Lecture Notes on C^*-algebras

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Preface

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Chapter 1

Basics of C^*-algebras

1.1 Definition

We begin with the definition of a C^*-algebra.

Definition 1.1.1. *A C^*-algebra A is a (non-empty) set with the following algebraic operations:*

1. *addition, which is commutative and associative*
2. *multiplication, which is associative*
3. *multiplication by complex scalars*
4. *an involution $a \mapsto a^*$ (that is, $(a^*)^* = a$, for all a in A)*

Both types of multiplication distribute over addition. For a, b in A, we have $(ab)^* = b^*a^*$. The involution is conjugate linear; that is, for a, b in A and λ in $\mathbb{C}$, we have $(\lambda a + b)^* = \bar{\lambda}a^* + b^*$. For a, b in A and λ, μ in $\mathbb{C}$, we have $\lambda(ab) = (\lambda a)b = a(\lambda b)$ and $(\lambda\mu)a = \lambda(\mu a)$.

In addition, A has a norm in which it is a Banach algebra; that is,

$$\begin{aligned} \|\lambda a\| &= |\lambda|\|a\|, \\ \|a + b\| &\leq \|a\| + \|b\|, \\ \|ab\| &\leq \|a\|\|b\|, \end{aligned}$$

for all a, b in A and λ in $\mathbb{C}$, and A is complete in the metric $d(a, b) = \|a - b\|$.

Finally, for all a in A, we have $\|a^*a\| = \|a\|^2$.

Very simply, A has an algebraic structure and a topological structure coming from a norm. The condition that A be a Banach algebra expresses a compatibility between these structures. The final condition, usually referred to as the C^*-condition, may seem slightly mysterious, but it is a very strong link between the algebraic and topological structures, as we shall see presently.

It is probably also worth mentioning two items which are *not* axioms. First, the algebra need not have a unit for the multiplication. If it does have a unit, we write it as 1 or 1_A and say that A is unital. Secondly, the multiplication is *not* necessarily commutative. That is, it is not generally the case that $ab = ba$, for all a, b. The first of these two issues turns out to be a relatively minor one (which will be dealt with in Section 1.8). The latter is essential and, in many ways, is the heart of the subject.

One might have expected an axiom stating that the involution is isometric. In fact, it is a simple consequence of the ones given, particularly the C^*-condition.

Proposition 1.1.2. *If a is an element of a C^*-algebra A, then $\|a\| = \|a^*\|$.*

Proof. As A is a Banach algebra $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$ and so $\|a\| \leq \|a^*\|$. Replacing a with a^* then yields the result. □

Next, we introduce some terminology for elements in a C^*-algebra. For a given a in A, the element a^* is usually called the *adjoint* of a. The first term in the following definition is then rather obvious. The second is much less so, but is used for historical reasons from operator theory. The remaining terms all have a geometric flavour. If one considers the elements in $\mathcal{B}(\mathcal{H})$, operators on a Hilbert space, each of these purely algebraic terms can be given an equivalent formulation in geometric terms of the action of the operator on the Hilbert space.

Definition 1.1.3. *Let A be a C^*-algebra.*

1. *An element a is* self-adjoint *if $a^* = a$.*

2. *An element a is* normal *if $a^*a = aa^*$.*

3. *An element p is a* projection *if $p^2 = p = p^*$; that is, p is a self-adjoint idempotent.*

4. *Assuming that A is unital, an element u is a* unitary *if $u^*u = 1 = uu^*$; that is, u is invertible and $u^{-1} = u^*$.*

5. *Assuming that A is unital, an element u is an* isometry *if $u^*u = 1$.*

6. *An element u is a* partial isometry *if u^*u is a projection.*

7. *An element a is* positive *if it may be written $a = b^*b$, for some b in A. In this case, we often write $a \geq 0$ for brevity.*

1.2 Examples

Example 1.2.1. $\mathbb{C}$*, the complex numbers. More than just an example, it is the prototype.*

Example 1.2.2. *Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted $< \cdot, \cdot >$. The collection of bounded linear operators on $\mathcal{H}$, denoted by $\mathcal{B}(\mathcal{H})$, is a C^*-algebra. The linear structure is clear. The product is by composition of operators. The $*$ operation is the adjoint; for any operator a on $\mathcal{H}$, its adjoint is defined by the equation $< a^*\xi, \eta > = < \xi, a\eta >$, for all ξ and η in $\mathcal{H}$. Finally, the norm is given by*

$$\|a\| = \sup\{\|a\xi\| \mid \xi \in \mathcal{H}, \|\xi\| \leq 1\},$$

for any a in $\mathcal{B}(\mathcal{H})$.

Example 1.2.3. *If n is any positive integer, we let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. It is a C^*-algebra using the usual algebraic operations for matrices. The $*$ operation is to take the transpose of the matrix and then take complex conjugates of all its entries. For the norm, we must resort back to the same definition as our last example*

$$\|a\| = \sup\{\|a\xi\|_2 \mid \xi \in \mathbb{C}^n, \|\xi\|_2 \leq 1\},$$

where $\| \cdot \|_2$ is the usual ℓ^2-norm on $\mathbb{C}^n$.

Of course, this example is a special case of the last using $\mathcal{H} = \mathbb{C}^n$, and using a fixed basis to represent linear transformations as matrices.

Example 1.2.4. *Let X be a compact Hausdorff space and consider*

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous }\}.$$

The algebraic operations of addition, scalar multiplication and multiplication are all point-wise. The $*$ is point-wise complex conjugation. The norm is the usual supremum norm

$$\|f\| = \sup\{|f(x)| \mid x \in X\}$$

for any f in $C(X)$. This particular examples has the two additional features that $C(X)$ is both unital and commutative.

Extending this slightly, let X be a locally compact Hausdorff space and consider

$$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous, vanishing at infinity }\}.$$

Recall that a function f is said to vanish at infinity if, for every $\epsilon > 0$, there is a compact set K such that $|f(x)| < \epsilon$, for all x in $X \setminus K$. The algebraic operations and the norm are done in exactly the same way as the case above. This example is also commutative, but is unital if and only if X is compact (in which case it is the same as $C(X)$).

Example 1.2.5. *Suppose that A B are C^*-algebras, we form their direct sum*

$$A \oplus B = \{(a, b) \mid a \in A, b \in B\}.$$

The algebraic operations are all performed coordinate-wise and the norm is given by

$$\|(a, b)\| = \max\{\|a\|, \|b\|\},$$

for any a in A and b in B.

There is an obvious extension of this notion to finite direct sums. Also, if $A_n, n \geq 1$ is a sequence of C^*-algebras, their direct sum is defined as

$$\oplus_{n=1}^{\infty} A_n = \{(a_1, a_2, \ldots) \mid a_n \in A_n, \text{ for all } n, \lim_n \|a_n\| = 0\}.$$

Aside from noting the condition above on the norms, there is not much else to add.

Exercise 1.2.1. *Let $A = \mathbb{C}^2$ and consider it as a $*$-algebra with coordinate-wise addition, multiplication and conjugation. (In other words, it is $C(\{1, 2\})$.)*

1. *Prove that with the norm*

$$\|(\alpha_1, \alpha_2)\| = |\alpha_1| + |\alpha_2|$$

A is not *a C^*-algebra.*

2. *Prove that*

$$\|(\alpha_1, \alpha_2)\| = \max\{|\alpha_1|, |\alpha_2|\}$$

is the only *norm which makes A into a C^*-algebra. (Hint: Proceed as follows. First prove that, in any C^*-algebra norm, $(1,0),(1,1),(0,1)$ all have norm one. Secondly, show that for any (α_1, α_2) in A, there is a unitary u such that $u(\alpha_1, \alpha_2) = (|\alpha_1|, |\alpha_2|)$. From this, deduce that $\|(\alpha_1, \alpha_2)\| = \|(|\alpha_1|, |\alpha_2|)\|$. Next, show that if a, b have norm one and $0 \leq t \leq 1$, then $\|ta + (1-t)b\| \leq 1$. Finally, show that any elements of the form $(1, \alpha)$ or $(\alpha, 1)$ have norm 1 provided $|\alpha| \leq 1$.)*

1.3 Spectrum

We begin our study of C^*-algebra with the basic notion of spectrum and the simple result that the set of invertible elements in a unital Banach algebra must be open. While it is fairly easy, it is interesting to observe that this is an important connection between the algebraic and topological structures.

Lemma 1.3.1. 1. *If a is an element of a unital Banach algebra A and $\|a - 1\| < 1$, then a is invertible.*

2. *The set of invertible elements of A is open.*

Proof. Consider the following series in A:

$$b = \sum_{n=0}^{\infty} (1-a)^n.$$

It follows from our hypothesis that the sequence of partial sums for this series is Cauchy and hence they converge to some element b of A. It is then a simple continuity argument to see that

$$ab = (1 - (1-a))(\lim_N \sum_{N=0}^{N} (1-a)^n) = \lim_N 1 - (1-a)^{N+1} = 1.$$

A similar argument shows $ba = 1$ and so a is invertible.

If a is invertible, the map $b \to ab$ is a homeomorphism of A, which preserves the set of invertibles. It also maps the unit to a, so the conclusion follows from the first part. □

We now come to the notion of spectrum. Just to motivate it a little, consider the C^*-algebra $C(X)$, where X is a compact Hausdorff space. If f is an element of this algebra and λ is in $\mathbb{C}$, the function $\lambda - f$ is invertible precisely when λ is not in the range of f. This gives us a simple algebraic description of the range of a function and so it can be generalized.

Definition 1.3.2. *Let A be a unital algebra and let a be an element of A. The* spectrum *of a, denoted $spec(a)$ is $\{\lambda \in \mathbb{C} \mid \lambda 1 - a$ is not invertible $\}$. The* spectral radius *of a, denoted $r(a)$ is $\sup\{|\lambda| \mid \lambda \in spec(a)\}$, which is defined provided the spectrum is non-empty, allowing the possibility of $r(a) = \infty$.*

Remark 1.3.3. *Let us consider for the moment the C^*-algebra $M_n(\mathbb{C})$, where n is some fixed positive integer. Recall the very nice fact from linear algebra that the following four conditions on an element a of $M_n(\mathbb{C})$ are equivalent:*

1. *a is invertible,*
2. *$a : \mathbb{C}^n \to \mathbb{C}^n$ is injective,*
3. *$a : \mathbb{C}^n \to \mathbb{C}^n$ is surjective,*
4. *$\det(a) \neq 0$.*

This fact allows us to compute the spectrum of the element a simply by finding the zeros of $\det(\lambda - a)$, which (conveniently) is a polynomial of degree n. The remark we make now is that in the C^*-algebra $\mathcal{B}(\mathcal{H})$, such a result fails miserably when $\mathcal{H}$ is not finite-dimensional. First of all, the determinant function simply fails to exist and while the first condition implies the next two, any other implication between the first three doesn't hold.

We quote two fundamental results from the theory of Banach algebras, neither of which will be proved.

Theorem 1.3.4. *Let A be a unital Banach algebra. The spectrum of any element is non-empty and compact.*

We will not give a complete proof of this result. Let us sketch the argument that the spectrum is compact. To see that it is closed one shows that the complement is open, as an easy consequence of Lemma 1.3.1. Secondly, if λ is strictly greater than $\|a\|$, then a computation similar to the one in the proof of Lemma 1.3.1 shows that $\sum_n \lambda^{-1-n} a^n$ is a convergent series and its sum is an inverse for $\lambda 1 - a$. Hence the spectrum of a is contained in the closed disc at the origin of radius $\|a\|$.

As for showing the spectrum is non-empty, the basic idea is as follows. If a is in A and $\lambda 1 - a$ is invertible, for all λ in $\mathbb{C}$, then we may take a non-zero linear functional ϕ and look at $\phi((\lambda 1 - a)^{-1})$. One first shows this function is analytic. Then by using the formula from the last paragraph for $(\lambda 1 - a)^{-1}$, at least for $|\lambda|$ large, it can be shown that the function is also bounded. By Liouville's Theorem, it is constant. From this it is possible to deduce a contradiction.

The second fundamental result is the following.

Theorem 1.3.5. *Let a be an element of a unital Banach algebra A. The sequence $\|a^n\|^{\frac{1}{n}}$ is bounded by $\|a\|$, decreasing and has limit $r(a)$. In particular, $r(a)$ is finite and $r(a) \leq \|a\|$.*

We will not give a complete proof (see [**?**]), but we will demonstrate part of the argument. First, as $\|a^n\| \leq \|a\|^n$ in any Banach algebra, the sequence is at least bounded. Furthermore, if λ is a complex number with absolute value greater than $\limsup_n \|a^n\|^{\frac{1}{n}}$, then it is a simple matter to show that the series

$$\sum_{n=0}^{\infty} \lambda^{-n-1} a^n$$

is convergent in A. Moreover, some basic analysis shows that

$$(\lambda 1 - a) \sum_{n=0}^{\infty} \lambda^{-n-1} a^n = \lim_N 1 - \lambda^{-N} a^N = 1 - 0 = 1.$$

It follows then that $(\lambda 1 - a)$ is invertible and λ is not in the spectrum of a. What the reader should take away from this argument is the fact that working in a Banach algebra (where the series has a sum) is crucial.

With these results available, we move on to consider C^*-algebras. Here, we see immediately important consequences of the C^*-condition.

Theorem 1.3.6. *If a is a self-adjoint element of a unital C^*-algebra A, then $\|a\| = r(a)$.*

Proof. As a is self-adjoint, we have $\|a^2\| = \|a^* a\| = \|a\|^2$. It follows by induction that for any positive integer k, $\|a^{2^k}\| = \|a\|^{2^k}$. By then passing to a subsequence, we have

$$r(a) = \lim_n \|a^n\|^{\frac{1}{n}} = \lim_k \|a^{2^k}\|^{2^{-k}} = \|a\|.$$

□

Here we see a very concrete relation between the algebraic structure (in the form of the spectral radius) and the topological structure. While this last result clearly depends on the C^*-condition in an essential way, it tends to look rather restrictive because it applies only to self-adjoint elements. Rather curiously, the C^*-condition *also* allows us to deduce information about the norm of an arbitrary element, a, since it expresses it as the square root of the norm of a self-adjoint element, a^*a. For example, we have the following two somewhat surprising results.

Corollary 1.3.7. *Let A and B be C^*-algebras and suppose that $\rho : A \to B$ is a $*$-homomorphism. Then ρ is contractive; that is, $\|\rho(a)\| \leq \|a\|$, for all a in A. In particular, we have $\|\rho\| \leq 1$.*

Proof. First consider the case that a is self-adjoint. As our map is a unital homomorphism, it carries invertibles to invertibles and it follows that $spec(a) \supset spec(\rho(a))$. Hence, we have

$$\|\rho(a)\| = r(\rho(a)) \leq r(a) = \|a\|.$$

For an arbitrary element, we have

$$\|\rho(a)\| = \|\rho(a)^*\rho(a)\|^{\frac{1}{2}} = \|\rho(a^*a)\|^{\frac{1}{2}} \leq \|a^*a\|^{\frac{1}{2}} = \|a\|.$$

□

Corollary 1.3.8. *If A is a C^*-algebra, then its norm is unique. That is, if a $*$-algebra possess a norm in which it is a C^*-algebra, then it possesses only one such norm.*

Proof. We combine the C^*-condition with the fact that a^*a is self-adjoint for any a in A and Theorem 1.3.6 to see that we have

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \left(\sup\{|\lambda| \mid (\lambda 1 - a^*a) \text{ not invertible }\}\right)^{\frac{1}{2}}.$$

The right hand side clearly depends on the algebraic structure of A and we are done. □

Exercise 1.3.1. *With a as in Lemma 1.3.1, prove that the sequence $s_N = \sum_{n=1}^{N}(1-a)^n$ is Cauchy.*

Exercise 1.3.2. *For any real number $t \geq 0$, consider the following element of $M_2(\mathbb{C})$:*

$$a = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Find the spectrum of a, the spectral radius of a and the norm of a. (Curiously, the last is the hardest. If you are ambitious, see if you can find two different methods: one using the definition and one using the C^*-condition and the last theorem.)

Exercise 1.3.3. *1. Let $A = \mathbb{C}[x]$, the $*$-algebra of complex polynomials in one variable, x. Find the spectrum of any non-constant polynomial in A.*

2. *Prove that there is no norm on A in which it is a C^*-algebra.*

3. *Let B be the $*$-algebra of rational functions over $\mathbb{C}$. That is, it is the field of quotients for the ring A of the last part. Find the spectrum of any non-constant rational function in B.*

4. *Prove that there is no norm on B in which it is a C^*-algebra.*

5. *Find two things wrong with the following: if A is any $*$-algebra, then the formula given in the proof of 1.3.8 defines a norm which makes A into a C^*-algebra.*

1.4 Commutative C^*-algebras

In section 1.2, we gave an example of a commutative, unital C^*-algebra by considering $C(X)$, where X is a compact Hausdorff space. In fact, all commutative, unital C^*-algebras arise in this way. That is, the main goal of this section will be to prove that every commutative, unital C^*-algebra A is isomorphic to $C(X)$, for some compact Hausdorff space X.

For the first part of this section, we will restrict our attention to commutative, unital C^*-algebras, ending with the theorem above. Following on, we will also consider non-commutative C^*-algebras and obtain some interesting consequences.

Definition 1.4.1. *Let A be an algebra over $\mathbb{C}$. We define $\mathcal{M}(A)$ to be the set of non-zero homomorphisms to $\mathbb{C}$.*

We remark that the notation comes from the fact that a homomorphism is a multiplicative linear map.

Lemma 1.4.2. *Let A be a unital commutative C^*-algebra and let ϕ be in $\mathcal{M}(A)$.*

1. *$\phi(a)$ is in $spec(a)$, for every a in A.*

2. *The map ϕ is bounded and $\|\phi\| = 1$.*

3. *For all a in A, we have $\phi(a^*) = \overline{\phi(a)}$.*

Proof. Consider the kernel of ϕ. It is clearly an ideal in A and since ϕ is non-zero, it is a proper ideal and hence contains no invertible elements. For any a in A, $\phi(a)1 - a$ is clearly in the kernel of ϕ and hence is not invertible. Thus, $\phi(a)$ is in $spec(a)$ and $|\phi(a)| \leq r(a) \leq \|a\|$, from which it follows that ϕ is bounded and $\|\phi\| \leq 1$. On the other hand, a non-zero multiplicative map must send the unit to a non-zero idempotent and the complex numbers has only one such element: $\phi(1) = 1$. Noting that $\|1\|^2 = \|1^*1\| = \|1\|$ implies that $\|1\| = 1$ and hence $\|\phi\| \geq |\phi(1)| = 1$.

For the last part, any a in A may be written $b + ic$, where $b = (a + a^*)/2$ and $c = (ia^* - ia)/2$ are both self-adjoint. It suffices then, to prove that $\phi(b)$ is real, whenever $b = b^*$. By using a power series for the exponential function, one sees that, for any real number t, $u_t = e^{itb}$ is a well-defined element of A. Moreover an easy continuity argument shows that $u_{-t} = u_t^*$ and $u_t u_{-t} = 1$. From this it follows that $\|u_t\| = \|u_t u_t^*\|^{1/2} = 1$. As ϕ is continuous, we have

$$1 \geq |\phi(u_t)| = |e^{it\phi(b)}|,$$

for all real numbers t. It follows that $\phi(b)$ is real as desired. □

Lemma 1.4.3. *Let A be a unital commutative C^*-algebra. The set $\mathcal{M}(A)$ is a weak-$*$ compact subset of the unit ball of the dual space A^*.*

Proof. The Alaoglu Theorem (2.5.2, page 70 of []) asserts that the unit ball is weak-$*$ compact, so it suffices to prove that $\mathcal{M}(A)$ is closed. The weak-$*$ topology is defined so that a net ϕ_α converges to ϕ if and only if the net $\phi_\alpha(a)$ converges to $\phi(a)$, for all a in A. But then it is clear that if each ϕ_α is multiplicative, so is ϕ. □

This space $\mathcal{M}(A)$ will be our candidate compact, Hausdorff space. That is, we will show that A is isomorphic to $C(\mathcal{M}(A))$. First, we stress that the topology on $\mathcal{M}(A)$ is the weak-* topology from the dual space of A. Let us remark that we can already see a map from the former to the latter: for any a in A, the formula $\hat{a}(\phi) = \phi(a)$ means that we can think of a as a function on $\mathcal{M}(A)$. It is the map $a \to \hat{a}$ that will be our isomorphism.

