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Euler characteristics and Gysin sequences for group actions on boundaries*

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Abstract. Let G be a locally compact group, let X be a universal proper G-space, and let $\bar{X}$ be a G-equivariant compactification of X that is H-equivariantly contractible for each compact subgroup $H \subseteq G$. Let $\partial X = \bar{X} \setminus X$. Assuming the Baum-Connes conjecture for G with coefficients $\mathbb{C}$ and $C(\partial X)$, we construct an exact sequence that computes the map on K-theory induced by the embedding $C^*_{\mathrm{r}} G \to C(\partial X) \rtimes_{\mathrm{r}} G$. This exact sequence involves the equivariant Euler characteristic of X, which we study using an abstract notion of Poincaré duality in bivariant K-theory. As a consequence, if G is torsion-free and the Euler characteristic $\chi(G\backslash X)$ is non-zero, then the unit element of $C(\partial X) \rtimes_{\mathrm{r}} G$ is a torsion element of order $|\chi(G\backslash X)|$. Furthermore, we get a new proof of a theorem of Lück and Rosenberg concerning the class of the de Rham operator in equivariant K-homology.

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1. Introduction

Let G be a locally compact group, let X be a proper G-space, and let $\bar{X}$ be a compact G-space containing X as a G-invariant open subset. Suppose that both X and $\bar{X}$ are H-equivariantly contractible for all compact subgroups H of G; we briefly say that they are *strongly contractible* and call the action of G on $\partial X := \bar{X} \setminus X$ a *boundary action*.

For example, the group $G = \mathrm{PSL}(2, \mathbb{Z})$ admits the following two distinct boundary actions. On the one hand, since G is a free product of finite cyclic groups, it acts properly on a tree X ([54]). Let ∂X be its set of ends, which is a Cantor set, and let $\bar{X}$ be its ends compactification. Then X and $\bar{X}$ are strongly contractible, and the action of G on ∂X is a boundary action. On the other hand, $\mathrm{PSL}(2, \mathbb{Z}) \subseteq$

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$\mathrm{PSL}(2,\mathbb{R})$ acts by Möbius transformations on the hyperbolic plane $\mathbb{H}^2$. We compactify $\mathbb{H}^2$, as usual, by a circle at infinity. Again, $\mathbb{H}^2$ and $\bar{\mathbb{H}}^2$ are strongly contractible and the action on the circle $\partial\mathbb{H}^2$ is a boundary action. Other examples are: a word-hyperbolic group acting on its Gromov boundary; a group of isometries of a CAT(0) space X acting on the visibility boundary of X; a mapping class group of a Riemann surface acting on the Thurston boundary of Teichmüller space; a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ acting on its limit set. We discuss these examples in Section 2.

The purpose of this article is to describe the map on K-theory induced by the obvious inclusion $u\colon C^*_\mathrm{r}G \to C(\partial X) \rtimes_\mathrm{r} G$, where $G \times \partial X \to \partial X$ is a boundary action and $C(\partial X) \rtimes_\mathrm{r} G$ is its reduced crossed product C^*-algebra. Our result is analogous to the classical Gysin sequence, which we recall first.

Let X be a locally compact space and let $\pi\colon V \to X$ be a vector bundle over X, say of rank n. Let BV and SV be the (closed) disk and sphere bundles of V, respectively (with respect to some choice of metric on the bundle). Let H^*_c denote cohomology with compact supports. Since the bundle projection $BV \to X$ is a proper homotopy equivalence, we have $H^*_c(BV) \cong H^*_c(X)$ and $\mathrm{K}^*(BV) \cong \mathrm{K}^*(X)$. We assume now that the bundle V is oriented or K-oriented, respectively. Then we get Thom isomorphisms $H^{*-n}_c(X) \cong H^*_c(V)$ or $\mathrm{K}^{*-n}(X) \cong \mathrm{K}^*(V)$, and excision for the pair (BV, SV) provides us with long exact sequences of the form

$$\cdots \to \mathrm{H}^{*-n}_\mathrm{c}(X) \xrightarrow{\varepsilon^*} \mathrm{H}^*_\mathrm{c}(X) \xrightarrow{\pi^*} \mathrm{H}^*_\mathrm{c}(SX) \xrightarrow{\delta} \mathrm{H}^{*-n+1}_\mathrm{c}(X) \to \cdots,$$
$$\cdots \to \mathrm{K}^{*-n}(X) \xrightarrow{\varepsilon^*} \mathrm{K}^*(X) \xrightarrow{\pi^*} \mathrm{K}^*(SX) \xrightarrow{\delta} \mathrm{K}^{*-n+1}(X) \to \cdots$$

These are the classical *Gysin sequences*. In cohomology, the map ε^* is the cup product with the *Euler class* $e_V \in H^n(X)$ of the oriented bundle V. In K-theory, it is the cup product with the *spinor class* (see [23, IV.1.13]).

We are only interested in the case where X is a smooth n-dimensional manifold and $V = TX$ is its tangent bundle. Then the map $\varepsilon^*\colon \mathrm{H}^{*-n}_\mathrm{c}(X) \to \mathrm{H}^*_\mathrm{c}(X)$ vanishes unless $* = n$, for dimension reasons; if X is not compact, then $\mathrm{H}^0_\mathrm{c}(X) = 0$ and hence $\varepsilon^* = 0$. If X is compact, then $e_{TX} \in H^n(X)$ has the property that $\langle e_{TX}, [X]\rangle$ is the *Euler characteristic* $\chi(X)$ of X (see [6]). Theorem 41 gives a similar description of the map $\varepsilon^*\colon \mathrm{K}^{*-n}(X) \to \mathrm{K}^*(X)$: it vanishes on $\mathrm{K}^1(X)$ and is given by $x \mapsto \chi(X)\dim(x)\cdot \mathrm{pnt}!$ on $\mathrm{K}^0(X)$; here the functional $\dim\colon \mathrm{K}^0(X) \to \mathbb{Z}$ sends a vector bundle to its dimension and $\mathrm{pnt}! \in \mathrm{KK}_{-n}\big(\mathbb{C}, C_0(X)\big) \cong \mathrm{K}^n(X)$ is the wrong way element corresponding to the inclusion of a point in X, which is a K-oriented map. Notice that $\dim = 0$ unless X is compact. Since the map ε^* factors through $\mathbb{Z}$, we can cut the above long exact sequence into a pair of shorter exact sequences.

Now we return to the situation of a boundary action. For expository purposes, we assume that G is a torsion-free discrete group, although we treat arbitrary locally compact groups later on. If G is torsion-free, then X is a universal free proper G-space, so that $G\backslash X$ is a model for the classifying space BG. We warn the reader that $\mathrm{K}^*(G\backslash X)$ depends on the particular choice of BG because K-theory is only functorial for *proper* maps; we discuss this for $\mathrm{PSL}(2,\mathbb{Z})$ at the end of Section 2 (see also Example 35).

The exact sequences in the following theorem are quite similar to the classical Gysin sequence for the tangent bundle.

Theorem 1. *Let G be a torsion-free discrete group and let $G \times \partial X \to \partial X$ be a boundary action, where X is a finite-dimensional simplicial complex with a simplicial action of G. Assume that G satisfies the Baum-Connes conjecture with coefficients $\mathbb{C}$ and $C(\partial X)$. Let $u\colon C^*_{\mathrm{r}}G \to C(\partial X) \rtimes_{\mathrm{r}} G$ be the embedding induced by the unit map $\mathbb{C} \to C(\partial X)$.*

If $G\backslash X$ is compact and $\chi(G\backslash X) \neq 0$, then there are exact sequences

$$0 \to \langle \chi(G\backslash X)[1_{C^*_{\mathrm{r}}G}]\rangle \xrightarrow{\subseteq} \mathrm{K}_0(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^0(G\backslash X) \xrightarrow{\dim} \mathbb{Z} \to 0.$$

Here $\langle \chi(G\backslash X)[1_{C^*_{\mathrm{r}}G}]\rangle$ denotes the free cyclic subgroup of $\mathrm{K}_0(C^*_{\mathrm{r}}G)$ that is generated by $\chi(G\backslash X)[1_{C^*_{\mathrm{r}}G}]$, and dim *maps a vector bundle to its dimension.*

If $G\backslash X$ is not compact or if $\chi(G\backslash X) = 0$, then there are exact sequences

$$0 \to \mathrm{K}_0(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^0(G\backslash X) \to 0.$$

Corollary 2. *The class of the unit element in $\mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G)$ is a torsion element of order $|\chi(G\backslash X)|$ if $G\backslash X$ is compact and $\chi(G\backslash X) \neq 0$, and not a torsion element otherwise.*

Several authors have already noticed various instances of this corollary ([1, 11, 12, 16, 40, 45, 46, 55]): for lattices in $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSL}(2,\mathbb{C})$, acting on the boundary of hyperbolic 2- or 3-space, respectively; for closed subgroups of $\mathrm{PSL}(2, \mathbb{F})$ for a non-Archimedean local field $\mathbb{F}$ acting on the projective space $\mathbb{P}^1(\mathbb{F})$, where X is the Bruhat-Tits tree of $\mathrm{PSL}(2,\mathbb{F})$; for free groups acting on their Gromov boundary. Moreover, Mathias Fuchs has simultaneously obtained the assertions of Theorem 1 and Corollary 2 for some subgroups of almost connected Lie groups, by a completely different method.

Comparing the classical and non-commutative Gysin sequences, we see that the inclusion $u\colon C^*_{\mathrm{r}}G \to C(\partial X) \rtimes_{\mathrm{r}} G$ plays the role of the embedding $C(M) \to C(SM)$ induced by the bundle projection $SM \to M$. Therefore, if we view $C^*_{\mathrm{r}}G$ as the algebra of functions on a non-commutative space $\hat{G}$, then $C(\partial X) \rtimes_{\mathrm{r}} G$ plays the role of the algebra of functions on the sphere bundle of $\hat{G}$. Such an analogy has already been advanced by Alain Connes and Marc Rieffel in [13,44] for rather different reasons (and for a different class of boundary actions).

For groups with torsion and, more generally, for locally compact groups, we must use an *equivariant* Euler characteristic in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$ instead of the Euler characteristic of $G\backslash X$. To define it, we use a general notion of Poincaré duality in bivariant Kasparov theory. An *abstract dual* for a space X consists of a G-C^*-algebra $\mathscr{P}$ and a class $\Theta \in \mathrm{RKK}_n^G(X; \mathbb{C}, \mathscr{P})$ for some $n \in \mathbb{Z}$ such that the map

$$\mathrm{PD}\colon \mathrm{RKK}_{*-n}^G(Y; A \hat{\otimes} \mathscr{P}, B) \to \mathrm{RKK}_*^G(X \times Y; A, B), \quad f \mapsto \Theta \hat{\otimes}_{\mathscr{P}} f,$$

is an isomorphism for all pairs of G-C^*-algebras A, B and all G-spaces Y (compare [25, Theorem 4.9]).

Let X be any G-space that has such an abstract dual. The diagonal embedding $X \to X \times X$ yields classes

$$\Delta_X \in \mathrm{RKK}_0^G(X; C_0(X), \mathbb{C}), \qquad \mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}_{-n}^G(C_0(X) \hat{\otimes} \mathscr{P}, \mathbb{C}).$$

Let $\bar{\Theta} \in \mathrm{KK}_n^G(C_0(X), C_0(X) \hat{\otimes} \mathscr{P})$ be obtained from Θ by forgetting the X-linearity. We define the *abstract equivariant Euler characteristic* by

$$\mathrm{Eul}_X := \bar{\Theta} \hat{\otimes}_{C_0(X) \hat{\otimes} \mathscr{P}} \mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}_0^G(C_0(X), \mathbb{C}).$$

Examples show that this class deserves to be called an Euler characteristic. We were led to this definition by the consideration of the Gysin sequence.

In order to compute Eul_X, we need an explicit formula for PD^{-1}. Therefore, it is useful to consider a richer structure than an abstract dual, which we call a *Kasparov dual*. Gennadi Kasparov constructs the required structure for a smooth Riemannian manifold in [25, Section 4], using for $\mathscr{P}$ the algebra of C_0-sections of the Clifford algebra bundle on X. A fairly simple computation shows that the associated equivariant Euler characteristic is the class in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$ of the de Rham operator on X, which we denote by $\mathrm{Eul}_X^{\mathrm{dR}}$ and call the *equivariant de-Rham-Euler characteristic of* X.

If X is a simplicial complex and G acts simplicially, then a Kasparov dual for X is constructed in [26]. Since the description of Θ in [26] is too indirect for our purposes, we give a slightly different construction where we can write down Θ very concretely. We describe the equivariant Euler characteristic that we get from this combinatorial dual; the result may be computed in terms of counting G-orbits on the set of simplices and is called the *equivariant combinatorial Euler characteristic* $\mathrm{Eul}_X^{\mathrm{cmb}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$.

We show that the (abstract) Euler characteristic of X does not depend on the choice of the abstract dual. Therefore, if X admits a smooth structure and a triangulation at the same time, then $\mathrm{Eul}_X^{\mathrm{dR}} = \mathrm{Eul}_X = \mathrm{Eul}_X^{\mathrm{cmb}}$. This result is due to Wolfgang Lück and Jonathan Rosenberg ([34,50]). In the non-equivariant case, the assertion is that for a connected smooth manifold M, we have $\mathrm{Eul}_M^{\mathrm{dR}} = \chi(M) \cdot \dim$, where $\dim \in \mathrm{K}_0(M)$ is the class of the point evaluation homomorphism.

Now we outline the proof of Theorem 1 and its analogues for general locally compact groups. The starting point is the extension of G-C^*-algebras

$$0 \to C_0(X) \to C(\bar{X}) \to C(\partial X) \to 0,$$

which yields a six term exact sequence for the functor $\mathrm{K}_*^{\mathrm{top}}(G, \llcorner\!\lrcorner)$. The strong contractibility of $\bar{X}$ implies that $\mathrm{K}_*^{\mathrm{top}}\big(G, C(\bar{X})\big) \cong \mathrm{K}_*^{\mathrm{top}}(G)$. The resulting map $\mathrm{K}_*^{\mathrm{top}}(G) \cong \mathrm{K}_*^{\mathrm{top}}\big(G, C(\bar{X})\big) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\partial X)\big)$ in the exact sequence is induced by the unital inclusion $\mathbb{C} \to C(\partial X)$. A purely formal argument shows that the map $\mathrm{K}_*^{\mathrm{top}}\big(G, C_0(X)\big) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\bar{X})\big) \cong \mathrm{K}_*^{\mathrm{top}}(G)$ in the exact sequence is given by the Kasparov product with the equivariant *abstract* Euler characteristic $\mathrm{Eul}_X \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$. The heart of the computation is the explicit description of Eul_X as $\mathrm{Eul}_X^{\mathrm{dR}}$ or $\mathrm{Eul}_X^{\mathrm{cmb}}$.

Our interest in the problem of calculating the K-theory of boundary crossed products was sparked by discussions with Guyan Robertson at a meeting in Oberwolfach in 2004. We would like to thank him for drawing our attention to this question. We also thank Wolfgang Lück for helpful suggestions regarding Euler characteristics.

1.1. General setup

We always require topological spaces and groups to be locally compact, Hausdorff, and second countable. Let G be such a group. A *G-space* is a locally compact space equipped with a continuous action of G. A *G-C^*-algebra* is a separable C^*-algebra equipped with a strongly continuous action of G by automorphisms. We denote *reduced crossed products* by $A \rtimes_{\mathrm{r}} G$. We always equip $\mathbb{C}$ with the trivial action of G. Thus $\mathbb{C} \rtimes_{\mathrm{r}} G$ is the *reduced group C^*-algebra* $C_{\mathrm{r}}^* G$ of G.

A G-space is *proper* if the set of $g \in G$ with $gK \cap K \neq \emptyset$ is compact for any compact subset $K \subseteq X$. A *universal proper G-space* is a proper G-space $\mathcal{E}G$ with the property that for any proper G-space X there is a continuous G-equivariant map $X \to \mathcal{E}G$, which is unique up to G-equivariant homotopy. Such a G-space exists for any G by [27], and any two of them are G-equivariantly homotopy equivalent.

Definition 3. *We call a G-space* strongly contractible *if it is H-equivariantly contractible for each compact subgroup $H \subseteq G$.*

A proper G-space is strongly contractible if and only if it is universal. It is easy to see that universality implies strong contractibility: look at maps $G/H \times \mathcal{E}G \to \mathcal{E}G$. The converse implication is proved in [33] for the easier case of G-CW-complexes, where it already suffices to assume all fixed-point subsets for compact subgroups to be contractible.

Definition 4. *Let X be a* proper *G-space and let $\bar{X}$ be a* compact *G-space that contains X as an open G-invariant subspace. Then $\partial X := \bar{X} \setminus X$ is another compact G-space. We call the induced action of G on ∂X a* boundary action *if both X and $\bar{X}$ are strongly contractible. The G-spaces X and $\bar{X}$ are part of the data of a boundary action.*

In all our examples, $\bar{X}$ is a compactification of X, that is, X is dense in $\bar{X}$. A boundary action yields an extension of G-C^*-algebras

$$0 \to C_0(X) \xrightarrow{\iota} C(\bar{X}) \xrightarrow{\pi} C(\partial X) \to 0. \tag{1}$$

Let $\bar{\upsilon}\colon \mathbb{C} \to C(\bar{X})$ and $\partial\upsilon\colon \mathbb{C} \to C(\partial X)$ be the unit maps. The map $\partial\upsilon$ induces the obvious embedding $C^*_{\mathrm{r}}G \to C(\partial X) \rtimes_{\mathrm{r}} G$, which we also denote by u. We are going to study the induced map

$$u_* = \partial\upsilon_*\colon \mathrm{K}_*(C^*_{\mathrm{r}}G) \to \mathrm{K}_*(C(\partial X) \rtimes_{\mathrm{r}} G). \tag{2}$$

2. Examples of boundary actions

Many examples of boundary actions are special cases of two general constructions: the visibility compactification of a CAT(0) space and the Gromov compactification of a hyperbolic space (see [7] for both).

Let X be a second countable, locally compact CAT(0) space and let G act properly and isometrically on X. We consider geodesic rays $\mathbb{R}_+ \to X$ parametrised by arc length. Two such rays are equivalent if they are at bounded distance from each other. The *visibility boundary* of X is the set ∂X_∞ of equivalence classes of geodesic rays in X and the *visibility compactification* $\bar{X}_\infty$ is $X \cup \partial X_\infty$. This is a compactification of X for a canonical compact metrisable topology on $\bar{X}_\infty$; it has the property that $r(t)$ converges towards $[r]$ for $t \to \infty$ for any geodesic ray $r\colon \mathbb{R}_+ \to X$. The obvious action of G on $\bar{X}_\infty$ is continuous.

Let $H \subseteq G$ be a compact subgroup. Then H has a fixed-point ξ_H in X. For any $x \in X$ there is a unique geodesic segment connecting x and ξ_H. We may contract the space X along these geodesics, so that X is strongly contractible. Similarly, $\bar{X}_\infty$ is strongly contractible because any point in ∂X_∞ is represented by a unique geodesic ray emanating from ξ_H.

For instance, if X is a simply connected Riemannian manifold of non-positive sectional curvature, then X is CAT(0). If $\dim X = n$, then there is a homeomorphism from $\bar{X}_\infty$ onto a closed n-cell that identifies X with the open n-disk and ∂X_∞ with S^{n-1}.

Let G be an almost connected Lie group whose connected component is linear and reductive, and let $K \subseteq G$ be a maximal compact subgroup. Then the homogeneous space G/K with any G-invariant Riemannian metric is a CAT(0) space and has non-positive sectional curvature ([7]). If G is semi-simple and has rank 1, then the visibility boundary is equivalent to the *Fürstenberg boundary* G/P of G, where P is a minimal parabolic subgroup. If G has higher rank, then the Fürstenberg boundary of G is not a boundary action in our sense. However, there are points in ∂X_∞ that are fixed by P. Hence the unit map $\mathbb{C} \to C(G/P)$ factors through $\partial \upsilon\colon \mathbb{C} \to C(\partial X_\infty)$.

Euclidean buildings and trees are CAT(0) spaces as well. For instance, let G be a reductive p-adic group. Then its affine Bruhat-Tits building is a CAT(0) space, on which G acts properly and isometrically ([57]). The visibility compactification of the Bruhat-Tits building is equivalent to the Borel-Serre compactification in this case (see [5] and [51, Lemma IV.2.1]). The relationship between the visibility boundary of the building and the Fürstenberg boundary of G is exactly as in the Lie group case.

Let X be a CAT(0) space on which G acts properly and isometrically and let $\bar{X}_\infty$ be its visibility compactification. The *limit set* $\Lambda X_\infty \subseteq \partial X_\infty$ is the set of all accumulation points in the boundary of $G \cdot x$ for some $x \in X$. Its definition is most familiar for classical hyperbolic space $\mathbb{H}^n$ (see [42, Section 12]). The limit set is independent of the choice of x and therefore G-invariant. Its complement $X' := \bar{X}_\infty \setminus \Lambda X_\infty$ is called the *ordinary set* of G. It is strongly contractible for the same reason as $\bar{X}_\infty$. If the action of G on X' is proper, then ΛX_∞ is a boundary action. This is always the case for classical hyperbolic space ([42]).

Let X be a (quasi-geodesic) hyperbolic metric space. Two quasi-geodesic rays in X are considered equivalent if they have bounded distance. The *Gromov boundary* ∂X of X is defined as the set of equivalence classes of quasi-geodesic rays in X. There is a canonical compact metrisable topology on $\bar{X} := X \cup \partial X$ so that this becomes a compactification of X. The construction is natural with respect to quasi-isometric equivalence. That is, if X and Y are hyperbolic metric spaces and $f\colon X \to Y$ is a quasi-isometric equivalence, then f extends in a unique way to a map $\bar{f}\colon \bar{X} \to \bar{Y}$, whose restriction to the boundary $\partial f\colon \partial X \to \partial Y$ is a homeomorphism.

If G is a word-hyperbolic group, then one may apply this construction to the metric space underlying G (with a word metric). Since the action of G by left translation on itself is isometric, this action extends to an action of G by homeomorphisms of its boundary. Now let $X = P_d(G)$ be the Rips complex of G with parameter d. This space is strongly contractible for sufficiently large d ([35]). We may equip X with a G-invariant metric for which any orbit map $G \to X$ is a quasi-isometric equivalence. Since hyperbolicity and the Gromov boundary are invariant under quasi-isometric equivalence, X is itself hyperbolic and there is a canonical G-equivariant homeomorphism $\partial G \cong \partial X$. It is shown in [50] that $\bar{X}$ is strongly contractible. Hence $G \times \partial G \to \partial G$ is a boundary action.

Now we come to a completely different example of a boundary action. Let Σ_g be a closed Riemann surface of genus $g \geq 2$ and let G be a torsion-free subgroup of the *mapping class group*,

$$\mathrm{Mod}(\Sigma_g) := \mathrm{Diff}(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g).$$

Let $\mathscr{T}$ be the *Teichmüller space* for Σ_g, and let $\partial\mathscr{T}$ and $\bar{\mathscr{T}} := \mathscr{T} \cup \partial\mathscr{T}$ be its *Thurston boundary* and *Thurston compactification*, respectively ([17]). It is well-known that $\mathscr{T}$ and $\bar{\mathscr{T}}$ are contractible. This means that $G \times \partial\mathscr{T} \to \partial\mathscr{T}$ is a boundary action because we require G to be torsion-free.

It seems plausible that $\mathscr{T}$ and $\bar{\mathscr{T}}$ are strongly contractible with respect to the group $\mathrm{Mod}(\Sigma_g)$ itself, so that $\partial\mathscr{T}$ would be a boundary action of $\mathrm{Mod}(\Sigma_g)$. That $\mathscr{T}$ is strongly contractible (hence a universal proper $\mathrm{Mod}(\Sigma_g)$-space) follows from the proof of the *Nielsen realisation problem* by Steven Kerckhoff in [28]. The main result is that every finite subgroup H of $\mathrm{Mod}(\Sigma_g)$ fixes a point of $\mathscr{T}$. Moreover, any two points of $\mathscr{T}$ are connected by a unique *earthquake path* (see [56,28]). If $x \in \mathscr{T}$ is fixed by H, then the contraction of Teichmüller space along earthquake paths emanating from x provides an H-equivariant contracting homotopy for $\mathscr{T}$. It seems likely that this contracting homotopy extends to one for the compactification $\bar{\mathscr{T}}$, but a proof does not seem to exist in the literature so far.

