gamma)$, we have $\imath(b \otimes f) = \tau(b)\pi(f \otimes 1)$ whereas $\imath'(b \otimes f) = \tau(b)\big(1 \otimes \lambda(f)\big)$. Thus it remains to prove that $\tau(b)\pi(1 \otimes \tilde{f} - f \otimes 1) = 0$ in the Calkin algebra $Q(A \otimes H)$ whenever $b \in B$, $f \in C(\partial\Gamma)$ and $\tilde{f}$ is an extension of f to $\overline{\Gamma}$. Since every $b \in B$ is a closed linear combination of elements of the form $\gamma \cdot F$, with $\gamma \in \Gamma$ and $F \in C_c(\partial^2\Gamma)$, without loss of generality $b = F \in C_c(\partial^2\Gamma)$ and the result follows from Lemma 49. $\square$

Corollary 50. *The class γ_A lies in the range of the descent map*

$$\lambda : RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C}),$$

i.e. there exists $\gamma_{\partial\Gamma} \in RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C})$ such that $\lambda(\gamma_{\partial\Gamma}) = \gamma_A$.

Proof. Regard (see Remark 38) the class $[D] \in KK^{-1}(\mathbb{C}, B)$ as given by a homomorphism $\nu : C^*(\mathbb{R}) \to B \otimes K(V)$ for some separable Hilbert space V. It follows that $[D] \otimes 1_A$ is represented by the homomorphism $\nu \otimes 1_A : C^*(\mathbb{R}) \otimes A \to B \otimes A \otimes K(V)$. Hence the class γ_A is represented by the homomorphism $C^*(\mathbb{R}) \otimes A \to Q(A \otimes H \otimes V)$ given by the composition

$$C^*(\mathbb{R}) \otimes A \xrightarrow{\nu \otimes 1_A} B \otimes A \otimes K(V) \xrightarrow{\imath \otimes 1_{K(V)}} Q(A \otimes H \otimes V).$$

Referring to Lemma 7 with Λ the trivial group, let h denote this composition, and put h' equal to the composition

$$C^*(\mathbb{R}) \xrightarrow{\nu} B \otimes K(V) \xrightarrow{\tau \otimes 1_{K(V)}} Q(A \otimes H \otimes V),$$

and h'' the composition

$$A \xrightarrow{1_A} B \otimes A \otimes K(V) \xrightarrow{\iota \otimes 1_{K(V)}} Q(A \otimes H \otimes V).$$

By Proposition 48, h'' lifts to a map $A \to B(A \otimes H \otimes V)$ by setting $\tilde{h}''(a) = \varphi(a) \otimes 1_V$. Therefore by Lemma 7, γ_A is represented by the cycle $(A \otimes H \otimes V, F + 1)$, where $A \otimes H \otimes V$ has the (A, A)-bimodule structure which is standard on the right and which on the left is given by the homomorphism $a \mapsto \varphi(a) \otimes 1_V$, and where F is any operator on $A \otimes H \otimes V$ such that $\pi(F) = (\tau \otimes 1_{K(V)})\big(v(\psi)\big)$.

Now, by construction we may take F to be a limit of finite linear combinations of operators on the Hilbert (A, A)-bimodule $A \otimes H \otimes V$ of the form

$$f \otimes e_x \otimes v \mapsto (h_1 \circ x^{-1}) h_3 f \otimes \tilde{h}_2(x^{-1}) e_x \otimes T(v) \tag{7}$$

where T is compact, and $h_1 \otimes h_2 \otimes h_3 \in C_0(\partial^2\Gamma \times \partial\Gamma)$, and where $\tilde{h}_2$ denotes a lift of h_2 to a continuous function on $\overline{\Gamma}$; and also of the right translation operators

$$f \otimes e_x \otimes v \mapsto f \otimes e_{x\gamma} \otimes T(v), \tag{8}$$

where $\gamma \in \Gamma$ and T is compact. Consider the Hilbert $\big(C(\partial\Gamma), C(\partial\Gamma)\big)$-bimodule $C(\partial\Gamma) \otimes H \otimes V$. Let Γ act on $C(\partial\Gamma) \otimes H \otimes V$ by $\gamma(f \otimes e_x \otimes v) = \gamma(f) \otimes e_{\gamma x} \otimes v$. Then it is easy to check that with this action, $C(\partial\Gamma) \otimes H \otimes V$ becomes a $\Gamma - \big(C(\partial\Gamma), C(\partial\Gamma)\big)$-bimodule. Note that the left and right actions of $C(\partial\Gamma)$ are in fact the same. From equations (7) and (8) it is clear that F is constructed from operators on $A \otimes H \otimes V$ which restrict to operators on $C(\partial\Gamma) \otimes H \otimes V$, hence the same is true of F. Clearly, as an operator on $C(\partial\Gamma) \otimes H \otimes V$, F commutes with the left action of $C(\partial\Gamma)$ on the module, since this action is the same as the right action. Finally, F commutes mod compacts with the action of Γ, since the operators of which F is built all do. Hence the pair $\big(C(\partial\Gamma) \otimes H \otimes V, F + 1\big)$ actually defines a cycle for the group $RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C})$. Checking the definition of the descent map (see [21]) it is easy to see that the image of this cycle under descent is precisely the cycle corresponding to γ_A described in the first paragraph. □

We will use the above corollary to make use of the following consequence of a theorem of Tu, which we state in a slightly more general context. Let Λ denote a discrete group, which for simplicity we assume acts co-compactly on its classifying space for proper actions, $\underline{E}\Lambda$ (as is the case for torsion-free hyperbolic Γ). Let X be a compact metrizable space on which Λ acts by homeomorphisms. Recall from Section 1 the map $p^*_{\underline{E}\Lambda} : RKK_\Lambda(X; \mathbb{C}, \mathbb{C}) \to RKK_\Lambda(X \times \underline{E}\Lambda; \mathbb{C}, \mathbb{C})$. Finally, recall that a $C_0(\underline{E}\Lambda \times X)$-algebra D is a C^*-algebra together with a non-degenerate, asymptotically unital homomorphism $C_0(\underline{E}\Lambda \times X) \to \mathscr{Z}\big(\mathscr{M}(D)\big)$, where $\mathscr{Z}$ denotes center. D is called a Γ-$C_0(\underline{E}\Lambda \times X)$-algebra if Γ acts by automorphisms on D and the homomorphism $C_0(\underline{E}\Lambda \times X) \to \mathscr{Z}\big(\mathscr{M}(D)\big)$ is Γ-equivariant. Note that such D can be in particular viewed as a $C(X)$ algebra, by means of the map $C(X) \to C_b(\underline{E}\Gamma \times X) \to \mathscr{Z}\big(\mathscr{M}(D)\big)$. Let us make the following definition.

Definition 51. Let D be a $\Lambda - C_0(\underline{E}\Gamma \times X)$-algebra. Define a map

$$\sigma_{\underline{E}\Lambda, D} : RKK_\Lambda(\underline{E}\Lambda \times X; \mathbb{C}, \mathbb{C}) \to \mathscr{R}KK_\Lambda(X; D, D)$$

by replacing a cycle (H, F) by the cycle $(H \otimes_{C_0(\underline{E}\Lambda \times X)} D, F \otimes 1)$.

The Hilbert (D, D)-bimodule structure on $H \otimes_{C_0(\underline{E}\Lambda \times X)} D$ is well-defined as functions in $C_0(\underline{E}\Lambda \times X)$ act as central multipliers of D.