Lemma 1.4.4. *Let a be an element in a unital commutative C^*-algebra A. Evaluation at a is a continuous map from $\mathcal{M}(A)$ onto spec(a).*

Proof. Continuity is direct consequence of the definition of the weak-$*$ topology. Secondly, for any ϕ in $\mathcal{M}(A)$, $\phi(a)$ is in *spec*(a) from Lemma 1.4.2.

Finally, we need to see that the map is onto. That is, let λ be in *spec*(a). A simple Zorn's Lemma argument shows that $\lambda 1 - a$ is contained in a maximal proper ideal of A, say I. We claim that I is closed. Its closure is clearly an ideal. Moreover, it cannot contain the unit since it is invertible, the invertibles are open and I contains no invertible. Thus, the closure of I is also a proper ideal and hence by maximality, it is equal to I. The quotient A/I is then a field and also a Banach algebra, using the quotient norm. Let b be any element of this Banach algebra. As its spectrum is non-empty (1.3.4), we have a complex number λ such that $\lambda 1 - b$ is not invertible. As we are in a field, we have $\lambda 1 - b = 0$. Hence, we see that every element of A/I is a scalar multiple of the unit. That is, $A/I \cong \mathbb{C}$. The quotient map from A to $\mathbb{C}$ is then a non-zero homomorphism which sends a to λ, since $\lambda 1 - a$ is in I. □

The next result strengthens the conclusion, in that the evaluation map at a is actually injective, by adding the hypothesis that the commutative C^*-algebra is generated by a, a^* and 1. It can either be described as the intersection of all C^*-subalgebras which contain a, or as the closure of the $*$-algebra which is formed by taking the linear span of the unit and all products of a and its adjoint. From the latter description, it is clear that this algebra is commutative exactly when a commutes with its adjoint.

Lemma 1.4.5. *Let a be an element in a unital, commutative C^*-algebra, A, and assume that A is the Banach algebra generated by a, a^* and 1. Then evaluation at a is homeomorphism from $\mathcal{M}(A)$ to spec(a).*

Proof. We know already that the map is continuous and surjective. It remains only to see that it is injective. But if ϕ and ψ are in $\mathcal{M}(A)$ and

$\phi(a) = \psi(a)$, then it follows from 1.4.2 that $\phi(a^*) = \psi(a^*)$ and also that $\phi(1) = 1 = \psi(1)$, and hence $\phi = \psi$ since a, a^* and the unit generate A as a Banach algebra. □

We are now ready to prove our main result.

Theorem 1.4.6. *Let A be a commutative, unital C^*-algebra. The function sending a in A to $\hat{a}$ in $C(\mathcal{M}(A))$ defined by*

$$\hat{a}(\phi) = \phi(a), \phi \in \mathcal{M}(A),$$

is an isometric $*$-isomorphism from A to $C(\mathcal{M}(A))$.

Proof. The fact that $\hat{a}$ is a continuous function is a simple consequence of the definition of the weak-$*$ topology. It is easy to see that the map $\hat{}$ is a homomorphism. That it is a $*$-homomorphism is a consequence of Lemma 1.4.2. Our next aim is to see that $\hat{}$ is isometric. First, suppose that a is self-adjoint. From 1.3.6, we have

$$\|a\| = r(a) = \sup\{|\phi(a)| \mid \phi \in \mathcal{M}(A)\} = \sup\{|\hat{a}(\phi)| \mid \phi \in \mathcal{M}(A)\} = \|\hat{a}\|.$$

For an arbitrary element, we then have

$$\|a\| = \|a^*a\|^{\frac{1}{2}} = \|\widehat{a^*a}\|^{\frac{1}{2}} = \|\hat{a}^*\hat{a}\|^{\frac{1}{2}} = \|\hat{a}\|.$$

We must finally show that $\hat{}$ is onto. The range is clearly a unital $*$-algebra. We will prove that the range separates the points of $\mathcal{M}(A)$. Assuming that this is true for the moment, we can apply the Stone-Weierstrass Theorem (4.3.4, page 146 of []) which states that a unital $*$-subalgebra of $C(\mathcal{M}(A))$ which separates the points is dense. Moreover, since $\hat{}$ is isometric and A is complete, the range is also closed and we are done.

As for the claim that the range separates the points, let $\phi \neq \psi$ be two elements of $\mathcal{M}(A)$. The fact that they are unequal means that there exists an a in A such that $\phi(a) \neq \psi(a)$. In other words, $\hat{a}(\phi) \neq \hat{a}(\psi)$ and so $\hat{a}$ separates ϕ and ψ. □

Now, let us again consider the situation where is B a unital (not necessarily commutative) C^*-algebra and let a be an element of B. We may consider, A, the C^*-subalgebra of B generated by a, a^* and 1. This subalgebra is commutative exactly when the three generators commute with each other, and this is exactly when a is normal: $a^*a = aa^*$. Alternately, an element a in

B is normal if and only if there exists a commutative, unital C^*-subalgebra $a \in A \subseteq B$.

We can apply the last Theorem to the C^*-algebra A, as before, and Lemma 1.4.5 provides us with a fine description of $\mathcal{M}(A)$. However, an interesting issues arises. If a is some element of A, it is conceivable that it has an inverse in B, but that inverse is not in A. In other words, whether or not a is invertible depends on whether we regard it as an element of A or B. In fact, this turns out not to be an issue: the answers are always the same. We will prove that in full generality - i.e. without assuming A is commutative - in the next section. But for the moment, we need to worry about it in the special case where A is commutative.

This same issue also means that the spectrum of a depends on whether we think of it in A or B. Let us briefly introduce some notation to deal with this. For a in A, let $spec_A(a)$ be the set of complex numbers λ such that $\lambda 1 - a$ has no inverse in A and let $spec_B(a)$ be the set of complex numbers λ such that $\lambda 1 - a$ has no inverse in B. Obviously, the existence of an inverse in A implies one in B and so $spec_B(a) \subseteq spec_A(a)$.

Proposition 1.4.7. *Let B be a unital C^*-algebra and let A be a commutative C^*-subalgebra of B which contains the unit of B. An element a of A has an inverse in A if and only if it has an inverse in B. In consequence, we have $spec_A(a) = spec_B(a)$.*

Proof. First, it is clear that if a has an inverse in A, it also has one in B. Hence, we have $spec_A(a) \supseteq spec_B(a)$. So let us assume that a has an inverse in B. It follows that a^* does also ($(a^*)^{-1} = (a^{-1})^*$) and so does a^*a. We know from part 1 of Lemma 1.4.2 and Lemma 1.4.4 that $spec_A(a^*a) = \{\phi(a^*a) \mid \phi \in \mathcal{M}(A)\}$. On the other hand, it follows from part 3 of Lemma 1.4.2

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a)$$

is a non-negative real number. Also, part 2 of Lemma 1.4.2 implies that $\phi(a^*a) \leq \|a^*a\|$. It follows that

$$spec_B(a^*a) \subseteq spec_A(a^*a) \subseteq [0, \|a^*a\|].$$

As we know that a^*a is invertible, 0 is not in the spectrum so

$$spec_B(a^*a) \subseteq [\delta, \|a^*a\|]$$

for some positive number δ.

As $\lambda 1 - (\|a^*a\|1 - a^*a) = -\left((\|a^*a\| - \lambda)1 - a^*a\right)$, we see that

$$spec_B(\|a^*a\|1 - a^*a) = \|a^*a\| - spec_B(a^*a) \subseteq [0, \|a^*a\| - \delta].$$

The element $\|a^*a\|1 - a^*a$ is clearly self-adjoint, so we conclude that

$$\|(\|a^*a\|1 - a^*a)\| \leq \|a^*a\| - \delta < \|a^*a\|.$$

It follows from part 2 of Lemma 1.4.2 that, for any ϕ in $\mathcal{M}(A)$, we have

$$|\phi(\|a^*a\|1 - a^*a)| < \|a^*a\|.$$

The left-hand-side is simply

$$\phi(\|a^*a\|1 - a^*a) = \|a^*a\| - |\phi(a)|^2.$$

From these facts, we conclude that $\phi(a) \neq 0$. Another application of Lemmas 1.4.2 amd 1.4.4 implies that 0 is not in $spec_A(a)$ and so a is invertible in A. □

Now we return to the situation that a is a normal element of the unital C^*-algebra B and we attempt to exploit the commutative C^*-algebra A generated by $a, a^*, 1$, using the results we have so far. For any element of A, we can return to writing $spec(a)$.

A very curious thing happens: the map $a \to \hat{a}$ of 1.4.6 is much less useful than its inverse! If f is any function in $C(spec(a))$, using Lemma 1.4.5 to identify $spec(a)$ with $\mathcal{M}(A)$, there is an element of A whose image under ˆ is exactly f. We will denote this element by $f(a)$.

Definition 1.4.8. *Let B be a unital C^*-algebra and let a be a normal element of B. Let A be the C^*-subalgebra of B generated by a and the unit. For each f in $C(spec(a))$, we let $f(a)$ be the unique element of A such that*

$$\phi(f(a)) = f(\phi(a)),$$

for all ϕ in $\mathcal{M}(A)$.

The following now amounts to a restatement of 1.4.6 (and 1.4.5).

Corollary 1.4.9. *Let B be a unital C^*-algebra and let a be a normal element of B. The map sending f to $f(a)$ is an isometric $*$-isomorphism from $C(spec(a))$ to the C^*-subalgebra of B generated by a and the unit. Moreover, if $f(z) = \sum_{k,l} a_{k,l} z^k \overline{z}^l$ is any polynomial in z and $\overline{z}$, then*

$$f(a) = \sum_{k,l} a_{k,l} a^k (a^*)^l.$$

We take a slight detour by noting the following result. There is a more basic proof of this fact, but with what we have here, it becomes quite simple.

Corollary 1.4.10. *Let a be a normal element of the unital C^*-algebra B. Then a is self-adjoint if and only if $spec(a) \subseteq \mathbb{R}$.*

Proof. Under the isomorphism of 1.4.9, the restriction of the function $f(z) = z$ to the spectrum of a is mapped to a. Then a is self-adjoint if and only if $f|spec(a) = \bar{f}|spec(a)$, which holds if and only if $spec(a) \subseteq \mathbb{R}$. □

Corollary 1.4.11. *Let A and B be unital C^*-algebras and let $\rho : A \to B$ be a unital C^*-homomorphism. If a is a normal element of A, then $spec(\rho(a)) \subseteq spec(a)$ and $f(\rho(a)) = \rho(f(a))$, for any f in $C(spec(a))$.*

Proof. As we observed earlier, it is clear that ρ must carry invertible elements to invertible elements and the containment follows at once.

For the second part, let C and D denote the C^*-subalgebras of A and B which are generated by a and the unit if A and $\rho(a)$ and the unit of B, respectively. It is clear then that $\rho|C : C \to D$ is a unital $*$-homomorphism. Suppose that ψ is in $\mathcal{M}(D)$. Then $\psi \circ \rho$ is in $\mathcal{M}(C)$ and we have

$$\psi(\rho(f(a))) = \psi \circ \rho(f(a)) = f(\psi \circ \rho(a)) = f(\psi(\rho(a))).$$

By definition, this means that $\rho(f(a)) = f(\rho(a))$.

□

Exercise 1.4.1. *Let X be a compact Hausdorff space. For each x in X, define $\phi_x(f) = f(x)$. Prove that ϕ_x is in $\mathcal{M}(C(X))$ and that $x \to \phi_x$ is a homeomorphism between X and $\mathcal{M}(C(X))$. (Hint to show ϕ is surjective: the Reisz representation theorem identifies the linear functionals on $C(X)$ for you. You just need to identify which are multiplicative.)*

Exercise 1.4.2. *Suppose that X and Y are compact Hausdorff spaces and $\rho : C(X) \to C(Y)$ is a unital $*$-homomorphism.*

1. *Prove that there exists a continuous function $h : Y \to X$ such that $\rho(f) = f \circ h$, for all f in $C(X)$.*

2. *Prove that this statement may be false if ρ is not unital.*

3. *Give a necessary and sufficient condition on h for ρ to be injective.*

4. *Give a necessary and sufficient condition on h for ρ to be surjective.*

Exercise 1.4.3. *Let B be the C^*-algebra of bounded functions on $[0,1]$ (with the supremum norm). Let $Q = \{k2^{-n} \mid k \in \mathbb{Z}, n \geq 0\} \cap [0,1)$.*

1. *Let $\mathcal{A}$ be the linear span of all functions of the form $\chi_{[s,t)}, \chi_{[t,1]}$, where s, t are in Q. Prove that $\mathcal{A}$ is a $*$-algebra.*

2. *Let A be the closure of $\mathcal{A}$ is the supremum norm, which is a C^*-algebra. Prove that A contains $C[0,1]$.*

3. *Prove that for all f in A and s in Q,*

$$\lim_{x \to s^+} f(x), \lim_{x \to s^-} f(x)$$

   both exist.

4. *Let $h : \mathcal{M}(A) \to [0,1]$ be the map given in the last exercise and the inclusion $C[0,1] \subseteq A$. Describe $\mathcal{M}(A)$ and h.*

Exercise 1.4.4. *Let B be a unital C^*-algebra and suppose a is a self-adjoint element and $\frac{1}{4} > \epsilon > 0$ such that $\|a - a^2\| < \epsilon$.*

1. *Let $f(x) = x - x^2$. Prove that*

$$|f(x)| \geq \min\left\{\frac{|x|}{2}, \frac{|1-x|}{2}\right\},$$

   for all x in $\mathbb{R}$.

2. *Prove that $spec(a) \subseteq [-2\epsilon, 2\epsilon] \cup [1-2\epsilon, 1+2\epsilon]$.*

3. *Find a continuous function g on $spec(a)$ with values in $\{0,1\}$ such that $|g(x) - x| < 2\epsilon$, for all x in $spec(a)$. Also explain why there is no such function if $spec(a)$ is replaced by $[0,1]$.*

4. *Prove that there is a projection p in B such that $\|a - p\| < 2\epsilon$.*

1.5 Further consequences of the C^*-condition

In this section, we will prove three more important results on the structure of C^*-algebras.

The first is a nice and somewhat surprising extension of our earlier result that any $*$-homomorphism between C^*-algebras is necessarily a contraction (Corollary 1.3.7). If we additionally assume the map is injective, then it is actually isometric.

Lemma 1.5.1. *Let A and B be unital C^*-algebras and $\rho : A \to B$ be an injective, unital $*$-homomorphism. Then for any normal element a in A, we have $spec(a) = spec(\rho(a))$.*

Proof. As ρ maps invertible elements to invertible elements, it is clear that $spec(\rho(a)) \subseteq spec(a)$.

Let us suppose that the containment $spec(\rho(a)) \subseteq spec(a)$ is proper. Then we may find a non-zero continuous function f defined on $spec(a)$ whose restriction to $spec(\rho(a))$ is zero. Then we have $f(a) \neq 0$ while, using Corollary 1.4.9, we have

$$\rho(f(a)) = f(\rho(a)) = 0$$

since $f|spec(\rho(a)) = 0$. This contradicts the hypothesis that ρ is injective. □

Theorem 1.5.2. *If A and B are unital C^*-algebras and $\rho : A \to B$ is an injective $*$-homomorphism, then ρ is an isometry; that is, $\|\rho(a)\| = \|a\|$, for all a in A.*

Proof. The equality $\|\rho(a)\| = \|a\|$ for a self-adjoint element a follows from Theorem 1.3.6 and Lemma 1.5.1. For arbitrary a, we have

$$\|a\|^2 = \|a^*a\| = \|\rho(a^*a)\| = \|\rho(a)^*\rho(a)\| = \|\rho(a)\|^2$$

and we are done. □

Earlier, we defined the spectrum of an element of an algebra. The fact that the definition depends on the algebra in question is implicit. As a very simple example, the function $f(x) = x^2 + 1$ is invertible when considered in the ring of continuous functions on the unit interval, $C[0, 1]$, but not as an element of the ring of polynomials, $\mathbb{C}[x]$. Rather surprisingly, this does not occur in C^*-algebras. More precisely, we have the following.

Theorem 1.5.3. *Let B be a unital C^*-algebra and let A be a C^*-subalgebra of B containing its unit. If a is any element of A, then its spectrum in B coincides with its spectrum in A.*

Proof. It suffices to show that if a is any element of A which has an inverse in B, then that inverse actually lies in A. In the case that a is normal, this follows from Lemma 1.5.1 applied to the inclusion map of A in B.

For arbitrary a, if a is invertible, then so is a^* (its inverse is $(a^{-1})^*$) and hence a^*a is also invertible. As a^*a is self-adjoint and hence normal and since it clearly lies in A, $(a^*a)^{-1}$ also lies in A. Then we observe $a^{-1} = (a^*a)^{-1}(a^*a)a^{-1} = (a^*a)^{-1}a^*$ which obviously lies in A. □

The property of the conclusion of this last theorem is usually called *spectral permanence.*

There is a simple, but useful consequence of this fact and Theorem 1.4.9, usually known as the Spectral Mapping Theorem.

Corollary 1.5.4. *Let a be a normal element of a unital C^*-algebra B. For any continuous function f on $spec(a)$, we have $spec(f(a)) = f(spec(a))$. That is, the spectrum of $f(a)$ is simply the range of f.*

Proof. We have already noted the fact that for any compact Hausdorff space X and for any f in $C(X)$, the spectrum of f is simply the range of f, $f(X)$. We apply this to the special case of f in $C(spec(a))$ to see that the spectrum of f is $f(spec(a))$. Since the map from $C(spec(a))$ to the C^*-algebra generated by a is an isomorphism and it carries f to $f(a)$ (by definition), we see that $spec(f) = spec(f(a))$. The conclusion follows from the fact that the spectrum of a is the same in the C^*-algebra B as it is in the C^*-subalgebra. □

Exercise 1.5.1. *Show that Theorem 1.5.3 is false if A and B are simply algebras over $\mathbb{C}$. In fact, find a maximal counter-example: $A \subseteq B$ such that for* every *element a of A (which is not a multiple of the identity), its spectrum in A is* as different as possible *from its spectrum in B.*

1.6 Positivity

Let us begin by recalling two things. The first is the definition of a positive element in a C^*-algebra given in 1.1.3: an element a in a C^*-algebra is positive if $a = b^*b$, for some b in A. Observe that this means that a is necessarily self-adjoint. Secondly, we recall Corollary 1.4.10: a normal element a of a unital C^*-algebra is self-adjoint if and only if $spec(a) \subseteq \mathbb{R}$. Our goal now is to provide a similar "spectral" characterization of positivity for normal elements.

Our main result (Theorem 1.6.5) states that a self-adjoint element a in a unital C^*-algebra is positive if and only if $spec(a) \subseteq [0, \infty)$. Along the way, we will prove a number of useful facts about positive elements in a unital C^*-algebra.

Lemma 1.6.1. *Let a be a self-adjoint element of a unital C^*-algebra A.*

1. *Let $f(x) = \max\{x, 0\}$ and $g(x) = \max\{-x, 0\}$, for x in $\mathbb{R}$. Then $f(a), g(a)$ are both positive. We have*

$$\begin{aligned} a &= f(a) - g(a), \\ af(a) &= f(a)^2, \\ ag(a) &= -g(a)^2, \\ f(a)g(a) &= 0. \end{aligned}$$

2. *If $spec(a) \subseteq [0, \infty)$, then a is positive.*

Proof. It is clear that f and g are continuous functions on the spectrum of a, each is real-valued, $f(x) - g(x) = x, xf(x) = f(x)^2, xg(x) = -g(x)^2$ and $f(x)g(x) = 0,$, for all x in $\mathbb{R}$. Corollary 1.4.9 then implies that $f(a), g(a)$ are self-adjoint elements of A, satisfying the desired equations. Since $f(x)$ is positive, it has a square root and it follows that $f(a) = (\sqrt{f}(a))^2 = (\sqrt{f}(a))^*(\sqrt{f}(a))$ is positive. The same argument shows $g(a)$ is positive.

For the second statement, it suffices to notice in the proof above that if $spec(a) \subseteq [0, \infty)$, then $g(a) = 0$. $\square$

The following statement has a very easy proof which is a nice application of the spectral theorem. Moreover, the result is a very useful tool in dealing with elements with positive spectrum.