A group may admit more than one boundary action. For example, consider the group $G = \mathrm{PSL}(2, \mathbb{Z})$. It has two natural models for $\mathscr{E}G$, namely, the hyperbolic plane $\mathbb{H}^2$, on which it acts by Möbius transformations, and the tree X which corresponds to the free-product decomposition $G \cong \mathbb{Z}/2 * \mathbb{Z}/3$ (see [54, §I.4]). Whereas $G\backslash X$ is compact, $G\backslash\mathbb{H}^2$ is not. Both X and $\mathbb{H}^2$ are simultaneously hyperbolic and CAT(0), and their Gromov and visibility compactifications coincide; they compactify $\mathbb{H}^2$ by the circle at infinity $\mathbb{P}^1(\mathbb{R})$ and X by its set of ends, which is a Cantor set. These two boundary actions of $\mathrm{PSL}(2, \mathbb{Z})$ are related by a well-known G-equivariant isometric embedding of the tree X in $\mathbb{H}^2$ (see [54, §I.4]). This embedding extends to a continuous map between the compactifications and hence yields a map $\partial X \to \partial\mathbb{H}^2$. It is not clear how different boundary actions are related in general.

3. Applying the Baum-Connes conjecture

We recall the definition of the Baum-Connes assembly map (see [4]). Let $\mathscr{E}G$ be a second countable, locally compact model for the universal proper G-space. Write $\mathscr{E}G = \bigcup_{n\in\mathbb{N}} \mathscr{E}G_n$ for some increasing sequence of G-compact subsets $\mathscr{E}G_n \subseteq \mathscr{E}G$. The maps $\mathscr{E}G_n \to \mathscr{E}G_{n+1}$ are proper, so that we get an associated projective system of G-C^*-algebras $\big(C_0(\mathscr{E}G_n)\big)_{n\in\mathbb{N}}$. Let

$$\mathrm{K}_*^{\mathrm{top}}(G, A) := \varinjlim \mathrm{KK}_*^G(C_0(\mathscr{E}G_n), A).$$

The *Baum-Connes assembly map* is the composite map

$$\mu_{G,A}\colon \mathrm{K}_*^{\mathrm{top}}(G, A) \to \varinjlim \mathrm{KK}_*(C_0(\mathscr{E}G_n) \rtimes_{\mathrm{r}} G, A \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G),$$

where the first map is descent and the second map is the Kasparov product with a certain natural class in $\varprojlim \mathrm{K}_0(C_0(\mathscr{E}G_n) \rtimes_{\mathrm{r}} G)$.

Let X, $\bar{X}$ and ∂X be as in Definition 4. The Baum-Connes conjecture always holds for proper coefficient algebras and, especially, for $C_0(X)$. *We assume from now on that G satisfies the Baum-Connes conjecture with coefficients $\mathbb{C}$ and $C(\partial X)$*. That is, the vertical maps in the diagram

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(G,\mathbb{C}) & \xrightarrow{\partial\upsilon_*} & \mathrm{K}_*^{\mathrm{top}}\big(G, C(\partial X)\big) \\ \downarrow{\scriptstyle \mu_{G,\mathbb{C}}} & & \downarrow{\scriptstyle \mu_{G,C(\partial X)}} \\ \mathrm{K}_*(C_{\mathrm{r}}^*G) & \xrightarrow{\partial\upsilon_*} & \mathrm{K}_*(C(\partial X) \rtimes_{\mathrm{r}} G) \end{array}$$

are isomorphisms. Thus the map (2) that we are interested in is equivalent to the map

$$\partial\upsilon_*\colon \mathrm{K}_*^{\mathrm{top}}(G,\mathbb{C}) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\partial X)\big). \tag{3}$$

Our assumption on the Baum-Connes conjecture is known to be valid in many examples. Closed subgroups of $\mathrm{Isom}(\mathbb{H}^n)$ satisfy it because they even satisfy the Baum-Connes conjecture with arbitrary coefficients ([24]). The same holds for closed subgroups of other semi-simple Lie groups of rank 1 by [21,22]. Word-hyperbolic groups satisfy the assumption as well: the Baum-Connes conjecture with trivial coefficients is proved in [39], and the Baum-Connes conjecture for the coefficients $C(\partial G)$ follows from [58] because the action of G on ∂G is amenable.

We will exclusively deal with the map in (3) in the following. We *only* need the Baum-Connes conjecture to relate it to (2).

It is shown in [27] that $\mathrm{K}_*^{\mathrm{top}}$ satisfies excision for arbitrary extensions of G-C^*-algebras. Hence (1) gives rise to a six term exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0^{\mathrm{top}}\big(G, C_0(X)\big) & \xrightarrow{\iota_*} & \mathrm{K}_0^{\mathrm{top}}\big(G, C(\bar{X})\big) & \xrightarrow{\pi_*} & \mathrm{K}_0^{\mathrm{top}}\big(G, C(\partial X)\big) \\ \uparrow{\scriptstyle \delta} & & & & \downarrow{\scriptstyle \delta} \\ \mathrm{K}_1^{\mathrm{top}}\big(G, C(\partial X)\big) & \xleftarrow{\pi_*} & \mathrm{K}_1^{\mathrm{top}}\big(G, C(\bar{X})\big) & \xleftarrow{\iota_*} & \mathrm{K}_1^{\mathrm{top}}\big(G, C_0(X)\big). \end{array} \tag{4}$$

We are going to modify it in several steps.

Lemma 5. *The map $\bar{\upsilon}_*\colon \mathrm{K}_*^{\mathrm{top}}(G,\mathbb{C}) \to \mathrm{K}_*^{\mathrm{top}}\big(G, C(\bar{X})\big)$ is an isomorphism if $\bar{X}$ is strongly contractible.*

Proof. Since $\bar{X}$ is strongly contractible, $[\bar{\upsilon}]$ is invertible in $\mathrm{KK}_0^H\big(\mathbb{C}, C(\bar{X})\big)$ for all compact subgroups $H \subseteq G$. That is, $\bar{\upsilon}$ is a *weak equivalence* in the notation of [38]. It is shown in [10,38] that such maps induce isomorphisms on $\mathrm{K}_*^{\mathrm{top}}(G, \text{␣})$. □

Plugging the isomorphism of Lemma 5 into (4), we get an exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0^{\mathrm{top}}\big(G, C_0(X)\big) & \xrightarrow{\bar{\upsilon}_*^{-1}\iota_*} & \mathrm{K}_0^{\mathrm{top}}(G,\mathbb{C}) & \xrightarrow{\partial\upsilon_*} & \mathrm{K}_0^{\mathrm{top}}\big(G, C(\partial X)\big) \\ \uparrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} \\ \mathrm{K}_1^{\mathrm{top}}\big(G, C(\partial X)\big) & \xleftarrow{\partial\upsilon_*} & \mathrm{K}_1^{\mathrm{top}}(G,\mathbb{C}) & \xleftarrow{\bar{\upsilon}_*^{-1}\iota_*} & \mathrm{K}_1^{\mathrm{top}}\big(G, C_0(X)\big) \end{array} \tag{5}$$

It contains the map (3) that we are interested in because $\pi \circ \bar{\upsilon} = \partial\upsilon$.

Example 6. Suppose that the action of G on ∂X admits a fixed-point ξ. Then evaluation at ξ provides a section for $\bar{\upsilon}$. Since this evaluation homomorphism annihilates $C_0(X) \subseteq C(\bar{X})$, we get $\bar{\upsilon}_*^{-1}\iota_* = 0$. Therefore, the long exact sequence (5) splits into two short exact sequences.

Proposition 7. *Let G be a locally compact group with non-compact centre. Suppose that X is G-compact and that $\bar{X}$ is admissible in the sense of [18], that is, compatible with the coarse geometric structure of X.*

Then ∂X contains a fixed-point for G. Hence $\bar{\upsilon}_*^{-1}\iota_* = 0$ in (5).

Proof. Let $x \in X$ and let $(g_i)_{i\in\mathbb{N}}$ be a sequence in the centre of G that leaves any compact subset of G. Since $\bar{X}$ is compact, we may assume that the sequence $g_i x$ converges towards some $\xi \in \bar{X}$. Since the sequences $(g_i x)$ and $(g \cdot g_i x) = (g_i g x)$ are uniformly close for any $g \in G$, the compatibility of the coarse structure with the compactification implies that they have the same limit point in $\bar{X}$. Thus $g\xi = \xi$ for all $g \in G$. □

Of course, the map $\bar{\upsilon}_*^{-1}\iota_*$ in (5) is non-zero is general. The following notation is needed in order to describe it.

Let Y be a locally compact G-space and let A and B be G-C^*-algebras. The graded Abelian group $\mathrm{RKK}_*^G(Y; A, B)$ is defined as in [25]; its cycles are cycles $(\mathcal{E}, F)$ for $\mathrm{KK}_*^G(C_0(Y) \hat{\otimes} A, C_0(Y) \hat{\otimes} B)$ that satisfy the additional condition that the left and right $C_0(Y)$-actions on $\mathcal{E}$ agree. We may think of these cycles as G-equivariant continuous families of $\mathrm{KK}_*^G(A, B)$-cycles parametrised by Y. There is a natural map

$$\mathrm{RKK}_*^G(Y; A, B) \to \mathrm{KK}_*^G(C_0(Y) \hat{\otimes} A, C_0(Y) \hat{\otimes} B) \tag{6}$$

which forgets the Y-structure and a natural map

$$p_Y^* \colon \mathrm{KK}_*^G(A, B) \to \mathrm{RKK}_*^G(Y; A, B), \tag{7}$$

which sends a Kasparov cycle $(\mathcal{E}, F)$ for $\mathrm{KK}_*^G(A, B)$ to the constant family of Kasparov cycles $(C_0(Y) \hat{\otimes} \mathcal{E}, 1 \hat{\otimes} F)$.

Definition 8. We let $\Delta_Y \in \mathrm{RKK}_0^G(Y; C_0(Y), \mathbb{C})$ be the class of the $Y \rtimes G$-equivariant $*$-homomorphism $C_0(Y \times Y) \to C_0(Y)$ that is induced by the diagonal embedding $Y \to Y \times Y$.

Recall that the space X is a universal proper G-space. It is shown in [38, Section 7] that the map in (7) for $Y = X$ is an isomorphism whenever A is a proper G-C^*-algebra. Hence $\Delta_X \in \mathrm{RKK}_0^G(X; C_0(X), \mathbb{C})$ has a pre-image in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$. Anticipating a little, we denote this pre-image by Eul_X. This agrees with our official definition of the abstract Euler characteristic in Definition 12 by Proposition 14 and Lemma 16.

Proposition 9. *Let G be a locally compact group and let $\partial X = \bar{X} \setminus X$ be a boundary action of G. Then there is an exact sequence*

$$\begin{array}{ccccc} \mathrm{K}_0^{\mathrm{top}}\big(G, C_0(X)\big) & \xrightarrow{\mathrm{Eul}_X} & \mathrm{K}_0^{\mathrm{top}}(G, \mathbb{C}) & \xrightarrow{\partial\upsilon_*} & \mathrm{K}_0^{\mathrm{top}}\big(G, C(\partial X)\big) \\ \uparrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} \\ \mathrm{K}_1^{\mathrm{top}}\big(G, C(\partial X)\big) & \xleftarrow{\partial\upsilon_*} & \mathrm{K}_1^{\mathrm{top}}(G, \mathbb{C}) & \xleftarrow{\mathrm{Eul}_X} & \mathrm{K}_1^{\mathrm{top}}\big(G, C_0(X)\big), \end{array}$$

where Eul_X *denotes the Kasparov product with* $\mathrm{Eul}_X \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$.

Proof. It only remains to identify the maps $\bar{\upsilon}_*^{-1}\iota_*$ in (5) with Eul_X. Recall that the map p_X^* in (7) is an isomorphism if A is proper. Therefore, p_X^* induces an isomorphism

$$\mathrm{K}_*^{\mathrm{top}}(G, A) \xrightarrow{\cong} \varinjlim \mathrm{RKK}_*^G(X; C_0(\mathscr{E}G_n), A).$$

Thus the Kasparov product in $\mathrm{RKK}_*^G(X)$ gives rise to natural bilinear maps

$$\mathrm{K}_*^{\mathrm{top}}(G, A) \times \mathrm{RKK}_0^G(X; A, B) \to \mathrm{K}_*^{\mathrm{top}}(G, B), \qquad (x, y) \mapsto x \bullet y. \tag{8}$$

It is shown in [38, Section 7] that $p_X^*(f)$ for $f \in \mathrm{KK}_*^G(A, B)$ is invertible if and only if f is a weak equivalence. Therefore, $p_X^*[\bar{\upsilon}]$ is invertible in $\mathrm{RKK}_0^G\big(X; \mathbb{C}, C(\bar{X})\big)$. The homomorphism $C_0(X) \to C_0(X \times \bar{X})$ induced by the coordinate projection $X \times \bar{X} \to X$ is a representative for $p_X^*[\bar{\upsilon}]$. Since the diagonal embedding $X \to X \times X \subseteq X \times \bar{X}$ is a section for the coordinate projection, the element in $\mathrm{RKK}_0^G(X; C(\bar{X}), \mathbb{C})$ associated to the diagonal embedding is inverse to $p_X^*[\bar{\upsilon}]$. Hence $p_X^*[\iota] \mathbin{\hat{\otimes}}_{X, C(\bar{X})} p_X^*[\bar{\upsilon}]^{-1} = \Delta_X$. It follows easily from the definition of the product in (8) that

$$\iota_*(x) = x \mathbin{\hat{\otimes}}_{C_0(X)} [\iota] = x \bullet p_X^*[\iota], \qquad \bar{\upsilon}_*(y) = y \mathbin{\hat{\otimes}} [\bar{\upsilon}] = y \bullet p_X^*[\bar{\upsilon}],$$

for all $x \in \mathrm{K}_*^{\mathrm{top}}\big(G, C_0(X)\big)$, $y \in \mathrm{K}_*^{\mathrm{top}}(G, \mathbb{C})$. This implies

$$\bar{\upsilon}_*^{-1}\iota_*(x) = x \bullet (p_X^*[\iota] \mathbin{\hat{\otimes}}_{X, C(\bar{X})} p_X^*[\bar{\upsilon}]^{-1}) = x \bullet \Delta_X = x \mathbin{\hat{\otimes}}_{C_0(X)} p_X^{-1}(\Delta_X)$$

because p_X is invertible and the map in (8) is natural. □

4. Abstract Euler characteristics via Kasparov duality

The element $\mathrm{Eul}_X \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$ that appears in the Gysin sequence in Proposition 9 is so far only defined if X is a universal proper G-space; furthermore, it is not clear how it should be computed. In this section, we extend its definition to a more general class of G-spaces, using a formulation of Poincaré duality due to Gennadi Kasparov ([25, Section 4]). In the following sections, we will compute Eul_X using this alternative definition.

Definition 10. Let X be a locally compact G-space (we require neither properness nor strong contractibility). Let $n \in \mathbb{Z}$. Let $\mathscr{P}$ be a (possibly $\mathbb{Z}/2$-graded) G-C^*-algebra, and let $\Theta \in \mathrm{RKK}_n^G(X; \mathbb{C}, \mathscr{P})$.

We call $(\mathscr{P}, \Theta)$ an (n-dimensional) *abstract dual* for X if the map

$$\mathrm{PD}\colon \mathrm{RKK}^G_{*-n}(Y; A_1 \hat{\otimes} \mathscr{P}, A_2) \to \mathrm{RKK}^G_*(X \times Y; A_1, A_2), \quad f \mapsto \Theta \hat{\otimes}_{\mathscr{P}} f, \tag{9}$$

is an isomorphism for all G-spaces Y and all G-C^*-algebras A_1, A_2.

Here we use the Kasparov product

$$\hat{\otimes}_{\mathscr{P}}\colon \mathrm{RKK}^G_i(X; A, B \hat{\otimes} \mathscr{P}) \times \mathrm{RKK}^G_j(Y; A' \hat{\otimes} \mathscr{P}, B') \to \mathrm{RKK}^G_{i+j}(X \times Y; A \hat{\otimes} A', B \hat{\otimes} B')$$

(see [25]). Observe that (9) is the most general form for a natural transformation $\mathrm{RKK}^G_{*-n}(Y; A \hat{\otimes} \mathscr{P}, B) \to \mathrm{RKK}^G_*(X \times Y; A, B)$ that is compatible with Kasparov products in the sense that

$$\mathrm{PD}(f_1 \hat{\otimes}_B f_2) = \mathrm{PD}(f_1) \hat{\otimes}_B f_2 \qquad \text{in } \mathrm{RKK}^G_{i+j}(Y; A_1 \hat{\otimes} A_3, A_2 \hat{\otimes} A_4) \tag{10}$$

for all $f_1 \in \mathrm{RKK}^G_{i-n}(Y; A_1 \hat{\otimes} \mathscr{P}, A_2 \hat{\otimes} B)$, $f_2 \in \mathrm{RKK}^G_j(Y; A_3 \hat{\otimes} B, A_4)$. Since exterior products are graded commutative, we also get

$$\mathrm{PD}(f_1 \hat{\otimes}_B f_2) = (-1)^{in} f_1 \hat{\otimes}_B \mathrm{PD}(f_2) \qquad \text{in } \mathrm{RKK}^G_{i+j}(Y; A_1 \hat{\otimes} A_3, A_2 \hat{\otimes} A_4) \tag{11}$$

for all $f_1 \in \mathrm{RKK}^G_i(Y; A_1, A_2 \hat{\otimes} B)$, $f_2 \in \mathrm{RKK}^G_{j-n}(Y; A_3 \hat{\otimes} B \hat{\otimes} \mathscr{P}, A_4)$; both sides are equal to $(\Theta \hat{\otimes} f_1) \hat{\otimes}_{\mathscr{P} \hat{\otimes} B} f_2$.

The space Y does not play any serious role. We have put it into our definitions because Kasparov works in this generality in [25, Theorem 4.9]. The dimension n is not particularly important either because we can always reduce to the case $n = 0$ by a suspension.

We are mainly interested in the case of complex C^*-algebras and therefore only formulate Gysin sequences in this case. However, the purely formal arguments in this section are independent of Bott periodicity and therefore also work in the real and "real" cases. This is why we are careful to distinguish between KK_n and KK_{-n} in our notation. Of course, in the real case one has to replace $\mathbb{C}$ by $\mathbb{R}$ everywhere and use real-valued function spaces.

Later, we shall introduce further structure in order to write down the inverse map PD^{-1} more concretely, which is important for applications. However, this additional structure involves some choices. Since the abstract Euler characteristic is supposed to be independent of the dual, we discuss the formal aspects of the duality in the situation of Definition 10.

Remark 11. There exist spaces that do not possess an abstract dual, even for trivial G. If X is compact then $\mathrm{RKK}_*(X; A, B) \cong \mathrm{KK}_*(A, C(X) \hat{\otimes} B)$ for all A, B. Hence an abstract dual for X is nothing but a KK-dual for $C(X)$. Duality in this context is investigated by Claude Schochet in [52, 53]. The Universal Coefficient Theorem holds for $C(X)$ and shows that it has a KK-dual if and only if $\mathrm{K}_*\big(C(X)\big)$ is finitely generated (recall that all spaces are assumed second countable). This fails, for example, if X is a Cantor set.

Definition 12. Let X be a G-space with an abstract dual $(\mathscr{P}, \Theta)$. Let

$$\bar{\Theta} \in \mathrm{KK}^G_n(C_0(X), C_0(X) \hat{\otimes} \mathscr{P}) \quad \text{and} \quad \Delta_X \in \mathrm{RKK}^G_0(X; C_0(X), \mathbb{C}),$$

respectively, be the image of Θ under the forgetful map (6) and the element induced by the diagonal embedding as in Definition 8. Thus $\mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}^G_{-n}(C_0(X) \hat{\otimes} \mathscr{P}, \mathbb{C})$. We call

$$\mathrm{Eul}_X := \bar{\Theta} \hat{\otimes}_{C_0(X) \hat{\otimes} \mathscr{P}} \mathrm{PD}^{-1}(\Delta_X) \in \mathrm{KK}^G_0(C_0(X), \mathbb{C})$$

the *G-equivariant abstract Euler characteristic of X*.

This name will be justified by the examples in the following sections.

Our first task is to analyse the uniqueness of abstract duals and to show that Eul_X does not depend on their choice. We consider the slightly more complicated issue of functoriality right away.

Let X and X' be two G-spaces with abstract duals $(\mathscr{P}, \Theta)$ and $(\mathscr{P}', \Theta')$ of dimension n and n', respectively, and let PD and PD′ be the associated duality isomorphisms. Let $f: X \to X'$ be a continuous G-map; we do *not* require f to be proper. Then f induces natural maps

$$f^*: \mathrm{RKK}^G_*(X' \times Y; A, B) \to \mathrm{RKK}^G_*(X \times Y; A, B) \tag{12}$$

for all Y, A, B. Hence we get $f^*\Theta' \in \mathrm{RKK}^G_{n'}(X; \mathbb{C}, \mathscr{P}')$ and

$$\alpha_f := \mathrm{PD}^{-1}(f^*\Theta') \in \mathrm{KK}^G_{n'-n}(\mathscr{P}, \mathscr{P}').$$

Equivalently, $\Theta \hat{\otimes}_{\mathscr{P}} \alpha_f = f^*\Theta'$; this property characterises α_f uniquely and implies

$$\mathrm{PD}(\alpha_f \hat{\otimes}_{\mathscr{P}'} h) = f^*\Theta' \hat{\otimes}_{\mathscr{P}'} h = f^*\big(\mathrm{PD}'(h)\big) \qquad \text{in } \mathrm{RKK}^G_*(X; A, B) \tag{13}$$

for all $h \in \mathrm{KK}^G_{*-n'}(A \hat{\otimes} \mathscr{P}', B)$. The map $f \mapsto \alpha_f$ is a covariant functor in the following sense. If $f = \mathrm{id}_X$ and $(\mathscr{P}, \Theta) = (\mathscr{P}', \Theta')$, then $\alpha_{\mathrm{id}} = 1_{\mathscr{P}}$. Given composable maps $f\colon X \to X'$, $f'\colon X' \to X''$ and abstract duals for the G-spaces X, X', X'', we get $\alpha_{f'\circ f} = \alpha_f \hat{\otimes}_{\mathscr{P}'} \alpha_{f'}$.

If two maps $f_1, f_2\colon X \to X'$ are G-equivariantly homotopic, then they induce the same maps $f_1^* = f_2^*$ in (12). Hence $\alpha_{f_1} = \alpha_{f_2}$. Moreover, if f is a G-homotopy equivalence then (12) is bijective for all Y, A, B. By functoriality of α_f, we conclude that α_f is invertible if f is a G-homotopy equivalence. In the special case $f = \mathrm{id}$, we get a canonical KK^G-equivalence between two duals $(\mathscr{P}, \Theta)$, $(\mathscr{P}', \Theta')$ for the same space X.

Proposition 13. *Let $f\colon X \to X'$ be a G-homotopy equivalence and proper; we do not assume its homotopy inverse to be proper. We denote the class of the induced map $f^*\colon C_0(X') \to C_0(X)$ in $\mathrm{KK}^G_0(C_0(X'), C_0(X))$ by $[f^*]$. Then*

$$[f^*] \hat{\otimes}_{C_0(X)} \mathrm{Eul}_X = \mathrm{Eul}_{X'}.$$

The abstract Euler characteristic is independent of the choice of the abstract dual.

Proof. We let $\Delta_{X'}\colon C_0(X' \times X') \to C_0(X')$ be the diagonal restriction homomorphism. Then $f^*(\Delta_{X'}) \in \mathrm{RKK}^G_0(X; C_0(X'), \mathbb{C})$ is the class of the G-equivariant $*$-homomorphism induced by the map $(\mathrm{id}, f)\colon X \to X \times X'$. Thus $f^*(\Delta_{X'}) = [f^*] \hat{\otimes}_{C_0(X)} \Delta_X$. Since the map in (12) is bijective and natural with respect to the Kasparov product, this is equivalent to

$$\Delta_{X'} = (f^*)^{-1}([f^*] \hat{\otimes}_{C_0(X)} \Delta_X) = [f^*] \hat{\otimes}_{C_0(X)} (f^*)^{-1}(\Delta_X).$$

Define $\alpha_f \in \mathrm{KK}^G_{n'-n}(\mathscr{P}, \mathscr{P}')$ as above. Equation (13) is equivalent to

$$\alpha_f^{-1} \hat{\otimes}_{\mathscr{P}} \mathrm{PD}^{-1}(h) = (\mathrm{PD}')^{-1}(f^*)^{-1}(h) \qquad \text{for all } h \in \mathrm{RKK}^G_*(X; A, B).$$

We shall use the forgetful functors defined in (6) for the spaces X and X' and denote them by $h \mapsto \overline{h}$. They satisfy the compatibility relation

$$[f^*] \hat{\otimes}_{C_0(X)} \overline{f^*h} = \overline{h} \hat{\otimes}_{C_0(X')} [f^*]$$

in $\mathrm{KK}^G_*(C_0(X') \hat{\otimes} A, C_0(X) \hat{\otimes} B)$ for all $h \in \mathrm{RKK}^G_*(X'; A, B)$; these Kasparov products are comparatively easy to compute because $[f^*]$ is represented by a $*$-homomorphism. Now we compute

$$\begin{aligned}
[f^*] \hat{\otimes}_{C_0(X)} \mathrm{Eul}_X &= [f^*] \hat{\otimes}_{C_0(X)} \overline{\Theta} \hat{\otimes}_{C_0(X)\hat{\otimes}\mathscr{P}} \mathrm{PD}^{-1}(\Delta_X) \\
&= [f^*] \hat{\otimes}_{C_0(X)} \overline{f^*\Theta'} \hat{\otimes}_{\mathscr{P}'} \alpha_f^{-1} \hat{\otimes}_{C_0(X)\hat{\otimes}\mathscr{P}} \mathrm{PD}^{-1}(\Delta_X) \\
&= \overline{\Theta'} \hat{\otimes}_{C_0(X')} [f^*] \hat{\otimes}_{C_0(X)\hat{\otimes}\mathscr{P}'} (\mathrm{PD}')^{-1} \circ (f^*)^{-1}(\Delta_X) \\
&= \overline{\Theta'} \hat{\otimes}_{C_0(X')\hat{\otimes}\mathscr{P}'} (\mathrm{PD}')^{-1}\bigl([f^*] \hat{\otimes}_{C_0(X)} (f^*)^{-1}(\Delta_X)\bigr) \\
&= \overline{\Theta'} \hat{\otimes}_{C_0(X')\hat{\otimes}\mathscr{P}'} (\mathrm{PD}')^{-1}(\Delta_{X'}) = \mathrm{Eul}_{X'}.
\end{aligned}$$

We use (11) in the step from the third to the fourth line.