Next, we quote Tu's theorem (see [30]):

Theorem 52. *Let the action of* Λ *on* X *be topologically amenable in the sense of* [2]. *Then there exist a* Λ-$C_0(\underline{E}\Lambda \times X)$*-algebra* D *and elements* $\alpha \in \mathscr{R}KK_\Lambda(X; C(X), D)$, *and* $\beta \in \mathscr{R}KK_\Lambda(X; D, C(X))$, *satisfying*

$$\alpha \otimes_{X,D} \beta = 1_X \in \mathscr{R}KK_\Lambda(X; C(X), C(X)) = RKK_\Lambda(X; \mathbb{C}, \mathbb{C}), \quad \textit{and}$$

$$\beta \otimes_{X,C(X)} \alpha = 1_{X,D} \in \mathscr{R}KK_\Lambda(X; D, D).$$

Using Theorem 52 we can define a map $q : RKK_\Lambda(\underline{E}\Lambda \times X; \mathbb{C}, \mathbb{C}) \to RKK_\Lambda(X; \mathbb{C}, \mathbb{C})$ inverse to $p^*_{\underline{E}\Lambda}$ as follows.

Definition 53. For $a \in RKK_\Lambda(\underline{E}\Lambda \times X; \mathbb{C}, \mathbb{C})$, define

$$q(a) = \alpha \otimes_{X,D} \sigma_{\underline{E}\Lambda, D}(a) \otimes_{X,D} \beta \in \mathscr{R}KK_\Lambda(X; C(X), C(X)) = RKK_\Lambda(X; \mathbb{C}, \mathbb{C}),$$

where α and β are as in Theorem 52 and $\sigma_{\underline{E}\Lambda, D}$ is as in Definition 51.

We show that q and $p^*_{\underline{E}\Lambda}$ are inverse to each other. Let π_1 and π_2 denote the projections $\underline{E}\Lambda \times \underline{E}\Lambda \to \underline{E}\Lambda$, and π_1^*, π_2^* the corresponding homomorphisms $C_0(\underline{E}\Lambda) \to C_b(\underline{E}\Lambda \times \underline{E}\Lambda)$. It is a direct consequence of the axioms for $\underline{E}\Lambda$ (see [8]) that π_1 and π_2 are Λ-invariantly homotopic.

Theorem 54. *The map* $p^*_{\underline{E}\Lambda}$ *defines a ring isomorphism*

$$RKK_\Lambda(X; \mathbb{C}, \mathbb{C}) \to RKK_\Lambda(X \times \underline{E}\Lambda; \mathbb{C}, \mathbb{C})$$

with inverse q.

Proof. Because the proof is simply an X-parameterized version of the corresponding statement for $X = \mathrm{pt}$ we prove the latter for simplicity of exposition. From this assumption we have a $\Lambda - C_0(\underline{E}\Lambda)$-algebra D, and $\alpha \in KK_\Lambda(\mathbb{C}, D)$, $\beta \in KK_\Lambda(D, \mathbb{C})$, satisfying $\alpha \otimes_D \beta = 1_\mathbb{C}$ and $\beta \otimes_\mathbb{C} \alpha = 1_D$. Let $a \in KK_\Lambda(\mathbb{C}, \mathbb{C})$. Then

$$q(p^*_{\underline{E}\Lambda}(a)) = \alpha \otimes_D \sigma_{\underline{E}\Lambda, D}(p^*_{\underline{E}\Lambda}(a)) \otimes_D \beta = \alpha \otimes_D \sigma_D(a) \otimes_D \beta,$$

as is easy to check. On the other hand, by commutativity of the external tensor product and the assumption on α and β, $\alpha \otimes_D \sigma_D(a) \otimes_D \beta = \alpha \otimes_D \beta \otimes_\mathbb{C} a = a$. Hence $q(p^*_{\underline{E}\Lambda}(a)) = a$.

The other composition is slightly more elaborate. Consider, for $b \in RKK_\Lambda(\underline{E}\Lambda; \mathbb{C}, \mathbb{C})$

$$p^*_{\underline{E}\Lambda}(q(b)) = p^*_{\underline{E}\Lambda}(\alpha) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\sigma_{\underline{E}\Lambda, D}(b)) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\beta),$$

and in particular the term $p^*_{\underline{E}\Lambda}(\alpha) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\sigma_{\underline{E}\Lambda, D}(b))$. We claim that this is equal to $b \otimes_{\underline{E}\Lambda} p^*_{\underline{E}\Lambda}(\alpha)$, whereupon we shall be done. We may assume that b is given by a pair $(\mathscr{E}, 0)$, where $\mathscr{E}$ is a $\Gamma - C_0(\underline{E}\Lambda)$-module, and that α is given by a pair (D, M) where D is a

$C_0(\underline{E}\Lambda)$-algebra, and M is a self-adjoint multiplier of D. Then the module for the product $p^*_{\underline{E}\Lambda}(\alpha) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\sigma_{\underline{E}\Lambda, D}(b))$ can be written $\mathscr{E} \otimes_{C_0(\underline{E}\Lambda)} (C_0(\underline{E}\Lambda) \otimes D)$, where the tensor product is over the homomorphism $C_0(\underline{E}\Lambda) \to C(\underline{E}\Lambda \times \underline{E}\Lambda) \to \mathscr{M}(C_0(\underline{E}\Lambda) \otimes D)$, $f \mapsto \pi_2^*(f)$. The operator for the Kasparov product is given by multiplication by M in the D-coordinate; note this is well defined, as M, being a multiplier of D, commutes with the actions of functions on D.

On the other hand, consider the product $b \otimes_{\underline{E}\Lambda} p^*_{\underline{E}\Lambda}(\alpha)$. One calculates the product of modules to be $\mathscr{E} \otimes_{C_0(\underline{E}\Lambda)} (C_0(\underline{E}\Lambda) \otimes D)$, where this time the tensor product is over the homomorphism $f \mapsto \pi_1^*(f)$. The operator is again M acting in the D-coordinate. Now, since π_1 and π_2 are Λ-equivariantly homotopic, the two modules are homotopic, through a homotopy in which the action of M remains the same.

More precisely, the two cycles corresponding to the Kasparov products $p^*_{\underline{E}\Lambda}(\alpha) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\sigma_{\underline{E}\Lambda, D}(b))$ and $b \otimes_{\underline{E}\Lambda} p^*_{\underline{E}\Lambda}(\alpha)$ are, as we have indicated, homotopic, whence $p^*_{\underline{E}\Lambda}(\alpha) \otimes_{\underline{E}\Lambda, D} p^*_{\underline{E}\Lambda}(\sigma_{\underline{E}\Lambda, D}(b)) = b \otimes_{\underline{E}\Lambda} p^*_{\underline{E}\Lambda}(\alpha)$. This proves the claim. (See [21], p. 179 for the same sort of argument.) □

Corollary 55. *Let $\gamma_{\partial\Gamma}$ be any element of $RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C})$ such that $\lambda(\gamma_{\partial\Gamma}) = \gamma_A$. Then to show $\gamma_A = 1_A$, and thus that $(\hat{\Delta}, \Delta)$ is a Poincaré duality pair, it suffices to show $p^*_{\underline{E}\Gamma}(\gamma_{\partial\Gamma}) = 1_{\partial\Gamma \times \underline{E}\Gamma}$.*

For $p^*_{\underline{E}\Gamma}$, being a ring isomorphism, takes a multiplicative unit to a multiplicative unit.

Fix $\gamma_{\partial\Gamma}$ to be the class of the cycle for $RKK_\Gamma(\partial\Gamma; \mathbb{C}, \mathbb{C})$ described in the proof of Corollary 50. Then by that corollary $\lambda(\gamma_{\partial\Gamma}) = \gamma_A$. Denote the class $p^*_{\underline{E}\Gamma}(\gamma_{\partial\Gamma}) \in RKK_\Gamma(\underline{E}\Gamma \times \partial\Gamma; \mathbb{C}, \mathbb{C})$ by $\gamma_{\underline{E}\Gamma \times \partial\Gamma}$. By Corollary 55 it remains for us to show that $\gamma_{\underline{E}\Gamma \times \partial\Gamma} = 1_{\underline{E}\Gamma \times \partial\Gamma}$.