Lemma 1.6.2. *Let a be a self-adjoint element of a unital C^*-algebra A. The following are equivalent.*

1. *$spec(a) \subseteq [0, \infty)$.*

2. *For all $t \geq \|a\|$, we have $\|t - a\| \leq t$.*

3. For some $t \geq \|a\|$, we have $\|t - a\| \leq t$.

Proof. Since the spectrum of a is a subset of the reals and the spectral radius of a is its norm, $spec(a) \subseteq [-\|a\|, \|a\|]$. We consider the function $f_t(x) = t - x$, for values of $t \geq \|a\|$. The function is positive and monotone decreasing on $[-\|a\|, \|a\|]$. Hence, the norm of the restriction of f_t to the spectrum of a is just its value at the infimum of $spec(a)$. In view of Corollary 1.4.9, $\|t - a\|$ is also the value of f_t at the infimum of $spec(a)$. Finally, notice that $f_t(x) \leq t$ if and only if $x \geq 0$. Putting this together, we see that the minimum of $spec(a)$ is negative if and only if $\|t - a\| > t$. Taking the negations of both statements, $spec(a) \subseteq [0, \infty)$ if and only if $\|t - a\| \leq t$. This holds for all $t \geq \|a\|$. □

Here is one very useful consequence of this result.

Proposition 1.6.3. *If a and b are self-adjoint elements of the unital C^*-algebra A and $spec(a), spec(b) \subseteq [0, \infty)$, then $spec(a + b) \subseteq [0, \infty)$.*

Proof. Let $t = \|a\| + \|b\|$ which is evidently at least $\|a + b\|$. Then we have

$$\begin{aligned} \|t - (a + b)\| &= \|(\|a\| - a) + (\|b\| - b)\| \\ &\leq \|(\|a\| - a)\| + \|(\|b\| - b)\| \\ &\leq \|a\| + \|b\| \\ &= t, \end{aligned}$$

where we have used the last lemma in moving from the second line to the third. The conclusion follows from another application of the last lemma. □

Before getting to our main result, we will need the following very basic fact regarding the spectrum.

Lemma 1.6.4. *Let a, b be two elements of a unital algebra A. Then we have*

$$spec(ab) \setminus \{0\} = spec(ba) \setminus \{0\}.$$

Proof. It clearly suffices to prove that if λ is a non-zero complex number such that $\lambda - ab$ is invertible, then $\lambda - ba$ is invertible also. Consider $x =$

$\lambda^{-1} + \lambda^{-1}b(\lambda - ab)^{-1}a$. We see that

$$\begin{aligned} x(\lambda - ba) &= (\lambda^{-1} + \lambda^{-1}b(\lambda - ab)^{-1}a)(\lambda - ba) \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda - ab)^{-1}(\lambda a - aba) \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda - ab)^{-1}(\lambda - ab)a \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}ba \\ &= 1. \end{aligned}$$

A similar computation which we omit shows that $(\lambda - ba)x = 1$ and we are done. □

We are now ready to prove our main result.

Theorem 1.6.5. *Let a be a self-adjoint element of a unital C^*-algebra A. Then a is positive if and only if $spec(a) \subseteq [0, \infty)$.*

Proof. The 'if' direction has already been done in Lemma 1.6.1. Let us now assume that $a = b^*b$, for some b in A. Using the notation of 1.6.1, consider $c = bg(a)$. Then we have $c^*c = g(a)b^*bg(a) = g(a)ag(a) = -g(a)^3$. Write $c = d + ie$, where d, e are self-adjoint elements of A. A simple computation shows that

$$cc^* = d^2 + e^2 - c^*c = d^2 + e^2 + g(a)^3.$$

As the functions x^2 and $g(x)^3$ are positive, it follows from Corollary 1.5.4 that each of d^2, c^2 and $g(a)^3$ has spectrum contained in $[0, \infty)$. By Lemma 1.6.3, so does cc^*. On the other hand, again using Corollary 1.5.4, we have $spec(c^*c) = spec(-g(a)^3) \subseteq (-\infty, 0]$.

We now appeal to Lemma 1.6.4 (using $a = c, b = c^*$) to conclude that $spec(c^*c) = spec(cc^*) = \{0\}$. But this means that $-g(a)^3 = c^*c = 0$ and it follows that $g(a) = 0$. This implies that the restriction of g to the spectrum of a is zero, which means $spec(a) \subseteq [0, \infty)$ and we are done. □

It is worth mentioning the following rather handy consequence.

Corollary 1.6.6. *If a is a normal element in a unital C^*-algebra B and f is a real-valued function on $spec(a)$, then $f(a)$ is self-adjoint. Similarly, if f is positive, then $f(a)$ is positive.*

Proof. We know that $spec(f(a)) = f(spec(a))$ from Theorem 1.5.4. Moreover, since $f(a)$ lies in the C^*-algebra generated by a and the unit, which is commutative, it itself is normal. The two statements now follow from Corollary 1.4.10 and Corollary 1.6.5. □

Exercise 1.6.1. *Let A be a unital C^*-algebra and suppose a is in A. Let $f(x) = \sqrt{x}, x \geq 0$ and define $|a| = f(a^*a)$.*

1. *Show that if a is invertible, so is $|a|$.*
2. *Show that if a is invertible, then $u = a|a|^{-1}$ is unitary.*
3. *Prove that if a is invertible, u and $|a|$ commute if and only if a is normal.*

The expression $a = u|a|$ is usually called the polar decomposition of a.

1.7 Finite-dimensional C^*-algebras

In this section, we investigate the structure of finite-dimensional C^*-algebras. The main objective will be the proof of the following theorem, which is a very satisfactory one.

Theorem 1.7.1. *Let A be a unital, finite-dimensional C^*-algebra. Then there exist positive integers K and $N_1, \ldots, N_K$ such that*

$$A \cong \oplus_{k=1}^{K} M_{N_k}(\mathbb{C}).$$

Moreover, K is unique and $N_1, \ldots, N_K$ are unique, up to a permutation.

The theorem is also valid without the hypothesis of the C^*-algebra being unital (in fact, every finite dimensional C^*-algebra *is* unital, as a consequence of the theorem), but we do not quite have the means to prove that yet, so we content ourselves with the version stated above.

At this point, the reader has a choice. The first option is to simply accept the result above as a complete classification of finite-dimensional C^*-algebras and then move on to the next section. The other option, obviously, is to keep reading to the completion of the section and see the proof. This brings up the question: is it worth it? Aside from the simple satisfaction of having seen a complete proof, there are two points in what follows which should be drawn to the reader's attention.

The first point is some simple calculus for rank-one operators on Hilbert space. This is not particularly deep, but the notation is useful and some simple facts will be assembled which will be used again later, beyond the finite-dimensional case.

The second point is that the proof of the main result is built around the existence of projections in a finite-dimensional C^*-algebra. In general, C^*-algebras may or may not have non-trivial projections. As an example, $C(X)$, where X is a compact Hausdorff space, has no projections other than 0 and 1 if and only if X is connected. However, the principle which is worth observing is that an abundance of projections in a C^*-algebra can be a great help in understanding its structure.

Before we get to a proof of this, we will introduce some useful general notation for certain operators on Hilbert space.

Definition 1.7.2. *If $\mathcal{H}$ is a Hilbert space and ξ, η are vectors in $\mathcal{H}$, then we define $\xi \otimes \eta^* : \mathcal{H} \to \mathcal{H}$ by*

$$\xi \otimes \eta^*(\zeta) =< \zeta, \eta > \xi, \zeta \in \mathcal{H}.$$

It is worth noting that if $\mathcal{H}$ is just $\mathbb{C}^N$, for some positive integer N, then we can make use of the fact that the matrix product of an $i \times j$ matrix and a $j \times k$ matrix exists. With this in mind, if ξ and η are in $\mathbb{C}^N$, which we regard as $N \times 1$ matrices, then $\xi^T \eta$ is a 1×1 matrix while $\xi \eta^T$ is an $N \times N$ matrix. (Here ξ^T denotes the transpose of ξ.) The formula above just encodes the simple consequence of associativity

$$(\xi \eta^T)\zeta = \xi(\eta^T \zeta).$$

Lemma 1.7.3. *Let ξ, η, ζ, ω be vectors in the Hilbert space $\mathcal{H}$ and let a be in $\mathcal{B}(\mathcal{H})$. We have*

1. *$\xi \otimes \eta^*$ is a bounded linear operator on $\mathcal{H}$ and $\|\xi \otimes \eta^*\| = \|\xi\|\|\eta\|$.*

2. *$(\xi \otimes \eta^*)\mathcal{H} = span\{\xi\}$, provided $\eta \neq 0$.*

3. *$(\xi \otimes \eta^*)^* = \eta \otimes \xi^*$,*

4. *$(\xi \otimes \eta^*)(\zeta \otimes \omega^*) =< \zeta, \eta > \xi \otimes \omega^*$,*

5. *$a(\xi \otimes \eta^*) = (a\xi) \otimes \eta^*$,*

6. *$(\xi \otimes \eta^*)a = \xi \otimes (a^*\eta)^*$.*

Moreover, if $\xi_1, \xi_2, \ldots, \xi_n$ *is an orthonormal basis for* $\mathcal{H}$*, then*

$$\sum_{i=1}^{n} a\xi_i \otimes \xi_i^* = a,$$

for any a *in* $\mathcal{B}(\mathcal{H})$*. In particular, the linear span of* $\{\xi_i \otimes \xi_j^* \mid 1 \leq i, j \leq n\}$ *is* $\mathcal{B}(\mathcal{H})$ *and*

$$\sum_{i=1}^{n} \xi_i \otimes \xi_i^* = 1.$$

We start toward a proof of the main theorem above by assembling some basic facts about finite-dimensional C^*-algebras. The main point of the following result is that general normal elements can be obtained as linear combinations of projections.

Lemma 1.7.4. *Let* A *be a unital finite-dimensional* C^**-algebra.*

1. *Every normal element in* A *has finite spectrum.*
2. *Every normal element of* A *is a linear combination of projections.*

Proof. For a normal, from 1.4.9, we know that $C(spec(a))$ is isomorphic to a C^*-subalgebra of A, which must also be finite-dimensional. It follows that $spec(a)$ is finite.

For each λ in $spec(a)$, let p_λ be the element of $C(spec(a))$ which is 1 on λ and zero elsewhere. As $spec(a)$ is finite, this function is continuous. As a function, this is clearly a self-adjoint idempotent, so the same is true of $p_\lambda(a)$. Moreover, it follows that

$$\sum_{\lambda \in spec(a)} \lambda p_\lambda(z) = z,$$

for all z in $spec(a)$. It follows from 1.4.9 that

$$\sum_{\lambda \in spec(a)} \lambda p_\lambda(a) = a.$$

□

Roughly speaking, we now know that a finite-dimensional C^*-algebra has a wealth of projections. Analyzing the structure of these will be the key point in our proof.

Lemma 1.7.5. *Let A be a C^*-algebra. The relation defined on projections by $p \geq q$ if $pq = q$ (and hence after taking adjoints $qp = q$ also) is a partial order.*

We leave the proof as an easy exercise.

Lemma 1.7.6. *Let A be a C^*-algebra.*

1. *For projections p, q in A, $p \geq q$ if and only if $pAp \supset qAq$. In particular, if $pAp = qAq$, then $p = q$*

2. *If pAp has finite dimension greater than 1, then there exists $q \neq 0$ with $p \geq q$.*

3. *If A is unital and finite-dimensional, then it has minimal non-zero projections in the order $\leq$.*

Proof. First, suppose that $p \geq q$. Then we have $qAq = pqAqp \subseteq pAp$. Conversely, suppose that $pAp \supset qAq$. Hence, we have $q = qqq \in qAq \subseteq pAp$, so $q = pap$, for some a in A. Then $pq = ppap = pap = q$. For the last statement, if $pAp = qAq$, then it follows from the first part that $p \leq q$ and $q \leq p$, hence $p = qp = q$.

A moment's thought shows that pAp is a $*$-algebra. It contains $ppp = p$ which is clearly a unit for this algebra. If it is finite-dimensional, then it is closed and hence is a C^*-subalgebra. We claim that if a self-adjoint element of pAp, say a, has only one point in its spectrum, then that element is a scalar multiple of p. Let $spec(a) = \{\lambda\}$. Then functions $f(z) = z$ and $g(z) = \lambda$ are equal on $spec(a)$. So by Corollary 1.4.9, $f(a) = g(a)$. On the other hand, Corollary 1.4.9 also asserts that $f(a) = a$ while $g(a) = \lambda p$. Our claim follows and this means that if every self-adjoint element of pAp has only one point in its spectrum that pAp is spanned by p and one dimensional. So if pAp has dimension greater than one, we must have a self-adjoint element, a, in pAp with at least two points in its spectrum. Choose a surjective function, f, from $spec(a)$ to $\{0, 1\}$. It is automatically continuous on $spec(a)$, $q = f(a)$ is a projection which is non-zero. As q is in pAp, we have $pq = q$.

For the last statement, we first notice that since A is unital, it contains non-zero projections. Suppose that p is any non-zero projection in A, so pAp has dimension at least one. If this dimension is strictly greater than one, we may find q as in part 2. So q is non-zero and since $q \neq p$, qAq is a proper linear subspace of pAp, so $dim(qAq) < dim(pAp)$. Continuing in this fashion, we may eventually find q such that $dim(qAq) = 1$. Hence, q is minimal in the order $\leq$ among non-zero projections.

□

We will consider non-zero projections p which are minimal with respect to the relation $\geq$. From the last result, we have $pAp = \mathbb{C}p$, for any such projection.

Lemma 1.7.7. *Suppose that $p_1, p_2, \ldots, p_K$ are minimal, non-zero projections in A with $p_i A p_j = 0$, for $i \neq j$. Then $p_1, \ldots, p_K$ are linearly independent. A finite set of minimal projections satisfying the hypothesis will be called* independent.

Proof. If some p_i can be written as a linear combination of the others, then we would have

$$\mathbb{C}p_i = p_i A p_i = p_i A(\sum_{j \neq i} \alpha_j p_j) \subseteq \sum_{j \neq i} p_i A p_j = 0,$$

a contradiction. □

Lemma 1.7.6 guarantees the existence of minimal, non-zero projections in a unital, finite-dimensional C^*-algebra A. Lemma 1.7.7 shows that an independent set of these cannot contain more than $dim(A)$ elements and it follows that we may take a maximal independent set of minimal, non-zero projections and it will be finite.

Theorem 1.7.8. *Let A be a finite-dimensional C^*-algebra and let $p_1, p_2, \ldots, p_K$ be a maximal set of independent minimal non-zero projections in A.*

1. *For each $1 \leq k \leq K$, Ap_k is a finite-dimensional Hilbert space with inner product $< a, b > p_k = b^* a$, for all a, b in Ap_k.*

2. *For each $1 \leq k \leq K$, Ap_kA (meaning the linear span of elements of the form ap_ka' with a, a' in A) is a unital C^*-subalgebra of A.*

3. *$\oplus_{k=1}^{K} Ap_kA = A$,*

4. *For each $1 \leq k \leq K$, define the map $\pi_k : A \to \mathcal{B}(Ap_k)$ defined by $\pi_k(a)b = ab$, for a in A and b in Ap_k. Then π_k is a $*$-homomorphism and its restriction to Ap_lA is zero for $l \neq k$ and an isomorphism for $l = k$.*

Proof. For the first statement, if a, b are in Ap_k, then $a = ap_k, b = bp_k$ and so $b^*a = (bp_k)^*ap_k = p_kb^*ap_k \in p_kAp_k = \mathbb{C}p_k$, so the scalar $< a, b >$ as described exists. This is clearly linear in a and conjugate linear in b. It is clearly non-degenerate, for if $< a, a >= 0$, then $a^*a = 0$ which implies $a = 0$.

Choose B_k to be an orthonormal basis for Ap_k which means that for any a in Ap_k, we have

$$\sum_{b\in B_k} bb^*a = \sum_{b\in B_k} b < a, b > p_k = \sum_{b\in B_k} < a, b > bp_k = \sum_{b\in B_k} < a, b > b = a.$$

Next, we define

$$q_k = \sum_{b\in B_k} bb^* = \sum_{b\in B_k} bp_kb^*.$$

It is clear that q_k is self-adjoint, in Ap_kA and, from above, $q_ka = a$, for any a in Ap_k.

Now if a, a' are in A, we have

$$q_k(ap_ka') = (q_kap_k)a' = ap_ka'$$

and

$$(ap_ka')q_k = (q_k(a')^*p_ka^*)^* = ((a')^*p_ka^*)^* = ap_ka',$$

so that q_k is a unit for Ap_kA. This completes the proof of part 2. In particular, we note that $q_kp_k = p_k$.

Since $p_kAp_l = 0$ for all $k \neq l$, we see that $Ap_kA \cdot Ap_lA = 0$ as well. From this, we also see that $q_kAp_l = 0$ and $q_kAp_lA = 0$ for $k \neq l$. In particular, the projections q_k are pairwise orthogonal.

Consider $q = \sum_{k=1}^{K} q_k$ which is clearly a central projection. We claim that it is the identity for A. If not, there is an element b in A with $bq \neq b$ and so $b - bq$ is a non-zero element in the set

$$q^\perp = \{a \in A \mid qa = 0\}.$$

It is a simple matter to verify that $q^\perp$ is a C^*-subalgebra of A and is obviously finite-dimensional. If it contains a non-zero element, then it contains a non-zero minimal projection, say p. Then for any $k = 1, 2, \ldots, K$, we have

$$pq_k \leq pq = 0$$

and so $pq_k = 0$. It then follows that

$$pAp_k \subseteq pAp_kA = pq_kAp_kA = 0.$$

The set of projections $p, p_1, \ldots, p_K$ is an independent set of non-zero minimal projections, contradicting the maximality of $p_1, \ldots, p_K$. We conclude that q is the identity of A, as desired. In consequence, we have

$$A = \left(\sum_k q_k \right) A = \oplus_{k=1}^K q_k A = \oplus_{k=1}^K A p_k A.$$

Now, we must establish the last part. We know already that Aq_lA acts trivially on Ap_k, if $l \neq k$. The only thing remaining to verify is that π_k is an isomorphism from Ap_kA to $\mathcal{B}(Ap_k)$. Every element of Ap_k is in the span of B_k. It follows that every element of $p_kA = (Ap_k)^*$ is a linear combination the the adjoints of elements of B_k. Therefore $ap_ka' = (ap_k)(a'p_k)^*$ is in the span of $bc^*, b, c \in B_k$. For any scalars $\alpha_{b,c}, b, c \in B_k$ and b_0, c_0 in B_k, we compute

$$< \pi_k \left(\sum_{b,c} \alpha_{b,c} bc^* \right) c_0, b_0 > p_k = \sum_{b,c} \alpha_{b,c} < bc^* c_0, b_0 > p_k = \sum_{b,c} \alpha_{b,c} b_0^* bc^* c_0.$$

Since B_k is an orthonormal basis, b_0^*b is zero unless $b = b_0$, in which case it is p_k. Similarly, c^*c_0 is zero unless $c = c_0$, in which case it is also p_k. We conclude that

$$< \pi_k \left(\sum_{b,c} \alpha_{b,c} bc^* \right) c_0, b_0 > p_k = \alpha_{b_0,c_0} p_k.$$

If $\pi_k(\sum_{b,c} \alpha_{b,c} bc^*) = 0$, it follows that each coefficient $\alpha_{b,c}$ is zero and from this it follows that the restriction of π_k to Ap_kA is injective.

We now show π_k is onto. Let a, b be in Ap_k. We claim that $\pi_k(ab^*) = a \otimes b^*$. For any c in Ap_k, we have

$$(a \otimes b^*)c =< c, b > a =< c, b > ap_k = a(< c, b > p_k) = a(b^*c) = \pi_k(ab^*)c.$$

We have shown that the rank one operator $a \otimes b^*$ is in the range of π_k and since the span of such operators is all of $\mathcal{B}(Ap_k)$, we are done. $\square$

Theorem 1.7.9. *If N is a positive integer, then the centre of $M_N(\mathbb{C})$ is the scalar multiples of the identity. If $N_1, N_2, \ldots, N_K$ are positive integers, then the centre $\oplus_{k=1}^K M_{N_k}(\mathbb{C})$ is isomorphic to $\mathbb{C}^K$ and is spanned by the identity elements of the summands.*

Proof. We may assume that $N \geq 2$, since the other case is trivial. We identify $M_N(\mathbb{C})$ and $\mathcal{B}(\mathbb{C}^N)$. Suppose that a is in the centre of $\mathcal{B}(\mathbb{C}^N)$ and let ξ be any vector in $\mathbb{C}^N$. Then we have

$$(a\xi \otimes \xi^*) = a(\xi \otimes \xi^*) = (\xi \otimes \xi^*)a = \xi \otimes (a^*\xi)^*.$$

Applying both sides to the vector ξ, we see that

$$a\xi < \xi, \xi >=< a^*\xi, \xi > \xi.$$

It follows that, for every vector ξ there is a scalar r such that $a\xi = r\xi$. If ξ and η are two linearly independent vectors we know that there are scalars, r, s, t, such that

$$r(\xi + \eta) = a(\xi + \eta) = a\xi + a\eta = s\xi + t\eta.$$

By linear independence, we see that $r = s = t$. We have shown that for every vector ξ, $a\xi$ is a scalar multiple of ξ. Moreover, the scalar is independent of ξ. Thus a is a multiple of the identity.