Especially, if $f = \mathrm{id}$ then the computation above shows that Eul_X does not depend on the choice of the abstract dual. □

Proposition 14. *If X is a universal proper G-space, then X has an abstract dual.*

Proof. Let $\mathsf{D} \in \mathrm{KK}_0^G(\mathsf{P}, \mathbb{C})$ be a Dirac morphism in the notation of [38]. Since D is a weak equivalence, $p_X^*(\mathsf{D})$ is invertible; let $\Theta \in \mathrm{RKK}_0^G(X; \mathbb{C}, \mathsf{P})$ be its inverse. In [38, Theorem 7.1], we may take $\pi \in \mathrm{KK}_0^G(\tilde{A}, A)$ to be $1_A \otimes \mathsf{D} \in \mathrm{KK}_0^G(A \otimes \mathsf{P}, A)$. Then [38, Theorem 7.1] implies that the map PD that we get from Θ is an isomorphism. □

Assume now that X and X' are *G-compact* universal proper G-spaces. Then they are G-homotopy equivalent in a canonical way, so that their abstract duals are canonically KK^G-equivalent. Any continuous G-map between X and X' is proper because both spaces are G-compact. Hence $C_0(X)$ and $C_0(X')$ are KK^G-equivalent. Moreover, we have natural isomorphisms

$$\mathrm{K}_*^{\mathrm{top}}(G) := \mathrm{K}_*^{\mathrm{top}}(G, \mathbb{C}) \cong \mathrm{KK}_*^G(C_0(X), \mathbb{C}).$$

It follows from Proposition 13 that the abstract Euler characteristics of X and X' agree as elements of $\mathrm{K}_0^{\mathrm{top}}(G)$. Hence we may give the following definition:

Definition 15. Let G be a locally compact group that has a G-compact universal proper G-space X. Identify $\mathrm{KK}_*^G(C_0(X), \mathbb{C}) \cong \mathrm{K}_*^{\mathrm{top}}(G)$ and view the abstract Euler characteristic Eul_X as an element of $\mathrm{K}_0^{\mathrm{top}}(G)$. We denote the result by $\mathrm{Eul}_{\mathscr{E}} G$ and call it the *abstract Euler characteristic of G*.

If X' is any universal proper G-space and X is a G-compact universal proper G-space, then any G-map $f: X \to X'$ is proper and a G-homotopy equivalence. Hence Proposition 13 yields $\mathrm{Eul}_{X'} = f^*(\mathrm{Eul}_{\mathscr{E}} G)$.

Let X again be an arbitrary G-space with an abstract dual $(\mathscr{P}, \Theta)$. Then we define $D \in \mathrm{KK}_{-n}^G(\mathscr{P}, \mathbb{C})$ by

$$\mathrm{PD}(D) := \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = 1_{\mathbb{C}} \qquad \text{in } \mathrm{RKK}_0^G(X; \mathbb{C}, \mathbb{C}). \tag{14}$$

Lemma 16. *Let X be a G-space with an abstract dual $(\mathscr{P}, \Theta)$. Then Θ is invertible if and only if $p_X^*(D) \in \mathrm{RKK}_{-n}^G(X; \mathscr{P}, \mathbb{C})$ is invertible, if and only if the map*

$$p_X^*: \mathrm{RKK}_*^G(Y; \mathscr{P} \hat{\otimes} A, B) \to \mathrm{RKK}_*^G(X \times Y; \mathscr{P} \hat{\otimes} A, B) \tag{15}$$

is invertible for all Y, A, B. In this case, the map

$$p_X^*: \mathrm{RKK}_*^G(Y; C_0(X) \hat{\otimes} A, B) \to \mathrm{RKK}_*^G(X \times Y; C_0(X) \hat{\otimes} A, B) \tag{16}$$

is invertible as well, and $\mathrm{Eul}_X = (p_X^*)^{-1}(\Delta_X)$ *in* $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$.

Proof. Since $\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = 1_{\mathbb{C}}$, Θ is a left inverse for $p_X^*(D)$ with respect to the Kasparov composition product in $\mathrm{RKK}_*^G(X)$. Hence one is invertible if and only if the other is, and they are inverse to each other in that case. By hypothesis, $\mathrm{PD}(f) := \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} p_X^*(f)$ defines an invertible map on $\mathrm{RKK}_*^G(Y; A \mathbin{\hat{\otimes}} \mathscr{P}, B)$ for all Y, A, B. If Θ is invertible, then so is the Kasparov product with Θ that appears in PD; hence the map in (15) is invertible. Conversely, if the map in (15) is invertible, then the Kasparov product with Θ is invertible as a map $\mathrm{RKK}_{*-n}^G(X; \mathscr{P}, B) \to \mathrm{RKK}_*^G(X; \mathbb{C}, B)$ for all B. This implies invertibility of Θ.

If Θ is invertible, so is $\bar{\Theta} \in \mathrm{KK}_n^G(C_0(X), C_0(X) \mathbin{\hat{\otimes}} \mathscr{P})$. Therefore, the map in (16) is equivalent to one of the form (15); thus it is invertible as well. Its inverse is computed as follows. We have

$$p_X^*(f) = (\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D) \mathbin{\hat{\otimes}} f = \mathrm{PD}(D \mathbin{\hat{\otimes}} f) = \mathrm{PD}(\bar{\Theta}^{-1} \mathbin{\hat{\otimes}_{C_0(X)}} f).$$

for all $f \in \mathrm{KK}_*^G(C_0(X) \mathbin{\hat{\otimes}} A, B)$ because $1_{C_0(X)} \mathbin{\hat{\otimes}} D$ and $\bar{\Theta}$ are inverse to each other and $\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = 1_{\mathbb{C}}$. Hence

$$(p_X^*)^{-1}(f') = \bar{\Theta} \mathbin{\hat{\otimes}_{C_0(X)\hat{\otimes}\mathscr{P}}} \mathrm{PD}^{-1}(f') \tag{17}$$

for all $f' \in \mathrm{RKK}_*^G(X; C_0(X) \mathbin{\hat{\otimes}} A, B)$. In particular, $(p_X^*)^{-1}(\Delta_X) = \mathrm{Eul}_X$ as desired. □

Proposition 14 and Lemma 16 show that our preliminary definition of Eul_X in Section 3 is a special case of Definition 12.

The following discussion has the purpose of motivating the definition of a Kasparov dual by explaining how it is related to an abstract dual.

Define $\nabla \in \mathrm{KK}_n^G(\mathscr{P}, \mathscr{P} \mathbin{\hat{\otimes}} \mathscr{P})$ by

$$\mathrm{PD}(\nabla) = \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} \nabla = \Theta \mathbin{\hat{\otimes}_X} \Theta \in \mathrm{RKK}_{2n}^G(X; \mathbb{C}, \mathscr{P} \mathbin{\hat{\otimes}} \mathscr{P}). \tag{18}$$

Let Φ_P for a G-C^*-algebra P be the flip automorphism

$$\Phi_P \colon P \mathbin{\hat{\otimes}} P \to P \mathbin{\hat{\otimes}} P, \qquad x_1 \mathbin{\hat{\otimes}} x_2 \mapsto (-1)^{|x_1|\cdot|x_2|} x_2 \mathbin{\hat{\otimes}} x_1,$$

where $|x| \in \mathbb{Z}/2$ denotes the degree of x. Of course, this sign only occurs if P is $\mathbb{Z}/2$-graded. Recall also the class $D \in \mathrm{KK}_n^G(\mathscr{P}, \mathbb{C})$ defined in (14).

Lemma 17. *The maps D and ∇ satisfy*

$$\nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} \Phi_{\mathscr{P}} = (-1)^n \nabla, \tag{19}$$

$$(-1)^n \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (\nabla \mathbin{\hat{\otimes}} 1_{\mathscr{P}}) = \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (1_{\mathscr{P}} \mathbin{\hat{\otimes}} \nabla), \tag{20}$$

$$(-1)^n \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (D \mathbin{\hat{\otimes}} 1_{\mathscr{P}}) = 1_{\mathscr{P}} = \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (1_{\mathscr{P}} \mathbin{\hat{\otimes}} D). \tag{21}$$

Equation (19) *holds in* $\mathrm{KK}_n^G(\mathscr{P}, \mathscr{P} \mathbin{\hat{\otimes}} \mathscr{P})$*, equation* (20) *holds in* $\mathrm{KK}_{2n}^G(\mathscr{P}, \mathscr{P}^{\hat{\otimes}3})$*, and* (21) *holds in* $\mathrm{KK}_0^G(\mathscr{P}, \mathscr{P})$.

For $n = 0$, this means that $\mathscr{P}$ with comultiplication ∇ and counit D is a cocommutative, counital coalgebra object in KK^G.

Proof. It is well-known that the exterior product in $\mathrm{RKK}^G_*(X)$ is graded commutative. Especially, $(\Theta \hat{\otimes}_X \Theta) \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} \Phi_{\mathscr{P}} = (-1)^n \Theta \hat{\otimes}_X \Theta$. This is equivalent to (19) because PD is compatible with Kasparov products and bijective. One checks easily that PD maps both $(-1)^n \nabla \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} (\nabla \hat{\otimes} 1_{\mathscr{P}})$ and $\nabla \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} (1_{\mathscr{P}} \hat{\otimes} \nabla)$ to $\Theta \hat{\otimes}_X \Theta \hat{\otimes}_X \Theta$ in $\mathrm{RKK}^G_{3n}(X; \mathscr{P}, \mathscr{P}^{\hat{\otimes}3})$. This yields (20). Similarly,

$$\mathrm{PD}(1_{\mathscr{P}}) = (-1)^n \mathrm{PD}\big(\nabla \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} (D \hat{\otimes} 1_{\mathscr{P}})\big) = \mathrm{PD}\big(\nabla \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} (1_{\mathscr{P}} \hat{\otimes} D)\big) = \Theta$$

implies (21). □

Now we define a natural transformation

$$\sigma'_{X,\mathscr{P}} \colon \mathrm{RKK}^G_*(X \times Y; A, B) \to \mathrm{RKK}^G_*(Y; A \hat{\otimes} \mathscr{P}, B \hat{\otimes} \mathscr{P}) \tag{22}$$

by $\sigma'_{X,\mathscr{P}}(f) := \nabla \hat{\otimes}_{\mathscr{P}} \mathrm{PD}^{-1}(f)$, where $\hat{\otimes}_{\mathscr{P}}$ operates on the *second* copy of $\mathscr{P}$ in the target $\mathscr{P} \hat{\otimes} \mathscr{P}$ of ∇. We have

$$\begin{aligned}\mathrm{PD}\big(\sigma'_{X,\mathscr{P}}(f)\big) := \Theta \hat{\otimes}_{\mathscr{P}} \sigma'_{X,\mathscr{P}}(f) &= \Theta \hat{\otimes}_{\mathscr{P}} \nabla \hat{\otimes}_{\mathscr{P}} \mathrm{PD}^{-1}(f) \\ &= \Theta \hat{\otimes}_X \Theta \hat{\otimes}_{\mathscr{P}} \mathrm{PD}^{-1}(f) = \Theta \hat{\otimes}_X f\end{aligned} \tag{23}$$

in $\mathrm{RKK}^G_*(X \times Y; A, A' \hat{\otimes} \mathscr{P})$ for all $f \in \mathrm{RKK}^G_{*-n}(X \times Y; A, A')$. It follows from the graded commutativity of exterior products and Lemma 17 that

$$(-1)^{ni} \sigma'_{X,\mathscr{P}}(f) \hat{\otimes}_{\mathscr{P}} D = (-1)^n \nabla \hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}} D \hat{\otimes} \mathrm{PD}^{-1}(f) = \mathrm{PD}^{-1}(f) \tag{24}$$

for $f \in \mathrm{RKK}^G_i(X \times Y; A, B)$. This may seem useless for computing PD^{-1} because the definition of $\sigma'_{X,\mathscr{P}}$ itself involves PD^{-1}. The point of the notion of a Kasparov dual is that we require $\sigma'_{X,\mathscr{P}}$ to agree with another map that is easy to compute.

Recall that an $X \rtimes G$-C^**-algebra* is a G-C^*-algebra P equipped with a G-equivariant essential $*$-homomorphism from $C_0(X)$ into the centre of the multiplier algebra of P. This is equivalent to a G-equivariant essential $*$-homomorphism $m \colon C_0(X) \hat{\otimes} P \to P$, which we call the *$X$-structure map for P*. Given any $X \rtimes G$-C^*-algebra P, we get natural maps

$$\sigma_{X,P} \colon \mathrm{RKK}^G_*(X; A, B) \to \mathrm{KK}^G_*(P \hat{\otimes} A, P \hat{\otimes} B), \tag{25}$$

which send the class of a cycle $(\mathcal{E}, F)$ to $[(P \hat{\otimes}_{C_0(X)} \mathcal{E}, 1 \hat{\otimes}_{C_0(X)} F)]$ (see [25]). It is clear from the definition that

$$\sigma_{X,P}\big(p_X^*(f)\big) = 1_P \hat{\otimes} f \tag{26}$$

for all $f \in \mathrm{KK}^G_*(A, B)$ and all P.

Definition 18. Let X be a locally compact G-space. An (n-dimensional) *Kasparov dual* for X is a triple $(\mathscr{P}, D, \Theta)$ where $\mathscr{P}$ is a (possibly $\mathbb{Z}/2$-graded) $X \rtimes G$-C^*-algebra, $D \in \mathrm{KK}^G_{-n}(\mathscr{P}, \mathbb{C})$, and $\Theta \in \mathrm{RKK}^G_n(X; \mathbb{C}, \mathscr{P})$, such that

18.1. $\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = 1_{\mathbb{C}}$ in $\mathrm{RKK}^G_0(X; \mathbb{C}, \mathbb{C})$;
18.2. $\Theta \mathbin{\hat{\otimes}_X} f = \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} \sigma_{X,\mathscr{P}}(f)$ in $\mathrm{RKK}^G_*(X \times Y; A, B \mathbin{\hat{\otimes}} \mathscr{P})$ for all $f \in \mathrm{RKK}^G_{*-n}(X \times Y; A, B)$;
18.3. $\sigma_{X,\mathscr{P}}(\Theta) \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} \Phi_{\mathscr{P}} = (-1)^n \sigma_{X,\mathscr{P}}(\Theta)$ in $\mathrm{KK}^G_n(\mathscr{P}, \mathscr{P} \mathbin{\hat{\otimes}} \mathscr{P})$.

This definition is abstracted from the arguments in [25, Section 4].

Proposition 19. *A triple $(\mathscr{P}, D, \Theta)$ as above is a Kasparov dual for X if and only if $(\mathscr{P}, \Theta)$ is an abstract dual for X (Definition 10), $\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = 1_{\mathbb{C}}$, and the maps $\sigma'_{X,\mathscr{P}}$ and $\sigma_{X,\mathscr{P}}$ defined in* (22) *and* (25) *agree.*

If $(\mathscr{P}, D, \Theta)$ is a Kasparov dual, then $\nabla = \sigma_{X,\mathscr{P}}(\Theta)$, and the duality isomorphisms

$$\mathrm{PD} : \mathrm{KK}^G_{*-n}(\mathscr{P} \mathbin{\hat{\otimes}} A, B) \to \mathrm{RKK}^G_*(X; A, B),$$
$$\mathrm{PD}^{-1} : \mathrm{RKK}^G_i(X; A, B) \to \mathrm{KK}^G_{i-n}(\mathscr{P} \mathbin{\hat{\otimes}} A, B),$$

are given by

$$\mathrm{PD}(f) = \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} f \qquad \text{for } f \in \mathrm{KK}^G_{*-n}(\mathscr{P} \mathbin{\hat{\otimes}} A, B),$$
$$\mathrm{PD}^{-1}(f') = (-1)^{ni} \sigma_{X,\mathscr{P}}(f') \mathbin{\hat{\otimes}_{\mathscr{P}}} D \qquad \text{for } f' \in \mathrm{RKK}^G_i(X; A, B).$$

Proof. First we show that an abstract dual with the additional properties required in the proposition is a Kasparov dual. Condition 18.1 is clear and 18.2 follows from (23) and $\sigma'_{X,\mathscr{P}} = \sigma_{X,\mathscr{P}}$. The formula for PD is part of the definition of an abstract dual, the one for PD^{-1} follows from (24). Since $\mathrm{PD}^{-1}(\Theta) = 1_{\mathscr{P}}$, we have $\sigma_{X,\mathscr{P}}(\Theta) = \sigma'_{X,\mathscr{P}}(\Theta) = \nabla$. Therefore, 18.3 follows from Lemma 17.

Suppose conversely that we have a Kasparov dual. Define PD and PD^{-1} as in the statement of the proposition. We must check that they are inverse to each other. The composite $\mathrm{PD} \circ \mathrm{PD}^{-1}$ sends $f \in \mathrm{RKK}^G_i(X; A, B)$ to

$$(-1)^{ni} \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} \sigma_{X,\mathscr{P}}(f) \mathbin{\hat{\otimes}_{\mathscr{P}}} D = (-1)^{ni} \Theta \mathbin{\hat{\otimes}_X} f \mathbin{\hat{\otimes}_{\mathscr{P}}} D = f \mathbin{\hat{\otimes}_X} \Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} D = f$$

as desired. Let $\nabla := \sigma_{X,\mathscr{P}}(\Theta) \in \mathrm{RKK}^G_n(\mathscr{P}, \mathscr{P} \mathbin{\hat{\otimes}} \mathscr{P})$. It follows from 18.1 that

$$\nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (\mathrm{id}_{\mathscr{P}} \mathbin{\hat{\otimes}} D) = \sigma_{X,\mathscr{P}}(\Theta \otimes_{\mathscr{P}} D) = \sigma_{X,\mathscr{P}}(1_X) = 1_{\mathscr{P}}.$$

Using 18.3, we also get $\nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (D \mathbin{\hat{\otimes}} \mathrm{id}_{\mathscr{P}}) = (-1)^n 1_{\mathscr{P}}$. Therefore, the composite $\mathrm{PD}^{-1} \circ \mathrm{PD}$ sends $f \in \mathrm{KK}^G_{i-n}(\mathscr{P} \mathbin{\hat{\otimes}} A, B)$ to

$$\begin{aligned}(-1)^{ni} \sigma_{X,\mathscr{P}}(\Theta \mathbin{\hat{\otimes}_{\mathscr{P}}} f) \mathbin{\hat{\otimes}_{\mathscr{P}}} D &= (-1)^{ni} \sigma_{X,\mathscr{P}}(\Theta) \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (f \mathbin{\hat{\otimes}} D)\\ &= (-1)^n \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (D \mathbin{\hat{\otimes}} f) = (-1)^n \nabla \mathbin{\hat{\otimes}_{\mathscr{P}\hat{\otimes}\mathscr{P}}} (D \mathbin{\hat{\otimes}} \mathrm{id}_{\mathscr{P}}) \mathbin{\hat{\otimes}_{\mathscr{P}}} f = f.\end{aligned}$$

In the second step we use graded commutativity of exterior products. Thus $(\mathscr{P}, \Theta)$ is an abstract dual for X. Condition 18.2 requires $\mathrm{PD}\big(\sigma_{X,\mathscr{P}}(f)\big) = \Theta \mathbin{\hat\otimes}_X f$ for all $f \in \mathrm{RKK}^G_*(X; A, B)$. The same equation holds for $\sigma'_{X,\mathscr{P}}$ by (23). We get $\sigma'_{X,\mathscr{P}} = \sigma_{X,\mathscr{P}}$ because PD is bijective. □

If we have a Kasparov dual for X, then we can improve the definition of the abstract Euler characteristic:

Lemma 20. *Let X be a locally compact G-space that admits a Kasparov dual $(\mathscr{P}, D, \Theta)$ and let $\bar\Theta \in \mathrm{KK}^G_n(C_0(X), C_0(X) \mathbin{\hat\otimes} \mathscr{P})$ be the image of Θ under the functor that forgets the X-structure. Let $m\colon C_0(X) \mathbin{\hat\otimes} \mathscr{P} \to \mathscr{P}$ be the X-structure map for $\mathscr{P}$. Then*

$$\mathrm{Eul}_X = \bar\Theta \mathbin{\hat\otimes}_{C_0(X)\hat\otimes\mathscr{P}} [m] \mathbin{\hat\otimes}_{\mathscr{P}} D.$$

Proof. We have

$$\mathrm{Eul}_X := \bar\Theta \mathbin{\hat\otimes}_{C_0(X)\hat\otimes\mathscr{P}} \mathrm{PD}^{-1}(\Delta_X) = \bar\Theta \mathbin{\hat\otimes}_{C_0(X)\hat\otimes\mathscr{P}} \sigma_{X,\mathscr{P}}(\Delta_X) \mathbin{\hat\otimes}_{\mathscr{P}} D.$$

It remains to check that $\sigma_{X,\mathscr{P}}(\Delta_X) = [m]$. We have $\Delta_X(f_1 \mathbin{\hat\otimes} f_2) = f_1 \cdot f_2$ for all $f_1, f_2 \in C_0(X)$. Hence the homomorphism $\sigma_{X,\mathscr{P}}(\Delta_X)\colon C_0(X) \mathbin{\hat\otimes} \mathscr{P} \to \mathscr{P}$ maps $f_1 \mathbin{\hat\otimes} f_2 \mapsto f_1 \cdot f_2$ for all $f_1 \in C_0(X)$, $f_2 \in \mathscr{P}$. This is the definition of the homomorphism m. □

It is useful for proofs to know that Definition 18.2 can be weakened as follows:

Lemma 21. *Let $\mathscr{P}$ be an $X \rtimes G$-C^*-algebra and let $\Theta \in \mathrm{RKK}^G_n(X; \mathbb{C}, \mathscr{P})$. Suppose that the formula $\Theta \mathbin{\hat\otimes}_X f = \Theta \mathbin{\hat\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)$ required in Definition 18.2 holds whenever f is the class of an $(X \times Y) \rtimes G$-equivariant $*$-homomorphism $C_0(X \times Y, A) \to C_0(X \times Y, B)$. Then 18.2 holds in complete generality.*

Proof. If $f = p_X^*(f')$, then (26) implies $\Theta \mathbin{\hat\otimes}_X f = \Theta \mathbin{\hat\otimes} f' = \Theta \mathbin{\hat\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)$. Similarly, 18.2 holds for $p_X^*(f_0) \mathbin{\hat\otimes}_{X,A} f \mathbin{\hat\otimes}_{X,B} p_X^*(f_1)$ with $f_0 \in \mathrm{KK}^G_*(A_1, A)$, $f_1 \in \mathrm{KK}^G_*(B, B_1)$ once it holds for f. Therefore, we are done if we show that every class in $\mathrm{RKK}^G_*(X \times Y; A, B)$ can be written as a product of this kind, where f is the class of an $(X \times Y) \rtimes G$-equivariant $*$-homomorphism. We deduce this from considerations related to the universal property of KK^G.