6. Alternative description of $\gamma_{\underline{E}\Gamma \times \partial\Gamma}$

We need first consider more closely the element $\gamma_{\partial\Gamma}$, as its description in Corollary 50 is unsatisfactory for our purposes, relying as it does on an inexplicit homomorphism $v : C^*(\mathbb{R}) \to B \otimes K(V)$. We would like to describe a cycle corresponding to $\gamma_{\partial\Gamma}$, whence to $\gamma_{\partial\Gamma \times \underline{E}\Gamma}$, in such a way as to incorporate the bimodule E associated to the space $G\Gamma$ of pseudogeodesics in a more explicit way. Actually, it is quite difficult to do this for $\gamma_{\partial\Gamma}$ because of dilatability issues, but easy to do it for $\gamma_{\partial\Gamma \times \underline{E}\Gamma}$. So we focus on the latter task. In this section we simply state what this new description of $\gamma_{\partial\Gamma \times \underline{E}\Gamma}$ is, constructing a certain geometric cycle for $RKK_\Gamma(\partial\Gamma \times \underline{E}\Gamma; \mathbb{C}, \mathbb{C})$ whose class we will denote by $\gamma'_{\partial\Gamma \times \underline{E}\Gamma}$. We can readily show that $\gamma'_{\underline{E}\Gamma \times \partial\Gamma} = 1_{\underline{E}\Gamma \times \partial\Gamma}$. In the last section we will verify that in fact $\gamma_{\underline{E}\Gamma \times \partial\Gamma} = \gamma'_{\underline{E}\Gamma \times \partial\Gamma}$. Taking these two results together, we will thus have proven $\gamma_{\underline{E}\Gamma \times \partial\Gamma} = 1_{\underline{E}\Gamma \times \partial\Gamma}$.

Recall that we are assuming $\partial\Gamma$ has a fixed point-free map S. By compactness of $\partial\Gamma$ there exists $\delta_0 > 0$ such that $d_{\bar{\Gamma}}(a, S(a)) \geqq \delta_0$ for all $a \in \partial\Gamma$. By abuse of notation, we also denote by S the *equivariant* map $\partial\Gamma \times \Gamma \to \partial\Gamma$ defined by $S(a, z) = z(S(z^{-1}a))$.

Lemma 56. *There exists an equivariant map $\partial\Gamma \times \Gamma \to G\Gamma$, $(a,z) \mapsto r_{a,z}$, satisfying $r_{a,z}(-\infty) = a$ and $r_{a,z}(+\infty) = S(a,z)$.*

Proof. For each $(a,b) \in \partial^2\Gamma$, let $r_{a,b}$ be a pseudogeodesic from a to b, such that the map $(a,b) \mapsto r_{a,b}$ is continuous (see Note 32). For $a \in \partial\Gamma$, let $r_a = r_{a,S(a)}$. We have $r_a(-\infty) = a$ and $r_a(+\infty) = S(a)$. To construct an equivariant map as required, we may set $r_{a,z} = z(r_{z^{-1}a})$. □

Recall that N is the parameter of the Rips complex, which we have fixed throughout.

Lemma 57. *There exists a continuous function Q on $\partial\Gamma \times \Gamma \times \overline{\Gamma}$ satisfying the following properties*:

(1) $0 \leqq Q(a,z,x) \leqq 1$ *for all* $(a,z,x) \in \partial\Gamma \times \Gamma \times \overline{\Gamma}$.

(2) *Q is invariant under the triple diagonal action of* Γ *on* $\partial\Gamma \times \Gamma \times \overline{\Gamma}$.

(3) *If x_n is a sequence in Γ, $z \in \Gamma$, and $x_n \to S(a,z)$, then for every $w \in B_N(z)$, we have $Q(a,w,x_n) \to 0$.*

(4) *If x_n is a sequence in Γ, $z \in \Gamma$, and $x_n \to a$, then for every $w \in B_N(z)$, we have $Q(a,w,x_n) \to 1$.*

Proof. Let $Q(a,x)$ be a continuous function on $\partial\Gamma \times \Gamma$ such that $0 \leqq Q \leqq 1$, $Q(a,x) = 1$ for $d_{\overline{\Gamma}}(a,x) < \delta/2$, and $Q(a,x) = 0$ for $d_{\overline{\Gamma}}(a,x) \geqq \delta$, where δ is to be determined later. Let then $Q(a,z,x)$ be the continuous function on $\partial\Gamma \times \Gamma \times \overline{\Gamma}$ defined by $Q(a,z,x) = Q(z^{-1}a, z^{-1}x)$ for $z \in \Gamma$. Q is invariant under the triple diagonal Γ action on $\partial\Gamma \times \overline{\Gamma} \times \Gamma$. We prove the statement (3); the statement (4) is similar.

We claim that to prove Q has the required property, we may assume $z = x_0$, where recall x_0 is the identity of the group Γ, regarded as a basepoint in $\underline{E}\Gamma$. For, assuming the result for $z = x_0$, let z be arbitrary. Let w be such that $d(z,w) \leqq N$, and let $x_n \to S(a,z) = zS(z^{-1}(a))$. Then $z^{-1}x_n \to S(z^{-1}(a))$. Now $d(z^{-1}w, x_0) \leqq N$. Hence $Q(z^{-1}(a), z^{-1}w, z^{-1}x_n) \to 0$ by what we have assumed proved. But $Q(z^{-1}(a), z^{-1}w, z^{-1}x_n) = Q(a,w,x_n)$, by equivariance of Q. This proves the claim.

Let δ_0 be as in the paragraph preceding Lemma 56, and let R_0 correspond to δ_0 as in Lemma 14. Thus, for every $a \in \partial\Gamma$ we have $d(x_0, [a, S(a)]) \leqq R_0$. Choose $R > 2N + 2R_0$, choose δ according to R as per Lemma 14, and then Q in the first paragraph as corresponding to δ. The result of these choices is that $Q(a,x) = 0$ unless $d(x_0, [x,a]) \geqq R$. Let then $x_n \to S(a)$ and let $w \in B_N(x_0)$. Then if $Q(a,w,x_n) = Q(w^{-1}(a), w^{-1}x_n)$ does not converge to 0 we may assume after extracting a subsequence if necessary that for all large n, $d(x_0, [w^{-1}a, w^{-1}x_n]) \geqq R$. Hence $d(w, [x_n, a]) \geqq R$. Since $x_n \to S(a)$ it follows that $d(w, [a, S(a)]) \geqq R/2$ and hence $d(x_0, [a, S(a)]) \geqq R/2 - N > R_0$, contradicting choice of R_0. □

Consider the function $Q(a,z,x)$ constructed in Lemma 57. It will be convenient to view Q as a function on $\partial\Gamma \times \Gamma \times \overline{\underline{E}\Gamma}$ satisfying the same properties as the original Q;

this is easy to arrange, by reproving Lemma 57 with $\overline{\Gamma}$ replaced by $\overline{\underline{E}\Gamma}$. Recall the map $G\Gamma \to \underline{E}\Gamma$, $r \mapsto r(0)$ whose existence was proved in Lemma 30. Define a function $\tilde{Q}$ on $\partial\Gamma \times \Gamma \times G\Gamma$ by the formula $\tilde{Q}(a, z, r) = Q\big(a, z, r(0)\big)$. Note $\tilde{Q}$ is Γ-invariant.

Define a $C_0(\partial\Gamma \times \underline{E}\Gamma)$-valued inner product on the linear space

$$C_c(\partial\Gamma \times \underline{E}\Gamma \times \Gamma \times G\Gamma)$$

by the formula

$$\langle \xi, \eta \rangle(a, \mu) = \int_\Gamma \int_{\mathbb{R}} \bar{\xi}(a, \mu, z, g_t r_{a,z})\, dt\, d\mu(z).$$

Note the above integral in the z-variable is simply a finite sum, as the support of $\mu \in \underline{E}\Gamma$ has diameter at most N. Clearly $C_c(\partial\Gamma \times \underline{E}\Gamma \times \Gamma \times G\Gamma)$ carries left and right actions of $C_0(\partial\Gamma \times \underline{E}\Gamma)$, and these two actions agree, and are compatible with the inner product.