The second statement follows immediately from the first. $\square$

Let us complete the proof of Theorem 1.7.1. In fact, almost everything is done. We know that from part 3 of Theorem 1.7.8 that $A = \oplus_k Ap_kA$ and from part 4 of the same theorem that, for each k, $\pi_k : Ap_kA \to \mathcal{B}(Ap_k)$ is an isomorphism. Therefore, we have

$$\oplus_{k=1}^K \pi_k : A = \oplus_{k=1}^K Ap_kA \to \oplus_{k=1}^K \mathcal{B}(Ap_k)$$

is an isomorphism.

It only remains for us to prove the uniqueness of K and $N_1, N_2, \ldots, N_K$. We see from the last result that K is equal to the dimension of the centre of A. Next, the units of the summands of A are exactly the minimal non-zero projections in the centre and if q_k is the unit of summand M_{N_k}, then N_k is the square root of the dimension of q_kA.

Exercise 1.7.1. *Let A be a C^*-algebra. Let e be in A and satisfy e^*e is a projection. Prove that*

1. *$ee^*e = e$. (Hint: compute $(ee^*e - e)^*(ee^*e - e)$.)*

2. *ee^* is also a projection.*

Exercise 1.7.2. *Let A be a C^*-algebra and let $n \geq 1$. Suppose that we have elements of A, $e_{i,j}, 1 \leq i, j \leq n$ which satisfy*

$$e_{i,j}e_{k,l} = \begin{cases} e_{i,l} & j = k \\ 0 & j \neq k \end{cases}$$

and $e_{i,j}^* = e_{j,i}$, for all i, j, k, l. Assuming that at least one $e_{i,j}$ is non-zero, prove that the span of the $e_{i,j}, 1 \leq i, j, \leq n$, is a C^*-subalgebra of A and is isomorphic to $M_n(\mathbb{C})$. (Hint: first show that all $e_{i,j}$ are non-zero, then show they are linearly independent.) Such a collection of elements is usually called a set of matrix units.

Exercise 1.7.3. *Suppose that $a_1, a_2, \ldots, a_N$ are elements in a C^*-algebra A and satisfy:*

1. *$a_1^*a_1 = a_2^*a_2 = \cdots = a_N^*a_N$ is a projection,*
2. *$a_ia_i^*a_ja_j^* = 0$ for all $1 \leq i \neq j \leq N$.*

Prove that $e_{i,j} = a_ia_j^*, 1 \leq i, j \leq N$ is a set of matrix units.

1.8 Non-unital C^*-algebras

The main result in this section establishes a very close connection between non-unital and unital C^*-algebras.

Theorem 1.8.1. *Let A be a C^*-algebra. There exists a C^*-algebra $\tilde{A}$ which is unital, contains A as a closed two-sided ideal and $\tilde{A}/A \cong \mathbb{C}$. Moreover, this C^*-algebra is unique.*

Proof. Let $\tilde{A} = \mathbb{C} \oplus A$, as a vector space. We define the product on $\tilde{A}$ by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab),$$

for a, b in A and λ, μ in $\mathbb{C}$. We also define

$$(\lambda, a)^* = (\bar{\lambda}, a^*),$$

for a in A and λ in $\mathbb{C}$. It is clear that $(1, 0)$ is a unit for this algebra. The map sending a in A to $(0, a)$ is obviously an injective $*$-homomorphism of A into $\tilde{A}$, whose image is an ideal. We will usually suppress this map in our notation; this means that, for a in A, $(0,a)$ and a mean the same thing. Moreover, the quotient of $\tilde{A}$ by A is obviously isomorphic to $\mathbb{C}$.

We turn now to the issue of a norm. In fact, it is quite easy to see that by defining

$$\|(\lambda,a)\|_1 = |\lambda| + \|a\|,$$

for (λ,a) in $\tilde{A}$, we obtain a Banach algebra with isometric involution. Getting the C^*-condition (which does not hold for this norm) is rather trickier.

Toward a definition of a good C^*-norm, let us make a nice little observation. If we recall only the fact that A is a Banach space, we can study $\mathcal{B}(A)$, the algebra of bounded linear operators on it. Recalling now the product on A, the formula $\pi(a)b = ab$, for a,b in A, defines $\pi(a)$ as an operator on A. The Banach algebra inequality $\|ab\| \leq \|a\|\|b\|$, simultaneously shows that $\pi(a)$ is in $\mathcal{B}(A)$ and $\|\pi(a)\| \leq \|a\|$. It is easy to see that $\pi : A \to \mathcal{B}(A)$ is a homomorphism. Finally, we consider

$$\|\pi(a)a^*\| = \|aa^*\| = \|a\|^2 = \|a\|\|a^*\|,$$

which implies that $\|\pi(a)\| = \|a\|$. In short, the norm on A can be seen as the operator norm of A acting on itself.

We are going to use a minor variation of that: since A is an ideal in $\tilde{A}$, we can think of $\tilde{A}$ as acting on A.

We define a norm by

$$\|(\lambda,a)\| = \sup\{|\lambda|, \|(\lambda,a)b\|, \|b(\lambda,a)\| \mid b \in A, \|b\| \leq 1\},$$

for all (λ,a) in $\tilde{A}$. Notice that $(\lambda,a)b = \lambda b + ab$ is in A so the norm involved on the right is that of A. It is easy to see the set on the right is bounded (by $\|(\lambda,a)\|_1$), so the supremum exists. We claim that $\|(0,a)\| = \|a\|$, for all a in A. The inequality $\leq$ follows from the fact that the norm on A is a Banach algebra norm. Letting $b = \|a\|^{-1}a^*$ yields the inequality $\geq$ (at least for non-zero a).

We next observe that $\|(\lambda,a)\| = 0$ implies (using $b = \|a\|^{-1}a^*$) that $|\lambda| = \|aa^*\| = 0$ which in turn means that $(\lambda,a) = (0,0)$. It is easy to see that $*$ is isometric. Next, it is trivial that $\|(\lambda,a)\| \leq |\lambda| + \|a\|$ and since both $\mathbb{C}$ and A are complete, so is $\tilde{A}$ in this norm. We leave the details that the norm satisfies the Banach space conditions to the reader.

This leaves us to verify the C^*-condition. Let (λ,a) be in $\tilde{A}$. The inequality $\|(\lambda,a)^*(\lambda,a)\| \leq \|(\lambda,a)\|^2$ follows from the Banach property and that $*$ is isometric. For the reverse, we first note that note that $\|(\lambda, a)^*(\lambda, a)\| \geq |\lambda|^2$. Next, for any b in the unit ball of A, we have

$$\begin{aligned}
\|(\lambda, a)^*(\lambda, a)b\| &\geq \|b^*\|\|(\lambda, a)^*(\lambda, a)b\| \\
&\geq \|b^*(\lambda, a)^*(\lambda, a)b\| \\
&= \|(\lambda, a)b\|^2.
\end{aligned}$$

Taking the supremum over all b in the unit ball yields $\|(\lambda, a)^*(\lambda, a)\| \geq \|(\lambda, a)\|^2$.

The last item is to prove the uniqueness of $\tilde{A}$. Suppose that B satisfies the desired conditions. We define a map from $\tilde{A}$ as above to B by $\rho(\lambda, a) = \lambda 1_B + a$, for all $\lambda \in \mathbb{C}$ and $a \in A$. It is a simple computation to see that ρ is a $*$-homomorphism. Let $q : B \to B/A$ be the quotient map. Since A is an ideal and is not all of B, it cannot contain the unit of B. Thus $q(1_B) \neq 0$. Let us prove that ρ is injective. If $\rho(\lambda, a) = 0$, then $0 = q \circ \rho(\lambda, a) = q(\lambda 1_B + a) = \lambda q(1_B)$ and we conclude that $\lambda = 0$. It follows immediately that $a = 0$ as well and so ρ is injective. It is clear that A is contained in the image of ρ and that $q \circ \rho$ is surjective as well. Since B/A is one-dimensional, we conclude that ρ is surjective. The fact that ρ is isometric follows from Theorem 1.5.2. □

We conclude with a few remarks concerning the spectral theorem in the case of non-unital C^*-algebras. Particularly, we would like generalizations of Theorem 1.4.6 and Corollary 1.4.9. To generalize Theorem 1.4.6, we would simply like to drop the hypothesis that the algebra be unital.

Before beginning, let us make a small remark. If x_0 is a point in some compact Hausdorff space X, there is a natural isomorphism $\{f \in C(X) \mid f(x_0) = 0\} \cong C_0(X \setminus \{x_0\})$ which simply restricts the function to $X \setminus \{x_0\}$.

Theorem 1.8.2. *Let A be a commutative C^*-algebra, let $\tilde{A}$ be as in Theorem 1.8.1 and let $\pi : \tilde{A} \to \mathbb{C}$ be the quotient map with kernel A. Then π is in $\mathcal{M}(\tilde{A})$. Moreover, the restriction of the isomorphism of 1.4.6 to A is an isometric $*$-isomorphism between A and $C_0(\mathcal{M}(\tilde{A}) \setminus \{\pi\})$.*

Proof. We give a sketch only. The first statement is clear. Secondly, if a is in A, then $\hat{a}(\pi) = \pi(a) = 0$, by definition of π. So $\hat{}$ maps A into $C_0(\mathcal{M}(\tilde{A}) \setminus \{\pi\})$ (using the indentification above) and is obviously an isometric $*$-homomorphism. The fact that it is onto follows from the facts that A has codimension one in $\tilde{A}$ while the same is true of $C_0(\mathcal{M}(\tilde{A}) \setminus \{\pi\})$ in $C(\mathcal{M}(\tilde{A}))$. □

We note the obvious corollary.

Corollary 1.8.3. *If A is a commutative C^*-algebra then there exists a locally compact Hausdorff space X such that $A \cong C_0(X)$. Moreover, X is compact if and only if A is unital.*

Generalizing Corollary 1.4.9 is somewhat more subtle. If a is a normal element of any (possibly non-unital) C^*-algebra B, we can always replace B by $\tilde{B}$ and apply 1.4.9. So the hypothesis that B is a unital C^*-algebra is rather harmless. The slight catch is that the isomorphism of 1.4.9 has range which is the C^*-subalgebra generated by a and the unit, which takes us outside of our original B, if it is not unital. The appropriate version here strengthens the conclusion by showing that, for functions f in $C(spec(a))$ satisfying $f(0) = 0$, $f(a)$ actually lies in the C^*-subalgebra generated by a. This fact is quite useful even in situations where B is unital, since it strengthens the conclusion of Theorem 1.4.9. This is, in fact, how it is stated.

Theorem 1.8.4. *Let a be a normal element of the unital C^*-algebra B. The isomorphism of 1.4.9 restricts to an isometric $*$-isomorphism between $C_0(spec(a) \setminus \{0\})$ and the C^*-subalgebra of B generated by a.*

Proof. Again, we provide a sketch. First, assume that 0 is not in the spectrum of a, so a is invertible. Notice that $C_0(spec(a)\setminus\{0\}) = C(spec(a))$. We would like to simply apply 1.4.9, but we need to see that the C^*-algebra generated by a and the unit is the same as that generated by a alone. Now, let f be the continuous function on $spec(a) \cup \{0\}$ which is 1 on $spec(a)$ and 0 at 0. Let $\epsilon > 0$ be arbitrary and use Weierstrass' Theorem to approximate f by a polynomial $p(z, \bar{z})$ to within ϵ. It follows that $p(z, \bar{z}) - p(0, 0)$ is a polynomial with no constant term which approximates f to within 2ϵ. Then $p(a, a^*)$ is within 2ϵ of $f(a) = 1$. On the other hand, the formula of 1.4.9 shows that $p(a, a^*)$ is in the C^*-subalgebra generated by a alone. In this case, the conclusion is exactly the same as for 1.4.9.

Now we assume that 0 is in the spectrum of a. Let f be in $C_0(spec(a) \setminus \{0\})$. We may view f as a continuous function of $spec(a)$ by defining $f(0) = 0$. Again let $\epsilon > 0$ be arbitrary and use Weierstrass' Theorem to approximate f by a polynomial $p(z, \bar{z})$ to within ϵ. It follows that $p(z, \bar{z}) - p(0, 0)$ is a polynomial with no constant term which approximates f to within 2ϵ. Again, we see that $f(a)$ can be approximated to within 2ϵ by an element of the C^*-algebra generated by a alone. From this we see that $C_0(spec(a) \setminus \{0\})$ is mapped to the C^*-subalgebra generated by a. The rest of the conclusion is straightforward. □

Exercise 1.8.1. *If A is a unital C^*-algebra, then $\tilde{A} \cong \mathbb{C} \oplus A$ as C^*-algebras (see Example 1.2.5). Write the isomorphism explicitly, since it isn't the obvious one!*

Exercise 1.8.2. *Let X be a locally compact Hausdorff space and let $A = C_0(X)$. Prove that $\mathcal{M}(\tilde{A})$ is homeomorphic to $X \cup \{\infty\}$, the one-point compactification of X. Discuss the overlap of this exercise and the last one.*

1.9 Ideals and quotients

In this section, we consider ideals in a C^*-algebra. Now ideals come in many forms; they can be either one-sided or two-sided and they may or may not be closed. We will concentrate here on closed, two-sided ideals. Generally, one-sided ideals are rather difficult to describe. (For a simple tractable case, see Exercise 1.9.2 below.) Ideals that are not closed can be even worse, although in specific situations, there are ones of interest that arise. For an example, the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$ is an ideal which is not closed. For another example, consider a locally compact, non-compact Hausdorff space X. The set of all compactly supported continuous functions on X is an ideal in $C_0(X)$.

If A is a C^*-algebra and I is a closed subspace, then we may form the quotient space A/I. We remark that the norm on the quotient A/I is defined by

$$\|a + I\| = \inf\{\|a + b\| \mid b \in I\}.$$

For the proofs that this is a well-defined norm and makes A/I into a Banach space, we refer the reader to [**?**].

If, in addition, I is also a two-sided ideal, then A/I is also an algebra, just as usual in a first course in ring theory and it is an easy matter to check that our norm above is actually a Banach algebra norm. At this point, A/I might fall short of being a C^*-algebra on two points. The first is having a $*$-operation. There is an obvious candidate, $(a+I)^* = a^* + I$, but to see this is well-defined, we need to know that I is closed under $*$. The second point is seeing that the norm satisfies C^*-condition. This all turns out to be true and we state our main result.

Theorem 1.9.1. *Let A be a C^*-algebra and suppose that I is a closed (meaning as a topological subset), two-sided ideal. Then I is also closed under the $*$-operation and A/I, with the quotient norm, is a C^*-algebra.*

In the course of the proof, we will need the following technical result.

Lemma 1.9.2. *Let a be an element in a C^*-algebra A. For any $\epsilon > 0$, there is a continuous function f on $[0, \infty)$ such that $e = f(a^*a)$ is in A, is positive, $\|e\| \leq 1$ and $\|a - ae\| < \epsilon$.*

Proof. We note from Theorem 1.13.1 that $spec(a^*a) \subseteq [0, \infty)$. Define the function f on $[0, \infty)$ by $f(t) = t(\epsilon + t)^{-1}$. Since $f(0) = 0$, we may apply Theorem 1.7.5 to see that $f(a^*a)$ is contained in the C^*-algebra generated by a^*a and hence in A. It is clear that $0 < f(t) < 1$, for all $0 \leq t < \infty$ and so $e = f(a^*a)$ is well-defined, positive and norm less than or equal to one. Notice in what follows, when we write expressions like $1 - e$, this can be regarded as an element of $\tilde{A}$, even if A is non-unital. We have

$$\begin{aligned} \|a - ae\|^2 &= \|a(1 - e)\| \\ &= \|a(1 - f(a^*a))\|^2 \\ &= \|(1 - f(a^*a))a^*a(1 - f(a^*a))\| \\ &= \|g(a^*a)\| \\ &\leq \|g\|_\infty, \end{aligned}$$

where $g(t) = t(1 - f(t))^2$. It is a simple calculus exercise to check that g attains its maximum (on the positive axis) at $t = \epsilon$ and its maximum is $\epsilon/4$. □

We now turn to the proof of the main result.

Proof. Let a be in I. We will show that a^* is also. Let $\epsilon > 0$ and apply the last lemma to a. Since a is in I, so is a^*a. We claim that $e = f(a^*a)$ is in I, also. This point is a little subtle in applying 1.7.5 since we do not yet know that I is a C^*-algebra. However, we are applying the function to the self-adjoint element a^*a. By approximating the function by polynomials in a^*a we see that the C^*-algebra generated by this element is the same as the closed algebra generated by the element and hence is contained in I.

Since e is self-adjoint and the $*$ operation is isometric, we have

$$\|a^* - ea^*\| = \|a - ae\| \leq \epsilon.$$

Since e is in I, so is ea^*. We see then that a^* is within ϵ of an element of I. As I is closed and ϵ is arbitrary, we conclude that a^* is in I.

We now prove that the norm satisfies the C^*-condition.

Let a be in A. First, we show that the $*$ operation is isometric since

$$\begin{aligned} \|a^* + I\| &= \inf\{\|a^* + b\| \mid b \in I\} \\ &= \inf\{\|a^* + b^*\| \mid b \in I\} \\ &= \inf\{\|a + b\| \mid b \in I\} \\ &= \|a + I\|. \end{aligned}$$

Now it suffices to prove that $\|a + I\|^2 \leq \|a^*a + I\|$. We claim that

$$\|a + I\| = \inf\{\|a + b\| \mid b \in I\} = \inf\{\|a(1 - e)\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

We first observe that any element e satisfying $e \geq 0$ (meaning that e is positive) and norm less than 1 must have spectrum contained in $[0, 1]$. It follows that the norm of $1 - e$ also has norm less than 1.

As $a(1-e) = a-ae$ and ae is in I for any e satisfying the given conditions, we see immediately that

$$\inf\{\|a + b\| \mid b \in I\} \leq \inf\{\|a(1 - e)\| \mid e \in I, e \geq 0, \|e\| \leq 1\}.$$

For the reverse, let b be in I let $\epsilon > 0$ and use the Lemma to obtain e as desired with $\|b - be\| < \epsilon$. Then we have

$$\|a + b\| \geq \|(a + b)(1 - e)\| \geq \|a(1 - e)\| - \|b - be\| \geq \|a(1 - e)\| - \epsilon.$$

As $\epsilon > 0$ was arbitrary, we have established the reverse inequality.

Having the claim, we now check that for a given e with conditions as above, we have

$$\|a(1 - e)\|^2 = \|(1 - e)a^*a(1 - e)\| \leq \|a^*a(1 - e)\|.$$

Taking infimum over all such e yields the result. $\square$

As a final remark, we note that if I is a closed, two-sided ideal in A, the natural quotient map $\pi(a) = a + I, a \in A$ from A to A/I is a $*$-homomorphism. The proof is a triviality, but this can often be a useful point of view.

Exercise 1.9.1. *1. Let X be a compact, Hausdorff space and let $A = C(X)$. Let $Z \subseteq X$ be closed and*

$$I = \{f \in C(X) \mid f(z) = 0, \text{ for all } z \in Z\}.$$

First prove that I is a closed, two-sided ideal in A. Next, since I and A/I are both clearly commutative C^*-algebras (although the former may not be unital) find $\mathcal{M}(I)$ and $\mathcal{M}(A/I)$.

2. *Let A be as above. Prove that any closed two-sided ideal I in A arises as above. (Hint: A/I is commutative and unital; use Exercise 1.4.2 on the quotient map.)*

Exercise 1.9.2. *Let $n \geq 2$ and $A = M_n(\mathbb{C})$.*

1. *Suppose that I is a right ideal in A. Show that $I\mathbb{C}^n = \{a\xi \mid a \in I, \xi \in \mathbb{C}^n\}$ is a subspace of $\mathbb{C}^n$. (Hint: if ξ, η are unit vectors and a is a matrix, then $a\xi = a(\xi \otimes \eta^*)\eta$.)*

2. *Prove the correspondence between the set of all right ideals in A and the set of all linear subspaces of $\mathbb{C}^n$ in the previous part is a bijection.*

3. *Prove that A is simple; that is, it has no closed two-sided ideals except 0 and A.*

Exercise 1.9.3. *Let $\rho : A \to B$ be a $*$-homomorphism between two C^*-algebras. Prove that $\rho(A)$ is closed and is a C^*-subalgebra of B. (Hint: Theorem 1.9.1 and Theorem 1.5.2.)*

1.10 Traces

As a C^*-algebra is a Banach space, it has a wealth of linear functionals, basically thanks to the Hahn-Banach Theorem. We can ask for extra properties in a linear functional. Indeed we did back in Chapter 1.4 when we looked at homomorphisms which are simply linear functionals which are multiplicative. That was an extremely successful idea in dealing with commutative C^*-algebras, but most C^*-algebras of interest will have none.