Using [36, Proposition 5.4], we identify $\mathrm{KK}^G_0(A, B)$ with the set of homotopy classes of G-equivariant $*$-homomorphisms

$$\chi\big(A \mathbin{\hat\otimes} \mathbb{K}(L^2G)\big) \to B \mathbin{\hat\otimes} \mathbb{K}(L^2(G \times \mathbb{N}) \oplus L^2(G \times N)^{\mathrm{op}});$$

here $\chi(A)$ is a certain universal algebra due to Ulrich Haag. The identification sends a homomorphism f to the Kasparov product $f_0 \mathbin{\hat\otimes}_{\chi(\dots)} f \mathbin{\hat\otimes}_{B\hat\otimes\mathbb{K}(\dots)} f_1$ for certain natural elements $f_0 \in \mathrm{KK}^G_0\big(A, \chi(\dots)\big)$, $f_1 \in \mathrm{KK}^G_0(B \mathbin{\hat\otimes} \mathbb{K}(\dots), B)$.

The same reasoning shows that the map

$$f \mapsto p_X^*(f_0) \mathbin{\hat{\otimes}}_{\chi(\dots)} f \mathbin{\hat{\otimes}}_{B \hat{\otimes} \mathbb{K}(\dots)} p_X^*(f_1)$$

identifies $\mathrm{RKK}_0^G(X \times Y; A, B)$ with the set of homotopy classes of $(X \times Y) \rtimes G$-equivariant $*$-homomorphisms

$$C_0(X \times Y) \mathbin{\hat{\otimes}} \chi\bigl(A \mathbin{\hat{\otimes}} \mathbb{K}(L^2G)\bigr) \\ \to C_0(X \times Y) \mathbin{\hat{\otimes}} B \mathbin{\hat{\otimes}} \mathbb{K}(L^2(G \times \mathbb{N}) \oplus L^2(G \times N)^{\mathrm{op}}).$$

Hence every element of $\mathrm{RKK}_0^G(X \times Y; A, B)$ can be decomposed in the required form. Putting in some more Clifford algebras, we get the same assertion for elements of $\mathrm{RKK}_*^G(X \times Y; A, B)$. □

5. The combinatorial Euler characteristic

Let X be a countable, locally finite, simplicial complex, equipped with a simplicial, continuous action of a locally compact group G. We are going to define the combinatorial Euler characteristic for such X. Although we only write down definitions for complex C^*-algebras, it is evident that everything we do here works in the real case as well.

We define simplicial complexes as in, say, [8, I.Appendix]. However, we do not consider the empty set as a simplex. The geometric realisation of X is a second countable, locally compact space because X is locally finite and countable. Since we want to denote the geometric realisation by X as well, it is convenient to write SX for the set of (non-empty) simplices of X. For each simplex $\sigma \in SX$, we write $|\sigma|$ for the corresponding subset of X, and we let $\xi_\sigma \in |\sigma|$ be its barycentre. The resulting map

$$\xi : SX \to X, \qquad \sigma \mapsto \xi_\sigma \tag{27}$$

identifies SX with a discrete G-invariant subset of X. We give SX the discrete topology and the induced action of G, which is of course continuous. Equivalently, the stabiliser G_σ of ξ_σ is open for all $\sigma \in SX$.

We require G_σ to act trivially on $|\sigma|$. This is crucial to get a correct formula for the Euler characteristic. However, this assumption involves no loss of generality because we may, if necessary, replace X by its barycentric subdivision, which clearly satisfies this condition.

We decompose SX into the subsets $S_\pm X$ of simplices of even and odd dimension. Let $\ell^2(S_\pm X)$ be the $\mathbb{Z}/2$-graded Hilbert space with orthonormal basis SX and even and odd subspaces $\ell^2(S_+X)$ and $\ell^2(S_-X)$, respectively. In Section 6, we will mostly be using the Hilbert space $\ell^2(SX)$ with trivial grading. We write $\ell^2\bigl(S_\pm X\bigr)$ here to emphasise the non-trivial grading. Representing $C_0(SX)$ by diagonal operators on $\ell^2\bigl(S_\pm X\bigr)$, we get a natural injective $*$-homomorphism $C_0(SX) \to$

$\mathbb{K}\big(\ell^2\big(S_\pm X\big)\big)$, which is G-equivariant and by operators of even parity. Moreover, the map (27) induces a G-equivariant $*$-homomorphism $\xi^*\colon C_0(X) \to C_0(SX)$.

Definition 22. Let $\mathrm{Eul}_X^{\mathrm{cmb}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$ be the class of the G-equivariant $*$-homomorphism

$$C_0(X) \to C_0(SX) \to \mathbb{K}\big(\ell^2\big(S_\pm X\big)\big)$$

described above. We call this the *combinatorial G-equivariant Euler characteristic of X.*

We now describe $\mathrm{Eul}_X^{\mathrm{cmb}}$ more explicitly. For a subgroup $H \subseteq G$, we let $X^H \subseteq X$ be the fixed-point subset. For each connected component A of X^H, pick a point $x \in A$ and let $\dim_{H,A} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$ be the class of the homomorphism $C_0(X) \to C_0(G/H) \subseteq \mathbb{K}\big(\ell^2(G/H)\big)$ that sends $f \in C_0(X)$ to the operator of multiplication by the function $gH \mapsto f(gx)$. This does not depend on the choice of x by homotopy invariance. Moreover, we have $\dim_{gHg^{-1},gA} = \dim_{H,A}$ for all $g \in G$ because the resulting Kasparov cycles are unitarily equivalent. In particular, $\dim_{H,gA} = \dim_{H,A}$ if g belongs to the normaliser $N(H)$ of H. Thus we may replace the connected components of X^H by the connected components of $N(H)\backslash X^H$.

Given an open subgroup $H \subseteq G$ and a connected component A of $N(H)\backslash X^H$, we let $S(H, A) \subseteq SX$ be the set of all simplices of A whose stabiliser is *exactly* equal to H. Let $\mathrm{Eul}_{X,H,A}^{\mathrm{cmb}}$ be the class in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$ of the map

$$C_0(X) \xrightarrow{\xi^*} C_0(SX) \to \mathbb{K}\Big(\ell^2\big(G \cdot S_\pm(H, A)\big)\Big),$$

where the second map is the representation by diagonal operators as above. Then

$$\mathrm{Eul}_X^{\mathrm{cmb}} = \sum_{(H),A} \mathrm{Eul}_{X,H,A}^{\mathrm{cmb}},$$

where (H) runs through the set of conjugacy classes of those open subgroups of G that occur as stabilisers of simplices in X, and, for each of these, A runs through the set of connected components of $N(H)\backslash X^H$.

Suppose first that $N(H)\backslash S(H, A)$ is finite. Let $\chi(X, H, A) \in \mathbb{Z}$ be the alternating sum of the numbers of n-simplices in $N(H)\backslash S(H, A)$. Then

$$\mathrm{Eul}_{X,H,A}^{\mathrm{cmb}} = \chi(X, H, A) \cdot \dim_{H,A}\,.$$

If $S(H, A)$ is infinite, we let $\chi(X, H, A) := 0$ and claim that $\mathrm{Eul}_{X,H,A}^{\mathrm{cmb}} = 0$ and $\dim_{H,A} = 0$. This is because there is a continuous path $(x_t)_{t\in\mathbb{R}_+}$ in A such that $\lim_{t\to\infty} \|f|_{Gx_t}\|_\infty = 0$ for all $f \in C_0(X)$. Thus

$$\mathrm{Eul}_X^{\mathrm{cmb}} = \sum_{(H),A} \chi(X, H, A) \cdot \dim_{H,A}, \tag{28}$$

where the summation runs over the same data (H), A as above. If this sum is infinite, we have to add the cycles, not just their classes. The summation in (28) is finite if and only if all fixed-point subspaces X^H have finitely many connected components and there are, up to conjugacy, only finitely many different subgroups of G that occur as the stabiliser of a simplex in X.

We are mainly interested in the case where X is strongly contractible. Then all fixed-point subsets X^H are contractible and *a fortiori* connected. Hence we may write $\dim_H$ and $\chi(X, H)$ instead of $\dim_{H,A}$ and $\chi(X, H, A)$.

Example 23. Consider $G := \mathrm{PSL}(2, \mathbb{Z})$. The free product decomposition $G \cong \mathbb{Z}/2 * \mathbb{Z}/3$ gives rise to a tree X on which G acts in such a way that the fundamental domain is an edge with stabilisers $\mathbb{Z}/2$ and $\mathbb{Z}/3$ at the end points and $\{1\}$ in the interior (see [54, §I.4, Theorem 7]). The action of G on SX has only three orbits in this case, two orbits on vertices and one on edges. We find

$$\mathrm{Eul}_X^{\mathrm{cmb}} = \dim_{\mathbb{Z}/2} + \dim_{\mathbb{Z}/3} - \dim_{\{1\}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C}).$$

Example 24. The case where G is discrete and acts freely on X is particularly simple. Then we have natural isomorphisms

$$\mathrm{KK}_0^G(C_0(X), \mathbb{C}) \cong \mathrm{KK}_0(C_0(X) \rtimes_{\mathrm{r}} G, \mathbb{C}) \cong \mathrm{KK}_0(C_0(G\backslash X), \mathbb{C}). \tag{29}$$

They map $\mathrm{Eul}_X^{\mathrm{cmb}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$ to $\mathrm{Eul}_{G\backslash X}^{\mathrm{cmb}} \in \mathrm{KK}_0(C_0(G\backslash X), \mathbb{C})$. We have

$$\mathrm{Eul}_{G\backslash X}^{\mathrm{cmb}} = \sum_{A \in \pi_0(G\backslash X)} \chi(A) \cdot \dim_A,$$

where $\chi(A)$ is the usual Euler characteristic of $A \subseteq G\backslash X$ and $\dim_A$ is the class in $\mathrm{KK}_0(C_0(G\backslash X), \mathbb{C})$ of the homomorphism $C_0(G\backslash X) \to \mathbb{C}$, $f \mapsto f(x)$, for any $x \in A$. If $G\backslash X$ is connected, we get $\mathrm{Eul}_{G\backslash X}^{\mathrm{cmb}} = \chi(G\backslash X) \cdot \dim$.

In [34], the relationship between various topological constructions of Euler characteristics is discussed. Here we consider another construction from representation theory that is related to the Euler characteristic of a group.

We assume that G is a totally disconnected locally compact group for which there is a G-compact universal proper G-space X. This holds, for instance, for reductive p-adic groups or hyperbolic groups, where we may take the affine Bruhat-Tits building or the Rips complex, respectively. We may then choose X to be a G-finite simplicial complex with simplicial action of G. As in Definition 15, we identify $\mathrm{KK}_*^G(C_0(X), \mathbb{C}) \cong \mathrm{K}_*^{\mathrm{top}}(G)$, so that we can view the combinatorial Euler characteristic of X as an element $\mathrm{Eul}_X^{\mathrm{cmb}} \in \mathrm{K}_0^{\mathrm{top}}(G)$. This class is independent of the choice of X; we omit the verification because our main theorem (Theorem 30) yields in any case that $\mathrm{Eul}_X^{\mathrm{cmb}} = \mathrm{Eul}_{\mathscr{E}} G$, where $\mathrm{Eul}_{\mathscr{E}} G$ is as in Definition 15.

We assume now that G satisfies the Baum-Connes conjecture, so that we lose nothing by mapping $\mathrm{Eul}_X \in \mathrm{K}_0^{\mathrm{top}}(G)$ to $\mathrm{K}_0(C_{\mathrm{r}}^* G)$. This class can be described as follows:

$$\mu_G(\mathrm{Eul}_X^{\mathrm{cmb}}) = \sum_{\sigma \in G \backslash SX} (-1)^{|\sigma|} [\tau(G_\sigma)] \in \mathrm{K}_0(C_{\mathrm{r}}^* G),$$

where $\tau(G_\sigma) \in C_{\mathrm{r}}^*(G_\sigma) \subseteq C_{\mathrm{r}}^*(G)$ is the projection onto the trivial representation of the compact-open subgroup G_σ. As a projection in the reduced C^*-algebra, $\tau(G)$ is given by $\mathrm{Vol}(G_\sigma)^{-1} \cdot 1_{G_\sigma}$, where 1_{G_σ} is the characteristic function of G_σ.

The class $\mu_G(\mathrm{Eul}_X^{\mathrm{cmb}}) \in \mathrm{K}_0(C_{\mathrm{r}}^* G)$ is related to the representation theory of G. For discrete groups, this idea goes back to Hyman Bass ([3]).

Recall that the *Hecke algebra* of G is the space of locally constant, compactly supported functions $G \to \mathbb{C}$ with the convolution product. If G is discrete, this is nothing but the group algebra of G. The projections $\tau(G_\sigma)$ all lie in $\mathscr{H}(G)$, so that their alternating sum actually lies in $\mathrm{K}_0^{\mathrm{alg}}\bigl(\mathscr{H}(G)\bigr)$.

Let $\mathsf{R}(G)$ be the category of smooth representations of G (always on complex vector spaces). We say that a smooth representation of G has *type (FP)* if it has a finite length resolution (P_n, δ_n) by finitely generated projective objects of $\mathsf{R}(G)$. This resolution is unique up to chain homotopy equivalence. We have $P_n \cong \mathscr{H}(G)^{k_n} \cdot p_n$ for certain projections $p_n \in M_{k_n}\bigl(\mathscr{H}(G)\bigr)$, which yield classes in $\mathrm{K}_0^{\mathrm{alg}}\bigl(\mathscr{H}(G)\bigr)$. Let

$$\chi(M) := \sum_{n=0}^{\infty} (-1)^n [p_n] \in \mathrm{K}_0^{\mathrm{alg}}\bigl(\mathscr{H}(G)\bigr).$$

This is well-defined, that is, independent of the choices of the resolution (P_n, δ_n) and the projections p_n.

We may consider the cellular chain complex of X with coefficients $\mathbb{C}$ as a chain complex of smooth representations of G. Its homology vanishes for $* > 0$ and is $\mathbb{C}$ with the trivial representation of G for $* = 0$. Since X is G-compact, $\mathbb{C}[SX]$ is a *finite* direct sum of $\mathscr{H}(G)$-modules of the form $\mathbb{C}[G/H]$ for certain compact-open subgroups $H \subseteq G$. The latter are finitely generated projective objects of $\mathsf{R}(G)$ because $\mathbb{C}[G/H] \cong \mathscr{H}(G) \cdot \tau(H)$ for all compact-open subgroups $H \subseteq G$. Thus the trivial representation of G has type (FP) and the natural map $\mathrm{K}_0^{\mathrm{alg}}\bigl(\mathscr{H}(G)\bigr) \to \mathrm{K}_0(C_{\mathrm{r}}^* G)$ maps $\chi(\mathbb{C})$ to $\mu_G(\mathrm{Eul}_{\mathscr{E}} G)$.

Representation theorists usually replace $\chi(\mathbb{C})$ by its Chern character, which belongs to $\mathrm{HH}_0\bigl(\mathscr{H}(G)\bigr) := \mathscr{H}(G)/[\mathscr{H}(G), \mathscr{H}(G)]$; it is represented by the function

$$\sum_{\sigma \in G \backslash SX} (-1)^{|\sigma|} \tau(G_\sigma) \in \mathscr{H}(G).$$

If G is a semi-simple p-adic group, another representative for the same class is the *Euler-Poincaré function* of Robert Kottwitz ([31, Section 2]); it is computed from the cellular chain complex of the affine Bruhat-Tits building with its natural polysimplicial structure. Recall that we refine this to a simplicial structure with the additional property that G_σ fixes the simplex σ pointwise. Both chain complexes produce the same class in $\mathrm{HH}_0\big(\mathscr{H}(G)\big)$, even in $\mathrm{K}_0^{\mathrm{alg}}\big(\mathscr{H}(G)\big)$, because they both come from finite projective resolutions of the trivial representation of G.

Peter Schneider and Ulrich Stuhler construct analogous Euler-Poincaré functions for general irreducible representations of semi-simple p-adic groups in [51]; this construction can be modified easily to produce elements of $\mathrm{K}_0^{\mathrm{alg}}\big(\mathscr{H}(G)\big)$, see [37]. Although the Borel-Serre compactification plays an important role in [51], it is not clear to us whether these more general Euler characteristics of irreducible representations are related to Kasparov duals or boundary actions.

6. A Kasparov dual for simplicial complexes

In this section, we assume X to be a *finite-dimensional*, locally finite, countable simplicial complex equipped with a simplicial, continuous action of a locally compact group G. We do not require the action to be proper.

Our main goal is to exhibit the combinatorial Euler characteristic as an abstract Euler characteristic. A Kasparov dual for X has already been constructed by Gennadi Kasparov and Georges Skandalis in [26]. However, they only describe Θ indirectly, which makes it hard to compute Eul_X. Therefore, we give an independent and completely explicit construction for Θ. We also modify their definitions of $\mathscr{P}$ and D slightly to get a simple formula for Θ.

We need some preparations before we can start the actual construction. As in Section 5, SX denotes the set of simplices of X, and we usually write X both for the simplicial complex and its geometric realisation. Let $S_0X \subseteq SX$ be the set of vertices, that is, 0-simplices of X. Suppose that X is at most n-dimensional and let

$$\underline{\mathbf{n}} := \{0, 1, \dots, n\}.$$

A *colouring* on X is a function $\gamma\colon S_0X \to \underline{\mathbf{n}}$ such that for any simplex $\sigma \in SX$, the images under γ of the vertices of σ are pairwise distinct. A *coloured simplicial complex* is a simplicial complex equipped with such a colouring. The action of G is compatible with the colouring if the function γ is G-invariant. (Coloured simplicial complexes are called *typed* in [26].) Most of our constructions only involve a single simplex in X at a time. The colouring allows us to piece these local constructions together.

Let X be any n-dimensional simplicial complex and let $X^{(1)}$ be its barycentric subdivision. Recall that the vertex set of $X^{(1)}$ is equal to the set of simplices of X; the simplices in $X^{(1)}$ are labelled bijectively by strictly increasing chains in the partially ordered set of simplices of X; here the partial order is defined by $\sigma \le \sigma'$ if σ is a face of σ'. The map $S_0X^{(1)} = SX \to \underline{\mathbf{n}}$ that sends a simplex to its dimension is a canonical colouring on $X^{(1)}$. Thus it is no loss of generality to assume X to carry a G-invariant colouring; we assume this in the following.

We shall use the affine Euclidean space

$$E := \Big\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\Big|\; \sum t_i = 1\Big\}. \tag{30}$$

Sometimes, we specify a point in E by *homogeneous coordinates*:

$$[t_0, \dots, t_n] := \Big(\sum t_i\Big)^{-1}(t_0, \dots, t_n) \tag{31}$$

provided $\sum t_i \neq 0$. We realise the standard n-simplex as the subset

$$\Sigma := \{(t_0, \dots, t_n) \in E \mid t_i \geq 0 \quad \text{for all } i \in \underline{\mathbf{n}}\}. \tag{32}$$

Let $\mathscr{S}(\underline{\mathbf{n}})$ be the partially ordered set of *non-empty* subsets of $\underline{\mathbf{n}}$. We extend the colouring γ to a map $SX \to \mathscr{S}(\underline{\mathbf{n}})$ by sending a simplex to the set of colours of its vertices. We also define $\gamma(\emptyset) := \emptyset$. We identify $\underline{\mathbf{n}}$ with the set of vertices of Σ. Since a face of Σ is determined by the set of vertices it contains, this identifies $\mathscr{S}(\underline{\mathbf{n}})$ with the partially ordered set of faces of Σ. Under this identification, $f \subseteq \underline{\mathbf{n}}$ corresponds to the face

$$\begin{aligned} |f| &:= \{(t_0, \dots, t_n) \in \Sigma \mid t_i = 0 \quad \text{for } i \in \underline{\mathbf{n}} \setminus f\} \\ &= \{(t_0, \dots, t_n) \in E \mid t_i \geq 0 \quad \text{for } i \in f \text{ and } t_i = 0 \text{ for } i \in \underline{\mathbf{n}} \setminus f\}. \end{aligned} \tag{33}$$

We may view the map $\gamma \colon SX \to \mathscr{S}(\underline{\mathbf{n}})$ as a G-invariant simplicial map; passing to geometric realisations, we get a G-invariant continuous map

$$|\gamma| \colon X \to \Sigma. \tag{34}$$

Any point $x \in X$ belongs to some simplex $\sigma \in SX$. The restriction of $|\gamma|$ to $|\sigma|$ is the unique affine map that sends a vertex of colour i to the corresponding vertex of Σ. If σ is of dimension k, then $\gamma(\sigma) \subseteq \underline{\mathbf{n}}$ has $k+1$ elements and hence defines a k-dimensional face of Σ. Hence the restriction of $|\gamma|$ to $|\sigma|$ is a *homeomorphism* from $|\sigma|$ to the face $|\gamma(\sigma)|$ of Σ.

For any $f \subseteq \underline{\mathbf{n}}$, we define a closed convex subset $R_f \subseteq E$ by

$$R_f := \{(t_0, \dots, t_n) \in E \mid t_i \geq 0 \quad \text{for } i \in f \text{ and } t_i \leq 0 \text{ for } i \in \underline{\mathbf{n}} \setminus f\}. \tag{35}$$

These regions are a crucial ingredient of our construction. Figure 1 illustrates them for $n = 2$. We have $R_{\underline{\mathbf{n}}} = \Sigma$ and $R_\emptyset = \emptyset$ because no point of E satisfies $t_i < 0$ for all $i \in \underline{\mathbf{n}}$. The sets R_f for $f \in \mathscr{S}(\underline{\mathbf{n}})$ cover E and have mutually disjoint interiors. We also define $R_S := \bigcup_{f \in S} R_f$ if $S \subseteq \mathscr{S}(\underline{\mathbf{n}})$ is a set of faces of Σ. We are mainly interested in

$$R_{\leq f} := \bigcup_{\{l \in \mathscr{S}(\underline{\mathbf{n}}) \mid l \leq f\}} R_l = \{(t_0, \dots, t_n) \in E \mid t_i \leq 0 \quad \text{for } i \in \underline{\mathbf{n}} \setminus f\} \tag{36}$$

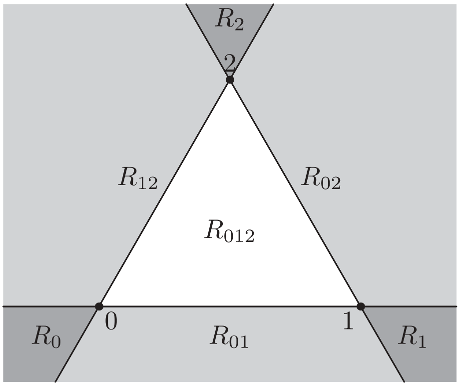

Fig. 1. The regions R_f

for $f \subseteq \underline{\mathbf{n}}$. It follows immediately that

$$R_{\leq f_1} \cap R_{\leq f_2} = R_{\leq (f_1 \cap f_2)} \tag{37}$$

for all $f_1, f_2 \in \mathscr{S}(\underline{\mathbf{n}})$.

We define a retraction $q\colon E \to \Sigma$ by

$$q(t_0, \dots, t_n) := [\max(t_0, 0), \dots, \max(t_n, 0)]. \tag{38}$$

Inspection of (33) and (36) yields

$$q^{-1}(|f|) = R_{\leq f}. \tag{39}$$

Kasparov and Skandalis use the nearest point retraction to Σ instead of q in [26]. We prefer q because of the more explicit formula.

We can now define the underlying C^*-algebra $\mathscr{P}$ of our Kasparov dual. We use the C^*-algebra of compact operators on $\ell^2(SX)$. The group G acts on this Hilbert space via its action on the basis SX. We equip $\ell^2(SX)$ with the trivial grading, as opposed to the grading by parity that we used in Section 5. We describe an operator T on $\ell^2(SX)$ by a matrix $(T_{\sigma\sigma'})_{\sigma,\sigma'\in SX}$. For a function $\varphi\colon Y \to \mathbb{B}\big(\ell^2(SX)\big)$, its matrix coefficients are functions $\varphi_{\sigma\sigma'}\colon Y \to \mathbb{C}$, defined by $\varphi_{\sigma\sigma'}(y) := \varphi(y)_{\sigma\sigma'}$ for $y \in Y$. For the remainder of this section, we abbreviate

$$\mathbb{K} := \mathbb{K}\big(\ell^2(SX)\big).$$

Let

$$\mathscr{P} := \{\varphi \in C_0(E, \mathbb{K}) \mid \operatorname{supp} \varphi_{\sigma\sigma'} \subseteq R_{\leq \gamma(\sigma\cap\sigma')} \quad \text{for all } \sigma, \sigma' \in SX\}. \tag{40}$$

Hence $\varphi_{\sigma\sigma'} = 0$ unless σ and σ' have a common face. We let G act on $C_0(E, \mathbb{K})$ by $g\varphi(t) := \pi_g \circ \varphi(t) \circ \pi_g^{-1}$ for all $g \in G$, $t \in E$ where π_g comes from the action of G on $\ell^2(SX)$. Obviously, $\mathscr{P}$ is a closed, self-adjoint, G-invariant subspace of

$C_0(E,\mathbb{K})$. We have to check that $\mathscr{P}$ is closed under multiplication. If $\varphi,\psi\in\mathscr{P}$, $\sigma,\sigma'\in SX$, then we have $(\varphi\cdot\psi)_{\sigma\sigma'}=\sum_{\tau\in SX}\varphi_{\sigma\tau}\psi_{\tau\sigma'}$. Using (37) and that the colouring is injective on the vertices of τ, we get

$$\operatorname{supp}\varphi_{\sigma\tau}\psi_{\tau\sigma'}\subseteq R_{\leq\gamma(\sigma\cap\tau)}\cap R_{\leq\gamma(\sigma'\cap\tau)}=R_{\leq\gamma(\sigma\cap\tau\cap\sigma')}\subseteq R_{\leq\gamma(\sigma\cap\sigma')}.$$

Hence each individual summand $\varphi_{\sigma\tau}\psi_{\tau\sigma'}$ satisfies the support condition (40). Thus $\mathscr{P}$ is a G-invariant C^*-subalgebra of $C_0(E,\mathbb{K})$.