Definition 58. Let $\tilde{\mathscr{E}}$ be the Hilbert $\big(C_0(\partial\Gamma \times \underline{E}\Gamma), C_0(\partial\Gamma \times \underline{E}\Gamma)\big)$-bimodule obtained by completing $C_c(\partial\Gamma \times \underline{E}\Gamma \times \Gamma \times G\Gamma)$ with respect to the above inner product.

Definition 59. Define an operator $\tilde{P}$ on the Hilbert $C_0(\partial\Gamma \times \underline{E}\Gamma)$-module $\tilde{\mathscr{E}}$ by

$$(\tilde{P}\xi)(a, \mu, z, r) = \int_\Gamma \tilde{Q}(a, w, r)\xi(a, \mu, w, r)\, d\mu(w).$$

Remark 60. It is possible to view $\tilde{\mathscr{E}}$ as the sections of a field of Hilbert spaces $\tilde{H}_{(a,\mu)}$ over $\partial\Gamma \times \underline{E}\Gamma$, and the operator $\tilde{P}$ as corresponding to a field of operators $\tilde{P}_{(a,\mu)}$, in the following manner. For distinct boundary points a and b, let us denote by $[a, b]$ the fiber over (a, b) in the map $G\Gamma \to \partial^2\Gamma$ provided by Theorem 27. Note that $[a, b]$ has a canonical affine structure, and hence there is in particular a canonical translation invariant measure on it corresponding to Lebesgue measure on $\mathbb{R}$. Now, if μ is a point mass corresponding to a point $z \in \Gamma$, set $\tilde{H}_{(a,z)} = L^2\big([a, S(a, z)]\big)$. If μ is an arbitrary point of $\underline{E}\Gamma$, set $\tilde{H}_{(a,\mu)}$ to be the completion of the linear space of functions $\Gamma \to \bigoplus_{z \in \mathrm{supp}(\mu) \subset \Gamma} \tilde{H}_{(a,z)}$ with respect to the inner product $\langle \xi, \eta \rangle_{(a,\mu)} = \int_\Gamma \langle \xi(z), \eta(z) \rangle_{\tilde{H}_{(a,z)}} d\mu(z)$. The operator $\tilde{P}$ corresponds to the following field of operators $\{\tilde{P}_{(a,\mu)}\}$. If μ is a point mass corresponding to a point $z \in \Gamma$, $\tilde{P}_{(a,z)}$ is given by pointwise multiplication by $\tilde{Q}(a, z, \cdot)$ in the variable $r \in [a, S(a, z)]$ on $\tilde{H}_{(a,z)}$. If μ is an arbitrary point of $\underline{E}\Gamma$, let $\tilde{P}_{(a,\mu)}$ be defined by $\tilde{P}_{(a,\mu)}\xi(z)(r) = \int_\Gamma \tilde{Q}(a, w, r)\xi(w)(r)\, d\mu(w)$.

Definition 61. Define a homomorphism $C^*(\mathbb{R}) \to B(\tilde{\mathscr{E}})$ by the unitary representation $t \mapsto U_t$, where $(U_t\xi)(a, \mu, z, r) = \xi(a, \mu, z, g_{-t}r)$.

As the left $C^*(\mathbb{R})$ action so defined commutes with the $C_0(\partial\Gamma \times \underline{E}\Gamma)$ action, we may view $\tilde{\mathscr{E}}$ as a $\big(C^*(\mathbb{R}) \otimes C_0(\partial\Gamma \times \underline{E}\Gamma), C_0(\partial\Gamma \times \underline{E}\Gamma)\big)$-bimodule. Next, note that the triple diagonal action of Γ on $\partial\Gamma \times \underline{E}\Gamma \times \Gamma \times G\Gamma$ induces an action of Γ on $C_0(\partial\Gamma \times \underline{E}\Gamma \times \Gamma \times G\Gamma)$ as linear maps. It is easy to check that this action is compatible with the inner product and right action. It will follow from our remarks below that the homomorphism $C^*(\mathbb{R}) \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \to B(\tilde{\mathscr{E}})$ is Γ-equivariant. Hence $\tilde{\mathscr{E}}$ is in fact a $\Gamma - \big(C^*(\mathbb{R}) \otimes C_0(\partial\Gamma \times \underline{E}\Gamma), C_0(\partial\Gamma \times \underline{E}\Gamma)\big)$-bimodule.

Remark 62. Note that from the field perspective, the fact that $\tilde{\mathscr{E}}$ is a $\Gamma - \big(C_0(\partial\Gamma \times \underline{E}\Gamma), C_0(\partial\Gamma \times \underline{E}\Gamma)\big)$-bimodule (which it is in particular, ignoring the left $C^*(\mathbb{R})$-action) may be re-stated as: $\gamma \in \Gamma$ maps $\tilde{H}_{(a,\mu)}$ isometrically onto $\tilde{H}_{(\gamma a, \gamma\mu)}$. We note also that we can identify $\tilde{H}_{(a,\mu)}$ in a Γ-equivariant fashion with $L^2(\mathbb{R}) \otimes L^2_\mu(\Gamma)$. Under this identification the action of $\gamma \in \Gamma$, $\tilde{H}_{(a,z)} \to \tilde{H}_{(\gamma a, \gamma z)}$ becomes trivial on the $L^2(\mathbb{R})$ factor, and the usual action on the $L^2_\mu(\Gamma)$ factor; and the $C^*(\mathbb{R})$ action becomes trivial on the $L^2_\mu(\Gamma)$ factor and the regular representation on the $L^2(\mathbb{R})$ factor. All this follows from using the Γ-equivariant section $(a,z) \mapsto r_{a,z}$ to identify Γ-equivariantly each $[a, S(a,z)]$ with $\mathbb{R}$.

Lemma 63. *The following hold*:

(1) *The map* $C^*(\mathbb{R}) \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \to B(\tilde{\mathscr{E}})$ *is* Γ *equivariant.*

(2) *For* $\varphi \in C^*(\mathbb{R})$, $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$, $[\varphi f, \tilde{P}] = f[\varphi, \tilde{P}]$ *is a compact operator.*

(3) *The operator* $\tilde{P}$ *is* Γ*-equivariant*: $[\gamma, \tilde{P}] = 0$ *for all* $\gamma \in \Gamma$.

(4) $\varphi f(\tilde{P}^2 - \tilde{P})$ *and* $\varphi f(\tilde{P}^* - \tilde{P})$ *are compact for all* $\varphi \in C^*(\mathbb{R})$ *and* $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$.

Proof. The first statement is clear. Using the field description, it is easy to see that to prove the second statement it suffices to prove that for each $(a,\mu) \in \partial\Gamma \times \underline{E}\Gamma$ the commutators $[\varphi, \tilde{P}_{(a,\mu)}]$ are compact operators on $\tilde{H}_{(a,\mu)}$, for $\varphi \in C^*(\mathbb{R})$. Under the identification $\tilde{H}_{(a,z)} \cong L^2(\mathbb{R})$ pointed out in Remark 62, the operators $\tilde{P}_{(a,z)}$ become multiplication by functions $\chi_{(a,z)}(t)$ which satisfy $\lim_{t\to-\infty} \chi_{(a,z)}(t) = 1$ and $\lim_{t\to+\infty} \chi_{(a,z)}(t) = 0$. From this it follows immediately that for $\varphi \in C^*(\mathbb{R})$, the commutator $[\varphi, \tilde{P}_{(a,z)}]$ on $\tilde{H}_{(a,z)}$ is compact. Indeed, if φ is a compactly supported function on $\mathbb{R}$, and χ is a function with $\lim_{t\to-\infty} \chi(t) = 1$ and $\lim_{t\to+\infty} \chi(t) = 0$, it is easy to check that the commutator of convolution with φ and pointwise multiplication by χ is a compact operator on $L^2(\mathbb{R})$. The result for the operators $\tilde{P}_{(a,\mu)}$ follows, since each $\tilde{P}_{(a,\mu)}$ is a convex combination of the $P_{(a,z)}$. In an exactly analogous way one proves that the operators $\varphi(\tilde{P}^2_{(a,\mu)} - \tilde{P}_{(a,\mu)})$, for $\varphi \in C^*(\mathbb{R})$, are compact operators on $\tilde{H}_{(a,\mu)}$, which is part of the fourth assertion; self-adjointness follows similarly. Equivariance of $\tilde{P}$ is a direct consequence of equivariance of the function $\tilde{Q}$. □