It turns out that there is some middle ground between being a linear functional and a homomorphism. This is the notion of a trace.

Definition 1.10.1. *Let A be a unital C^*-algebra. A linear functional ϕ on A is said to be* positive *if $\phi(a^*a) \geq 0$, for all a in A. A* trace *on A is a positive linear functional $\tau : A \to \mathbb{C}$ with $\tau(1) = 1$ satisfying*

$$\tau(ab) = \tau(ba),$$

for all a, b in A. This last condition is usually called the trace property. The trace is said to be faithful if $\tau(a^*a) = 0$ occurs only for $a = 0$.

Notice that the trace property is satisfied by any homomorphism, but we will see that it is strictly weaker. For the moment, it is convenient to regard it as something stronger than mere linearity (or even positivity) but weaker than being a homomorphism.

Example 1.10.2. *If A is commutative, every positive linear functional, ϕ, with $\phi(1) = 1$, is a trace.*

Theorem 1.10.3. *Let $\mathcal{H}$ be a Hilbert space of (finite) dimension n. If $\{\xi_1, \ldots, \xi_n\}$ is an orthonormal basis for $\mathcal{H}$, then*

$$\tau(a) = n^{-1} \sum_{i=1}^{n} < a\xi_i, \xi_i >,$$

for any a in $\mathcal{B}(\mathcal{H})$, defines a faithful trace on $\mathcal{B}(\mathcal{H})$. The trace is unique. If we identify $\mathcal{B}(\mathcal{H})$ with $M_n(\mathbb{C})$, then this trace is expressed as

$$\tau(a) = n^{-1} \sum_{i=1}^{n} a_{i,i}$$

for any a in $M_n(\mathbb{C})$.

Proof. First, it is clear that τ is a linear functional. Next, we see that, for any a in $\mathcal{B}(\mathcal{H})$, we have

$$\tau(a^*a) = n^{-1} \sum_{i=1}^{n} < a^*a\xi_i, \xi_i >= n^{-1} \sum_{i=1}^{n} < a\xi_i, a\xi_i >= n^{-1} \sum_{i=1}^{n} \|a\xi_i\|^2.$$

It follows at once that τ is positive. Moreover, if $\tau(a^*a) = 0$, then $a\xi_i = 0$, for all i. As a is zero on a basis, it is the zero operator. We also see from this, using $a = 1$, that $\tau(1) = 1$.

It remains for us to verify the trace property. We do this by first considering the case of rank one operators. We observe that, for any ξ, η in $\mathcal{H}$, we have

$$\begin{aligned}\tau(\xi \otimes \eta) &= n^{-1} \sum_{i=1}^{n} < (\xi \otimes \eta)\xi_i, \xi_i > \\ &= n^{-1} \sum_{i=1}^{n} < \xi_i, \eta >< \xi, \xi_i > \\ &= n^{-1} < \sum_{i=1}^{n} < \xi, \xi_i > \xi_i, \eta > \\ &= n^{-1} < \xi, \eta > .\end{aligned}$$

Now let ξ, η, ζ, ω be in $\mathcal{H}$ and $a = \xi \otimes \eta^*, b = \zeta \otimes \omega^*$. We have $ab =< \zeta, \eta > \xi \otimes \omega^*$ and $ba =< \xi, \omega > \zeta \otimes \eta^*$. It follows then from the computation above that

$$\begin{aligned}\tau(ab) &= < \zeta, \eta > \tau(\xi \otimes \omega^*) \\ &= n^{-1} < \zeta, \eta >< \xi, \omega > \\ &= < \xi, \omega > \tau(\zeta \otimes \eta^*) \\ &= \tau(ba).\end{aligned}$$

Since the linear span of such operators is all of $\mathcal{B}(\mathcal{H})$, the trace property holds.

We finally turn to the uniqueness of the trace. Suppose that ϕ is any trace on $\mathcal{B}(\mathcal{H})$. We will show that ϕ and τ agree on $\xi_i \otimes \xi_j^*$, first considering the case $i \neq j$. Let $a = \xi_i \otimes \xi_i^*$ and $b = \xi_i \otimes \xi_j^*$. It follows that $ab = b$ while $ba = 0$. Therefore, we have $\phi(b) = \phi(ab) = \phi(ba) = 0 = \tau(b)$. Next, for any i, j, let $v = \xi_i \otimes \xi_j^*$. Then $v^*v = \xi_j \otimes \xi_j^*$ while $vv^* = \xi_i \otimes \xi_i^*$. Therefore, we have $\phi(\xi_i \otimes \xi_i^*) = \phi(\xi_j \otimes \xi_j^*)$. Finally, we have

$$1 = \phi(1) = \phi\left(\sum_{i=1}^{n} \xi_i \otimes \xi_i^*\right) = n\phi(\xi_1 \otimes \xi_1^*).$$

Putting these together, we see that

$$\phi(\xi_i \otimes \xi_i^*) = \phi(\xi_1 \otimes \xi_1^*) = n^{-1} = \tau(\xi_i \otimes \xi_i^*).$$

We have shown that τ and ϕ agree on a spanning set, $\xi_i \otimes \xi_j^*, 1 \leq i, j \leq n$, hence they are equal. $\square$

We mention in passing the following handy fact. Basically, it is telling us that, when applied to projections, the trace recovers the geometric notion of the dimension of the range.

Theorem 1.10.4. *Let τ be the unique trace on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a finite dimensional Hilbert space. If p is a projection, then $dim(p\mathcal{H}) = \tau(p)dim(\mathcal{H})$.*

Proof. Choose an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ for $\mathcal{H}$ is such a way that $\{\xi_1, \ldots, \xi_k\}$ is a basis for $p\mathcal{H}$. We omit the remaining computation. $\square$

Exercise 1.10.1. *Prove that, for $n > 1$, there is no non-zero $*$-homomorphism from $M_n(\mathbb{C})$ to $\mathbb{C}$. In fact, give two different proofs, one using 1.10.3 above, and one using Exercise 1.9.2.*

1.11 Representations

The study of C^*-algebras is motivated by the prime example of closed $*$-algebras of operators on Hilbert space. With this in mind, it is natural to find ways that a given abstract C^*-algebra may act as operators on Hilbert space. Such an object is called a representation of the C^*-algebra.

Definition 1.11.1. *Let A be a $*$-algebra. A* representation *of A is a pair, $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism. We also say that π is a representation of A on $\mathcal{H}$.*

Now the reader should prepare for a long list of simple properties, constructions and results concerning representations.

The first important notion for representations is that of unitary equivalence. One should consider unitarily equivalent representations as being 'the same'.

Definition 1.11.2. *Let A be a $*$-algebra. Two representations of A, $(\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2)$, are* unitarily equivalent *if there is a unitary operator $u : \mathcal{H}_1 \to \mathcal{H}_2$ such that $u\pi_1(a) = \pi_2(a)u$, for all a in A. In this case, we write $(\pi_1, \mathcal{H}_1) \sim_u (\pi_2, \mathcal{H}_2)$ or $\pi_1 \sim_u \pi_2$.*

The most important operation on representations of a fixed $*$-algebra is the direct sum. The following definition is stated in some generality, but at first pass, the reader can assume the collection of representations has exactly two elements.

Definition 1.11.3. *Let A be a $*$-algebra and $(\pi_\iota, \mathcal{H}_\iota), \iota \in I$, be a collection of representations of A. Their direct sum is $(\oplus_{\iota \in I} \pi_\iota, \oplus_{\iota \in I} \mathcal{H}_\iota)$, where $\oplus_{\iota \in I} \mathcal{H}_\iota$ consists of tuples, $\xi = (\xi_\iota)_{\iota \in \mathcal{H}}$ satisfying $\sum_{\iota \in I} \|\xi_\iota\|^2 < \infty$ and*

$$(\oplus_{\iota \in I} \pi_\iota(a)\xi)_\iota = \pi_\iota(a)\xi_\iota, \iota \in I.$$

Definition 1.11.4. *Let A be a $*$-algebra and let $(\pi, \mathcal{H})$ be a representation of A. A subspace $\mathcal{N} \subset \mathcal{H}$ is said to be* invariant *if $\pi(a)\mathcal{N} \subset \mathcal{N}$, for all a in A.*

For the most part, we will be interested in subspaces that are also *closed.*

It is a fairly simple matter to find a closed invariant subspace for an operator on a Hilbert space whose orthogonal complement is *not* invariant for that same operator. However, when dealing with an entire self-adjoint collection of operators, this is not the case.

Proposition 1.11.5. *Let A be a $*$-algebra and let $(\pi, \mathcal{H})$ be a representation of A. A closed subspace $\mathcal{N}$ is invariant if and only if $\mathcal{N}^\perp$ is.*

Proof. For the 'only if' direction, it suffices to consider ξ in $\mathcal{N}^\perp$ and a in A and show that $\pi(a)\xi$ is again in $\mathcal{N}^\perp$. To this end, let η be in $\mathcal{N}$. We have

$$< \pi(a)\xi, \eta > = < \xi, \pi(a)^*\eta > = < \xi, \pi(a^*)\eta > = 0,$$

since $\pi(a^*)\mathcal{N} \subset \mathcal{N}$.

The 'if' direction follows since $(\mathcal{N}^\perp)^\perp = \mathcal{N}$. □

In the case of the Proposition above, it is possible to define two representations of A by simply restricting the operators to either $\mathcal{N}$ or $\mathcal{N}^\perp$. That is, we define

$$\pi|_{\mathcal{N}}(a) = \pi(a)|\mathcal{N}, a \in A.$$

Moreover, the direct sum of these two representations is unitarily equivalent to the original. That is, we have

$$(\pi, \mathcal{H}) \sim_u (\pi|_{\mathcal{N}}, \mathcal{N}) \oplus (\pi|_{\mathcal{N}^\perp}, \mathcal{N}^\perp).$$

One can, in some sense, consider the notion of reducing a representation to an invariant subspace and its complement as an inverse to taking direct sums.

Definition 1.11.6. *A representation, $(\pi, \mathcal{H})$, of a $*$-algebra, A, is* non-degenerate *if the only vector ξ in $\mathcal{H}$ such that $\pi(a)\xi = 0$ for all a in A, is $\xi = 0$. Otherwise, the representation is* degenerate.

The following is a trivial consequence of the definitions and we leave the proof for the reader.

Proposition 1.11.7. *A representation $(\pi, \mathcal{H})$ of a unital $*$-algebra is non-degenerate if and only if $\pi(1) = 1$.*

In fact, we can easily restrict our attention to non-degenerate representations, as the following shows.

Theorem 1.11.8. *Every representation of a $*$-algebra is the direct sum of a non-degenerate representation and the zero representation (on some Hilbert space).*

A particularly nice class of representations are those which are cyclic. For the moment, we only give the definition, but their importance will emerge in the next section.

Definition 1.11.9. *Let $(\pi, \mathcal{H})$ be a representation of a $*$-algebra A. We say that a vector ξ in $\mathcal{H}$ is* cyclic *if the linear space $\pi(A)\xi$ is dense in $\mathcal{H}$. We say that the representation is cyclic if it has a cyclic vector.*

Notice that every cyclic representation is non-degenerate as follows. Let ξ be the cyclic vector. If $\pi(a)\eta = 0$, for all a in A, then we have

$$< \pi(a)\xi, \eta > = < \xi, \pi(a^*)\eta > = < \xi, 0 > = 0.$$

As $\pi(a)\xi$ is dense in $\mathcal{H}$, we conclude that $\eta = 0$.

The next notion is a somewhat obvious one following our discussion of invariant subspaces. An irreducible representation is one that cannot be decomposed into smaller ones.

Definition 1.11.10. *A representation of a $*$-algebra is* irreducible *if the only closed invariant subspaces are 0 and $\mathcal{H}$. It is* reducible *otherwise.*

The following furnishes a handy link between irreducible representations and cyclic ones.

Proposition 1.11.11. *A non-degenerate representation of a $*$-algebra is irreducible if and only if every non-zero vector is cyclic.*

Proof. First assume that $(\pi, \mathcal{H})$ is irreducible. Let ξ be a non-zero vector, then $\pi(A)\xi$ is evidently an invariant subspace and its closure is a closed invariant subspace. If it is 0, then the representation is degenerate, which is impossible. Otherwise, it must be $\mathcal{H}$, meaning that ξ is a cyclic vector for π.

Conversely, suppose that π is non-degenerate, but reducible. Let $\mathcal{N}$ be a closed invariant subspace which is neither 0 nor $\mathcal{H}$. If ξ is any non-zero vector in $\mathcal{N}$, then $\pi(A)\xi$ is clearly contained in $\mathcal{N}$ and hence cannot be dense in $\mathcal{H}$. Hence, we have found a non-zero vector which is not cyclic. □

Next, we give a more useful criterion for a representation to be reducible.

Proposition 1.11.12. *A non-degenerate representation of a $*$-algebra is irreducible if and only if the only positive operators which commute with its image are scalars.*

Proof. Let $(\pi, \mathcal{H})$ be a representation of the $*$-algebra A.

First, we suppose that there is a non-trivial closed invariant subspace, $\mathcal{N}$. Let p be the orthogonal projection onto $\mathcal{N}$. That is, $p\xi = \xi$, for all ξ in $\mathcal{N}$ and $p\xi = 0$, for all ξ in $\mathcal{N}^{\perp}$. It is easy to check that $p = p^* = p^2$, which means that p is positive. Moreover, as both $\mathcal{N}$ and $\mathcal{N}^{\perp}$ are non-empty, this operator is not a scalar. We check that it commutes with $\pi(a)$, for any a in A. If ξ is in $\mathcal{N}$, we know that $\pi(a)\xi$ is also and so

$$(p\pi(a))\xi = p(\pi(a)\xi) = \pi(a)\xi = \pi(a)(p\xi) = (\pi(a)p)\xi.$$

On the other hand, if ξ is in $\mathcal{N}^{\perp}$, then so is $\pi(a)\xi$ and

$$(p\pi(a))\xi = p(\pi(a)\xi) = 0 = \pi(a)(0) = \pi(a)(p\xi) = (\pi(a)p)\xi.$$

Since every vector in $\mathcal{H}$ is the sum of two as above, we see that $p\pi(a) = \pi(a)p$.

Conversely, suppose that h is some positive, non-scalar operator on $\mathcal{H}$, which commutes with every element of $\pi(A)$. If the spectrum of h consists of a single point, then it follows from 1.4.9 that h is a scalar. As this is not the case, the spectrum consists of at least two points. We may then find non-zero continuous functions f, g on $spec(h)$ whose product is zero. Since f is non-zero on the spectrum of h, the operator $f(h)$ is non-zero. Let $\mathcal{N}$ denote the closure of its range, which is a non-zero subspace of $\mathcal{H}$. On the other hand, $g(h)$ is also a non-zero operator, but it is zero on the range of $f(h)$ and hence on $\mathcal{N}$. This implies that $\mathcal{N}$ is a proper subspace of $\mathcal{H}$.

Next, we claim that, as h commutes with $\pi(A)$, so does $f(h)$. Let a be in A. For any $\epsilon > 0$, we may find a polynomial $p(x)$ such that $\|p - f\|_\infty < \epsilon$ in $C(spec(h))$ and this means that $\|p(h) - f(h)\| < \epsilon$. On the other hand, it is clear that $p(h)$ will commute with $\pi(a)$, since h does. Finally, we claim that $\mathcal{N}$ is invariant under $\pi(a)$. In fact, it suffices to check that the range of $f(h)$ is invariant. But if ξ is in $\mathcal{H}$, we have

$$\pi(a)(f(h)\xi) = \pi(a)f(h)\xi = f(h)\pi(a)\xi \in f(h)\mathcal{H},$$

and we are done. □

Exercise 1.11.1. *Fix a positive integer n and consider $A = M_n(\mathbb{C})$. For $K \geq 1$, let*

$$\mathcal{H}_K = \oplus_{k=1}^K \mathbb{C}^n.$$

Define π_K by

$$\pi_K(a)(\xi_1, \xi_2, \ldots, \xi_K) = (a\xi_1, a\xi_2, \ldots, a\xi_K).$$

This is a non-degenerate representation of A on $\mathcal{H}_K$.

1. *Give a necessary and sufficient condition on a vector $(\xi_1, \xi_2, \ldots, \xi_K)$ for it to be a cyclic vector for π_K.*

2. *For which values of K is the representation π_K cyclic?*

3. *For which values of K is the representation π_K irreducible? (You should be able to give two proofs; one using the definition and one applying Proposition 1.11.11.)*

Exercise 1.11.2. *Fix $n \geq 1$ and let $A = M_n(\mathbb{C})$. There is an obvious representation which we call ρ of A on $\mathbb{C}^n$:*

$$\rho(a)\xi = a\xi, a \in A, \xi \in \mathbb{C}^n.$$

We are using the usual multiplication of matrices (or matrix and vector).

Now let $(\pi, \mathcal{H})$ be any non-degenerate representation of A. (If you like, you may start by assuming that $\mathcal{H}$ is finite dimensional, although it makes little difference.)

1. *Use the fact that A is simple (Exercise 1.9.2) to show that $\pi(e_{1,1}) \neq 0$.*

2. *Let ξ be any unit vector in $\pi(e_{1,1})\mathcal{H}$. Prove that*
$$\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi$$
is an orthonormal set.

3. *Prove that*
$$\pi(A)\xi = span\{\xi, \pi(e_{2,1})\xi, \ldots, \pi(e_{n,1})\xi\}.$$

4. *Letting $\mathcal{H}_\xi$ denote the subspace of the last part, prove that $\pi|_{\mathcal{H}_\xi}$ is unitarily equivalent to ρ.*

5. *Let $\mathcal{B}$ be an orthonormal basis for $\pi(e_{1,1})\mathcal{H}$. Prove that, for any $\xi \neq \eta$ in $\mathcal{B}$, $\mathcal{H}_\xi$ and $\mathcal{H}_\eta$ are orthogonal.*

6. *Prove that $\oplus_{\xi\in\mathcal{B}}\mathcal{H}_\xi = \mathcal{H}$.*

7. *Prove that $(\pi, \mathcal{H})$ is unitarily equivalent to $\oplus_{\xi\in\mathcal{B}}(\rho, \mathbb{C}^n)$.*

Exercise 1.11.3. *Prove that if $(\pi, \mathcal{H})$ is an irreducible representation of a commutative C^*-algebra, then $\mathcal{H}$ is one-dimensional.*

Exercise 1.11.4. *Let $A = C[0, 1]$. Consider the representation $(\pi, L^2([0, 1], \lambda))$, where λ is Lebesgue measure and $(\pi(f)\xi)(x) = f(x)\xi(x)$, for f in A, ξ in $L^2([0, 1], \lambda)$ and x in $[0, 1]$. (If you like, you may assume that ξ actually lies is $C[0, 1]$, although we regard it as a vector in $L^2([0, 1], \lambda)$. This makes things somewhat easier.)*

1. *Give a necessary and sufficient condition on a vector ξ in $L^2([0, 1], \lambda)$ to be a cyclic vector.*

2. *Prove that π is faithful; that is, it is injective.*

3. *Prove that this representation is* not *the direct sum of a collection of irreducible representations.*

4. *Find a countable collection of irreducible representation of A whose direct sum is injective.*

Exercise 1.11.5. *Suppose $(\pi, \mathcal{H})$ is finite-dimensional irreducible representation of a C^*-algebra A. Prove that $\pi(A) = \mathcal{B}(\mathcal{H})$. (Hint: Theorem 1.7.1, Theorem 1.7.9 and Proposition 1.11.12.)*

1.12 The GNS construction

In the last section, we have discussed a number of properties of representations of a given C^*-algebra. Of course, what is missing at the moment is that we don't know we have any. We turn to this problem now.

The situation is similar to the one we encounter with groups. As groups naturally appear as symmetries, one looks for ways that abstract groups may act that way. The simplest way of obtaining something of this type is to take advantage of the group product to let the group act as permutations of itself. The result in this case is Cayley's Theorem.

The same basic idea works here: the multiplication allows one to see the elements of a C^*-algebra acting as linear transformations of itself. The problem is, of course, that the C^*-algebra is a fine vector space, but does not usually have the structure of a Hilbert space. To produce an inner product or bilinear form, we use the linear functionals on the C^*-algebra in a clever way, called the GNS (for Gelfand-Naimark-Segal) construction.