We may interpret the algebra $\mathscr{P}$ physically as follows. The simplices σ are possible states of a system. For $t\in E$, the system may only be in the state σ if $t_i<0$ for all $i\in\underline{\mathbf{n}}\setminus\gamma(\sigma)$; two such states σ,σ' may interact if $t_i<0$ for all $i\in\underline{\mathbf{n}}\setminus\gamma(\sigma\cap\sigma')$.

Next we define the X-structure map $m\colon C_0(X)\otimes\mathscr{P}\to\mathscr{P}$. Recall that the map $|\gamma|$ defined in (34) restricts to a homeomorphism $|\gamma|_\sigma\colon|\sigma|\to|\gamma(\sigma)|$ for each $\sigma\in SX$ and that $q(R_{\leq f})\subseteq|f|$ by (39). Hence we may define a continuous G-equivariant map

$$E\times SX\supseteq\bigcup_{\sigma\in SX}R_{\leq\gamma(\sigma)}\times\{\sigma\}\xrightarrow{\bar q}X,\qquad \bar q(t,\sigma):=|\gamma|_\sigma^{-1}\big(q(t)\big).\tag{41}$$

We extend $\bar q$ to all of $E\times SX$ by $\bar q(t,\sigma):=|\gamma|_\sigma^{-1}\circ a_{\gamma(\sigma)}\circ q(t)$, where we choose simplicial retractions $a_f\colon\Sigma\to|f|$ for $f\in\mathscr{S}(\underline{\mathbf{n}})$. By construction, $\bar q(t,\sigma)\in|\sigma|$ for all $t\in E$, $\sigma\in SX$. With this extension of $\bar q$ we get a G-equivariant essential $*$-homomorphism

$$m'\colon C_0(X)\xrightarrow{\bar q^*}C_b(E\times SX)\to\mathcal{M}\big(C_0(E,\mathbb{K})\big),$$

where the second map is the representation by diagonal operators on $\ell^2(SX)$.

Lemma 25. *We have $m'(\varphi_1)\circ\varphi_2=\varphi_2\circ m'(\varphi_1)\in\mathscr{P}$ for all $\varphi_1\in C_0(X)$, $\varphi_2\in\mathscr{P}$.*

Proof. It follows from the definitions that

$$(m'(\varphi_1)\circ\varphi_2)_{\sigma\sigma'}(t)=\varphi_1\big(\bar q(t,\sigma)\big)\cdot\varphi_2(t)_{\sigma\sigma'},$$
$$\big(\varphi_2\circ m'(\varphi_1)\big)_{\sigma\sigma'}(t)=\varphi_1\big(\bar q(t,\sigma')\big)\cdot\varphi_2(t)_{\sigma\sigma'}$$

for all $\sigma,\sigma'\in SX$, $t\in E$. The function $(\varphi_2)_{\sigma\sigma'}$ is supported in the region $R_{\leq\gamma(\sigma\cap\sigma')}$ because $\varphi_2\in\mathscr{P}$, see (40). Therefore, $(m'(\varphi_1)\varphi_2)_{\sigma\sigma'}$ and $\big(\varphi_2 m'(\varphi_1)\big)_{\sigma\sigma'}$ are supported in this region as well, so that $m'(\varphi_1)\varphi_2\in\mathscr{P}$ and $\varphi_2 m'(\varphi_1)\in\mathscr{P}$. It remains to check that the two matrix coefficients above agree. This follows if $\bar q(t,\sigma)=\bar q(t,\sigma')$ for all $t\in R_{\leq\gamma(\sigma\cap\sigma')}$ because both matrix coefficients are supported in this region.

If $t\in R_{\leq\gamma(\sigma\cap\sigma')}$, then $t\in R_{\leq\gamma(\sigma)}$ and $t\in R_{\leq\gamma(\sigma')}$ by (37). Thus $\bar q(t,\sigma)$ and $\bar q(t,\sigma')$ are both defined by (41). We have $q(t)\in|\gamma(\sigma\cap\sigma')|$ by (39). Both $|\gamma|_\sigma$ and $|\gamma|_{\sigma'}$ extend $|\gamma|_{\sigma\cap\sigma'}$, which is a homeomorphism onto the face $|\gamma(\sigma\cap\sigma')|$. Therefore, $\bar q(t,\sigma)=\bar q(t,\sigma\cap\sigma')=\bar q(t,\sigma')$ as desired. □

Hence there is a unique $*$-homomorphism $m\colon C_0(X)\otimes\mathscr{P}\to\mathscr{P}$ with $m(\varphi_1\otimes\varphi_2):=m'(\varphi_1)\circ\varphi_2$; since $C_0(X)$ is nuclear, it does not matter which tensor product we choose here. The map m is G-equivariant and essential, so that $\mathscr{P}$ becomes an $X\rtimes G$-C^*-algebra. If we view $\varphi\in C_0(X)\otimes\mathscr{P}$ as a function $X\times E\to\mathbb{K}$, we can describe m explicitly in terms of matrix coefficients:

$$m(\varphi)_{\sigma\sigma'}(t)=\varphi_{\sigma\sigma'}(\bar{q}(t,\sigma\cap\sigma'),t)=\varphi_{\sigma\sigma'}(|\gamma|^{-1}_{\sigma\cap\sigma'}\circ q(t),t). \tag{42}$$

The last expression has to be taken with a grain of salt because $|\gamma|^{-1}_{\sigma\cap\sigma'}\circ q(t)$ is only defined for $t\in R_{\le\gamma(\sigma\cap\sigma')}$; for other values of t, we have $\varphi_{\sigma\sigma'}(x,t)=0$ regardless of the value of x because of the definition of $\mathscr{P}$ in (40).

Next, we define $D\in\mathrm{KK}^G_n(\mathscr{P},\mathbb{C})$. Let $[i]\in\mathrm{KK}^G_0\big(\mathscr{P},C_0(E)\big)$ be the class of the inclusion map $i\colon\mathscr{P}\to C_0(E,\mathbb{K})$. Since $E\cong\mathbb{R}^n$, we have canonical invertible elements

$$\beta_E\in\mathrm{KK}_n(C_0(E),\mathbb{C}),\qquad \hat{\beta}_E\in\mathrm{KK}_{-n}\big(\mathbb{C},C_0(E)\big),$$

such that

$$\begin{aligned}\beta_E\otimes\hat{\beta}_E&=1_{C_0(E)}&&\text{in }\mathrm{KK}^G_0(C_0(E),C_0(E)),\\ \hat{\beta}_E\otimes_{C_0(E)}\beta_E&=1_{\mathbb{C}}&&\text{in }\mathrm{KK}^G_0(\mathbb{C},\mathbb{C}).\end{aligned}$$

We set

$$D:=[i]\otimes_{C_0(E)}\beta_E\in\mathrm{KK}^G_n(\mathscr{P},\mathbb{C}). \tag{43}$$

We will construct $\Theta\in\mathrm{RKK}^G_{-n}(X;\mathbb{C},\mathscr{P})$ as

$$\Theta:=\hat{\beta}_E\otimes_{C_0(E)}[\vartheta],$$

where $[\vartheta]\in\mathrm{RKK}^G_0(X;C_0(E),\mathscr{P})$ is the class of an $X\rtimes G$-equivariant $*$-homomorphism $\vartheta\colon C_0(X)\otimes C_0(E)\to C_0(X)\otimes\mathscr{P}$. The latter is, of course, equivalent to a G-equivariant continuous family of $*$-homomorphisms $\vartheta_x\colon C_0(E)\to\mathscr{P}$ for $x\in X$. Its construction is rather involved. This is the point where we deviate most seriously from [26].

The first ingredient for ϑ is a certain G-equivariant function from X to the unit sphere of $\ell^2(SX)$. For this we need the barycentric subdivision $X^{(1)}$ of X. Recall that the vertices of this subdivision are in bijection with SX. Let $x\in X$ and let $\sigma^{(1)}$ be some simplex of the barycentric subdivision that contains x. The vertices of $\sigma^{(1)}$ form a strictly increasing chain $\sigma_0\subset\cdots\subset\sigma_k$ in SX; we view these as basis vectors of $\ell^2(SX)$. Any point of $|\sigma^{(1)}|$ can be written uniquely as a convex combination of the vertices σ_j; formally, $x=\sum_{j=0}^k t_j\sigma_j$, where $t_j\ge 0$ for all $j\in\underline{\mathbf{k}}$ and $\sum t_j=1$. The barycentric subdivision of a 2-simplex is illustrated in Figure 2; we have represented the vertices by their colours in $\{0,1,2\}$. The shaded maximal simplex of the barycentric subdivision is labelled by the chain $0,01,012$.

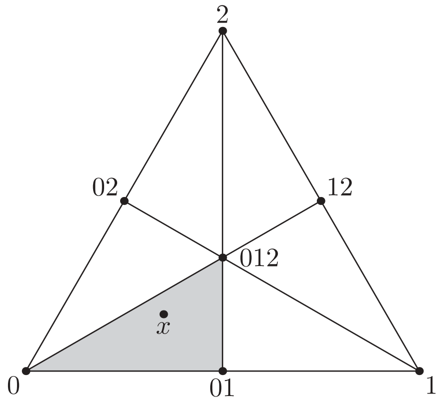

Fig. 2. barycentric subdivision of the standard 2-simplex

Let

$$v'(x) = v'\left(\sum_{j=0}^{k} t_j\sigma_j\right) := \sum_{j=0}^{k} \sqrt{t_j}\sigma_j \in \ell^2(SX). \tag{44}$$

This defines a continuous map from $|\sigma^{(1)}|$ to the unit sphere of $\ell^2(SX)$. If some of the coordinates t_j of x vanish, then we may replace $\sigma^{(1)}$ by the face that is spanned by the σ_j with $t_j \neq 0$. Since this does not change $v'(x)$, the maps v' may be glued together to a continuous map $v' \colon X \to \ell^2(SX)$, whose range is contained in the unit sphere. Now let $P' \colon X \to \mathbb{K}$ be the function whose value at $x \in X$ is the rank-1-projection onto $\mathbb{C} \cdot v'(x)$. The maps v' and P' are evidently G-equivariant.

An important point about this definition is that the basis vectors involved in $v'(x)$ form a chain in SX; hence there is some region in E where $P'(x)$ is a possible value for an element of $\mathscr{P}$. Observe that $P'(x)$ is a diagonal operator in the basis SX if and only if $v'(x)$ is a basis vector, if and only if x is a vertex of the barycentric subdivision; equivalently, $x = \xi_\sigma$ for some $\sigma \in SX$. In order to proceed with the construction, we need a projection-valued function that is diagonal not merely *at* these points but *near* them. Therefore, we replace v' and P' by $v := v' \circ \mathscr{C}$ and $P := P' \circ \mathscr{C}$ with a certain *collapsing map* $\mathscr{C} \colon X \to X$.

Choose $L \in \big(0, 1/(n+1)\big)$. We first define a map $\mathscr{C}_\Sigma \colon \Sigma \to \Sigma$ on the standard simplex by

$$\mathscr{C}_\Sigma\big((t_0, \dots, t_n)\big) := [\min\{t_0, L\}, \dots, \min\{t_n, L\}], \tag{45}$$

where $[\dots]$ denotes homogeneous coordinates as in (31). If $t_j = 0$, then $\min\{t_j, L\} = 0$ as well. This means that $\mathscr{C}_\Sigma(|f|) \subseteq |f|$ for each face f of Σ (these faces are defined in (33)). Therefore, if $\sigma \in SX$ we may define

$$\mathscr{C} \colon |\sigma| \to |\sigma|, \qquad x \mapsto |\gamma|_\sigma^{-1} \circ \mathscr{C}_\Sigma \circ |\gamma|_\sigma(x),$$

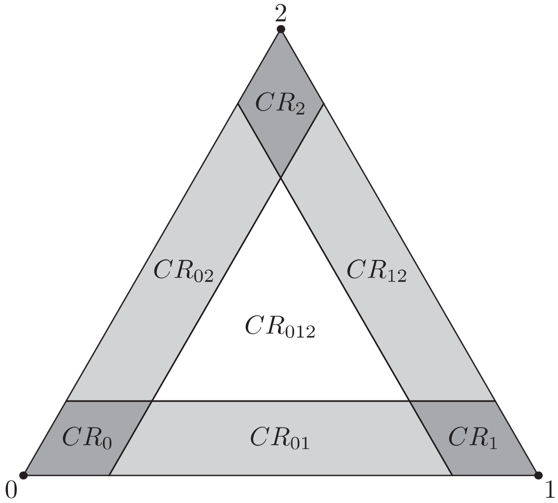

Fig. 3. The regions CR_f

using the homeomorphisms $|\gamma|_\sigma$ defined after (34). These maps on simplices match on $|\sigma \cap \sigma'|$, so that we get a global map $\mathscr{C}: X \to X$. Now we put $v := v' \circ \mathscr{C}$ and $P := P' \circ \mathscr{C}$; thus $P(x)$ is the rank-1-projection onto $\mathbb{C} \cdot v(x)$ for all $x \in X$.

In order to formulate some properties of the collapsing map, we define

$$CR_f := \{(t_0, \dots, t_n) \in \Sigma \mid t_i \geq L \quad \text{for } i \in f,\ t_i \leq L \quad \text{for } i \in \underline{\mathbf{n}} \setminus f\}. \tag{46}$$

for $f \in \mathscr{S}(\underline{\mathbf{n}})$. These regions cover Σ because $L < 1/(n+1)$. Figure 3 illustrates them for the 2-simplex. Combining (33) and (46), we get

$$|f| \cap CR_f = \{(t_0, \dots, t_n) \in \Sigma \mid t_i \geq L \quad \text{for } i \in f,\ t_i = 0 \quad \text{for } i \in \underline{\mathbf{n}} \setminus f\}. \tag{47}$$

Hence $\mathscr{C}_\Sigma(|f| \cap CR_f)$ consists of a single point (t_i'), with homogeneous coordinates $t_i' = L$ for $i \in f$ and $t_i' = 0$ for $i \in \underline{\mathbf{n}} \setminus f$. The rescaling replaces L by $1/\#f$ and thus produces the barycentre ξ_f of the face f; that is,

$$\mathscr{C}_\Sigma(|f| \cap CR_f) = \{\xi_f\}. \tag{48}$$

Lemma 26. *If $x \in X$ satisfies $|\gamma|(x) \in CR_f$, then we have $P(x)_{\sigma\sigma'} = 0$ or $f \subseteq \gamma(\sigma \cap \sigma')$.*

Proof. Let τ be some simplex of X that contains x. Since $v(x)$ only has non-zero coefficients at the faces of τ and since the restriction of $|\gamma|$ to $|\tau|$ is injective, we may assume without loss of generality that $X = \Sigma$ and $|\gamma| = \mathrm{id}_\Sigma$. Moreover, the assertion is invariant under simplicial automorphisms of Σ, that is, permutations of coordinates. (We transform both x and f, of course.) We can achieve that $x_0 \geq x_1 \geq \cdots \geq x_n$ by a coordinate permutation. Since $x \in CR_f$, the only possibilities for f are $f = \{0, \ldots, k\}$ for some $k \in \underline{\mathbf{n}}$ and we have $x_k \geq L \geq x_{k+1}$. Let $x' := \mathscr{C}_\Sigma(x)$, then we get $P(x) = P'(x')$ and

$$x'_0 = x'_1 = \cdots = x'_k \geq x'_{k+1} \geq \cdots \geq x'_n.$$

Let $\alpha_j \in \Sigma$ be the vertex of the barycentric subdivision of Σ that is labelled by the simplex $\{0, \ldots, j\}$; equivalently, α_j is the barycentre of that simplex; explicitly, the first $j + 1$ coordinates of α_j are $1/(j + 1)$, the remaining ones vanish. We have

$$x' = x'_n(n + 1)\alpha_n + \sum_{j=k}^{n-1}(x'_j - x'_{j+1})(j + 1)\alpha_j,$$

so that x' is a convex combination of α_j with $j \geq k$. Such convex combinations form a single simplex in the barycentric subdivision. Hence the vector $v(x) = v''(x') \in \ell^2(SX)$ only contains the basis vectors α_j with $j \geq k$. Therefore, if $P(x)_{\sigma,\sigma'} \neq 0$, then σ and σ' are among the α_j with $j \geq k$. This implies $f \subseteq \gamma(\sigma \cap \sigma')$ as asserted. □

For $\lambda > 1$, let $r_\lambda\colon E \to E$ be the radial expansion map around the barycentre of Σ. Explicitly,

$$r_\lambda(t_0, \ldots, t_n) = \left(\lambda t_0 - \frac{\lambda - 1}{n + 1}, \ldots, \lambda t_n - \frac{\lambda - 1}{n + 1}\right). \tag{49}$$

If $(\lambda - 1)/(n + 1) = \lambda L$, that is, $\lambda = (1 - (n + 1)L)^{-1}$, then we get

$$CR_f = r_\lambda^{-1}(R_f) \tag{50}$$

and hence $r_\lambda(CR_f) \subseteq R_f \subseteq R_{\leq f}$; this follows immediately from the definitions (35) and (46), see also Figures 1 and 3. We shall need a slightly different result, as follows. For $\delta > 0$, let

$$B(\delta) := \left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum t_j = 0 \quad \text{and} \quad |t_j| < \delta \quad \text{for all } j = 0, \ldots, n\right\}.$$

Lemma 27. *If* $\lambda > (1 - (n + 1)L)^{-1}$, *then there exists* $\delta > 0$ *such that* $r_\lambda(s) + B(\delta) \subseteq R_{\leq f}$ *for all* $f \in \mathscr{S}(\underline{\mathbf{n}})$, $s \in CR_f$.

Proof. This follows immediately from the definition of r_λ and the definitions (36) and (46) of the regions $R_{\leq f}$ and CR_f. □

Choose λ and $\delta > 0$ as in Lemma 27 and choose an orientation-preserving diffeomorphism $h\colon E \xrightarrow{\cong} B(\delta)$. Let E_+ be the one-point compactification of E. Extend h^{-1} to a map $h^{-1}\colon E_+ \to E_+$ by $h^{-1}(t) := \infty$ for $t \notin B(\delta)$ and extend

$\varphi \in C_0(E)$ to E_+ by $\varphi(\infty) := 0$. We get a continuous family of $*$-homomorphisms

$$h_s!\colon C_0(E) \to C_0(E), \qquad h_s!\varphi(t) := \varphi \circ h^{-1}\big(t - r_\lambda(s)\big) \tag{51}$$

for $s \in \Sigma$, where $r_\lambda\colon E \to E$ is defined in (49). Our notation stems from the fact that $h_s!$ is the wrong-way map associated to the open embedding

$$h_s\colon E \to E, \qquad t \mapsto h(t) + r_\lambda(s).$$

By construction, $h_s!(\varphi)$ vanishes outside $r_\lambda(s) + B(\delta)$ for all $\varphi \in C_0(E)$. Using the map $|\gamma|\colon X \to \Sigma$ defined in (34), we get a G-invariant continuous family of $*$-homomorphisms $h_{|\gamma|(x)}\colon C_0(E) \to C_0(E)$ parametrised by $x \in X$.

Lemma 28. *The formula*

$$\vartheta_x(\varphi) := h_{|\gamma|(x)}!(\varphi) \otimes P(x)$$

for $x \in X$ defines a G-equivariant continuous family of $*$-homomorphisms $C_0(E) \to \mathscr{P}$ and hence a class $[\vartheta] \in \mathrm{RKK}_0^G(X; C_0(E), \mathscr{P})$. We define

$$\Theta := \hat{\beta}_E \otimes_{C_0(E)} [\vartheta] \in \mathrm{RKK}_{-n}^G(X; \mathbb{C}, \mathscr{P}).$$

Proof. It is clear that ϑ_x is a G-equivariant continuous family of $*$-homomorphisms into $C_0(E, \mathbb{K}) \supseteq \mathscr{P}$. We must check that its range is contained in $\mathscr{P}$. Fix $\sigma, \sigma' \in SX$ and $x \in X$ such that $P(x)_{\sigma\sigma'} \neq 0$. We have $|\gamma|(x) \in CR_f$ for some $f \in \mathscr{S}(\underline{\mathbf{n}})$ because these regions cover Σ. Lemma 26 yields $f \subseteq \gamma(\sigma \cap \sigma')$. Let $V := r_\lambda|\gamma|(x) + B(\delta)$, then $V \subseteq R_{\le f} \subseteq R_{\le \gamma(\sigma\cap\sigma')}$ by Lemma 27. Since $h_{|\gamma|(x)}!\varphi$ is supported in V for all $\varphi \in C_0(E)$, we get $\operatorname{supp} \vartheta_x(\varphi)_{\sigma\sigma'} \subseteq R_{\le\gamma(\sigma\cap\sigma')}$. This means that $\vartheta_x(\varphi) \in \mathscr{P}$ (see (40)). □

Theorem 29. *The triple $(\mathscr{P}, D, \Theta)$ defined above is a Kasparov dual for X of dimension $-n$.*

Proof. First we check condition 18.1, that is, $\Theta \otimes_{\mathscr{P}} D = 1_{\mathbb{C}}$. Let $\vartheta = (\vartheta_x)_{x\in X}$ be as in Lemma 28. Let i be the embedding $\mathscr{P} \to C_0(E) \otimes \mathbb{K}$; it defines a class $[i] \in \mathrm{KK}_0^G\big(\mathscr{P}, C_0(E)\big)$. Then

$$\Theta \otimes_{\mathscr{P}} D = \hat{\beta}_E \otimes_{C_0(E)} [\vartheta] \otimes_{\mathscr{P}} [i] \otimes_{C_0(E)} \beta_E.$$

We are done if we show $[i \circ \vartheta] = 1_{C_0(E)}$ in $\mathrm{RKK}_0^G(X; C_0(E), C_0(E))$ because $\hat{\beta}_E \otimes_{C_0(E)} \beta_E = 1_{\mathbb{C}}$. Since we no longer impose any support restrictions on the range of $i \circ \vartheta_x$, the family of maps

$$\vartheta_x^s(\varphi)(t) := \varphi \circ h^{-1}\Big(t - r_{s\lambda}\big(|\gamma|(x)\big)\Big) P(x)$$

for $s \in [0,1]$ provides a natural homotopy between $i \circ \vartheta_x = \vartheta_x^1$ and the map

$$\vartheta_x^0 \varphi(t) := \varphi \circ h^{-1}(t - \xi) P(x),$$

where ξ is the barycentre of Σ. Since h is an orientation-preserving homeomorphism $E \to B(\delta)$, the endomorphism $\varphi \mapsto \varphi \circ h^{-1}(t-\xi)$ of $C_0(E)$ is homotopic to the identity map. Thus $[\vartheta]$ is the exterior product of $1_{C_0(E)}$ and the class $[P] \in \mathrm{RKK}_0^G(X; \mathbb{C}, \mathbb{C})$ determined by the continuous family of projections $P(x)$, $x \in X$. The continuous family of unit vectors $v(x)$ may be viewed as a G-equivariant continuous family of isometries $\hat{v}(x)\colon \mathbb{C} \to \ell^2(SX)$ with $\hat{v}(x)\hat{v}^*(x) = P(x)$. This means that $[P] = [1_{\mathbb{C}}]$. This finishes the proof that $\Theta \otimes_{\mathscr{P}} D = 1_{\mathbb{C}}$.

Next we verify 18.2, which asserts that $\Theta \hat{\otimes}_X f = \Theta \hat{\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)$ in $\mathrm{RKK}_*^G(X \times Y; A, B \hat{\otimes} \mathscr{P})$ for all $f \in \mathrm{RKK}_{*+n}^G(X \times Y; A, B)$ and all Y, A, B. Since the classes β_E and $\hat{\beta}_E$ are inverse to each other, this is equivalent to $[\vartheta] \hat{\otimes}_X f = [\vartheta] \hat{\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)$ in $\mathrm{RKK}_*^G(X \times Y; A \hat{\otimes} C_0(E), B \hat{\otimes} \mathscr{P})$. By Lemma 21, it suffices to prove this in the special case where f is an $(X \times Y) \rtimes G$-equivariant $*$-homomorphism. Thus both factors in our product are now classes of equivariant $*$-homomorphisms.