We have shown the following:

Corollary 64. *The pair* $(\tilde{\mathscr{E}}, \tilde{P})$ *defines a cycle for the group*

$$RKK^1_\Gamma\big(\partial\Gamma \times \underline{E}\Gamma; C^*(\mathbb{R}), \mathbb{C}\big).$$

Definition 65. Let

$$\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([\hat{d}_\mathbb{R}]) \otimes_{\partial\Gamma\times\underline{E}\Gamma, C^*(\mathbb{R})} [(\tilde{\mathscr{E}}, \tilde{P})] \in RKK_\Gamma(\partial\Gamma \times \underline{E}\Gamma; \mathbb{C}, \mathbb{C}).$$

Proposition 66. *We have*:

$$\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = 1_{\partial\Gamma\times\underline{E}\Gamma}.$$

Proof. We first deform the cycle corresponding to $\gamma'_{\partial\Gamma\times\underline{E}\Gamma}$ as follows. Identifying the field of Hilbert spaces $(a,\mu)\mapsto \tilde{H}_{(a,\mu)}$ with the field $(a,\mu)\mapsto L^2_\mu(\Gamma; L^2(\mathbb{R}))$ as in Remark 62, form a homotopy of operators $\tilde{P}^t_{(a,\mu)}$ by the formula

$$[\tilde{P}^t_{(a,\mu)}\xi](z) = \int_\Gamma [(1-t)\chi_{(a,w)} + t\chi_{(-\infty,0]}]\xi(w)\,d\mu(w),$$

where the functions $\chi_{a,w}$ are as in the proof of Lemma 63. It is easy to check this formula defines an operator homotopy in $RKK^1_\Gamma(\partial\Gamma\times\underline{E}\Gamma; C^*(\mathbb{R}),\mathbb{C})$, deforming the cycle corresponding to $\gamma'_{\partial\Gamma\times\underline{E}\Gamma}$ to the cycle given by the same field of Hilbert spaces, but with the field of operators given on $\tilde{H}_{(a,\mu)}$ by $\chi_{(-\infty,0]}\otimes P_\mu$. Now, P_μ is a rank one projection which is in addition Γ-invariant. Let $\mu\mapsto\xi_\mu$ denote a continuous selection of a unit vector in $L^2_\mu(\Gamma)$ for which $P_\mu\xi_\mu=\xi_\mu$ and for which $\gamma\xi_\mu=\xi_{\gamma\mu}$, for any $\gamma\in\Gamma$. We have

$$\tilde{H}_{(a,\mu)} = L^2(\mathbb{R})\otimes[\xi_\mu]\oplus L^2(\mathbb{R})\otimes L^2_\mu(\Gamma)^0,$$

where $L^2_\mu(\Gamma)^0$ denotes the functions in $L^2_\mu(\Gamma)$ with μ-integral 0, and $[\xi_\mu]$ denotes the one dimensional linear subspace generated by ξ_μ. With respect to this decomposition, the operator corresponding to our new deformed cycle is simply $\chi_{(-\infty,0]}\otimes 1\oplus 0$, and the $C^*(\mathbb{R})$-action is diagonal. It follows that the deformed cycle is the direct sum of a degenerate cycle and the cycle given by the *constant* field of Hilbert spaces $L^2(\mathbb{R})$, and operators $\chi_{(-\infty,0]}$, with the usual $C^*(\mathbb{R})$-action. The class of the latter is $1_{\partial\Gamma\times X}$ by a $\partial\Gamma\times\underline{E}\Gamma$-parameterized version of Corollary 3, and the class of the former is 0 in RKK, and so we are done: $\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = 1_{\partial\Gamma\times\underline{E}\Gamma}$. □

7. Proof that $\gamma_{\underline{E}\Gamma\times\partial\Gamma} = \gamma'_{\underline{E}\Gamma\times\partial\Gamma}$

We now pass to proving $\gamma_{\underline{E}\Gamma\times\partial\Gamma} = \gamma'_{\underline{E}\Gamma\times\partial\Gamma}$. Our strategy for doing this is to define an element $b\in RKK^{-1}_\Gamma(\partial\Gamma\times\underline{E}\Gamma;\mathbb{C},B)$ such that $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])\otimes_{\partial\Gamma\times\underline{E}\Gamma} b$. We will then separately verify that the axioms for a Kasparov product of $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])$ and b are satisfied by the cycle for $\gamma'_{\partial\Gamma\times\underline{E}\Gamma}$ of the previous section, from which we will conclude that $\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = \gamma_{\partial\Gamma\times\underline{E}\Gamma}$.

We first recall the homomorphism $\iota: B\otimes A\to Q(A\otimes H)$, which in Lemma 48 we showed has the form $\iota(b\otimes a) = \tau(b)\pi(\varphi(a))$, with $\varphi(f) = f\otimes 1\in B(A\otimes H)$ and $\varphi(\gamma) = \gamma\otimes u_\gamma\in B(A\otimes H)$. Let Γ act on $C(\partial\Gamma)\otimes H$ diagonally. Then it is clear that ι restricts to a Γ-equivariant homomorphism $B\otimes C(\partial\Gamma)\to Q(C(\partial\Gamma)\otimes H)$ having the form $b\mapsto\tau(b)$, $f\mapsto\pi(f\otimes 1)$. We denote this latter Γ-equivariant homomorphism $B\otimes C(\partial\Gamma)\to Q(C(\partial\Gamma)\otimes H)$ by $\iota_{\partial\Gamma}$.

Remark 67. A great deal of the complication in this part of the argument arises from the difficulty in representing the class $\gamma_{\partial\Gamma}$ as a product of two equivariant classes, even whilst knowing that $\gamma_{\partial\Gamma}$ itself is an equivariant class. Specifically, we do not know whether or not the homomorphism $\iota_{\partial\Gamma}$ is dilatable in the sense of Definition 5. The idea is that this problem will vanish when inflating everything over $\underline{E}\Gamma$. After doing this, the inflated map, which we will call $\iota_{\partial\Gamma\times\underline{E}\Gamma}$, will in fact become dilatable, and $\gamma_{\partial\Gamma\times\underline{E}\Gamma}$ (though not $\gamma_{\partial\Gamma}$) will become, as we would like, a product of two equivariant classes, specifically as $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])\otimes_{\partial\Gamma\times\underline{E}\Gamma}[\iota_{\partial\Gamma\times\underline{E}\Gamma}]$. The class b mentioned in the first paragraph of this section will be simply the dilation of $\iota_{\partial\Gamma\times\underline{E}\Gamma}$; i.e. $b = [\iota_{\partial\Gamma\times\underline{E}\Gamma}]$.

Recall the module E of Definition 33. Choose an embedding of E as a direct summand of a trivial B-module $B \otimes V$ for some Hilbert space V, and denote by v the homomorphism $C^*(\mathbb{R}) \to B \otimes K(V)$ obtained by the composition

$$C^*(\mathbb{R}) \to K(E) \to K(B \otimes V) \cong B \otimes K(V).$$

Let $v_{\partial\Gamma}$ denote the homomorphism $C^*(\mathbb{R}) \otimes C(\partial\Gamma) \to B \otimes C(\partial\Gamma) \otimes K(V)$ obtained by tensoring v with the identity on $C(\partial\Gamma)$ and re-arranging factors. Finally, let $\iota_{\partial\Gamma,V}$ denote the homomorphism $B \otimes C(\partial\Gamma) \otimes K(V) \to Q\big(C(\partial\Gamma) \otimes H \otimes V\big)$ obtained by tensoring $\iota_{\partial\Gamma}$ by the identity homomorphism on $K(V)$ and re-arranging factors. We have essentially already proved the following lemma (see the proof of Corollary 50), but we restate it for the sake of emphasis. Recall the function $\psi \in C^*(\mathbb{R})$ of Section 1.