One added bonus is that we will see that every abstract C^*-algebra is isomorphic to a closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space. (Note, however, that neither the Hilbert space nor the isomorphism are unique.)

The key property for the functionals we will consider is a notion of positivity, which was already introduced when we discussed traces. Probably this is not very surprising, given the important part that positivity has played in the theory so far.

Let us recall the first part of Definition 1.10.1 and add a little more.

Definition 1.12.1. *Let A be a C^*-algebra. A linear functional ϕ on A is* positive *if $\phi(a^*a) \geq 0$, for all a in A. In the case that A is unital, the linear functional ϕ is a* state *if it is positive and, in addition, $\phi(1) = 1$.*

We begin with a rather remarkable simple characterization of states.

Proposition 1.12.2. *Let ϕ be a linear functional on a unital C^*-algebra A with $\phi(1) = \|\phi\| = 1$. Then ϕ is a state.*

Proof. We begin by showing that if a is self-adjoint, then $\phi(a)$ is real. Suppose that $\Im(\phi(a)) \neq 0$ and without loss of generality, assume it is positive. Then select $0 < r < \Im(\phi(a)) \leq \|a\|$. The function $f(s) = \sqrt{\|a\|^2 + s^2} - s$ is clearly greater than r at $s = 0$. By multiplying and dividing by $\sqrt{\|a\|^2 + s^2} + s$ and simplifying, we see that $f(s)$ tends to 0 as s tends to infinity. Thus we can find s such that $f(s) = r$. A simple geometric argument shows that the disc with centre $-is$ and radius $s + r$ then contains the spectrum of a, but does not contain $\phi(a)$. That is, $\|a - is\| \leq r + s$ while $|\phi(a) - is| > r + s$. But now we can compute

$$|\phi(a - is)| = |\phi(a) - is\phi(1)| = |\phi(a) - is| > r + s,$$

since $\phi(1) = 1$,and this contradicts $\|\phi\| = 1$.

Now we let a be positive and show that $\phi(a) \geq 0$. From Lemma 1.13.2, we know that $\|\|a\| - a\| \leq \|a\|$. Applying ϕ to $\|a\| - a$ and using the facts that $\phi(1) = 1 = \|\phi\|$, we have

$$|\|a\| - \phi(a)| \leq \|a\|,$$

which implies that $\phi(a) \geq 0$. We have shown that ϕ is positive, as desired. □

Lemma 1.12.3. *Let ϕ be a positive linear functional on a unital C^*-algebra A.*

1. *For all a, b in A, we have*

$$|\phi(b^*a)|^2 \leq \phi(a^*a)\phi(b^*b).$$

2. *For all a in A, we have $\phi(a^*) = \overline{\phi(a)}$.*

3. *$\phi(1) = \|\phi\|$.*

4. *For all a, b in A, we have*

$$\phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b).$$

Proof. Let a, b be in A. For any complex number λ, we know that

$$0 \leq \phi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\phi(a^*a) + \bar{\lambda}\phi(a^*b) + \lambda\phi(b^*a) + \phi(b^*b).$$

The first and last terms are obviously positive, so the sum of the middle two is real. Using $b = 1$ and $\lambda = i$ shows that the imaginary parts of $\phi(a)$ and $\phi(a^*)$ are opposite, while $b = 1$ and $\lambda = 1$ shows the real parts of $\phi(a)$ and $\phi(a^*)$ are equal. The second part follows.

Since the identity element of A has norm one, we have $|\phi(1)| \leq \|\phi\|$ and as $1^*1 = 1$, $\phi(1)$ is positive. From the equation above, and the second part applied to b^*a, we see that

$$-2\Re(\lambda\phi(b^*a)) \leq |\lambda|^2\phi(a^*a) + \phi(b^*b).$$

If $\phi(a^*a) = 0$, the fact that this holds for arbitrary λ means that $\phi(b^*a) = 0$ also. For $\phi(a^*a) \neq 0$, choose $z \in \mathbb{C}$ such that $|z| = 1$, $z\phi(b^*a) = |\phi(b^*a)|$ and using $\lambda = -z\phi(a^*a)^{-\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}}$ leads us to

$$|\phi(b^*a)| \leq \phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}},$$

and the proof of the first statement is complete.

For the third part, $\|1\| = 1$, so that $|\phi(1)| \leq \|\phi\|$. Using $b = 1$ in part 1, we have

$$|\phi(a)| \leq \phi(a^*a)^{\frac{1}{2}}\phi(1) \leq \|\phi\|^{\frac{1}{2}}\|a^*a\|^{\frac{1}{2}}\phi(1)^{\frac{1}{2}} = \|\phi\|^{\frac{1}{2}}\|a\|\phi(1)^{\frac{1}{2}}.$$

Taking supremum over all a of norm one, we get

$$\|\phi\| \leq \|\phi\|^{\frac{1}{2}}\phi(1)^{\frac{1}{2}}$$

and it follows that $\|\phi\| \leq \phi(1)$. This completes the proof of part 3.

For the last part, we consider the functional $\psi(c) = \phi(b^*cb)$, for any c in A. This is clearly another positive linear functional and so we have $\|\psi\| = \psi(1) = \phi(b^*b)$. Then we have

$$\phi(b^*a^*ab) = \psi(a^*a) \leq \|\psi\|\|a^*a\| = \phi(b^*b)\|a\|^2.$$

$\square$

We are now ready to give our main result of this section.

Theorem 1.12.4. *Let A be a unital C^*-algebra and let ϕ be a state on A.*

1. *The set*
$$N_\phi = \{a \in A \mid \phi(a^*a) = 0\}$$
is a closed left ideal in A.

2. *The bilinear form $< a + N_\phi, b + N_\phi >= \phi(b^*a)$ is well-defined and non-degenerate on A/N_ϕ and the completion of A/N_ϕ is a Hilbert space, denoted $\mathcal{H}_\phi$.*

3. *The formula*

$$\pi_\phi(a)(b + N_\phi) = ab + N_\phi,$$

   for a, b in A extends to define $\pi_\phi(a)$ as a bounded linear operator on $\mathcal{H}_\phi$.

4. *The function π_ϕ is a representation of A on $\mathcal{H}_\phi$.*

5. *The vector $\xi_\phi = 1 + N_\phi \in A/N_\phi \subset \mathcal{H}_\phi$ is a cyclic vector for π_ϕ with norm one.*

Proof. The fact that N_ϕ is closed follows easily from the fact that ϕ is continuous. Next, suppose that a is in A and b is in N_ϕ. The fact that ab is in N_ϕ follows immediately from the last part of the last Lemma.

For the second part, it follows from part one of the last Lemma that $\phi(b^*a) = 0$ if either a or b is in N_ϕ and so the bilinear form is well defined. It is also clearly positive definite.

The fact that N_ϕ is a left ideal means that, for any a in A, the formula given for $\pi_\phi(a)$ yields a well-defined linear transformation on A/N_ϕ. The fact that it is continuous (and hence extends to a bounded linear operator on $\mathcal{H}_\phi$) follows immediately from the last part of the last Lemma.

It is clear that π_ϕ is linear and multiplicative. Let us check it preserves adjoints. For a, b, c in A, we have

$$\begin{aligned} < \pi_\phi(a^*)b + N_\phi, c + N_\phi > &= < a^*b + N_\phi, c + N_\phi > \\ &= \phi(c^*(a^*b)) = \phi((ac)^*b) \\ &= < b + N_\phi, ac + N_\phi > \\ &= < b + N_\phi, \pi_\phi(a)c + N_\phi > \\ &= < \pi_\phi(a)^*b + N_\phi, c + N_\phi > . \end{aligned}$$

Since this holds for arbitrary b and c in A, we conclude that $\pi_\phi(a^*) = \pi_\phi(a)^*$.

For the last part, if b is any element of A, it is clear that

$$\pi_\phi(b)\xi_\phi = b \cdot 1 + N_\phi = b + N_\phi.$$

It follows then that $\pi_\phi(A)\xi_\phi$ contains A/N_ϕ and is therefore dense in $\mathcal{H}_\phi$.

Finally, we compute

$$\|\xi_\phi\| = \|1 + N_\phi\| = \phi(1^*1)^{1/2} = \phi(1)^{1/2} = 1.$$

□

Definition 1.12.5. *Let A be a unital C^*-algebra and let ϕ be a state on A. The triple $(\mathcal{H}_\phi, \pi_\phi, \xi_\phi)$ is called the* GNS representation *of ϕ.*

In brief, the GNS construction takes a state and produces a representation and a unit cyclic vector. In fact, it may be reversed as follows.

Theorem 1.12.6. *Let A be a unital C^*-algebra and suppose that π is a representation of A on the Hilbert space $\mathcal{H}$ with cyclic vector, ξ, of norm one. Then*

$$\phi(a) =< \pi(a)\xi, \xi >,$$

for all a in A, defines a state on A. Moreover, the GNS representation of ϕ is unitarily equivalent to π in the sense that there is a unitary operator $u : \mathcal{H} \to \mathcal{H}_\phi$ satisfying $u\pi(a)u^* = \pi_\phi(a)$, for all a in A and $u\xi = \xi_\phi$.

We have shown a correspondence between states and cyclic representations. Under this, the irreducible representations are characterized by a nice geometric property, namely that they are extreme points among the set of states. Such points are often called *pure states*.

Theorem 1.12.7. *Let ϕ be a state on the unital C^*-algebra A. The GNS representation $(\pi_\phi, \mathcal{H}_\phi)$ is irreducible if and only if ϕ is not a non-trivial convex combination of two other states. That is, if there are states ϕ_0 and ϕ_1 and $0 < t < 1$ such that $\phi = t\phi_0 + (1-t)\phi_1$, then $\phi_0 = \phi_1 = \phi$.*

Proof. First, we suppose that $\phi = t\phi_0 + (1-t)\phi_1$ with $\phi_0 \neq \phi_1$ and $0 < t < 1$ and show that the GNS representation is not irreducible. Define a bilinear form on A/N_ϕ by $(a + N_\phi, b + N_\phi) = t\phi_0(b^*a)$. We have

$$\begin{aligned}
|(a + N_\phi, b + N_\phi)|^2 &= t^2|\phi_0(b^*a)|^2 \\
&\leq |t\phi_0(a^*a)||t\phi_0(b^*b)| \\
&\leq |\phi(a^*a)||\phi(b^*b)| \\
&= \|a + N_\phi\|^2\|b + N_\phi\|^2.
\end{aligned}$$

It follows that our bilinear form is well-defined on the quotient A/N_ϕ and also extends to one on $\mathcal{H}_\phi$. Moreover, there exists a unique positive bounded linear positive operator h on $\mathcal{H}_\phi$ such that

$$t\phi_0(b^*a) = (a + N_\phi, b + N_\phi) =< h(a + N_\phi), b + N_\phi > .$$

We will show that h is not a scalar multiple of the identity and commutes with $\pi_\phi(a)$, for all a in A. It will then follow from Proposition 1.11.12 that π_ϕ is not irreducible.

If h is a multiple of the identity, then using $b = 1$ in the formula above, we see that ϕ is a multiple of ϕ_0. As both are states and take the same value at the unit, they are then equal. Since $\phi = t\phi_0 + (1-t)\phi_1$, we also have $\phi_1 = \phi$, and this is a contradiction.

Now, let a, b, c be in A. We have

$$\begin{aligned} < \pi_\phi(a)h(b+N_\phi), c+N_\phi > &= < h(b+N_\phi), \pi_\phi(a)^*(c+N_\phi) > \\ &= < h(b+N_\phi), a^*c+N_\phi > \\ &= t\phi_0((a^*c)^*b) \\ &= t\phi_0(c^*ab) \\ &= < h(ab+N_\phi), c+N_\phi > \\ &= < h\pi_\phi(a)(b+N_\phi), c+N_\phi > . \end{aligned}$$

As this equality holds on a dense set of pairs of vectors $b+N_\phi, c+N_\phi$, we conclude that $\pi_\phi(a)h = h\pi_\phi(a)$. This completes the proof.

In the other direction, suppose that π_ϕ is reducible. That is, we may find a non-trivial invariant subspace $\mathcal{N} \in \mathcal{H}_\phi$. Write $\xi_\phi = \xi_0 + \xi_1$, where ξ_0 is in $\mathcal{N}$ and ξ_1 is orthogonal to $\mathcal{N}$. We claim that neither is zero, for if $\xi_1 = 0$, then $\xi_\phi = \xi_0$ is in $\mathcal{N}$. As this subspace is invariant, we see that $\pi_\phi(A)\xi$ is contained in $\mathcal{N}$. On the other hand, since ξ_ϕ is cyclic, this subspace must be dense in $\mathcal{H}_\phi$ which is a contradiction. The other case is similar.

Define

$$\phi_i(a) = \|\xi_i\|^{-2} < \pi_\phi(a)\xi_i, \xi_i >,$$

for a in A and $i = 0, 1$. Then ϕ_0 and ϕ_1 are states on A. We also claim that $\phi = \|\xi_0\|^2\phi_0 + \|\xi_1\|^2\phi_1$. To see this, we have

$$\begin{aligned} \phi(a) &= < \pi_\phi(a)\xi_\phi, \xi_\phi > \\ &= < \pi_\phi(a)(\xi_0+\xi_1), (\xi_0+\xi_1) > \\ &= < \pi_\phi(a)\xi_0, \xi_0 > + < \pi_\phi(a)\xi_1, \xi_0 > \\ &\quad + < \pi_\phi(a)\xi_0, \xi_1 > + < \pi_\phi(a)\xi_1, \xi_1 > \\ &= \|\xi_0\|^2\phi_0(a) + < \pi_\phi(a)\xi_1, \xi_0 > \\ &\quad + < \pi_\phi(a)\xi_0, \xi_1 > + \|\xi_1\|^2\phi_1(a). \end{aligned}$$

The two central terms are both zero since $\mathcal{N}$ and $\mathcal{N}^\perp$ are both invariant under $\pi_\phi(a)$ and ξ_0 is in the former and ξ_1 is in the latter.

We claim that $\phi_0 \neq \phi_1$. First, let $C = \min\{\|\xi_0\|, \|\xi_1\|\} > 0$. Since the vector ξ_ϕ is cyclic, we may find a in A with $\|\pi_\phi(a)\xi_\phi - \xi_0\| < 2^{-1}C$. Writing $\pi_\phi(a)\xi_\phi = \pi_\phi(a)\xi_0 + \pi_\phi(a)\xi_1$ and noting the first vector is in $\mathcal{N}$ and the second is in $\mathcal{N}^\perp$, we see that

$$\|\pi_\phi(a)\xi_0 - \xi_0\|, \|\pi_\phi(a)\xi_1\| < 2^{-1}C.$$

Then we have

$$\begin{aligned} |\phi_0(a) - 1| &\leq \left| \|\xi_0\|^{-2} < \pi_\phi(a)\xi_0, \xi_0 > - \|\xi_0\|^{-2} < \xi_0, \xi_0 > \right| \\ &\leq \|\xi_0\|^{-2} \|\pi_\phi(a)\xi_0 - \xi_0\| \|\xi_0\| \\ &< \|\xi_0\| 2^{-1} C \\ &\leq 2^{-1}. \end{aligned}$$

We also have

$$\begin{aligned} |\phi_1(a)| &\leq \left| \|\xi_1\|^{-2} < \pi_\phi(a)\xi_1, \xi_1 > \right| \\ &\leq \|\xi_1\|^{-2} \|\pi_\phi(a)\xi_1\| \|\xi_1\| \\ &< \|\xi_1\| 2^{-1} C \\ &\leq 2^{-1}. \end{aligned}$$

We conclude that $\phi_0(a) \neq \phi_1(a)$ and we are done. □

We have already done everything we will need about representations, but there remains one extremely important application of the GNS construction and it would be negligent not to cover it. Let us begin by stating the result.

Theorem 1.12.8. *Let A be a C^*-algebra. Then there exists a Hilbert space $\mathcal{H}$ and C^*-subalgebra $B \subset \mathcal{B}(\mathcal{H})$ which is $*$-isomorphic to A.*

The importance should be obvious: every abstract C^*-algebra is isomorphic to a C^*-algebra of operators. In some sense, that statement is probably a little backwards. The real importance of this result is the *definition* of a C^*-algebra as an abstract object. That is, it is possible to give a definition which does not rely on operators on Hilbert space.

Obviously the C^*-algebra B is unique up to isomorphism, but one might imagine trying to make some stronger uniqueness result along the lines of having a unitary operator between the two Hilbert spaces which conjugates one algebra to the other. Any such statement is hopelessly wrong, in general.

Let us indicate briefly the proof of Theorem 1.12.8, beginning with the following Lemma.

Lemma 1.12.9. *Let A be a unital C^*-algebra and let a be a self-adjoint element of A. There exists an irreducible representation π of A such that $\|\pi(a)\| = \|a\|$.*

Proof. Let B be the C^*-subalgebra of A generated by a and the unit. It is unital and commutative. We know that $\mathcal{M}(B)$ is homeomorphic to $spec(a)$ via $\phi \to \phi(a)$ and that $spec(a)$ is a compact subset of the real numbers. Choose ϕ_0 in $\mathcal{M}(B)$ such that $|\phi_0(a)| = \sup\{|x| \mid x \in spec(a)\} = \|a\|$. Of course, we have $\phi_0(1) = 1$.

Next, we appeal to the Hahn-Banach Theorem [**?**] to find a linear functional ϕ on A which extends ϕ_0 and has $\|\phi\| = \|\phi_0\|$. It follows at once that

$$\phi(1) = \phi_0(1) = 1 = \|\phi_0\| = \|\phi\|.$$

By Lemma 1.11.3, ϕ is a state. We also have

$$|\phi(a)| = |\phi_0(a)| = \|a\|.$$

At this point, if we let π_ϕ be the GNS representation of ϕ, then since ξ_ϕ is a unit vector, we have

$$\|\pi_\phi\| \geq | < \pi_\phi(a)\xi_\phi, \xi_\phi > | = |\phi(a)| = \|a\|.$$

The reverse inequality follows since π_ϕ is contractive. The only thing we are missing is that π_ϕ should be irreducible. To achieve this, we consider the set of all states ϕ which satisfy $|\phi(a)| = |\phi_0(a)| = \|a\|$. We have shown this set is non-empty. It is also not hard to prove it is a weak-$*$ closed, convex subset of the unit ball of the dual space of A. Then the Krein-Milman theorem asserts that it is the closed convex hull of its extreme points. In particular, it has extreme points and if ϕ is chosen from among them, we can show that such a point is also extreme among the set of states and so we have that π_ϕ is irreducible by 1.12.7. $\square$

For the proof of Theorem 1.12.8, we proceed as follows. For each a in the unit ball of A, we find an irreducible representation π_a of A such that

$\|\pi_a(a^*a)\| = \|a^*a\|$. Of course, it follows that $\|\pi_a(a)\| = \|a\|$, as well. We finally take the direct sum over all a in the unit ball of A of the representations π_a. It follows that this representation is isometric and we are done by letting B denote its range.

Exercise 1.12.1. *Let X be a compact, Hausdorff space and let $A = C(X)$. The Riesz representation theorem states that every linear functional ϕ on A is given as*

$$\phi(f) = \int_X f(x)d\mu(x),$$

for some finite Borel measure μ on X.

1. *Give a necessary and sufficient condition on μ as above for the associated ϕ to be a state.*
2. *Prove that N_ϕ is a closed two-sided ideal in A.*
3. *Describe $\mathcal{H}_\phi$ is terms of μ.*
4. *Prove that N_ϕ is also the kernel of π_ϕ.*

Exercise 1.12.2. *Let $A = M_N(\mathbb{C})$ and let τ be the trace of Theorem 1.9.1. Let π be the obvious representation of A on $\mathbb{C}^N$: $\pi(a)\xi = a\xi$. Describe the relationship between π and π_τ. (More explicitly, describe π_τ in terms of π, up to unitary equivalence. Hint: let $\{\xi_1, \xi_2, \ldots, \xi_N\}$ be an orthonormal basis for $\mathbb{C}^N$. First observing that τ is faithful, what is N_τ? Then use the fact that $\{\xi_i \otimes \xi_j^* \mid 1 \leq i, j \leq N\}$ is a linear basis for A. Can you find invariant subspaces for π_τ?)*

1.13 von Neumann algebras

The first thing to say on the topic is that this course is not about von Neumann algebras. Certainly one can easily give such a course; they are a deep and fascinating subject. We are not trying to give such a course here, but it would seem to be negligent if a reader were to come out of this course without knowing a little bit about them. Our approach will be to give a general sense of what they are, why they are similar to C^*-algebras and why they are different.