We view f as a G-equivariant continuous family of $C_0(Y)$-linear $*$-homomorphisms $f_x\colon C_0(Y, A) \to C_0(Y, B)$ for $x \in X$. Then $\vartheta \hat{\otimes}_X f$ corresponds to the continuous family of maps $\vartheta_x \hat{\otimes} f_x\colon C_0(E) \hat{\otimes} C_0(Y, A) \to \mathscr{P} \hat{\otimes} C_0(Y, B)$, $x \in X$. Explicitly,

$$\begin{aligned}(\vartheta \hat{\otimes}_X f)_x(\varphi \hat{\otimes} a)_{\sigma\sigma'}(t) &:= \vartheta_x(\varphi)_{\sigma\sigma'}(t) f_x(a) \\ &= (h_{|\gamma|(x)}!\varphi)(t) P(x)_{\sigma\sigma'} f_x(a) \qquad \text{in } C_0(Y, B)\end{aligned} \tag{52}$$

for all $\varphi \in C_0(E)$, $a \in C_0(Y, A)$, $t \in E$, $\sigma, \sigma' \in SX$.

By definition, we have $\sigma_{X,\mathscr{P}}(f)(\varphi_1 \cdot \varphi_2 \hat{\otimes} a) = \varphi_1 \cdot f(\varphi_2 \hat{\otimes} a)$ for all $\varphi_1 \in \mathscr{P}$, $\varphi_2 \in C_0(X)$, $a \in C_0(Y, A)$. Using (42), we rewrite this as

$$\sigma_{X,\mathscr{P}}(f)(\varphi \hat{\otimes} a)_{\sigma\sigma'}(t) = \varphi_{\sigma\sigma'}(t) f_{\bar{q}(\sigma,t)}(a) \qquad \text{in } C_0(Y, B) \tag{53}$$

for all $\varphi \in \mathscr{P}$, $a \in C_0(Y, A)$, $\sigma, \sigma' \in SX$, $t \in E$. Composition with ϑ yields the continuous family of maps $\big(\vartheta \hat{\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)\big)_x = \sigma_{X,\mathscr{P}}(f) \circ (1_A \hat{\otimes} \vartheta_x)$ from $C_0(E) \hat{\otimes} C_0(Y, A)$ to $\mathscr{P} \hat{\otimes} C_0(Y, B)$ for $x \in X$. Thus

$$\big(\vartheta \hat{\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)\big)_x(\varphi \hat{\otimes} a)_{\sigma\sigma'}(t) = (h_{|\gamma|(x)}!\varphi)(t) P(x)_{\sigma\sigma'} f_{\bar{q}(\sigma,t)}(a). \tag{54}$$

The only difference between the two families in (52) and (54) is that we use f_x and $f_{\bar{q}(\sigma,t)}$, respectively. Whenever $P(x)_{\sigma\sigma'} \neq 0$, x and $\bar{q}(\sigma, t)$ lie in the same simplex $\sigma \cap \sigma'$ of X. Arguments as in the proofs of Lemmas 25 and 28 show that we still get a homomorphism from $C_0(E) \hat{\otimes} C_0(Y, A)$ to $\mathscr{P} \hat{\otimes} C_0(Y, B)$ if we replace $f_{\bar{q}(\sigma,t)}$ in (54) with $f_{(1-s)x+s\bar{q}(\sigma,t)}$ for $s \in [0,1]$. Thus $\vartheta \hat{\otimes}_X f$ and $\vartheta \hat{\otimes}_{\mathscr{P}} \sigma_{X,\mathscr{P}}(f)$ are homotopic. This finishes the proof of 18.2.

It remains to verify 18.3. Since β_E is invertible, we may replace $\sigma_{X,\mathscr{P}}(\Theta)$ with $\sigma_{X,\mathscr{P}}([\vartheta])$ in this statement. This is the class of a G-equivariant $*$-homomorphism $\sigma_{X,\mathscr{P}}(\vartheta)\colon \mathscr{P}\otimes C_0(E)\to\mathscr{P}\otimes\mathscr{P}$. We must check

$$[\Phi_{\mathscr{P}}\circ\sigma_{X,\mathscr{P}}(\vartheta)] = (-1)^n[\sigma_{X,\mathscr{P}}(\vartheta)] \qquad \text{in } \mathrm{KK}_0^G(\mathscr{P}\otimes C_0(E),\mathscr{P}\otimes\mathscr{P});$$

here $\Phi_{\mathscr{P}}$ denotes the flip automorphism on $\mathscr{P}\otimes\mathscr{P}$. We describe $\sigma_{X,\mathscr{P}}(\vartheta)$ by specifying its matrix coefficients with respect to the basis $SX\times SX$ of $\ell^2(SX)\hat{\otimes}\ell^2(SX)\cong\ell^2(SX\times SX)$. Equation (53) yields

$$\begin{aligned}&\sigma_{X,\mathscr{P}}(\vartheta)(\varphi_1\hat{\otimes}\varphi_2)(t_1,t_2)_{(\sigma_1,\sigma_2),(\sigma_1',\sigma_2')}\\ &=\varphi_1(t_1)_{\sigma_1\sigma_1'}\cdot\varphi_2\circ h^{-1}\bigl(t_2-r_\lambda\circ|\gamma|\circ\bar{q}(t_1,\sigma_1)\bigr)\cdot P\bigl(\bar{q}(\sigma_1,t_1)\bigr)_{\sigma_2\sigma_2'}\end{aligned}\tag{55}$$

for all $\varphi_1\in\mathscr{P}$, $\varphi_2\in C_0(E)$, $t_1,t_2\in E$, $\sigma_1,\sigma_1',\sigma_2,\sigma_2'\in SX$. Fix $t_1,t_2\in E$ and choose $f\subseteq\underline{\mathbf{n}}$ minimal such that t_1 belongs to the interior of $R_{\le f}$. Thus $q(t_1)\in|f|$ by (39). If $\varphi_1(t_1)_{\sigma_1\sigma_1'}\neq 0$, then $f\subseteq\gamma(\sigma_1\cap\sigma_1')$ by the definition of $\mathscr{P}$, see (40). Hence (41) yields $\bar{q}(\sigma_1,t)=|\gamma|_{\sigma_1}^{-1}q(t_1)$. Thus $|\gamma|\bar{q}(\sigma_1,t_1)=q(t_1)$ and we can rewrite the right hand side of (55) as

$$\varphi_1(t_1)_{\sigma_1\sigma_1'}\cdot\varphi_2\circ h^{-1}\bigl(t_2-r_\lambda q(t_1)\bigr)\cdot P\bigl(\bar{q}(\sigma_1,t_1)\bigr)_{\sigma_2\sigma_2'}.$$

For $\sigma\in SX$, $f\subseteq\underline{\mathbf{n}}$, we let

$$\begin{aligned}SX_{\ge\sigma}&:=\{\sigma'\in SX\mid\sigma'\ge\sigma\},\\ SX_{\ge f}&:=\{\sigma'\in SX\mid\gamma(\sigma')\supseteq f\},\\ SX_{=f}&:=\{\sigma'\in SX\mid\gamma(\sigma')=f\},\end{aligned}\tag{56}$$

Since γ is a colouring, any simplex in $SX_{\ge f}$ contains a unique face σ with $\gamma(\sigma)=f$. This means that $SX_{\ge f}$ is the disjoint union of the subsets $SX_{\ge\sigma}$, where $\sigma\in SX_{=f}$. We write $\sigma_{1,f}:=\tau$ if $\gamma(\tau)=f$ and $\sigma_1\in SX_{\ge\tau}$. With f as defined above, we have $\bar{q}(\sigma_1,t_1)\in|\sigma_{1,f}|$. Thus $P\bigl(\bar{q}(\sigma_1,t_1)\bigr)_{\sigma_2\sigma_2'}=0$ unless σ_2 and σ_2' are faces of $\sigma_{1,f}$.

We also choose f_2 such that $q(t_1)\in CR_{f_2}$. Then it follows from Lemma 26 that $f_2\subseteq\gamma(\sigma_2\cap\sigma_2')$ for all $\sigma_2,\sigma_2'\in SX$ with $P\bigl(\bar{q}(\sigma_1,t_1)\bigr)_{\sigma_2\sigma_2'}\neq 0$. Since both σ_2 and σ_2' are faces of σ_1, this is equivalent to $\sigma_2,\sigma_2'\in SX_{\ge\sigma_{1,f_2}}$. Moreover, Lemma 27 yields that t_2 belongs to the interior of $R_{\le f_2}$.

It follows from the definition of $\mathscr{P}$ that the possible values of $\varphi_1(t_1)$ for $\varphi_1\in\mathscr{P}$ are exactly the elements of

$$\bigoplus_{\sigma\in SX_{=f}}\mathbb{K}\bigl(\ell^2(SX_{\ge\sigma})\bigr)\subseteq\mathbb{K}\bigl(\ell^2(SX)\bigr)=\mathbb{K}.$$

A similar description is available for the possible values at t_2, of course; the relevant face $f' \subseteq \underline{\mathbf{n}}$ is the minimal subset for which t_2 is an interior point of $R_{\leq f'}$. We have $f' \leq f_2$ by Lemma 27. The map $\sigma_{X,\mathscr{P}}(\vartheta)$ gives rise to an embedding

$$\bigoplus_{\tau \in SX_{=f}} \mathbb{K}\big(\ell^2(SX_{\geq\tau})\big) \to \bigoplus_{\tau \in SX_{=f}} \mathbb{K}\big(\ell^2(SX_{\geq\tau} \times SX_{\geq\tau_{f'}})\big),$$

which is induced by the family of isometries

$$J(\tau, t_1)\colon \ell^2(SX_{\geq\tau}) \to \ell^2(SX_{\geq\tau} \times SX_{\geq\tau_{f'}}), \qquad \eta \mapsto \eta \otimes v\big(\bar{q}(\tau, t_1)\big),$$

for $\tau \in SX_{=f}$. Here we use the definition of $P(x)$ as the rank-1-projection onto the span of the unit vector $v(x)$.

Since the coefficients of $v(x)$ are non-negative for all $x \in X$, we have $(1 - s)v(x) + s\sigma \neq 0$ for any $s \in [0, 1]$, $\sigma \in SX$ (we view σ as a basis vector of $\ell^2(SX)$). Therefore, we may deform the isometry $J(\tau, t_1)$ by a continuous path of isometries

$$J^s(\tau, t_1)(\sigma)\colon \ell^2(SX_{\geq\tau}) \to \ell^2(SX_{\geq\tau} \times SX_{\geq\tau_{f'}}),$$
$$J^s(\tau, t_1)(\sigma) := \sigma \otimes \frac{(1-s)v\big(\bar{q}(\tau, t_1)\big) + s\sigma}{\|(1-s)v\big(\bar{q}(\tau, t_1)\big) + s\sigma\|} \qquad \text{for } \sigma \in SX_{\geq\tau}.$$

These isometries yield a homotopy of $*$-homomorphisms

$$\operatorname{Ad} J^s(t_1)\colon \bigoplus_{\tau \in SX_{=f}} \mathbb{K}\big(\ell^2(SX_{\geq\tau})\big) \to \bigoplus_{\tau \in SX_{=f}} \mathbb{K}\big(\ell^2(SX_{\geq\tau} \times SX_{\geq\tau_{f'}})\big).$$

Letting t_1, t_2 vary again, we get a homotopy of G-equivariant $*$-homomorphisms $\mathscr{P} \otimes C_0(E) \to \mathscr{P} \otimes \mathscr{P}$ by sending $\varphi_1 \otimes \varphi_2$ to the function

$$(t_1, t_2) \mapsto \operatorname{Ad} J^s(t_1)\varphi_1(t_1) \cdot \varphi_2 \circ h^{-1}\big(t_2 - r_\lambda q(t_1)\big).$$

Thus $\sigma_{X,\mathscr{P}}(\vartheta)$ is homotopic to the G-equivariant $*$-homomorphism

$$\alpha\colon \mathscr{P} \otimes C_0(E) \to \mathscr{P} \otimes \mathscr{P} \text{ defined by}$$
$$\alpha(\varphi_1 \otimes \varphi_2)(t_1, t_2) := \operatorname{Ad} J^1\varphi_1(t_1) \cdot \varphi_2 \circ h^{-1}\big(t_2 - r_\lambda q(t_1)\big),$$

where $J^1\colon \ell^2(SX) \to \ell^2(SX \times SX)$ is the diagonal embedding that sends the basis vector $\sigma \in SX$ to $\sigma \otimes \sigma$.

Fix t_1 once again and let $f \subseteq \underline{\mathbf{n}}$ be as above. Then $\operatorname{Ad} J^1\varphi(t_1)$ is an allowed value for functions in $\mathscr{P} \otimes \mathscr{P}$ if t_2 belongs to the interior of $R_{\geq f}$. Since this holds for the points in $r_\lambda((1-s)q(t_1) + st_1) + B(\delta)$ for $s \in [0, 1]$, the map α is homotopic to

$$\alpha'(\varphi_1 \otimes \varphi_2)(t_1, t_2) := \operatorname{Ad} J^1\varphi_1(t_1) \cdot \varphi_2 \circ h^{-1}(t_2 - r_\lambda t_1).$$

Observe that $\varphi \mapsto \varphi \circ r_{\lambda^{-s}}$ for $s \geq 0$ defines an endomorphism of $\mathscr{P}$. Hence the homotopy $\alpha'_s(\varphi)(t_1, t_2) := \alpha'(\varphi)(r_{\lambda^{-s}} t_1, t_2)$ connects α' with

$$\alpha''(\varphi)(t_1, t_2) := \operatorname{Ad} J^1 \varphi\big(r_{\lambda^{-1}} t_1, h^{-1}(t_2 - t_1)\big).$$

Since $[\sigma_{X,\mathscr{P}}(\vartheta)] = [\alpha'']$, condition 18.3 is equivalent to $[\Phi_{\mathscr{P}} \circ \alpha''] = (-1)^n [\alpha'']$. We have

$$\Phi_{\mathscr{P}} \circ \alpha''(\varphi)(t_1, t_2) = \operatorname{Ad} J^1 \varphi\big(r_{\lambda^{-1}} t_2, h^{-1}(t_1 - t_2)\big)$$

because the range of J^1 is invariant under $\Phi_{\mathbb{K}}$. We define yet another homotopy of homomorphisms $\mathscr{P} \otimes C_0(E) \to \mathscr{P} \otimes \mathscr{P}$ by

$$\alpha''_s(\varphi)(t_1, t_2) := \operatorname{Ad} J^1 \varphi\big(r_{\lambda^{-1}}(s t_1 + (1-s) t_2), h^{-1}(t_1 - t_2)\big).$$

It connects $\Phi_{\mathscr{P}} \circ \alpha''$ and $\alpha'' \circ (\mathrm{id}_{\mathscr{P}} \otimes f)$, where $f \colon C_0(E) \to C_0(E)$ is induced by the map $t \mapsto -t$ on E; here we assume that h is an even function, as we may. Of course, $[f] = (-1)^n$ in $\mathrm{KK}_0(C_0(E), C_0(E))$. This finishes the proof of 18.3. □

Theorem 30. *Let X be a simplicial complex equipped with a simplicial action of G. Then $\mathrm{Eul}_X = \mathrm{Eul}_X^{\mathrm{cmb}}$ in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$.*

Proof. Lemma 20 asserts that

$$\mathrm{Eul}_X = \bar{\Theta} \mathbin{\hat{\otimes}}_{C_0(X) \hat{\otimes} \mathscr{P}} [m] \mathbin{\hat{\otimes}}_{\mathscr{P}} D \in \mathrm{KK}_0^G(C_0(X), \mathbb{C});$$

here $\bar{\Theta} \in \mathrm{KK}_{-n}^G(C_0(X), C_0(X) \mathbin{\hat{\otimes}} \mathscr{P})$ is obtained from the class Θ defined in Lemma 28 by forgetting the X-structure; $m \colon C_0(X) \mathbin{\hat{\otimes}} \mathscr{P} \to \mathscr{P}$ is the multiplication homomorphism, which is described in (42); and D is defined in (43). Since the Bott periodicity classes β_E and $\hat{\beta}_E$, which appear in Θ and D, are inverse to each other, our assertion is equivalent to

$$\mathrm{id}_{C_0(E)} \mathbin{\hat{\otimes}} \mathrm{Eul}_X^{\mathrm{cmb}} = \beta_E \mathbin{\hat{\otimes}} \mathrm{Eul}_X \mathbin{\hat{\otimes}} \hat{\beta}_E = [\vartheta] \mathbin{\hat{\otimes}}_{C_0(X) \hat{\otimes} \mathscr{P}} [m] \mathbin{\hat{\otimes}}_{\mathscr{P}} [i],$$

where $\vartheta \colon C_0(X) \mathbin{\hat{\otimes}} C_0(E) \to C_0(X) \mathbin{\hat{\otimes}} \mathscr{P}$ is the continuous family of $*$-homomorphisms defined in Lemma 28 and $[i] \in \mathrm{KK}_0^G\big(\mathscr{P}, C_0(E)\big)$ is the class of the inclusion map $i \colon \mathscr{P} \to C_0(E, \mathbb{K})$. The above Kasparov product is the class of the composite homomorphism

$$i \circ m \circ \vartheta \colon C_0(X \times E) \to C_0(E, \mathbb{K}).$$

Plugging in the definition of ϑ and (42), we get

$$i \circ m \circ \vartheta(\varphi)_{\sigma\sigma'}(t) = \varphi\Big(\bar{q}(t, \sigma), h^{-1}\big(t - r_\lambda q(t)\big)\Big) \cdot P\big(\bar{q}(t, \sigma)\big)_{\sigma\sigma'}.$$

for all $\sigma, \sigma' \in SX$, $t \in E$, $\varphi \in C_0(X \times E)$. We want to simplify this expression. Assume that it is non-zero. Let $t' := q(t) \in \Sigma$ and $x' := \bar{q}(t, \sigma)$. Since the regions CR_f cover Σ, we have $t' \in CR_f$ for some $f \in \mathscr{S}(\underline{\mathbf{n}})$. Lemma 27 yields $r_\lambda t' + B(\delta) \subseteq R_{\leq f}$; hence we must have $t \in R_{\leq f}$ in order for $\varphi\big(x', h^{-1}(t - r_\lambda t')\big)$ to be non-zero. Since $t \in R_{\leq f}$, we get $t' \in |f| \cap CR_f$ by (39) and $\mathscr{C}_\Sigma(t') = \xi_f$ by (48). Therefore, $P(x') = P' \circ \mathscr{C}(x')$ is the projection onto a basis vector of $\ell^2(SX)$. Hence $P(x')_{\sigma\sigma'} = 0$ unless $\sigma = \sigma'$ and $\gamma(\sigma) = f$.

Summing up, $i \circ m \circ \vartheta(\varphi)(t) \in \mathbb{K}$ is diagonal with respect to the basis SX; the diagonal entry for the basis vector σ is supported in

$$D_{\gamma(\sigma)} := q^{-1}\big(CR_{\gamma(\sigma)} \cap |\gamma(\sigma)|\big)$$

and given there by the formula

$$\Lambda_\sigma(\varphi)(t) := \varphi\Big(\bar{q}(t,\sigma), h^{-1}\big(t - r_\lambda q(t)\big)\Big) = \varphi\Big(|\gamma|_\sigma^{-1} q(t), h^{-1}\big(t - r_\lambda q(t)\big)\Big).$$

The last term is defined for $t \in D_{\gamma(\sigma)}$ because $q(t) \in |\gamma(\sigma)|$ (see the definition of $\bar{q}$ in (41)). Let $\Lambda := (\Lambda_\sigma)\colon C_0(X \times E) \to C_0(SX \times E)$.

In the definition of $\mathrm{Eul}_X^{\mathrm{cmb}}$ we use the map $\xi^*\colon C_0(X) \to C_0(SX)$ induced by (27). Moving $|\gamma|_\sigma^{-1} q(t) \in |\sigma|$ linearly towards ξ_σ, we get a homotopy between Λ and the $*$-homomorphism $\Lambda' \circ (\xi^* \otimes \mathrm{id}_{C_0(E)})$, where we define $\Lambda'\colon C_0(SX \times E) \to C_0(SX \times E)$ by

$$\Lambda'\varphi(\sigma, t) := \begin{cases} \varphi\Big(\sigma, h^{-1}\big(t - r_\lambda q(t)\big)\Big) & \text{for } t \in D_{\gamma(\sigma)}, \\ 0 & \text{otherwise.} \end{cases} \tag{57}$$

We may describe Λ' by a family of maps $\Lambda'_\sigma\colon C_0(E) \to C_0(E)$ for $\sigma \in SX$. Equation (57) shows that Λ'_σ only depends on $\gamma(\sigma)$, so that we also denote it by $\Lambda'_{\gamma(\sigma)}$. Partitioning SX into the subsets $SX_{=f}$ defined in (56), we obtain

$$[i \circ m \circ \vartheta] = \sum_{f \in \mathscr{S}(\underline{\mathbf{n}})} [\xi^* \mathbin{\hat{\otimes}} \Lambda'_f] \mathbin{\hat{\otimes}}_{C_0(SX)} \big[C_0(SX) \to \mathbb{K}\big(\ell^2(SX_{=f})\big)\big]$$

in $\mathrm{KK}_0^G(C_0(X) \mathbin{\hat{\otimes}} C_0(E), C_0(E))$. The combinatorial Euler characteristic is defined by

$$\mathrm{Eul}_X^{\mathrm{cmb}} := \sum_{f \in \mathscr{S}(\underline{\mathbf{n}})} (-1)^{\dim f} [\xi^*] \mathbin{\hat{\otimes}}_{C_0(SX)} \big[C_0(SX) \to \mathbb{K}\big(\ell^2(SX_{=f})\big)\big].$$

Therefore, we get the desired equation $[i \circ m \circ \vartheta] = \mathrm{id}_{C_0(E)} \mathbin{\hat{\otimes}} \mathrm{Eul}_X^{\mathrm{cmb}}$ if we show that $[\Lambda'_f] = (-1)^{\dim f}$ in $\mathrm{KK}_0(C_0(E), C_0(E))$. It remains to verify the latter assertion.

Since all our constructions are invariant under coordinate permutations, we may assume $f = \{0, \dots, k\}$ with $k = \dim f$. If $t \in D_f$, then $t_i > 0$ for $i \in f$ and $t_i \leq 0$ for $i \in \underline{\mathbf{n}} \setminus f$. Hence q is given by

$$q(t)_i = \begin{cases} \big(1 - \sum_{j \in \underline{\mathbf{n}} \setminus f} t_j\big)^{-1} t_i & \text{for } i \in f, \\ 0 & \text{for } i \in \underline{\mathbf{n}} \setminus f. \end{cases}$$

The point $r_\lambda \xi_f$ belongs to D_f and satisfies $q r_\lambda(\xi_f) = \xi_f$. Hence it is a fixed-point of the map $r_\lambda \circ q$. We reparametrise our maps and consider $\psi_f(t) := r_\lambda \xi_f + t - r_\lambda q(r_\lambda \xi_f + t)$, where $t \in D_f - r_\lambda \xi_f$. Thus $\psi_f(0) = 0$. Since points in the range of ψ_f satisfy $\sum \psi_f(t)_i = 0$, we may drop one coordinate; we choose the 0th coordinate, which belongs to f.

It is easy to see that $\psi_f(t)_i = t_i$ for $i \in \underline{\mathbf{n}} \setminus f$. Moreover, if we fix the coordinates t_j with $j \in \underline{\mathbf{n}} \setminus f$, then $\psi_f(t)_i = -a\big(\sum_{j\in\underline{\mathbf{n}}\setminus f} t_j\big) \cdot t_i + b\big(\sum_{j\in\underline{\mathbf{n}}\setminus f} t_j\big)$ with certain rational functions a, b of one variable. Explicitly,

$$a(s) = \frac{(k+1)(\lambda - 1) + (n+1)s}{(n-k)\lambda + k + 1 - (n+1)s}.$$

The important point here is that $a(0) > 0$.