Lemma 68. *$\gamma_{\partial\Gamma}$ is represented by any cycle of the form $\big(C(\partial\Gamma) \otimes H \otimes V, F + 1\big)$ where $F \in B\big(C(\partial\Gamma) \otimes H \otimes V\big)$ is any operator for which $\pi(F) = \iota_{\partial\Gamma,V}\big(v_{\partial\Gamma}(\psi \otimes 1)\big)$.*

Now we tensor all the above data with $\underline{E}\Gamma$ as follows. Let firstly $\iota_{\partial\Gamma\times\underline{E}\Gamma}$ denote the homomorphism $B \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \to Q\big(C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H\big)$ obtained by tensoring $\iota_{\partial\Gamma}$ with the identity on $C_0(\underline{E}\Gamma)$ and re-arranging factors. Let $\iota_{\partial\Gamma\times\underline{E}\Gamma,V}$ denote the homomorphism $B \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes K(V) \to Q\big(C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H \otimes V\big)$ obtained by tensoring $\iota_{\partial\Gamma\times\underline{E}\Gamma}$ with the identity on $K(V)$ and re-arranging factors. Finally, let $v_{\partial\Gamma\times\underline{E}\Gamma}$ denote the homomorphism $C^*(\mathbb{R}) \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \to B \otimes C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes K(V)$ similarly obtained by tensoring with the identity on $C_0(\partial\Gamma \times \underline{E}\Gamma)$ and re-arranging factors. Then just as above we have:

Lemma 69. *$\gamma_{\partial\Gamma\times\underline{E}\Gamma}$ is represented by any cycle of the form*

$$\big(C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H \otimes V, G + 1\big),$$

where G is any operator on $C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H \otimes V$ satisfying

$$\pi(G) = \iota_{\partial\Gamma\times\underline{E}\Gamma,V}\big(v_{\partial\Gamma\times\underline{E}\Gamma}(\psi \otimes 1)\big).$$

Now, suppose we knew that $\iota_{\partial\Gamma\times\underline{E}\Gamma}$ was dilatable. Then $\iota_{\partial\Gamma\times\underline{E}\Gamma}$ would define a class $b = [\iota_{\partial\Gamma\times\underline{E}\Gamma}]$ in $RKK^1_\Gamma(\partial\Gamma \times \underline{E}\Gamma; B, \mathbb{C})$, and the class $\gamma_{\partial\Gamma\times\underline{E}\Gamma}$ would then factor in the equivariant category as $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma,B} b$. For emphasis, we state this all explicitly as a proposition, leaving the proof, which is a standard exercise in Kasparov theory, to the reader.

Proposition 70. *Let $(\mathscr{E}, P)$ be a cycle for $RKK^1_\Gamma(\partial\Gamma \times \underline{E}\Gamma; B, \mathbb{C})$ for which there exists an isometry $U : C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H \to \mathscr{E}$ of Hilbert $C_0(\partial\Gamma \times \underline{E}\Gamma)$-modules such that for every $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$ and $b \in B$:*

$$\pi\big(U^* P \phi(f \otimes b) P U\big) = \iota_{\partial\Gamma\times\underline{E}\Gamma}(f \otimes b),$$

where $\phi : C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes B \to B(\mathscr{E})$ is the left $C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes B$-structure of $\mathscr{E}$. Then

$$\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma,B} b,$$

where b denotes the class of $(\mathscr{E}, P)$.

Remark 71. After constructing such b, it will be possible to describe $\gamma_{\partial\Gamma\times\underline{E}\Gamma}$ without mention of the inexplicit homomorphism v. For $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])$, in addition to being represented by the homomorphism $v_{\partial\Gamma\times\underline{E}\Gamma}$, is alternatively represented simply by the pair $\big(C_0(\partial\Gamma\times\underline{E}\Gamma)\otimes E, 0\big)$. Hence the product $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])\otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b$ will be represented by the cycle $(E\otimes_B\mathscr{E}, R)$, where R is a P-connection. This is how we shall show that $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = \gamma'_{\partial\Gamma\times\underline{E}\Gamma}$. We will find a cycle $(\mathscr{E}, P)$ as in the hypothesis of the Proposition 70, such that the resulting cycle $(E\otimes_B\mathscr{E}, R)$ is homotopic to the cycle $(\tilde{E}, \tilde{P})$ described in the previous section. Since the latter cycle is homotopic to the cycle for $1_{\partial\Gamma\times\underline{E}\Gamma}$, we will conclude $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])\otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b = \gamma'_{\partial\Gamma\times\underline{E}\Gamma} = 1_{\partial\Gamma\times\underline{E}\Gamma}$.

We now set about construction of the cycle $(\mathscr{E}, P)$ and the embedding of $C_0(\partial\Gamma\times\underline{E}\Gamma)\otimes H$ into $\mathscr{E}$ as above.

Define a $C_0(\partial\Gamma\times\underline{E}\Gamma)$-valued inner product on the linear space $C_c(\partial\Gamma\times\underline{E}\Gamma\times\Gamma; H)$ by the formula

$$\langle\xi,\eta\rangle(a,\mu) = \int_\Gamma \langle\xi(a,\mu,z),\eta(a,\mu,z)\rangle\, d\mu(z).$$

Note that the integral is a finite sum, as the support of μ has diameter at most N, where N is the parameter of the Rips complex.

Definition 72. Let $\mathscr{E}$ be the right Hilbert $C_0(\partial\Gamma\times\underline{E}\Gamma)$-module obtained by completion of $C_c(\partial\Gamma\times\underline{E}\Gamma\times\Gamma; H)$ with respect to the above inner product.

Definition 73. Define an operator P on $\mathscr{E}$ as follows: let

$$P\xi(a,\mu,z)(x) = \int_\Gamma Q(a,w,x)\xi(a,\mu,w,x)\, d\mu(w).$$

Once again the integral is a finite sum.

Definition 74. Define a map $\phi : C_0(\partial\Gamma\times\underline{E}\Gamma)\otimes B\to B(\mathscr{E})$ by the following covariant pair. Let $F\in C_0(\partial^2\Gamma)$ and $f\in C_0(\partial\Gamma\times\underline{E}\Gamma)$. Define then

$$\big(\phi(f\otimes F)\xi\big)(a,\mu,z)(x) = f(a,\mu)F\big(x^{-1}(a), x^{-1}S(a,z)\big)\xi(a,\mu,z)(x).$$

For $\gamma\in\Gamma$, define $\phi(\gamma)\xi(a,\mu,z)(x) = \xi(a,\mu,z)(x\gamma)$.

Remark 75. As we did in the previous section, we can give a somewhat more intuitive description of the above data in terms of fields. From this point of view, $\mathscr{E}$ can be understood as sections of the continuous, equivariant field of Hilbert spaces $H_{(a,\mu)} = L^2_\mu(\Gamma; H)$. Note that for μ a point mass at a point $z\in\Gamma\subset\underline{E}\Gamma$, $H_{(a,\mu)}$ is simply H. The homomorphism ϕ can be understood as a field of homomorphisms $\phi_{(a,\mu)} : B\to B(H_{(a,\mu)})$ as follows: first define, for $(a,z)\in\partial\Gamma\times\Gamma$, a homomorphism $\phi_{(a,z)} : B\mapsto B(H)$ by

$$\phi_{(a,z)}(F)(x) = F\big(x^{-1}(a), x^{-1}S(a,z)\big) \quad \text{and} \quad \phi_{(a,z)}(\gamma) = \lambda^{\mathrm{op}}(\gamma^{-1}).$$

Then define, for $(a,\mu)\in\partial\Gamma\times\underline{E}\Gamma$, the homomorphism

$$\phi_{(a,\mu)} : B\to B(H_{(a,\mu)})$$

by $\phi_{(a,\mu)}(b)(\xi)(z)(x) = \phi_{(a,z)}(b)(\xi(z))(x)$. There is a similar description of the operator P as a field of operators $P_{(a,\mu)} : (P_{(a,\mu)}\xi)(z)(x) = \int_\Gamma Q(a,w,x)\xi(z)(x)\,d\mu(w)$.