The starting point is the study of the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$ and its subalgebras. The case that $\mathcal{H}$ is finite-dimensional is too restrictive, but we will assume that $\mathcal{H}$ is separable, for technical reasons.

Let A be a subalgebra of $\mathcal{B}(\mathcal{H})$. The first question is whether we would like to assume that A is $*$-closed. That is, if a is in A, is a^* also? There is a substantial amount of work done on non-self-adjoint subalgebras, but we will not discuss that at all here. Let us go on, assuming that $A^* = A$.

The next issue is to consider A with the relative topology of $\mathcal{B}(\mathcal{H})$ and whether we should insist that A is closed. This gets us into deep waters almost at once: $\mathcal{B}(\mathcal{H})$ has eight different topologies! (We'll name a couple in a minute.) Of course, the norm topology stands out as special and the study of C^*-algebras takes this as its starting point. In this context, an important idea is that there is an abstract definition of a C^*-algebra which does not refer to anything about operators or Hilbert spaces and we have shown that every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Turning to some of the other topologies on $\mathcal{B}(\mathcal{H})$, let us just mention a couple of them. First, there is the *weak operator topology.* A sequence of operators, $a_n, n \geq 1$, converges to an operator a in this topology if $\lim_n < a_n\xi, \eta >=< a\xi, \eta >$, for all vectors ξ, η in $\mathcal{H}$. As a concrete example, let us consider the Hilbert space ℓ^2 of square summable sequences. Define the unilateral shift operator

$$S(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots),$$

whose adjoint is

$$S^*(\xi_1, \xi_2, \ldots) = (\xi_2, \xi_3, \ldots),$$

for any $(\xi_1, \xi_2, \ldots)$ in ℓ^2.

Exercise 1.13.1. *The sequence $S^n, n \geq 1$, converges to 0 in the weak operator topology, but not in norm.*

Another example is the *strong operator topology.* Here a sequence a_n converges to a if $\lim_n \|a_n\xi - a\xi\| = 0$ (or $\lim_n a_n\xi = a\xi$ in $\mathcal{H}$), for all ξ in $\mathcal{H}$.

Exercise 1.13.2. *The sequence $S^n, n \geq 1$, does* not *converge to 0 in the* strong operator topology.

While this topology looks fairly natural, it has some unexpected problems: the map $a \to a^*$ is not continuous!

Exercise 1.13.3. *The sequence $(S^*)^n, n \geq 1$, converges to 0 in the strong operator topology.*

On the positive side, a convex set in $\mathcal{B}(\mathcal{H})$ is strongly closed (meaning closed in the strong operator topology) if and only if it is weakly closed.

Before we get too carried away or get too concerned about these matters, we mention that the situation is not unlike what one finds in a measure theory. There are a number of different types of convergence for functions on a measure space.

Definition 1.13.1. *A von Neumann algebra is a strongly closed (weakly closed), unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.*

The point here is that one uses the weak or strong operator topologies in an essential way in the study of von Neumann algebras. Moreover, there is no way to give a 'non-spatial' version of this definition: A von Neumann algebra acts on a specific Hilbert space.

Let us give a few (very simple) examples.

Example 1.13.2. $\mathcal{B}(\mathcal{H})$.

Example 1.13.3. *Let (X, M, μ) be a standard measure space. Let $L^2(X, M, \mu)$ be the Hilbert space of square summable measurable functions on X and let $L^\infty(X, M, \mu)$ act as multiplication operators on this Hilbert space. Then this is a commutative von Neumann algebra.*

The weak operator topology is weaker than the norm topology. Hence, every von Neumann algebra is also a C^*-algebra. One needs to be a little careful with that statement. For example, if one refers to a "separable von Neumann", it is usually assumed that this refers to the weak operator topology. Indeed, the only von Neumann algebras which are separable in the norm topology are finite-dimensional.

Our analysis begins with that of a single self-adjoint or normal operator. Recall that Theorem 1.4.9 was obtained for a normal element of a C^*-algebra, not necessarily acting on a Hilbert space. If this operator is acting on a Hilbert space, we get considerably more information.

If X is a compact, Hausdorff space, we define $B(X)$ to be the set of bounded Borel functions on X.

Theorem 1.13.4. *Let a be a normal operator on the Hilbert space $\mathcal{H}$ and let $W^*(a)$ denote the smallest von Neumann algebra that contains a and the identity operator. Then there is a contractive $*$-homomorphism from $B(spec(a))$ to $W^*(a)$ which extends the map of Theorem 1.4.9. We denote the image of f by $f(a)$. Moreover, if f_n is a bounded increasing sequence of real-valued functions in $B(spec(a))$ with $f = lub\{f_n \mid n \geq 1\}$, then $f(a)$ is the least self-adjoint operator such that $f(a) \geq f_n(a)$, for all $n \geq 1$, and $f_n(a)$ converges to $f(a)$ in the strong operator topology.*

Let us look at what's good and what's less than good in this new version. Obviously, the big improvement is in the domain of our map. No longer are we restricted to continuous functions on the spectrum, but we can use Borel functions. In particular, we can use the characteristic function of any Borel subset of the spectrum of a and since such functions are projections, their images will be so also. This means that $W^*(a)$ contains a wealth of projections. We've already seen in finite-dimensional C^*-algebras that projections are very useful and that goes here as well, even more so. The map also comes with some nice continuity properties. On the range side, the continuity is in terms of the strong operator topology (not surprisingly). The only real downside is that this map is not a $*$-isomorphism. It is not difficult to see why. It turns out (although we won't prove it) that if f is the characteristic function of a single point λ, then $f(a)$ is non-zero if and only if λ is an eigenvalue of the operator a.

The theory of von Neumann algebras started with a seminal series of papers by Murray and von Neumann. Their starting point was the realization that the existence of many projections was crucial. Of course, there are certainly too many projections, so they introduced a notion of equivalence: in a von Neumann algebra $\mathcal{M}$, projections p, q are equivalent if there exists v in $\mathcal{M}$ such that $v^*v = p$ and $vv^* = q$. It is not difficult to check this is an equivalence relation which we denote by $p \sim q$ amd we let $[p]$ denote the equivalence class of p. In our section on finite-dimensional C^*-algebras, we also introduced a notion of order on projections. This can be adapted to the equivalence relation as follows. If p, q are projections, we write $p \succeq q$ if there exist $p' \sim p, q' \sim q$ and $p'q' = q'p' = q'$. Of course, it is necessary to see this is well-defined and a partial order of the equivalence classes of projections in $\mathcal{M}$.

Next, we make a very simple observation: if two projections p, q satisfy $pq = 0$, then their sum $p + q$ is again a projection. Once again, we need to extend this notion to equivalence classes: for any projections p, q in $\mathcal{M}$, if there exist $p \sim p'$ and $q \sim q'$ such that $p'q' = 0$, then we define $[p] + [q] = [p' + q']$. Once again, we need to see this yields a well-defined, partially defined binary operation of the set of equivalence classes of projections.

A von Neumann algebra $\mathcal{M}$ is called a *factor* if its centre consists only of scalar multiples of the identity. These are the von Neumann algebras which irreducible; i.e. they cannot be broken down into simpler pieces. We will not give a precise statement only because it is slightly technical.

In the case of a factor acting on a separable Hilbert space, Murray and von Neumann showed that the set of equivalence classes of projections, with their order and partially defined addition, had exactly one of the following types:

1. $\{0, 1, \ldots, N\}$, for some positive integer N.
2. $\{0, 1, 2, \ldots, \infty\}$,
3. $[0, 1]$,
4. $[0, \infty]$,
5. $\{0, \infty\}$.

von Neumann algebras in the first class are called Type I_N and each is isomorphic to $M_N(\mathbb{C})$. Those in the second class are called Type I_∞ and each is isomorphic to $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space. Those in the third and fourth classes are called Type II_1 and II_∞, respectively. (The reader should note that if one looks only at the order, these are isomorphic. But the addition in the latter is defined on all pairs, which is not the case in the former.) Finally, the last class are called Type III. Here, every non-zero projection is equivalent to the identity.

More is actually true, the isomorphism between the set of equivalence classes of projections and the sets above is actually made by a trace functional on the algebra, which is essentially unique. Since this functional is obviously taking on the value ∞, there are some technical subtleties which we will not discuss here.

Chapter 2

Group C^*-algebras

In this chapter, we consider the construction of C^*-algebras from groups. The first section gives some preliminary ideas about unitary representations of groups. We then discuss the complex group algebra of a group. This is a purely algebraic construction which is a forerunner to the C^*-algebras we will consider later.

The reader will notice (at a certain point) that many statements begin with 'if G is a discrete group ...'. In rigorous terms, this simply means that we are considering a group and giving it the discrete topology. Of course, that doesn't really make much sense. What this should convey to the reader is that there is a notion of a topological group and whatever result or definition is about to be stated has a version for topological groups. What is about to appear is for the special case when the group's topology is discrete. Often this is a significant simplification; in the general case there may be some statement about a function on the group being continuous, which is automatically satisfied when the group is discrete. In the last section of the chapter, we will spend a little time discussing the general case.

2.1 Group representations

The first item to tackle before our construction of a C^*-algebra is the notion of a unitary representation of a group. The subject of group representations is vast and deep. Here, we give a few basic definitions and results.

Definition 2.1.1. *Let G be a discrete group. A* unitary representation *of G is a pair $(u, \mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space and u is a group ho-*

momorphism from G to the group of unitary operators on $\mathcal{H}$, usually written $\mathcal{U}(\mathcal{H})$, with product as group operation. We usually write u_g for the image of an element g in G under the map u. We also say that u is a unitary representation of G on $\mathcal{H}$. Unitary representations $(u, \mathcal{H})$ and $(v, \mathcal{K})$ are unitarily equivalent *if there is a unitary operator $w : \mathcal{H} \to \mathcal{K}$ such that $wu_gw^{-1} = v_g$, for all g in G.*

Recall that a unitary operator u satisfies $u^*u = uu^* = I$, the identity operator. Hence, if $(u, \mathcal{H})$ is a unitary representation of G, we have $u_{g^{-1}} = (u_g)^{-1} = u_g^*$, for any g in G. In particular, the range of the function u, while not even a linear space, is, at least, $*$-closed.

For any Hilbert space $\mathcal{H}$, there is the trivial representation of G on $\mathcal{H}$ which sends each group element to the identity operator. A rather more interesting example is the left regular representation of G.

Definition 2.1.2. *Let G be a discrete group. Its* left regular representation *is the unitary representation of G on the Hilbert space $\ell^2(G)$ defined by*

$$\lambda_g \xi(h) = \xi(g^{-1}h),$$

for all g, h in G and ξ in $\ell^2(G)$.

Throughout the chapter we will let δ_h be the function that is 1 at h and zero elsewhere, for some fixed group element h. Notice that the set $\delta_h \mid h \in G\}$ forms an orthonormal basis for $\ell^2(G)$. Also observe that for any g, h in G, we have $\lambda_g \delta_h = \delta_{gh}$.

Definition 2.1.3. *Let $(u, \mathcal{H})$ be a unitary representation of the discrete group G. A closed subspace $\mathcal{N} \subset \mathcal{H}$ is* invariant *for u if $u_g\mathcal{N} \subset \mathcal{N}$, for all g in G. We say that the representation is* irreducible *if its only invariant subspaces are* 0 *and $\mathcal{H}$ and is* reducible *otherwise.*

We have an exact analogue of the direct sum of representations of a $*$-algebra.

Definition 2.1.4. *If $(u_\alpha, \mathcal{H}_\alpha), \alpha \in A$ is a collection of unitary representations of a group G, then their direct sum $(\oplus_\alpha u_\alpha, \oplus_\alpha \mathcal{H}_\alpha)$ is the unitary representation defined by $(\oplus_\alpha u_\alpha)_g = \oplus_\alpha (u_\alpha)_g$, for all g in G.*

The following result is an analogue of Proposition 1.10.1 for invariant subspaces for representations of $*$-algebras. In fact, the only property of the representation which the proof of 1.10.1 used is that the range is $*$-closed. Hence, the same proof works equally well here.

Proposition 2.1.5. *If $\mathcal{N}$ is an invariant subspace for the unitary representation $(u, \mathcal{H})$, then so is $\mathcal{N}^\perp$. In this case, u is unitarily equivalent to $u|_{\mathcal{N}} \oplus u|_{\mathcal{N}^\perp}$.*

2.2 Group algebras

The following is a basic definition from algebra. For the moment, we do not need to assume the group is finite, but rather our construction has an obvious finiteness condition built in.

Definition 2.2.1. *Let G be a group. Its (complex) group algebra, denoted $\mathbb{C}G$, consists of all formal sums $\sum_{g\in G} a_g g$, where each a_g is a complex number and $a_g = 0$, for all but finitely many g in G. Defining $a_g g \cdot a_h h = a_g a_h gh$, for all $g, h \in G$ and $a_g, a_h \in \mathbb{C}$ and extending by linearity, $\mathbb{C}G$ becomes a complex algebra. Moreover, defining $g^* = g^{-1}$ and extending to be conjugate linear, $\mathbb{C}G$ becomes a complex $*$-algebra.*

If we adopt the notation $1g = g$, for any g in G, then $G \subset \mathbb{C}G$. The identity element of G is also the identity of this algebra. Moreover, each element g in G is a unitary in $\mathbb{C}G$.

Let us just note that

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g'\in G} b_{g'} g'\right) = \sum_{g,g'\in G} a_g b_{g'} g g' = \sum_{h\in G}\left(\sum_{g\in G} a_g b_{g^{-1}h}\right) h.$$

Proposition 2.2.2. *Let G be a group. Its group algebra is commutative if and only if G is abelian.*

Theorem 2.2.3. *Let G be a discrete group. If $u : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G on the Hilbert space $\mathcal{H}$, then u has a unique extension to a unital representation of $\mathbb{C}G$ on $\mathcal{H}$ defined by*

$$\pi_u\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g u_g,$$

for $\sum_{g\in G} a_g g$ in $\mathbb{C}G$. Moreover, if $\pi : \mathbb{C}G \to \mathcal{B}(\mathcal{H})$ is a unital representation, then its restriction to G is a unitary representation of G. Finally, the representation u is irreducible if and only if π_u is.

Proof. The first part of the statement is immediate from the definition and some simple calculations which we leave to the reader.

For the last statement, we proceed as follows. For each group element g, we may regard g as an element of $\mathbb{C}G$. We define $u_g = \pi(g)$. We must verify that u_g is unitary. To this end, we use the facts that π is a $*$-representation and that $g^* = g^{-1}$ in $\mathbb{C}G$ to obtain

$$\pi(g)^*\pi(g) = \pi(g^*)\pi(g) = \pi(g^{-1})\pi(g) = \pi(g^{-1}g) = \pi(1) = 1.$$

A similar computation proves that $\pi(g)\pi(g)^* = 1$ as well, so $\pi(g)$ is unitary. The fact that u is a homomorphism follows from the fact that π is multiplicative.

For the last statement, if u is reducible, then there is a non-trivial invariant subspace for all of the operators $u_g, g \in G$. But then that subspace is also clearly invariant for any linear combination of these operators, hence for every element of $\pi_u(\mathbb{C}G)$. Conversely, if the collection of operators $\pi_u(a), a \in \mathbb{C}G$ has a common non-trivial invariant subspace, then $u_g = \pi(g)$, for every group element g, also leaves this space invariant. □

Proposition 2.2.4. *Let G be a discrete group. Let $(\lambda, \ell^2(G))$ be the left regular representation of G. The associated representation of $\mathbb{C}G$ is*

$$\left[\pi_\lambda\left(\sum_{g\in G} a_g g\right)\xi\right](h) = \sum_{g\in G} a_g \xi(g^{-1}h),$$

for all $\sum_{g\in G} a_g g$ in $\mathbb{C}G$, h in G and ξ in $\ell^2(G)$.

Theorem 2.2.5. *Let G be a group. The left regular representation π_λ of $\mathbb{C}G$ is injective.*

Proof. Let $a = \sum_{g\in G} a_g g$ be an element of $\mathbb{C}G$ and assume that it is in the kernel of π_λ. Fix g_0 in G and let δ_{g_0}, δ_e be as described earlier. An easy computation shows that

$$0 =< \pi_\lambda(a)\delta_{g_0}, \delta_e >=< \pi_\lambda(\sum_{g\in G} a_g g)\delta_{g_0}, \delta_e >= \sum_h \sum_g a_g \delta_{g_0}(g^{-1}h)\delta_e(h) = a_{g_0}.$$

As g_0 was arbitrary, we conclude $a = 0$. □

Theorem 2.2.6. *Let G be a group. The map from $\mathbb{C}G$ to $\mathbb{C}$ defined by $\tau(\sum_{g\in G} a_g g) = a_e$, for any $\sum_{g\in G} a_g g$ in $\mathbb{C}G$, is a faithful trace. (Recall that a trace τ is faithful if $\tau(a^*a) = 0$ only if $a = 0$.)*

Proof. First of all, it is clear that τ is conjugate linear. Next, we check that it is positive and faithful. For any $a = \sum_{g \in G} a_g g$ in $\mathbb{C}G$, we have

$$\begin{aligned} a^* a &= \left(\sum_{g \in G} a_g g\right)^* \left(\sum_{h \in G} a_h h\right) \\ &= \left(\sum_{g \in G} \overline{a_g} g^{-1}\right) \left(\sum_{h \in G} a_h h\right) \\ &= \sum_{g,h} \overline{a_g} a_h g^{-1} h. \end{aligned}$$

Hence $\tau(a^*a)$ is the coefficient of e which is the sum over $g^{-1}h = e$, or $g = h$:

$$\tau(a^*a) = \sum_{g \in G} \overline{a_g} a_g = \sum_{g \in G} |a_g|^2 \geq 0,$$

and $\tau(a^*a) = 0$ implies $a = 0$.

Finally, we verify the trace property. Consider simply $\tau(gh)$, where g and h are in G. This is clearly one when $g^{-1} = h$ and zero otherwise. Exactly the same argument applies to $\tau(hg)$ and so we see that $\tau(gh) = \tau(hg)$, for any g, h in G. The trace property follows from linearity. $\square$

Exercise 2.2.1. *Let G be a discrete group and τ be the trace on the group algebra as above. What is the result of applying the GNS construction to the trace? (Warning: nice answer, trivial proof.)*

2.3 Finite groups

We are aiming to construct a C^*-algebra from a discrete group. So far, the group algebra is doing well. What it lacks is a norm. This turns out to be quite a subtle and deep problem. The simplest case is when the group is finite, which we consider now.

Theorem 2.3.1. *Let G be a finite group. Then there is a unique norm on $\mathbb{C}G$ in which it is a C^*-algebra.*

Proof. The formula $\|a\| = \|\pi_\lambda(a)\|$ (operator norm) defines a semi-norm on $\mathbb{C}G$, where π_λ is the left regular representation as above. The fact that it is a norm follows from the last theorem. It satisfies the C^*-condition since π_λ is a $*$-homomorphism and the operator norm satisfies the C^*-condition. Finally, it is complete in this norm because $\mathbb{C}G$ is finite dimensional. The uniqueness is just a restatement of 1.3.8. $\square$

With the information that we already have available from Section 1.7 about the structure of finite-dimensional C^*-algebras, we can completely describe the C^*-algebra of a finite group, at least in principle. Recall that two elements g_1, g_2 of G are *conjugate* if there is an element h such that $hg_1h^{-1} = g_2$. Conjugacy is an equivalence relation and the equivalence class of an element g, $\{hgh^{-1} \mid h \in G\}$, is called a conjugacy class.

Theorem 2.3.2. *Let G be a finite group with conjugacy classes $C_1, C_2, \ldots, C_K$. For each $1 \leq k \leq K$, define*

$$c_k = \sum_{g \in C_k} g \in \mathbb{C}G.$$

The set $\{c_1, \ldots, c_K\}$ is linearly independent and its span is the centre of $\mathbb{C}G$. In particular, $\mathbb{C}G$ is isomorphic to $\oplus_{k=1}^K M_{n_k}(\mathbb{C})$ and

$$\sum_{k=1}^{K} n_k^2 = \#G.$$

Proof. It is clear that G is a linear basis for $\mathbb{C}G$, so $\mathbb{C}G$ is finite dimensional. Hence by Theorem 1.6.1, $\mathbb{C}G$ is isomorphic to $\oplus_{k=1}^K M_{n_k}(\mathbb{C})$ for some positive integers $n_1, \ldots, n_K$.