Since $h^{-1} \circ \psi_f(t) = \infty$ unless $|\psi_f(t)_i| < \delta$ for all $i \in \underline{\mathbf{n}}$, we may restrict attention to t with $|t_j| < \delta$ for $j \in \underline{\mathbf{n}} \setminus f$, so that $|s| < (n-k)\delta$. We may choose δ as small and λ as great as we like. Therefore, the difference between a and the constant function $a(0)$ is negligible. Hence the maps $a(rs) \cdot t_i + b(rs)$ for $r \in [0, 1]$ give rise to an isotopy between ψ_f and the invertible linear map

$$\psi'_f(t)_i = \begin{cases} t_i & \text{for } i \in \underline{\mathbf{n}} \setminus f, \\ -a(0) \cdot t_i & \text{for } i \in f, i \neq 0. \end{cases}$$

Recall that we have dropped one coordinate, so that we do not have to worry about the condition $\sum t_i = 0$ any more. This also means that there only remain $\dim f$ relevant coordinates in f, which are multiplied by a negative number. Hence $[\psi'_f] = (-1)^{\dim f}$ in $\mathrm{KK}_0(C_0(\mathbb{R}^n), C_0(\mathbb{R}^n))$. Since h is orientation-preserving, we have $[h^{-1}] = 1$. Therefore, $[\Lambda'_f] = [h^{-1} \circ \psi_f] = [h^{-1} \circ \psi'_f] = (-1)^{\dim f}$. □

7. Gysin sequence in the simplicial case

Theorem 31. *Let G be a locally compact group and let $\partial X = \bar{X} \setminus X$ be a boundary action as in Definition 4. Suppose that X is a finite-dimensional, locally finite simplicial complex with a simplicial action of G. Suppose also that G satisfies the Baum-Connes conjecture with coefficients $\mathbb{C}$ and $C(\partial X)$. Then there is an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{K}_1(C^*_{\mathrm{r}} G) & \xrightarrow{u_*} & \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) & \xrightarrow{\delta} & \mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \\ & & & & & & \downarrow{\scriptstyle \mathrm{Eul}^{\mathrm{cmb}}_X} \\ 0 & \longleftarrow & \mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) & \xleftarrow{\delta} & \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) & \xleftarrow{u_*} & \mathrm{K}_0(C^*_{\mathrm{r}} G) \end{array}$$

where $\mathrm{Eul}_X^{\mathrm{cmb}}$ *denotes the Kasparov product with the combinatorial equivariant Euler characteristic* $\mathrm{Eul}_X^{\mathrm{cmb}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$. *More explicitly,*

$$\mathrm{Eul}_X^{\mathrm{cmb}}(x) = \sum_{(H)} \chi(X, H) \dim_H(x),$$

where the summation runs over the conjugacy classes in G of stabilisers of simplices in X, and $\dim_H \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$ *and* $\chi(X, H) \in \mathbb{Z}$ *are defined as on page 19.*

Proof. We plug the formula $\mathrm{Eul}_X = \mathrm{Eul}_X^{\mathrm{cmb}}$ of Theorem 30 into the abstract Gysin sequence of Proposition 9. By definition, $\mathrm{Eul}_X^{\mathrm{cmb}}$ factors through the homomorphism $\xi^*\colon C_0(X) \to C_0(SX)$ induced by the barycentre embedding (27). Writing SX as a disjoint union of G-orbits, we get

$$\mathrm{K}_*^{\mathrm{top}}\bigl(G, C_0(SX)\bigr) \cong \mathrm{K}_*(C_0(SX) \rtimes_{\mathrm{r}} G) \\ \cong \bigoplus\nolimits_{\sigma \in G\backslash SX} \mathrm{K}_*(C_0(G/G_\sigma) \rtimes_{\mathrm{r}} G) \cong \bigoplus\nolimits_{\sigma \in G\backslash SX} \mathrm{K}_*\bigl(C_{\mathrm{r}}^*(G_\sigma)\bigr)\,.$$

Here G_σ denotes the stabiliser of the simplex σ, which is a compact-open subgroup of G. The map $\mathrm{Eul}_X^{\mathrm{cmb}}\colon \mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_1(C_{\mathrm{r}}^* G)$ vanishes because it factors through $\mathrm{K}_1(C_0(SX) \rtimes_{\mathrm{r}} G) = 0$. This yields the asserted long exact sequence. Equation (28) yields the formula for $\mathrm{Eul}_X^{\mathrm{cmb}}(x)$. □

We now describe the map $\dim_H\colon \mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0(C_{\mathrm{r}}^* G)$ for a compact-open subgroup $H \subseteq G$, which occurs in Theorem 31. It factors through the map $\mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0(C_0(G/H) \rtimes_{\mathrm{r}} G) \cong \mathrm{Rep}(H)$ that is induced by an orbit restriction map $X \to G/H$. The composite map

$$\mathrm{Rep}(H) \cong \mathrm{K}_0(C_{\mathrm{r}}^* H) \cong \mathrm{KK}_0^G(C_0(G/H), \mathbb{C}) \to \mathrm{K}_0^{\mathrm{top}}(G) \to \mathrm{K}_0(C_{\mathrm{r}}^* G)$$

is equal to the induction map $i_H^G\colon\ \mathrm{Rep}(H) \to \mathrm{K}_0(C_{\mathrm{r}}^* G)$, which is induced by the embedding $C_{\mathrm{r}}^* H \subseteq C_{\mathrm{r}}^* G$. Thus $\dim_H(x)$ is the composite of an orbit restriction map and the induction map. Since there are relations between the orbit restriction and induction maps for different H, it is hard to describe the range and kernel of $\mathrm{Eul}_X^{\mathrm{cmb}}$ in general.

The following corollary is equivalent to Theorem 1 and Corollary 2. Let $1_{C_{\mathrm{r}}^* G}$ be the unit projection in $C_{\mathrm{r}}^* G$.

Corollary 32. *In the situation of Theorem 31, suppose in addition that G is discrete and torsion-free. If* $G\backslash X$ *is compact and* $\chi(G\backslash X) \neq 0$, *then there are exact sequences*

$$0 \to \langle \chi(G\backslash X)[1_{C_{\mathrm{r}}^* G}] \rangle \xrightarrow{\subseteq} \\ \mathrm{K}_0(C_{\mathrm{r}}^* G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^0(G\backslash X) \xrightarrow{\dim} \mathbb{Z} \to 0,$$

and the class of the unit element in $\mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G)$ *has torsion of order* $|\chi(G\backslash X)|$. *Otherwise, there are exact sequences*

$$0 \to \mathrm{K}_0(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^1(G\backslash X) \to 0,$$

$$0 \to \mathrm{K}_1(C^*_{\mathrm{r}}G) \xrightarrow{u_*} \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}^0(G\backslash X) \to 0,$$

and the class of the unit element in $\mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G)$ *has no torsion.*

Proof. Since the action on X is free and proper, $C_0(X) \rtimes_{\mathrm{r}} G$ is strongly Morita equivalent to $C_0(G\backslash X)$. Hence $\mathrm{K}_*(C_0(X) \rtimes_{\mathrm{r}} G) \cong \mathrm{K}^*(G\backslash X)$. Furthermore, we have an isomorphism $\mathrm{KK}^G_0(C_0(X), \mathbb{C}) \cong \mathrm{KK}_0(C_0(G\backslash X), \mathbb{C})$. It maps

$$\mathrm{Eul}^{\mathrm{cmb}}_X \mapsto \mathrm{Eul}^{\mathrm{cmb}}_{G\backslash X} = \chi(G\backslash X) \cdot \dim .$$

The Kasparov product with $\dim \in \mathrm{KK}^G_0(C_0(X), \mathbb{C})$ factors through $\mathrm{KK}^G_0(C_0(G), \mathbb{C}) \cong \mathbb{Z}$; one checks easily that it corresponds to the map

$$\mathrm{K}^0(G\backslash X) \to \mathbb{Z} \to \mathrm{K}_0(C^*_{\mathrm{r}}G), \qquad x \mapsto \dim(x) \cdot [1_{C^*_{\mathrm{r}}G}];$$

equivalently, $\dim \in \mathrm{KK}^G_0(C_0(G), \mathbb{C}) \to \mathrm{K}^{\mathrm{top}}_0(G)$ is a pre-image for $[1_{C^*_{\mathrm{r}}G}]$ under the Baum-Connes assembly map. Hence the range and kernel of the map $\mathrm{K}^0(G\backslash X) \to \mathrm{K}_0(C^*_{\mathrm{r}}G)$ are $\langle \chi(G\backslash X)[1_{C^*_{\mathrm{r}}G}]\rangle$ and the kernel of $\chi(G\backslash X)\dim$, respectively. Now the exact sequences follow from Theorem 31. The assertions about the unit element follow because $u_*[1_{C^*_{\mathrm{r}}G}] = [1_{C(\partial X) \rtimes_{\mathrm{r}} G}]$. □

Example 33. Let $\mathbb{F}_n$ be the non-Abelian free group on n generators for $n \geq 2$. Let X be the Cayley graph of $\mathbb{F}_n$, which is a $2n$-regular tree, and let $\overline{X}$ be its ends compactification. Let $\partial X := \overline{X} \setminus X$ be the set of ends of X, which is a Cantor set. This compactification is the Gromov compactification of the hyperbolic group $\mathbb{F}_n$ and the visibility compactification of the CAT(0) space X. Of course, the group $\mathbb{F}_n$ is torsion-free, so that we are in the situation of Corollary 32. The group $\mathbb{F}_n$ satisfies the Baum-Connes conjecture with arbitrary coefficients by [19].

The orbit space $\mathbb{F}_n\backslash X$ is a wedge of n circles, hence compact. Therefore, $\mathrm{K}_*(C^*_{\mathrm{r}}G) \cong \mathrm{K}^{\mathrm{top}}_*(G) \cong \mathrm{K}_*(\mathbb{F}_n\backslash X)$ and $\mathrm{Eul}_X \in \mathrm{K}^{\mathrm{top}}_0(G)$ is the Euler characteristic of G. The K-homology and K-theory of $\mathbb{F}_n\backslash X$ are isomorphic to $\mathbb{Z}$ in degree 0 and $\mathbb{Z}^n$ in degree 1, and $\chi(\mathbb{F}_n\backslash X) = 1 - n$. Corollary 32 yields

$$\mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} \mathbb{F}_n) \cong \mathbb{Z}/\langle n-1\rangle \oplus \mathbb{Z}^n, \qquad \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} \mathbb{F}_n) \cong \mathbb{Z}^n.$$

Therefore, the class of the unit element in $\mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} \mathbb{F}_n)$ is a torsion element of order $n-1$. This example is also studied in [55].

If $n = 1$, we get $\mathbb{F}_1 = \mathbb{Z}$, $X = \mathbb{R}$, and $\partial X = \{\pm\infty\}$ with $\mathbb{Z}$ acting trivially. In this case, the Euler characteristic vanishes; this already follows from Example 6.

Example 34. Let Σ_g be a closed surface of genus $g \geq 2$ and let Γ_g be its fundamental group. Equip Σ_g with a hyperbolic metric and identify its universal cover with $\mathbb{H}^2$; this identifies Γ_g with a discrete torsion-free subgroup of $\mathrm{Isom}(\mathbb{H}^2)$. It follows that Γ_g satisfies the Baum-Connes conjecture with coefficients ([24]). The usual compactification of $\mathbb{H}^2$ by a circle at infinity $\partial\mathbb{H}^2 \cong S^1$ is both the visibility compactification and the Gromov compactification of $\mathbb{H}^2$ and therefore produces a boundary action of Γ_g.

As in the previous example, Γ_g is torsion-free and $\Gamma_g\backslash\mathbb{H}^2 \cong \Sigma_g$ is compact, so that $\mathrm{Eul}_{\mathbb{H}^2} \in \mathrm{K}_0^{\mathrm{top}}(G)$ is the Euler characteristic of G. The K-theory and K-homology of Σ_g are isomorphic to $\mathbb{Z}^2$ in degree 0 and $\mathbb{Z}^{2g}$ in degree 1, and $\chi(\Sigma_g) = 2 - 2g$. Therefore, Corollary 32 yields

$$\mathrm{K}_0(C(\partial\mathbb{H}^2) \rtimes_{\mathrm{r}} \Gamma_g) \cong \mathbb{Z}/\langle 2g-2\rangle \oplus \mathbb{Z}^{2g+1}, \qquad \mathrm{K}_1(C(\partial\mathbb{H}^2) \rtimes_{\mathrm{r}} \Gamma_g) \cong \mathbb{Z}^{2g+1}.$$

Explicit generators, as well as a dynamical proof of these assertions, can be found in [16]. This example is also studied in [1,11,12,41].

Example 35. Consider $G := \mathrm{PSL}(2,\mathbb{Z})$, acting properly on the tree X discussed in Example 23. Let $\bar{X}$ be the ends compactification of X and let $\partial X := \bar{X} \setminus X$; this is the same as the Gromov or the visibility boundary of the tree X. The group G satisfies the Baum-Connes conjecture with arbitrary coefficients because it is a closed subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ [24].

Since $G\backslash X$ is compact, we have $\mathrm{KK}_0^G(C_0(X),\mathbb{C}) \cong \mathrm{K}_0^{\mathrm{top}}(G)$, and the Euler characteristic $\mathrm{Eul}_X \in \mathrm{KK}_0^G(C_0(X),\mathbb{C}) \cong \mathrm{K}_0^{\mathrm{top}}(G)$ is the Euler characteristic of G. We have already computed $\mathrm{Eul}_X^{\mathrm{cmb}} = \mathrm{Eul}_X$ in Example 23. Hence

$$\mathrm{Eul}_{\mathscr{E}} G = \dim_{\mathbb{Z}/2} + \dim_{\mathbb{Z}/3} - \dim_{\{1\}} \in \mathrm{K}_0^{\mathrm{top}}(G).$$

Functions vanishing on the vertices form a G-invariant ideal in $C_0(X)$ that is isomorphic to $C_0(\mathbb{R} \times G)$ with the free action of G; the quotient C^*-algebra is isomorphic to $C_0(G/\mathbb{Z}/2) \oplus C_0(G/\mathbb{Z}/3)$. The corresponding long exact sequences for $\mathrm{K}_*(\llcorner \rtimes_{\mathrm{r}} G)$ and $\mathrm{KK}_*^G(\llcorner,\mathbb{C})$ are

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) & \longrightarrow & \mathrm{Rep}(\mathbb{Z}/2) \oplus \mathrm{Rep}(\mathbb{Z}/3) \\
\uparrow & & & & \downarrow{\scriptstyle (\dim,-\dim)} \\
0 & \longleftarrow & \mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) & \longleftarrow & \mathbb{Z},
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Rep}(\mathbb{Z}/2) \oplus \mathrm{Rep}(\mathbb{Z}/3) & \longrightarrow & \mathrm{KK}_0^G(C_0(X),\mathbb{C}) & \longrightarrow & 0 \\
\uparrow & & & & \downarrow \\
\mathbb{Z} & \longleftarrow & \mathrm{KK}_1^G(C_0(X),\mathbb{C}) & \longleftarrow & 0.
\end{array}$$

The vertical map in the second exact sequence sends $1 \in \mathbb{Z}$ to $(\rho, -\rho)$, where ρ denotes the regular representation. Thus $\mathrm{K}_1^{\mathrm{top}}(G) \cong 0$, $\mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) \cong 0$, $\mathrm{K}_0^{\mathrm{top}}(G) \cong \mathbb{Z}^4$, $\mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \cong \mathbb{Z}^4$. One can check that multiplication by $\mathrm{Eul}_{\mathscr{E}} G$ is a bijective map $\mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0^{\mathrm{top}}(G)$. Hence $\mathrm{K}_*(C(\partial X) \rtimes_{\mathrm{r}} G) \cong 0$.

If we consider the boundary action on $\partial\mathbb{H}^2$, then we replace $C_0(X)$ by $C_0(\mathbb{H}^2)$ in the Gysin sequence. Equivariant Bott periodicity applies here and yields $\mathrm{K}_*(C_0(\mathbb{H}^2) \rtimes_{\mathrm{r}} G) \cong \mathrm{K}_*^{\mathrm{top}}(G)$. Hence this group is concentrated in degree 0. One can check that the map $\mathrm{K}_0(C_0(\mathbb{H}^2) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G)$ that is induced by the embedding $X \to \mathbb{H}^2$ has kernel and cokernel isomorphic to $\mathbb{Z}$. Since the map $K_0(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0^{\mathrm{top}}(G)$ in the Gysin sequence for ∂X is invertible, the map $K_0(C_0(\mathbb{H}^2) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_0^{\mathrm{top}}(G)$ in the Gysin sequence for $\partial\mathbb{H}^2$ has kernel and cokernel $\mathbb{Z}$ as well. Thus $\mathrm{K}_*(C(\partial\mathbb{H}^2) \rtimes_{\mathrm{r}} G) \cong \mathbb{Z}$ for $* = 0, 1$.

Example 36. Let G be a reductive p-adic group and let $\Gamma \subseteq G$ be a torsion-free discrete subgroup. Let X be the affine Bruhat-Tits building of G and let ∂X_∞ be its visibility boundary. Recall that this is a boundary action of G. The Fürstenberg boundary is G/P, where $P \subseteq G$ is a minimal parabolic subgroup. Since there exist points in ∂X_∞ that are fixed by P, we get an embedding $G/P \subseteq \partial X_\infty$, which induces a map $\varphi\colon C(\partial X_\infty) \to C(G/P)$.

We assume that $\Gamma \subseteq G$ is cocompact or, equivalently, that $\Gamma\backslash X$ is compact, and that $\chi(\Gamma\backslash X) \neq 0$. We want to show that the class of the unit element in $\mathrm{K}_0(C(G/P) \rtimes_{\mathrm{r}} \Gamma)$ is a torsion element whose order divides $\chi(\Gamma\backslash X)$. For certain buildings, this result and much sharper estimates for the order have been obtained previously by Guyan Robertson in [29,45–48]. We remark that we get no information about the torsion of the unit element if Γ fails to be cocompact or if $\chi(\Gamma\backslash X) = 0$.

Observe first that $\dim \in \mathrm{K}_0^{\mathrm{top}}(\Gamma)$ is a canonical choice of a pre-image for the class of the unit element in $C_{\mathrm{r}}^*\Gamma$. As in Corollary 32, we find that the image of $\dim$ in $\mathrm{K}_0^{\mathrm{top}}\big(\Gamma, C(\partial X_\infty)\big)$ is a torsion element of order exactly equal to $|\chi(\Gamma\backslash X)|$. Mapping further via $\varphi\colon C(\partial X_\infty) \to C(G/P)$, we find that the image $\dim'$ of $\dim$ in $\mathrm{K}_0^{\mathrm{top}}\big(\Gamma, C(G/P)\big)$ is a torsion element whose order divides $\chi(\Gamma\backslash X)$. It is easy to see that the Baum-Connes assembly map sends $\dim'$ to the class [1] of the unit element in $C(G/P) \rtimes_{\mathrm{r}} \Gamma$. Hence $\chi(\Gamma\backslash X)[1] = 0$ as asserted.

Similarly, let G be an almost connected Lie group whose connected component is reductive and linear, and let $\Gamma \subseteq G$ be a torsion-free, cocompact discrete subgroup. Let $X := G/K$, where K is the maximal compact subgroup of G, and let ∂X_∞ be its visibility boundary. Again, this is a boundary action of G, and there exists an embedding $G/P \subseteq \partial X_\infty$ of the Fürstenberg boundary because there exist points in ∂X_∞ that are fixed by P. Corollary 32 applies because X has a Γ-invariant triangulation (see [20]). Arguing as above, we see that the class of the unit element in $\mathrm{K}_0(C(G/P) \rtimes_{\mathrm{r}} \Gamma)$ is a torsion element whose order divides $\chi(\Gamma\backslash X)$ if the latter is non-zero.

8. Equivariant Euler characteristics for smooth manifolds

If a locally compact group G acts properly by diffeomorphisms on a smooth manifold M, then there exists a complete G-invariant Riemannian metric on M. Hence the action of G factors through $\mathrm{Isom}(M)$, which is a Lie group unless M has infinitely many connected components. Throughout this section, we consider the situation where M is a complete Riemannian manifold and G is a locally compact group that acts isometrically on M. It does not matter whether or not this action is proper because $\mathrm{Isom}(M)$ acts properly on M in any case.

We recall the construction of a Kasparov dual for M in [25, Section 4].

Let $\mathbb{C}\mathrm{liff}$ be the bundle whose fibre at $x \in M$ is the Clifford algebra for the vector space T_x^*M with inner product given by the Riemannian metric. This is a bundle of $\mathbb{Z}/2$-graded finite-dimensional C^*-algebras, on which G acts in a canonical way. Let $\mathscr{P} := C_0(M, \mathbb{C}\mathrm{liff})$ be its $\mathbb{Z}/2$-graded C^*-algebra of C_0-sections, equipped with the canonical action of G. ($\mathscr{P}$ is denoted $C_\tau(M)$ in [25, 4.1].) We have a central embedding $C_0(M) \to \mathscr{P}$ as scalar-valued functions, so that $\mathscr{P}$ becomes a $\mathbb{Z}/2$-graded $M \rtimes G$-C^*-algebra.

Now we describe $D \in \mathrm{KK}_0^G(\mathscr{P}, \mathbb{C})$ (see [25, 4.2]). Let $\Lambda^*M = \bigoplus_n \Lambda^n M$ be the bundle of differential forms on M, graded by parity. Let $C_c^\infty(\Lambda^*M)$ be the space of smooth, compactly supported sections of Λ^*M, and let $L^2(\Lambda^*M)$ be the Hilbert space completion of $C_c^\infty(\Lambda^*M)$ with respect to the standard inner product given by the Riemannian metric. We let $\mathscr{P}$ act on $L^2(\Lambda^*M)$ by Clifford multiplication. Let $d\colon C_c^\infty(\Lambda^*M) \to C_c^\infty(\Lambda^*M)$ be the de Rham differential. The operator $d+d^*$ is an essentially self-adjoint, G-invariant unbounded operator on $L^2(\Lambda^*M)$. Together with the representation of $\mathscr{P}$, it defines a class $D \in \mathrm{KK}_0^G(\mathscr{P}, \mathbb{C})$. Here we use the unbounded picture of Kasparov theory by Saad Baaj and Pierre Julg ([2]).

Next we define $\Theta \in \mathrm{RKK}_0^G(M; \mathbb{C}, \mathscr{P})$ as in [25, 4.3–4.4]. The basic ingredients are the geodesic distance function $\rho\colon M \times M \to M$ and a G-invariant function $r\colon M \times M \to \mathbb{R}_{>0}$ such that any $x, y \in M$ with $\rho(x, y) < r(x)$ are joined by a unique geodesic. Let

$$U := \{(x, y) \in M \times M \mid \varrho(x, y) < r(x)\}$$

and pull T^*M back to a bundle $\pi_2^*T^*M$ on U via the coordinate projection $\pi_2\colon (x, y) \mapsto y$. Let $J_U \subseteq C_0(M) \hat{\otimes} \mathscr{P}$ be the ideal of sections that vanish outside U. We view J_U as a G-equivariant $\mathbb{Z}/2$-graded Hilbert module over $C_0(M) \hat{\otimes} \mathscr{P}$.

Define a covector field F on U by

$$F(x, y) := \frac{\rho(x, y)}{r(x)} \cdot d_2\rho(x, y) \in C_0(U, \pi_2^* T^* M).$$

where d_2 is the exterior derivative in the second variable y. The covector field F defines a G-invariant self-adjoint, odd multiplier of J_U; it satisfies $(f \hat{\otimes} 1) \cdot (1 - F^2) \in J_U$ for all $f \in C_0(M)$. Thus $\Theta = (J_U, F)$ is a cycle for $\mathrm{RKK}_0^G(M; \mathbb{C}, \mathscr{P})$. It is asserted in [25, 4.5, 4.6, 4.8] that $(\mathscr{P}, D, \Theta)$ is a Kasparov dual for M in the sense of Definition 18.

Definition 37. Let $\mathrm{Eul}_M^{\mathrm{dR}} \in \mathrm{KK}_0^G(C_0(M), \mathbb{C})$ be the class determined by the representation of $C_0(M)$ on $L^2(\Lambda^* M)$ and the operator $d + d^*$ described above. We call $\mathrm{Eul}_M^{\mathrm{dR}}$ the *G-equivariant de-Rham-Euler characteristic of M*.

Equivalently, we get $\mathrm{Eul}_M^{\mathrm{dR}}$ from $D \in \mathrm{KK}_0^G(\mathscr{P}, \mathbb{C})$ by restricting the representation of $\mathscr{P}$ to $C_0(M) \subseteq \mathscr{P}$.

Theorem 38. *Let M be a complete Riemannian manifold and let G be a locally compact group acting isometrically on M. Then the abstract G-equivariant Euler characteristic* Eul_M *is equal to* $\mathrm{Eul}_M^{\mathrm{dR}}$.