Next, note that Γ acts on $C_c(\partial\Gamma \times \underline{E}\Gamma \times \Gamma; H)$, and the action is compatible with the $(C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes B, C_0(\partial\Gamma \times \underline{E}\Gamma))$-bimodule structure and the inner product. Hence $\mathscr{E}$ has the structure of a $\Gamma - (C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes B, C_0(\partial\Gamma \times \underline{E}\Gamma))$-bimodule. We have furthermore:

Lemma 76. *If $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$ and $b \in B$, then $[P, \phi(f \otimes b)]$ is compact.*

Proof. Let $F \in C_c(\partial^2\Gamma)$ and $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$, and fix $(a,\mu) \in \partial\Gamma \times \underline{E}\Gamma$ and $z \in \operatorname{supp}(\mu)$. Then we have:

$$\big(P\phi(f \otimes F)\big)\xi(a,\mu,z)(x) = f(a,\mu)\int_\Gamma Q(a,w,x)F\big(x^{-1}(a), x^{-1}S(a,w)\big)\xi(a,\mu,w)(x)\,d\mu(w)$$

and

$$\big(\phi(f \otimes F)P\big)\xi(a,\mu,z)(x) = f(a,\mu)F\big(x^{-1}(a), x^{-1}S(a,z)\big)\int_\Gamma Q(a,w,x)\xi(a,\mu,w)(x)\,d\mu(w).$$

Let $x \to \infty$. Note that for any $w \in \operatorname{supp}(\mu)$ we have $d(z,w) \leqq N$. Fix such w. Now if the scalar $F\big(x^{-1}(a), x^{-1}S(a,w)\big) - F\big(x^{-1}(a), x^{-1}S(a,z)\big)$ does not converge to 0, it follows from the fact that $F \in C_c(\partial^2\Gamma)$ and the usual argument, that the distance from x to the geodesic $[S(a,z), S(a,w)]$ remains bounded, and hence that either $x \to S(a,z)$ or $x \to S(a,w)$. But in either case it follows from Lemma 57 and the fact that $d(z,w) \leqq N$ that both $Q(a,z,x) \to 0$ and $Q(a,w,x) \to 0$. We have shown that the difference $Q(a,w,x)\big(F\big(x^{-1}a, x^{-1}S(a,w)\big) - F\big(x^{-1}a, x^{-1}S(a,z)\big)\big)$ converges to 0 as $x \to \infty$ and with z and w fixed. It follows this difference converges to 0 uniformly in z and w, as the latter range over a finite set. From this it follows immediately that the difference of the above two expressions represents a compact operator on $\mathscr{E}$.

Finally, to show the commutator $[\phi(f \otimes \gamma), P]$ is compact, observe that

$$\big(\phi(\gamma)P\phi(\gamma^{-1}) - P\big)\xi(a,\mu,w,x) = \int_\Gamma \big(Q(a,w,x\gamma) - Q(a,w,x)\big)\xi(a,\mu,w,x)\,d\mu(w).$$

For every a and every w the function $x \mapsto Q(a,w,x\gamma) - Q(a,w,x)$ lies in $c_0(\Gamma)$, since Q is continuous in the x-variable. The result follows immediately. □

The proof of the following lemma follows the same strategy as that of the previous one, and we omit it.

Lemma 77. *$\phi(f \otimes b)(P^2 - P)$ and $\phi(f \otimes b)(P^* - P)$ are both compact operators, for all $b \in B$ and $f \in C_0(\partial\Gamma \times \underline{E}\Gamma)$.*

We have shown:

Corollary 78. *The pair $(\mathscr{E}, P)$ defines a cycle for $RKK^1_\Gamma(\partial\Gamma \times \underline{E}\Gamma; B, \mathbb{C})$.*

Definition 79. Let $b \in RKK^1_\Gamma(\partial\Gamma \times \underline{E}\Gamma; B, \mathbb{C})$ denote the class of the cycle $(\mathscr{E}, P)$ above.

We next embed $C_0(\partial\Gamma \times \underline{E}\Gamma) \otimes H$ into $\mathscr{E}$ as follows.

Definition 80. Define a map $U : C_0(\partial\Gamma \times \underline{E}\Gamma; H) \to \mathscr{E}$ by the formula $(U\xi)(a, \mu, w) = \xi(a, \mu)$.

U is clearly an isometric map of $C_0(\partial\Gamma \times \underline{E}\Gamma)$-modules.

Remark 81. From the field perspective, U consists of the field of isometries $U_{(a,\mu)} : H \to H_{(a,\mu)}$ sending ξ to the constant function $z \mapsto \xi$. Since each μ is a probability measure, U is indeed isometric.

Proposition 82. *The hypothesis of Proposition* 70 *holds for the pair* $(\mathscr{E}, P)$, *and the isometry* U *above.*

Proof. For simplicity of exposition we work with fields. From this point of view it is easy to see that the homomorphism $\iota_{\partial\Gamma\times\underline{E}\Gamma}$ is given by the field of homomorphisms $\{(\iota_{\partial\Gamma\times\underline{E}\Gamma})_{(a,\mu)} : B \to Q(H)\}$ over $\partial\Gamma \times \underline{E}\Gamma$, with $(\iota_{\partial\Gamma\times\underline{E}\Gamma})_{(a,\mu)}(F)$ the element of $Q(H)$ corresponding to multiplication by the function $x \mapsto \tilde{F}(x^{-1}a, x^{-1})$, where $\tilde{F}$ is an extension of F to a continuous function on $\partial\Gamma \times \overline{\Gamma}$. Secondly, $(\iota_{\partial\Gamma\times\underline{E}\Gamma})_{(a,\mu)}(\gamma) = \lambda^{\mathrm{op}}(\gamma^{-1})$. As mentioned above, the isometric module map U becomes the family of isometries $U_{(a,\mu)} : H \to H_{(a,\mu)}$, $U_{(a,\mu)}\xi(w) = \xi$ for all $w \in \operatorname{supp}(\mu)$. Recall the homomorphisms $\phi_{(a,\mu)}$ defined in the construction of the cycle corresponding to the class b, and the projections $P_{(a,\mu)}$. We now wish to show that, for any $b \in B$, the elements

$$T_b = \pi\big(U^*_{(a,\mu)} P_{(a,\mu)} \phi_{(a,\mu)}(b) P_{(a,\mu)} U_{(a,\mu)}\big) - (\iota_{\partial\Gamma\times\underline{E}\Gamma})_{(a,\mu)}(b)$$

are zero in the Calkin algebra of H. If $b = \gamma \in \Gamma$, it is easy to check that T_b is the zero operator, and so we can pass to the case $b = F \in C_c(\partial^2\Gamma)$. In this case, a short calculation shows that T_b corresponds to a diagonal operator, and, moreover, that to show it is 0 in the Calkin algebra, it is enough to show that as $x \to \infty$,

$$\int_\Gamma Q(a, w, x)\tilde{F}\big(x^{-1}(a), x^{-1}S(a, w)\big)\, d\mu(w) - \tilde{F}\big(x^{-1}(a), x^{-1}\big) \to 0,$$

where $\tilde{F}$ is an extension of F to a continuous function on $\partial\Gamma \times \overline{\Gamma}$. Firstly, if $x \to a$, then for large enough x, $Q(a, w, x) = 1$ for all $w \in \operatorname{supp}(\mu)$, and hence the difference between the above integral and the integral