Proof. Lemma 20 asserts that

$$\mathrm{Eul}_M = \bar{\Theta} \hat{\otimes}_{C_0(M)\hat{\otimes}\mathscr{P}} [m] \hat{\otimes}_{\mathscr{P}} D \in \mathrm{KK}_0^G(C_0(M), \mathbb{C}),$$

where $\bar{\Theta} \in \mathrm{KK}_0^G(C_0(M), C_0(M) \hat{\otimes} \mathscr{P})$ is obtained from Θ by forgetting the M-structure and where $m\colon C_0(M) \hat{\otimes} \mathscr{P} \to \mathscr{P}$ is the M-structure homomorphism of $\mathscr{P}$. We first compute

$$\bar{\Theta} \hat{\otimes}_{C_0(M)\hat{\otimes}\mathscr{P}} [m] = m_*(\bar{\Theta}) \in \mathrm{KK}_0^G(C_0(M), \mathscr{P}).$$

Its underlying Hilbert module is $J_U \hat{\otimes}_{C_0(M)\hat{\otimes}\mathscr{P}} \mathscr{P}$; this is isomorphic to $\mathscr{P}$ because J_U is an ideal in $C_0(M) \hat{\otimes} \mathscr{P}$ that contains all functions supported in some neighbourhood of the diagonal, and the multiplication homomorphism m restricts to the diagonal. The action of $C_0(M)$ is by multiplication on J_U; this corresponds to the embedding $m'\colon C_0(M) \to \mathscr{P}$ by scalar-valued functions. Since this homomorphism maps into $\mathscr{P}$ and not just into $\mathcal{M}(\mathscr{P})$, the Fredholm operator is irrelevant. Thus $m^*(\bar{\Theta}) = m'$.

Taking the Kasparov product with $D \in \mathrm{KK}_0^G(\mathscr{P}, \mathbb{C})$, we get

$$\mathrm{Eul}_M = \bar{\Theta} \hat{\otimes}_{C_0(M)\hat{\otimes}\mathscr{P}} [m] \hat{\otimes}_{\mathscr{P}} D = [m'] \hat{\otimes}_{\mathscr{P}} D = (m')^*(D).$$

This is equal to $\mathrm{Eul}_M^{\mathrm{dR}} \in \mathrm{KK}_0^G(C_0(M), \mathbb{C})$ by definition. □

Theorem 39 (Lück and Rosenberg [34]). *Let M be a smooth manifold and let G be a discrete group acting on M properly by diffeomorphisms. Then M has a G-equivariant triangulation. The de-Rham-Euler characteristic and the combinatorial Euler characteristic agree:*

$$\mathrm{Eul}_M^{\mathrm{dR}} = \mathrm{Eul}_M = \mathrm{Eul}_M^{\mathrm{cmb}} \in \mathrm{KK}_0^G(C_0(M), \mathbb{C}).$$

Proof. We have seen in Section 4 that the (abstract) equivariant Euler characteristic is independent of the Kasparov dual. Hence the assertion follows by combining Theorems 30 and 38. Recall that a smooth manifold equipped with a proper action of a discrete group G always admits a Riemannian metric for which the group acts isometrically and a triangulation for which the group acts simplicially ([20]). □

Remark 40. The analogues of Theorems 38 and 39 in real K-homology also hold, by exactly the same arguments.

As before, let M be an n-dimensional Riemannian manifold equipped with an isometric action of G. Assume, in addition, that M is G-equivariantly K-oriented. This means that its tangent bundle has a G-equivariant complex spinor bundle Spinor (see [15]). This bundle gives rise to a G-equivariant Morita equivalence between $\mathscr{P} = C_0(M, \mathbb{C}\mathrm{liff})$ and the trivial Clifford algebra bundle $C_0(M) \hat{\otimes} \mathbb{C}\mathrm{liff}(\mathbb{R}^n)$ (see [41]). Therefore, $C_0(M)$ and $\mathscr{P}$ are $\mathrm{KK}^{M \rtimes G}$-equivalent with a dimension shift of n. We may transport the structure of a Kasparov dual from $\mathscr{P}$ to $C_0(M)$. It is easy to see that $D \in \mathrm{KK}_0^G(\mathscr{P}, \mathbb{C})$ corresponds to the Dirac operator $\not{D}_M \in \mathrm{KK}_n^G(C_0(M), \mathbb{C})$, which acts on sections of Spinor. The map $\sigma_{M,C_0(M)}$ is simply the forgetful map as in (6). Hence the inverse

$$\mathrm{PD}^{-1}\colon \mathrm{RKK}_i^G(M; \mathbb{C}, \mathbb{C}) \to \mathrm{KK}_{i+n}^G(C_0(M), \mathbb{C}),$$

of the Poincaré duality map is given by

$$\mathrm{PD}^{-1}(f) := (-1)^{in} \bar{f} \hat{\otimes}_{C_0(M)} \not{D}_M.$$

We also get a class $[\mathrm{Spinor}] \in \mathrm{RKK}_{-n}^G(M; \mathbb{C}, \mathbb{C})$ by taking C_0-sections of Spinor with $F = 0$. Using $\mathrm{Spinor} \hat{\otimes} \mathrm{Spinor} \cong \Lambda^* M$, one shows easily that

$$\begin{aligned} \mathrm{PD}^{-1}[\mathrm{Spinor}] &= (-1)^n \overline{[\mathrm{Spinor}]} \hat{\otimes}_{C_0(M)} \not{D}_M \\ &= (-1)^n \mathrm{Eul}_M^{\mathrm{dR}} = (-1)^n \mathrm{Eul}_M. \end{aligned} \tag{58}$$

Together with Theorem 39, this allows us to compute the classical, commutative Gysin sequence in K-theory for the tangent bundle. We have seen in the introduction how to get a long exact Gysin sequence of the form

$$\cdots \to \mathrm{K}^{*-n}(M) \xrightarrow{\epsilon^*} \mathrm{K}^*(M) \xrightarrow{\pi^*} \mathrm{K}^*(SM) \xrightarrow{\delta} \mathrm{K}^{*-n+1}(M) \to \cdots,$$

where $\varepsilon^*(x) = x \hat{\otimes} \mathrm{Spinor}$ and $\pi\colon SM \to M$ is the bundle projection (see [23, IV.1.13] for more details). Now (58) yields:

Theorem 41. *Let M be a* K*-oriented, connected n-dimensional manifold. Let SM be its sphere bundle and let $\pi : SM \to M$ be the bundle projection. If M is compact and $\chi(M) \neq 0$, then there are exact sequences*

$$0 \to \langle \chi(M)\mathrm{pnt!}\rangle \xrightarrow{\subseteq} \mathrm{K}^n(M) \xrightarrow{\pi^*} \mathrm{K}^n(SM) \to \mathrm{K}^1(M) \to 0,$$

$$0 \to \mathrm{K}^{n+1}(M) \xrightarrow{\pi^*} \mathrm{K}^{n+1}(SM) \to \mathrm{K}^0(M) \xrightarrow{\dim} \mathbb{Z} \to 0,$$

where $\mathrm{pnt!} \in \mathrm{KK}_{-n}(C(*), C(M)) \cong \mathrm{K}^n(M)$ *is the wrong way element associated to the inclusion of a point in M.*

If M is not compact or if $\chi(M) = 0$, then there are exact sequences

$$0 \to \mathrm{K}^n(M) \xrightarrow{\pi^*} \mathrm{K}^n(SM) \to \mathrm{K}^1(M) \to 0,$$

$$0 \to \mathrm{K}^{n+1}(M) \xrightarrow{\pi^*} \mathrm{K}^{n+1}(SM) \to \mathrm{K}^0(M) \to 0.$$

Proof. By Theorem 39 we have $\mathrm{Eul}_M^{\mathrm{dR}} = \mathrm{Eul}_M^{\mathrm{cmb}}$. Since there is no group action, we have $\mathrm{Eul}_M^{\mathrm{cmb}} = \chi(M)\cdot\dim \in \mathrm{KK}_0(C(M), \mathbb{C})$ if M is compact, and $\mathrm{Eul}_M^{\mathrm{cmb}} = 0$ otherwise. Hence (58) yields [Spinor] $= 0$ if M is not compact or if $\chi(M) = 0$ and finishes the proof in that case. Otherwise, [Spinor] $= (-1)^n \chi(M) \cdot \mathrm{PD}(\dim)$. It is easy to see that $\mathrm{PD}(\dim) = \mathrm{pnt!}$; recall that pnt! is the class of the map $C_0(\mathbb{R}^n) \to C_0(M)$ given by a diffeomorphism from $\mathbb{R}^n$ onto some (small) open ball in M. Since any bundle restricts to a trivial bundle on this open ball, we get $x \hat{\otimes} \mathrm{pnt!} = 0$ for $x \in \mathrm{K}^1(M)$ and $x \hat{\otimes} \mathrm{pnt!} = \dim(x) \cdot \mathrm{pnt!}$ for $x \in \mathrm{K}^0(M)$. Hence the range and kernel of the map $\mathrm{K}^0(M) \to \mathrm{K}^n(M)$ in the Gysin sequence are equal to $\langle \chi(M)\mathrm{pnt!}\rangle$ and the kernel of dim, respectively. This yields the desired exact sequences. □

Now we return to the situation of boundary actions.

Theorem 42. *Let $\partial X = \bar{X} \setminus X$ be a boundary action of a locally compact group G as in Definition 4. Suppose that X is a complete Riemannian manifold on which G acts isometrically. Suppose also that G satisfies the Baum-Connes conjecture with coefficients in $\mathbb{C}$ and $C(\partial X)$. Then there is an exact sequence*

$$\begin{array}{ccccc}
\mathrm{K}_1(C_\mathrm{r}^* G) & \xrightarrow{u_*} & \mathrm{K}_1(C(\partial X) \rtimes_\mathrm{r} G) & \xrightarrow{\delta} & \mathrm{K}_0(C_0(X) \rtimes_\mathrm{r} G) \\
\uparrow{\scriptstyle \mathrm{Eul}_X^{\mathrm{dR}}} & & & & \downarrow{\scriptstyle \mathrm{Eul}_X^{\mathrm{dR}}} \\
\mathrm{K}_1(C_0(X) \rtimes_\mathrm{r} G) & \xleftarrow{\delta} & \mathrm{K}_0(C(\partial X) \rtimes_\mathrm{r} G) & \xleftarrow{u_*} & \mathrm{K}_0(C_\mathrm{r}^* G),
\end{array}$$

where $\mathrm{Eul}_X^{\mathrm{dR}}$ denotes the Kasparov product with the equivariant de-Rham-Euler characteristic $\mathrm{Eul}_X^{\mathrm{dR}} \in \mathrm{KK}_0^G(C_0(X), \mathbb{C})$.

Proof. We may replace $\mathrm{K}_*(C^*_{\mathrm{r}}G)$ and $\mathrm{K}_*(C(\partial X) \rtimes_{\mathrm{r}} G)$ by $\mathrm{K}^{\mathrm{top}}_*(G)$ and $\mathrm{K}^{\mathrm{top}}_*\big(G, C(\partial X)\big)$ because of our assumptions about the Baum-Connes conjecture. Hence the result follows from Theorem 38 and Proposition 9. □

The map $\mathrm{Eul}^{\mathrm{dR}}_X\colon \mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) \to \mathrm{K}_1(C^*_{\mathrm{r}}G)$ in Theorem 42 vanishes if G is discrete, compare Theorem 31. We do not know whether this still holds for non-discrete groups. In the following, we will examine some cases where $\mathrm{Eul}^{\mathrm{dR}}_X$ vanishes, so that the long exact sequence in Theorem 42 splits into two short exact sequences.

Proposition 43. *Let M be an oriented Riemannian manifold of odd dimension and suppose that G acts on M by orientation-preserving isometries. Then* $\mathrm{Eul}^{\mathrm{dR}}_M = 0$ *in* $\mathrm{KK}^G_0(C_0(M), \mathbb{C})$.

Proof. Let Vol be the canonical volume form. Let $n := \dim M$ and write $n = 2k + 1$ with $k \in \mathbb{N}$. We shall use the *Hodge $\star$ operation* ([43, §24–25]). It is a $C_0(M)$-linear map $\star\colon L^2(\Lambda^p M) \to L^2(\Lambda^{n-p} M)$ for all $p \in \underline{\mathbf{n}}$; it is defined by $\beta \wedge \star\alpha = (\alpha, \beta) \cdot \mathrm{Vol}$ for all $\alpha, \beta \in L^2(\Lambda^p M)$, where $(\alpha, \beta) \in C_0(M)$ denotes the pointwise inner product induced by the Riemannian metric. The operator $\star$ is unitary on $L^2(\Lambda^* M)$ and satisfies

$$\star\star\alpha = (-1)^{pn+p}\alpha, \qquad d^*(\alpha) = (-1)^{pn+n+1} \star d \star \alpha$$

for all $\alpha \in L^2(\Lambda^p M)$ (see [43, §24–25]). Consider the operator

$$\varepsilon\colon L^2(\Lambda^* M) \to L^2(\Lambda^* M), \qquad \alpha \mapsto \mathrm{i}^{k+p(p-1)} \star \alpha \quad \text{for } \alpha \in L^2(\Lambda^p M).$$

Straightforward computations show that $\varepsilon^2 = 1$ and $\varepsilon d \varepsilon = -d^*$; this implies that ε anti-commutes with $d + d^*$.

Since ε is still unitary and odd, it generates a grading-preserving representation of the Clifford algebra $\mathrm{Cliff}(\mathbb{R})$ on $L^2(\Lambda^* M)$. It commutes with the representations of G and $C_0(M)$ because G acts by orientation-preserving maps and $\star$ is $C_0(M)$-linear. Thus $(L^2(\Lambda^* M), d + d^*)$ becomes a cycle D_1 for $\mathrm{KK}^G_0(C_0(M) \hat{\otimes} \mathrm{Cliff}(\mathbb{R}), \mathbb{C})$. We have $\mathrm{Eul}^{\mathrm{dR}}_X = [u_{\mathrm{Cliff}(\mathbb{R})}] \hat{\otimes}_{\mathrm{Cliff}(\mathbb{R})} D_1$, where $u_{\mathrm{Cliff}(\mathbb{R})}\colon \mathbb{C} \to \mathrm{Cliff}(\mathbb{R})$ is the unit map. The Kasparov cycle $[u_{\mathrm{Cliff}(\mathbb{R})}]$ is evidently degenerate. Hence $\mathrm{Eul}^{\mathrm{dR}}_X = 0$. □

In the real case, the same argument still goes through if $\dim M \equiv 1 \pmod 4$ because then $\mathrm{i}^{k+p(p-1)} = \pm 1$ for all p.

The assumption that the action of G be orientation-preserving is necessary in Proposition 43. For a counterexample, take $M = S^1$ and $G = \mathbb{Z}/2$ acting on S^1 by reflection in the x-axis (equivalently, by complex conjugation). A straightforward computation shows that $\mathrm{Eul}^{\mathrm{cmb}}_{S^1} \neq 0$ in $\mathrm{KK}^{\mathbb{Z}/2}_0(C(S^1), \mathbb{C})$. Another counterexample is Example 44 below.

Now we consider the following situation. Let G be an almost connected Lie group and let $K \subseteq G$ be a maximal compact subgroup. Then the homogeneous space $X := G/K$ is a smooth manifold on which G acts properly and smoothly. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively, with K acting by the adjoint representation, and let $\mathfrak{p} := \mathfrak{g}/\mathfrak{k}$ with the induced action of K. Thus $\mathfrak{p}$ is K-equivariantly isomorphic to the tangent space of X at $K \in X$. We may equip $\mathfrak{p}$ with a K-invariant inner product. This inner product generates a G-invariant Riemannian metric on X, and any such Riemannian metric is complete. Hence the above construction of a Kasparov dual applies to X and we get $\mathrm{Eul}_X^{\mathrm{dR}} = \mathrm{Eul}_X$ in $\mathrm{KK}_0^G(C_0(X), \mathbb{C})$ by Theorem 38.

It is known that X is a universal proper G-space and that

$$\mathrm{K}_*^{\mathrm{top}}(G) \cong \mathrm{KK}_*^G(C_0(X), \mathbb{C}) \cong \mathrm{KK}_*^K(C_0(X), \mathbb{C}); \tag{59}$$

the first isomorphism follows because X is G-compact, the second is proved in [25]. Thus the Euler characteristic $\mathrm{Eul}_{\mathscr{E}} G$ of G is equal to $\mathrm{Eul}_X^{\mathrm{dR}}$ (see Definition 15).

Example 44. If $\dim X$ is odd and K acts on X by orientation-preserving maps, then we get $\mathrm{Eul}_{\mathscr{E}} G = 0$ from Proposition 43.

This fails if the action of K is not orientation-preserving; the semi-direct product group $\mathbb{Z}/2 \ltimes \mathbb{R}$ with $\mathbb{Z}/2$ acting on $\mathbb{R}$ by reflection at 0 provides a counterexample. By (59) we only have to compute the $\mathbb{Z}/2$-equivariant Euler characteristic. This agrees with the $\mathbb{Z}/2$-equivariant combinatorial Euler characteristic of $\mathbb{R}$. We use the standard triangulation of $\mathbb{R}$ with vertex set $\mathbb{Z}$. The combinatorial Euler characteristic turns out to be the class in $\mathrm{KK}_0^{\mathbb{Z}/2}(C_0(\mathbb{R}), \mathbb{C})$ of the homomorphism $C_0(\mathbb{R}) \to \mathbb{C}$, $f \mapsto f(0)$. This class is non-zero, only its image in $\mathrm{KK}_0(C_0(\mathbb{R}), \mathbb{C})$ vanishes.

From now on, we assume that X is even-dimensional and that the action of G preserves the orientation. Even more, we want X to be G-equivariantly K-oriented. This is equivalent to the existence of a K-equivariant complex spinor bundle $S_{\mathfrak{p}}$ for $\mathfrak{p}$. This automatically exists if G is simply connected. In general, we can get such a spinor bundle if we replace G by an appropriate two-fold covering $\tilde{G}$. The computation of $\mathrm{K}_*^{\mathrm{top}}(G)$ in [14, Section 4] identifies $\mathrm{K}_*^{\mathrm{top}}(G)$ with a direct summand in $\mathrm{K}_*^{\mathrm{top}}(\tilde{G})$. Therefore, we do not loose information if we work $\tilde{G}$-equivariantly instead of G-equivariantly; hence it is no loss of generality to assume that X is G-equivariantly K-oriented.

As we have observed above, the existence of a K-orientation implies that $C_0(X)$ is a 0-dimensional Kasparov dual for X, so that we get a Poincaré duality isomorphism

$$\mathrm{KK}_*^G(C_0(X), \mathbb{C}) \cong \mathrm{RKK}_*^G(X; \mathbb{C}, \mathbb{C}),$$

which maps $\mathrm{Eul}_{\mathscr{E}} G$ to [Spinor] by (58). Moreover, since the groupoid $G \ltimes X$ is Morita equivalent to the group K, we get an isomorphism

$$\mathrm{RKK}^G_*(X; \mathbb{C}, \mathbb{C}) \cong \mathrm{KK}^K_*(\mathbb{C}, \mathbb{C}) \cong \begin{cases} \mathrm{Rep}(K) & \text{for } * = 0, \\ 0 & \text{for } * = 1, \end{cases}$$

see [25]. The resulting isomorphism $\mathrm{K}_0^{\mathrm{top}}(G) \cong \mathrm{Rep}(K)$ maps $\mathrm{Eul}_{\mathscr{E}} G$ to $[S_{\mathfrak{p}}]$.

By construction, we have $S_{\mathfrak{p}} \hat{\otimes} S^*_{\mathfrak{p}} \cong \Lambda^* \mathfrak{p}$. Let $\chi : G \to \mathbb{C}$ be the character of the representation $S_{\mathfrak{p}}$. Then $|\chi|^2$ is the character of $\Lambda^*\mathfrak{p}$. We have $[S_{\mathfrak{p}}] = 0$ if and only if $[\Lambda^*\mathfrak{p}] = 0$, if and only if $|\chi|^2 = 0$.

We denote the representation of K on $\mathfrak{p}$ by $\alpha_{\mathfrak{p}}$. For $g \in G$, let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\alpha_{\mathfrak{p}}(g)$, counted with multiplicity; then we can describe the eigenvalues of g acting on $\Lambda^*\mathfrak{p}$ as well and get

$$|\chi(g)|^2 = \prod_{j=1}^{n} (1 - \lambda_j) = \det\bigl(1 - \alpha_{\mathfrak{p}}(g)\bigr).$$

Hence $|\chi|^2$ vanishes at g if and only if 1 is an eigenvalue of $\alpha_{\mathfrak{p}}(g)$.

Assume now, in addition, that K is connected, and let $T \subseteq K$ be a maximal torus, with Lie algebra $\mathfrak{t}$; thus $\mathfrak{t}$ is a maximal Abelian subspace of $\mathfrak{k}$. Then any conjugacy class in K meets T. Therefore, $[S_{\mathfrak{p}}] = 0$ if and only if $|\chi|^2$ vanishes on T. This holds if and only if $\alpha_{\mathfrak{p}}|_T$ contains the trivial representation, if and only if $\mathfrak{t}$ is *not* maximal Abelian in $\mathfrak{g}$. We have proved:

Proposition 45. *Let G be a connected Lie group and $K \subseteq G$ a maximal compact subgroup. If* $\dim \mathfrak{g} \not\equiv \dim \mathfrak{k} \bmod (2)$*, then we always have* $\mathrm{Eul}_{\mathscr{E}} G = 0$*. If* $\dim \mathfrak{g} \equiv \dim \mathfrak{k} \bmod (2)$*, then* $\mathrm{Eul}_{\mathscr{E}} G \neq 0$ *if and only if maximal Abelian subspaces of* $\mathfrak{k}$ *remain maximal Abelian in* $\mathfrak{g}$.

The *complex rank* of a reductive Lie algebra is, by definition, the rank of a maximal Abelian subspace. Therefore, if $\mathfrak{g}$ is reductive (and $\dim X$ is even), then $\mathrm{Eul}_{\mathscr{E}} G \neq 0$ if and only if $\mathfrak{g}$ and $\mathfrak{k}$ have the same complex rank. This never happens if $\mathfrak{g}$ is a complex Lie algebra. Semi-simple real Lie algebras that are not complex are classified in [30] using their Vogan diagrams. One can check that $\mathfrak{g}$ and $\mathfrak{k}$ have the same complex rank if and only if the order-2 automorphism that is part of the Vogan diagram of $\mathfrak{g}$ is trivial.

We return to the situation where G is a connected Lie group and $X := G/K$ is G-equivariantly K-oriented and even-dimensional. Suppose, in addition, that $\bar{X}$ is a strongly contractible G-equivariant compactification of X, so that we get a boundary action of G on $\partial X := \bar{X} \setminus X$. We have a Morita equivalence $C_0(X) \rtimes_{\mathrm{r}} G \sim C^*(K)$, so that $\mathrm{K}_0(C_0(X) \rtimes_{\mathrm{r}} G) \cong \mathrm{Rep}(K)$ and $\mathrm{K}_1(C_0(X) \rtimes_{\mathrm{r}} G) = 0$. Thus the Gysin sequence in Theorem 42 simplifies to an exact sequence

$$0 \to \mathrm{K}_1(C(\partial X) \rtimes_{\mathrm{r}} G) \to \mathrm{Rep}(K) \to \mathrm{Rep}(K) \to \mathrm{K}_0(C(\partial X) \rtimes_{\mathrm{r}} G) \to 0,$$

which contains the map $\mathrm{Rep}(K) \to \mathrm{Rep}(K), \pi \mapsto \pi \hat{\otimes} [S_\mathfrak{p}]$. If $[S_\mathfrak{p}] = 0$, then we get $\mathrm{K}_0(C(\partial X) \rtimes_\mathrm{r} G) \cong \mathrm{Rep}(K)$ and $\mathrm{K}_1(C(\partial X) \rtimes_\mathrm{r} G) \cong \mathrm{Rep}(K)$. If $[S_\mathfrak{p}] \neq 0$, then the map $\mathrm{Rep}(K) \to \mathrm{Rep}(K)$ is injective; here we use that the ring $\mathrm{Rep}(K)$ has no zero-divisors because K is connected. Hence

$$\mathrm{K}_0(C(\partial X) \rtimes_\mathrm{r} G) \cong \mathrm{Rep}(K)/(S_\mathfrak{p}), \qquad \mathrm{K}_1(C(\partial X) \rtimes_\mathrm{r} G) \cong 0,$$

where $(S_\mathfrak{p})$ denotes the ideal generated by the virtual representation $S_\mathfrak{p}$.

Finally, let $\Gamma \subseteq G$ be a cocompact lattice. Then X is a cocompact universal proper Γ-space. The restriction map

$$\mathrm{K}_*^{\mathrm{top}}(G) \cong \mathrm{KK}_*^G(C_0(X), \mathbb{C}) \to \mathrm{KK}_*^\Gamma(C_0(X), \mathbb{C}) \cong \mathrm{K}_*^{\mathrm{top}}(\Gamma)$$

evidently maps $\mathrm{Eul}_\mathscr{E} G \mapsto \mathrm{Eul}_{\mathcal{E}\Gamma}$. Hence $\mathrm{Eul}_\mathscr{E} G = 0$ implies $\mathrm{Eul}_{\mathcal{E}\Gamma} = 0$. If Γ is also torsion-free, then $\mathrm{Eul}_{\mathcal{E}\Gamma} = 0$ is equivalent to $\chi(B\Gamma) = 0$.

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