$$\int \tilde{F}\big(x^{-1}(a), x^{-1}S(a, w)\big) - \tilde{F}\big(x^{-1}(a), x^{-1}\big)\, d\mu(w)$$

converges to 0 as $x \to a$. Considering the latter integral, for every w in the integrand we certainly have $d_{\overline{\Gamma}}\big(x^{-1}S_w(a), x^{-1}\big) \to 0$ as $x \to \infty$, else we would have by the usual argument that for some w, the distance from x to the ray $\big[e, S_w(a)\big)$ remains bounded, which would imply $x \to S_w(a)$, thus contradicting $x \to a$ and $a \neq S_w(a)$. Hence for every w in the integrand $d_{\overline{\Gamma}}\big(x^{-1}S_w(a), x^{-1}\big) \to 0$ as claimed, and so the integral converges to 0 by continuity of $\tilde{F}$ in the second variable. If x does not converge to a, it follows that $\tilde{F}\big(x^{-1}(a), x^{-1}\big) \to 0$, and we need only show the integral also converges to 0. If it does not, for at least one w, say w_1, $d_{\overline{\Gamma}}\big(x^{-1}(a), x^{-1}S_{w_1}(a)\big)$ does not converge to 0, whence $x \to a$ or $S_{w_1}(a)$. By assumption x does not converge to a so it converges to $S_{w_1}(a)$. But then by

Lemma 57, for *all* w in the support of μ, $Q(a, w, x) \to 0$, since for any such w, $d(w, w_1) \leqq N$, and we are done. $\square$

By Proposition 70 we conclude that $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b$. To show that $\gamma_{\partial\Gamma\times\underline{E}\Gamma} = \gamma'_{\partial\Gamma\times\underline{E}\Gamma}$ it therefore suffices to show that also $\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b$, which we will do by verifying that $\gamma'_{\partial\Gamma\times\underline{E}\Gamma}$ satisfies the axioms for a Kasparov product of $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D])$ and b.

Recall from the discussion in Remark 71 that the product $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b$ is given by the cycle $(E \otimes_B \mathscr{E}, R)$ where R is a P-connection. We first observe:

Lemma 83. *$E \otimes_B \mathscr{E} \cong \tilde{\mathscr{E}}$ equivariantly, and under this isomorphism the $C^*(\mathbb{R})$ action on $E \otimes_B \mathscr{E}$ becomes the action of $C^*(\mathbb{R})$ on $\tilde{\mathscr{E}}$ defined in Definition* 61.

Proof. To see this, we work from the field point of view, whereapon our statement becomes: for every $(a, \mu) \in \partial\Gamma \times \underline{E}\Gamma$, we have $E \otimes_B H_{(a,\mu)} \cong \tilde{H}_{(a,\mu)}$, and that furthermore, under this isomorphism, the action of $C^*(\mathbb{R})$ on $E \otimes_B H_{(a,\mu)}$ corresponds to the action of $C^*(\mathbb{R})$ on $\tilde{H}_{(a,\mu)}$ described in Remark 62.

The isomorphism is defined on the dense subset $C_c(G\Gamma) \otimes_B L^2_\mu(\Gamma; \mathbb{C}\Gamma)$ of $E \otimes_B L^2_\mu(\Gamma; H)$ by the composition of linear maps

$$C_c(G\Gamma) \otimes_B L^2_\mu(\Gamma; \mathbb{C}\Gamma) \cong \big(C_c(G\Gamma) \otimes_B \mathbb{C}\Gamma\big) \otimes L^2_\mu(\Gamma)$$

$$\cong C_c(G\Gamma) \otimes_{C_0(\partial^2\Gamma)} L^2_\mu(\Gamma) \to L^2_\mu\Big(\Gamma; \bigoplus_{z\in \operatorname{supp}\mu} C_c\big([a, S_z(a)]\big)\Big) \to \tilde{H}_{(a,\mu)}.$$

The penultimate map is induced by the restriction map $C_c(G\Gamma) \to \bigoplus_{z\in\operatorname{supp}(\mu)} C_c\big([a, S(a, z)]\big)$. This composition is isometric with respect to the various Hilbert module norms. The statement regarding the $C^*(\mathbb{R})$ actions is obvious. $\square$

Proposition 84. *We have*: $p^*_{\partial\Gamma\times\underline{E}\Gamma}([D]) \otimes_{\partial\Gamma\times\underline{E}\Gamma, B} b = \gamma'_{\partial\Gamma\times X}$.

Proof. We shall prove this by showing that the operator $\tilde{P}$ is a P-connection. We work with fields. Taking the product pointwise of the modules results in the field of modules $\tilde{H}_{(a,\mu)}$ by Lemma 83. We show that the operator $\tilde{P}_{(a,\mu)}$ described in Remark 60 is a $P_{(a,\mu)}$-connection. Let $\xi \in C_c(G\Gamma) \subset E$ and θ_ξ denote the operator $H \otimes L^2_\mu(\Gamma) \to E \otimes_B H \otimes L^2_\mu(\Gamma)$, $\eta \mapsto \xi \otimes_B \eta$. By [20] we need show that the operator $H \otimes L^2_\mu(\Gamma) \to \tilde{H}_{(a,\mu)}$,

$$A_{(a,\mu,\xi)}(\eta) = \tilde{P}_{(a,\mu)}(\xi \otimes_B \eta) - \xi \otimes_B P_{(a,\mu)}(\eta)$$

is a compact operator, and show as well that an adjointed version of this equation also represents a compact operator. We shall show the first; the second is verified analogously. To calculate explicitly the operator $A_{(a,\mu,\xi)}$, assume $\eta \in H \otimes L^2_\mu(\Gamma)$ has the simple form $\eta = e_x \otimes \alpha$ for $\alpha \in L^2_\mu(\Gamma)$ and $x \in \Gamma$. We have

$$(A_{(a,\mu,\xi)}\eta)(z)(r) = \int_\Gamma \big(Q\big(a, w, r(0)\big) - Q(a, w, x)\big)\alpha(w)\xi\big(x^{-1}(r)\big)\, d\mu(w)$$

and from this it is evident that it suffices to show that for $x \to \infty$ and $w \in \operatorname{supp}(\mu)$, the L^2-norm of the function $h = h(r) = \big(Q(a, w, r(0)) - Q(a, w, x)\big)\xi\big(x^{-1}(r)\big)$ of $\tilde{H}_{(a,w)} = L^2\big([a, S_w(a)]\big) \cong L^2(\mathbb{R})$ converges to 0, since this will express $A_{(a,\mu,\xi)}$ as a norm limit of finite rank operators.

Choose $\varepsilon > 0$. Then there exists $R > 0$ such that if x is large enough and $r(0) \in B_R(x)$, then $\big|Q(a, w, r(0)) - Q(a, w, x)\big| < \varepsilon$, by uniform continuity of $Q(a, w, \cdot)$ and the fact that the Gromov compatification of Γ, and also $\underline{E}\Gamma$, is 'good' (metric balls in the word metric become small in the topology of $\overline{\Gamma}$ near the boundary). Also, as $\xi \in C_c(G\Gamma)$, there exists some R for which $\xi(r) = 0$ unless $r(0) \in B_R(x_0)$. It follows that for x large enough and $r \in [a, S(a, w)]$, either $h(r) = 0$ or $|h(r)| < \varepsilon\big|\xi\big(x^{-1}(r)\big)\big|$. Consequently, for x sufficiently large, $\|h\|_{\tilde{H}_{(a,w)}} < \varepsilon\|\xi\|_E$, and we are done. □

Corollary 85. *We have*

$$\gamma'_{\partial\Gamma\times\underline{E}\Gamma} = \gamma_{\partial\Gamma\times\underline{E}\Gamma} \in RKK_\Gamma(\partial\Gamma \times \underline{E}\Gamma; \mathbb{C}, \mathbb{C})$$

and hence $\gamma_{\underline{E}\Gamma\times\partial\Gamma} = 1_{\partial\Gamma\times\underline{E}\Gamma}$.

This concludes the proof of Theorem 41.

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