Now suppose that $a = \sum_g a_g g$ is in the centre of $\mathbb{C}G$. Let h be any other element of G. Considering h as an element of $\mathbb{C}G$, it is invertible and its inverse is h^{-1}. As a is in the centre of G, we have

$$\sum_g a_g g = a = hah^{-1} = \sum_g a_g hgh^{-1} = \sum_g a_{h^{-1}gh} g.$$

For any g in G, comparing coefficients, we have $a_g = a_{h^{-1}gh}$. This means that the function a is constant on conjugacy classes in G. Hence it is a linear combination of the c_k.

Conversely, the same computation shows that each c_k commutes with every group element h. Since the group elements span the group algebra, each c_k is in the centre.

The fact that the set of all c_k is linearly independent is clear. $\square$

Example 2.3.3. $\mathbb{C}S_3 \cong M_2 \oplus \mathbb{C} \oplus \mathbb{C}$.

It is a simple matter to check that there are three conjugacy classes: the identity, the three transpositions and the two cycles of length three. So we need to find n_1, n_2, n_3 whose squares sum to 6.

Exercise 2.3.1. *Give another proof of Example 2.3.3 which doesn't consider conjugacy classes but only linear dimension and Theorem 2.2.2.*

Exercise 2.3.2. *Describe $\mathbb{C}S_4$ in terms of Theorem 1.6.1.*

2.4 The C^*-algebra of a discrete group

We begin with a quick review of the last section. To any discrete group, we can associate its group algebra. Unitary representations of the group correspond to unital representations of the group algebra in an easy way. And we found an injective representation of the group algebra by considering the left regular representation. When the group is finite, the group algebra is finite dimensional and this allows us to find an injective map from the group algebra into the operators on a Hilbert space. Using this, the operator norm on the image makes it into a C^*-algebra. A key point in this development is that the algebra is finite-dimensional. If it were not, we would not know that the group algebra is complete in the given norm. This is a subtle and important point, but it is not insurmountable. We just need to do some analysis.

The first step is to give the group algebra a norm. Even though it will not be complete, nor will the completion be a C^*-algebra, this is still a useful step.

Definition 2.4.1. *Let G be a discrete group. The ℓ^1-norm on $\mathbb{C}G$ is defined by*

$$\| \sum_{g \in G} a_g g \|_1 = \sum_{g \in G} |a_g|,$$

where $a_g, g \in G$ are complex numbers with only finitely many non-zero.

Now we have reached a notational impasse. Elements of the group algebra are typically denoted by $\sum_{g \in G} a_g g$ while the norm we have defined above is usually defined on functions from the group to the complex numbers. Of course, it is very easy to go back and forth between the two. The element above is associated with the function sending g in G to the complex number a_g, for all g in G.

Theorem 2.4.2. *Let G be a discrete group. The completion of $\mathbb{C}G$ in the ℓ^1-norm is*

$$\ell^1(G) = \{a : G \to \mathbb{C} \mid \sum_{g \in G} |a(g)| < \infty\}.$$

Moreover, the product

$$(ab)(g) = \sum_{h \in G} a(h)b(h^{-1}g),$$

for a, b in $\ell^1(G)$ and g in G, is well-defined, associative, extends the product on $\mathbb{C}G$ and makes $\ell^1(G)$ a Banach algebra. The involution

$$a^*(g) = \overline{a(g^{-1})}$$

a in $\ell^1(G)$ and g in G, is isometric, conjugate linear and satisfies $(ab)^* = b^*a^*$, for all a, b in $\ell^1(G)$.

Proof. The proof of the first statement is a standard fact in functional analysis. We refer the reader to []. To see that the product is well-defined, we note that the functions in $\ell^1(G)$ are bounded since they are summable and then the sum $\sum_{h \in G} a(h)b(h^{-1}g)$ is the product of a summable sequence and a bounded one and so is summable. To see it is in $\ell^1(G)$, for any a, b in $\ell^1(G)$, we have

$$\sum_{(h,g) \in G \times G} |a(h)b(h^{-1}g)| = \sum_{h \in G} |a(h)| \sum_{g \in G} |b(h^{-1}g)| = \sum_{h \in G} |a(h)| \|b\|_1 = \|a\|_1 \|b\|_1.$$

It follows that $\|ab\|_1 \leq \|a\|_1 \|b\|_1$. It is clear that the involution is isometric. It is clearly conjugate linear and satisfies $(ab)^* = b^*a^*$ on functions that are supported on a single element. By taking linear combinations, it is also satisfied for finitely supported functions. It then holds on all elements of $\ell^1(G)$ by continuity. □

Theorem 2.4.3. *Let G be a discrete group. The linear functional $\tau(a) = a(e), a \in \ell^1(G)$ is bounded and is a trace on $\ell^1(G)$.*

Proof. We note that for any a in $\ell^1(G)$, we have $\|a\| \geq |a(e)| = |\tau(a)|$, so τ is bounded. Since we have already proved that it is a trace on $\mathbb{C}G$ and is continuous, the trace property extends to $\ell^1(G)$. □

We now observe a relatively easy, but rather important result.

Theorem 2.4.4. *Let G be a discrete group. There are bijective correspondences between*

1. *$(u, \mathcal{H})$, unitary representations of G,*
2. *$(\pi, \mathcal{H})$, non-degenerate representations of $\mathbb{C}G$,*
3. *$(\pi, \mathcal{H})$, non-degenerate representations of $\ell^1(G)$.*

The correspondences from the third to the second and from the second to the first are both obtained by restriction. Moreover, these correspondences preserve the irreducible representations. Finally, every representation of $\ell^1(G)$ is contractive.

Proof. We are using the facts that $G \subset \mathbb{C}G \subset \ell^1(G)$ to take restrictions. Going back, unitary representations of G extend to unital representations of $\mathbb{C}G$ by linearity exactly as we saw already in 2.2.3. Unital representations of $\mathbb{C}G$ extend to $\ell^1(G)$ by continuity as follows.

Suppose that π is a unital representation of $\mathbb{C}G$. If a is in $\ell^1(G)$, we define

$$\pi(a) = \sum_{g \in G} a(g)\pi(g).$$

Of course, the reason the sum converges is simply because the coefficients are absolutely summable and since π is a $*$-homomorphism, $\pi(g)$ is unitary and hence has norm one, for every g in G. In fact, we have

$$\|\pi(a)\| = \|\sum_{g \in G} a(g)\pi(g)\| \leq \sum_{g \in G} |a(g)|\|\pi(g)\| = \sum_{g \in G} |a(g)| = \|a\|_1$$

and this also proves that π is contractive.

If π is a representation of $\ell^1(G)$ having a non-trivial invariant subspace, then its restriction to $\mathbb{C}G$ has the same non-trivial invariant subspace. Similarly, the restriction of a reducible representation $\mathbb{C}G$ to G will also be reducible. In the other direction, if a unitary representation of G has a non-trivial invariant subspace, then extending it to $\mathbb{C}G$ by linearity has the same non-trivial invariant subspace. Moreover, if a_n is any sequence in $\mathbb{C}G$ converging to a in $\ell^1(G)$ and $\pi(a_n), n \geq 1$ all leave $\mathcal{N}$ invariant, then so does $\pi(a)$. □

We are now ready to turn to the issue of the norm on $\mathbb{C}G$.

Definition 2.4.5. *Let G be a discrete group. We define a norm on $\mathbb{C}G$ by*

$$\|a\| = \sup\{\|\pi(a)\| \mid \pi, \text{ a representation of } \mathbb{C}G\},$$

for any a in $\mathbb{C}G$. We denote the completion of $\mathbb{C}G$ in this norm by $C^*(G)$, which we refer to as the group C^*-algebra *of G.*

Theorem 2.4.6. *The norm given in Definition 2.4.5 is well-defined and $C^*(G)$ is a C^*-algebra containing $\mathbb{C}G$ as a dense $*$-subalgebra.*

Proof. Begin by noting that every representation π of $\mathbb{C}G$ has a unique extension to $\ell^1(G)$ and that every representation of $\ell^1(G)$ is contractive. This implies that $\|\pi(a)\| \leq \|a\|_1$ and so the set on the right is bounded and the supremum exists, for any element a of $\mathbb{C}G$. Next, the left regular representation of $\mathbb{C}G$ is faithful and so this norm is also faithful.

The fact that on $\mathbb{C}G$ it satisfies $\|ab\| \leq \|a\|\|b\|$, $a, \in \mathbb{C}G$ and the C^*-condition follow from the fact that it is the supremum of norms all satisfying these conditions. It is then easy to see its completion is a C^*-algebra. □

Let us begin by observing a couple of simple things about our C^*-algebra.

Proposition 2.4.7. *If G is a discrete group, then $C^*(G)$ is a unital C^*-algebra and it is abelian if and only if G is abelian.*

Proof. The identity of the group G is also the identity of the group algebra $\mathbb{C}G$ and is therefore the identity of any completion. We have also already seen in Proposition 2.2.2 that G is abelian if and only if $\mathbb{C}G$ is abelian. It is easy to see that a dense subalgebra of a C^*-algebra is abelian if and only if the whole C^*-algebra is also. □

Let us remark that Theorem 2.4.4 can be extended to include non-degenerate representations of $C^*(G)$ as well. The relationship between $\ell^1(G)$ and $C^*(G)$ is somewhat subtle. Since the C^*-algebra norm of 2.4.5 is less than or equal to the norm on $\ell^1(G)$, there is a natural $*$-homomorphism from the latter to the former which extends the identity map on $\mathbb{C}G$. This is even injective (see Exercise 2.4.2).

In general, the collection of representations involved in the definition of the norm can be rather unwieldy. It will turn out to be extremely helpful if we can restrict to a more tractable subclass. The following theorem is very useful in this way. Its proof is a trivial consequence of Theorem 2.2.4 and Lemma 1.11.9.

Theorem 2.4.8. *Let G be a discrete group. For any element a in $\mathbb{C}G$, we have*

$$\|a\| = \sup\{\|\pi_u(a)\| \mid u \textit{ an irreducible representation of } G\}.$$

However, $C^*(G)$ is not the end of the story: recall that we have one extremely natural representation of $\mathbb{C}G$, the left regular representation.

Definition 2.4.9. *Let G be a discrete group. We define the* reduced norm *on $\mathbb{C}G$ by*

$$\|a\|_r = \|\pi_\lambda(a)\|,$$

for any a in $\mathbb{C}G$. We denote the completion of $\mathbb{C}G$ in this norm by $C_r^*(G)$, which we refer to as the reduced group C^*-algebra *of G.*

It is worth stating the following easy result.

Theorem 2.4.10. *Let G be a discrete group. There exists a canonical $*$-homomorphism $\rho : C^*(G) \to C_r^*(G)$ which extends the identity map on the group algebra $\mathbb{C}G$.*

Proof. If λ denotes the left regular representation, then π_λ is among the collection of representations of $\mathbb{C}G$ considered when defining the norm. Hence, we have $\|a\|_r = \|\pi_\lambda(a)\| \leq \|a\|$. The rest follows from the definitions. □

Our last main result is to see that the trace functional on the group algebra will extend to both full and reduced group C^*-algebras. In fact, this is a consequence of a nice little formula which expresses the trace in terms of the left regular representation.

Theorem 2.4.11. *Let G be a discrete group. For each a in $\mathbb{C}G$, we have*

$$\tau(a) = < \pi_\lambda(a)\delta_e, \delta_e > .$$

The trace τ extends continuously to both $C^*(G)$ and $C_r^*(G)$ and the same formula holds for a in either of these. Both extensions are traces on the respective C^*-algebras.

Proof. We compute

$$\begin{aligned}
< \pi_\lambda(a)\delta_e, \delta_e > &= < \pi_\lambda(\sum_{g\in G} a(g)g)\delta_e, \delta_e > \\
&= \sum_{g\in G} a(g) < \lambda(g)\delta_e, \delta_e > \\
&= \sum_{g\in G} a(g) < \delta_g, \delta_e > \\
&= a(e) \\
&= \tau(a).
\end{aligned}$$

It follows immediately from this that

$$|\tau(a)| \leq \|\pi_\lambda(a)\| = \|a\|_r \leq \|a\|,$$

and so τ extends as claimed. The fact that τ satisfies the trace properties on dense subalgebras of $C_r^*(G)$ and $C^*(G)$ implies the extensions will also. □

There is a general condition on a group to be *amenable*. It is a little technical and there are many equivalent forms, but the name is fairly descriptive: these are the groups which are most easily analyzed. If a group is amenable, then the full and reduced norms agree and the full and reduced C^*-algebras are identical, or more accurately, the map ρ of 2.4.10 is an isomorphism. In fact, the converse also holds.

Later, we will see an example of a non-amenable group, and just how different the two C^*-algebras can be.

Exercise 2.4.1. *Let G be a discrete group. For each group element g, we let δ_g be the function which is 1 at g and 0 elsewhere. We regard this as an element of $\ell^2(G)$. Let a be in $\mathcal{B}(\ell^2(G))$.*

1. *Prove that if a is in the closure of $\pi_\lambda(\mathbb{C}G)$, then $(a\delta_e)(g) =< a\delta_h, \delta_{gh} >$, for all g, h in G.*

2. *Prove that if a is in the centre of the closure of $\pi_\lambda(\mathbb{C}G)$, then the function $a\delta_e$ is constant on conjugacy classes in G.*

3. *In the last two problems, which topologies can you use when taking the closure?*

4. *Prove that if every conjugacy class in G is infinite, except that of the identity, then $C_r^*(G)$ has a trivial centre.*

5. *Prove that the groups $\mathbb{F}_2$ and S_∞ have the infinite conjugacy class property.*

Exercise 2.4.2. *As we noted above, the identity map on $\mathbb{C}G$ extends to a $*$-homomorphism from $\ell^1(G)$ to $C^*(G)$. Prove that the composition of this map with ρ of 2.4.10 is injective. (Hint: try to mimic the proof of Theorem 2.2.5.)*

2.5 Abelian groups

The simplest case of a group C^*-algebra to consider is when the group is abelian. As we've seen (Proposition 2.4.7), this means that the C^*-algebra will be abelian also and we have a very complete understanding of them (Theorem 1.4.6).

Definition 2.5.1. *Let G be a discrete abelian group. Its* dual group*, denoted $\hat{G}$, consists of all group homomorphisms $\chi : G \to \mathbb{T}$.*

Here is a result that follows immediately from Theorem 2.2.3 and Exercise 1.11.3, but it is worth stating in any case.

Theorem 2.5.2. *If G is a discrete, abelian group, then $\hat{G}$ is exactly the set of irreducible representation of G (with Hilbert space $\mathbb{C}$ in each case.)*

We comment that often notationally, the value of χ in $\hat{G}$ on an element g of G is written as $< g, \chi >$. This tends to emphasize a symmetry which we will explore further once we remove the hypothesis that G is discrete in Section 2.8.

Theorem 2.5.3. *Let G be a discrete abelian group. Its dual group, $\hat{G}$, is a group when endowed with the pointwise product of functions. Moreover, the collection of sets*

$$U(\chi_0, F, \epsilon) = \{\chi \in \hat{G} \mid |\chi(g) - \chi_0(g)| < \epsilon, g \in F\},$$

where χ_0 is in $\hat{G}$, $F \subset G$ is finite and $\epsilon > 0$, forms a neighbourhood base for a topology on $\hat{G}$. In this, $\hat{G}$ is compact and Hausdorff.

Proof. The first statement is clear. For the second, from its definition, it is clear that $\hat{G}$ is a subset of $\mathbb{T}^G$. The sets we have defined above are simply the intersections of a fairly standard basis for the product topology on $\mathbb{T}^G$ with $\hat{G}$. Moreover, it is a trivial consequence of the definitions that $\hat{G}$ is a closed subset of $\mathbb{T}^G$. As $\mathbb{T}^G$ is both Hausdorff and compact, so is $\hat{G}$. □

It is worth pausing a moment to see what happens in the case of one of the simplest abelian groups, the integers.

Proposition 2.5.4. *The map which sends χ to $\chi(1)$ is an isomorphism and homeomorphism from $\hat{\mathbb{Z}}$ to $\mathbb{T}$.*

Proof. Since $\mathbb{Z}$ is cyclic, any χ in $\hat{\mathbb{Z}}$ is uniquely determined by its value at 1, the generator of $\mathbb{Z}$. This means our map is injective. Similarly, the fact that $\mathbb{Z}$ is free means that, for any z in $\mathbb{T}$, there is a homomorphism χ with $\chi(1) = z$: specifically, the map $\xi(n) = z^n$. This means our map is onto. It follows easily from the definitions that it is a group isomorphism.

The map is clearly continuous and since both spaces are compact and Hausdorff, it is a homeomorphism as well. □

Of course, since $C^*(G)$ is unital and abelian, it must be isomorphic to $C(X)$, for some compact Hausdorff space X. Probably, some readers will already have guessed that X is $\hat{G}$.

Theorem 2.5.5. *If G is a discrete abelian group, then $C^*(G)$ is isomorphic to $C(\hat{G})$. The isomorphism takes a group element $g \in \mathbb{C}G \subset C^*(G)$ to the function $\hat{g}(\chi) = \chi(g)$.*

Proof. We know that $C^*(G)$ is isomorphic to $C(\mathcal{M}(C^*(G)))$. Moreover, this map sends group element g to the function $\hat{g}(\phi) = \phi(g)$. We will be done if we can identify $\mathcal{M}(C^*(G))$ with $\hat{G}$. First, if ϕ is in $\mathcal{M}(C^*(G))$, we may restrict it to $G \subset \mathbb{C}G \subset C^*(G)$. The fact that ϕ is multiplicative means that $\phi|_G$ is a group homomorphism. This means that restriction defines a map from $\mathcal{M}(C^*(G))$ to $\hat{G}$. Two multiplicative linear functionals that agree on G are clearly equal since the elements of G span a dense subset of $C^*(G)$, so this map is injective.

Now we claim this map is surjective. Let χ be in $\hat{G}$. If we regard χ as a unitary representation on a one-dimensional Hilbert space, then by Theorem 2.2.3, it is the restriction of a unique representation of $\mathbb{C}G$ on this same Hilbert space. It is obviously continuous in the C^*-algebra norm on $\mathbb{C}G$ and hence extends to $C^*(G)$. But such a map is simply a multiplicative linear functional. This establishes the claim.

It is clear from the definition that the map is continuous, but as both spaces are compact and Hausdorff, it is a homeomorphism as well.

$\square$

Exercise 2.5.1. *Let n be a positive integer. Find $\hat{\mathbb{Z}_n}$, the dual group of the cyclic group of order n.*

Exercise 2.5.2. *Using the answer to exercise 2.5.1 above and Theorem 2.5.5, write each element of $\mathbb{Z}_n$ as a continuous function on $\hat{\mathbb{Z}_n}$. Also, find an element of $\mathbb{C}\mathbb{Z}_n$ (written as a linear combination of group elements) which is a minimal central projection.*

Exercise 2.5.3. *Let*

$$G = \mathbb{Z}[1/2] = \left\{ \frac{n}{2^k} \mid n, k \in \mathbb{Z}, k \geq 0 \right\}.$$

Describe $\hat{G}$. (Remark: the best one can do here is try to describe an element χ in $\hat{G}$ in terms of its restrictions to the subgroups

$$\mathbb{Z} \subset \frac{1}{2}\mathbb{Z} \subset \frac{1}{4}\mathbb{Z} \subset \cdots$$

in G.)

Exercise 2.5.4. *With G as above, describe $\widehat{G/\mathbb{Z}}$.*

2.6 The infinite dihedral group

In this section, we are going to analyze the structure of the simplest infinite, non-abelian group: the infinite dihedral group. It is the semi-direct product of the group of integers, $\mathbb{Z}$, by the cyclic group of order two, written here as $\{1, -1\}$ with multiplication as group operation, with the action given by $n \cdot j = nj$, for n in $\mathbb{Z}$ and $j = \pm 1$. We will find it most convenient to regard it as the group with two generators, a, b, subject to the relations, $b^2 = e$ and $bab = a^{-1}$. We will denote the group by D_∞.

Before stating the result, let us introduce some useful concepts. If X is a compact Hausdorff space and A is any C^*-algebra, then we denote by

$C(X, A)$ the set of continuous functions from X to A. It becomes a $*$-algebra by taking pointwise operations. The norm is given by

$$\|f\| = \sup\{\|f(x)\| \mid x \in X\}.$$

It is a relatively easy matter to check that it is a C^*-algebra.

In the special case that $A = M_n(\mathbb{C})$, for some positive integer n, it is also easy to see that this is naturally isomorphic to $M_n(C(X))$, the $n \times n$ matrices with entries from $C(X)$. This is just the observation that a continuous function into matrices can also be regarded as a matrix of continuous functions.

In $M_2(\mathbb{C})$, we let D_0 denote the subalgbera of diagonal matrices and D_1 be the set of all matrices of the form $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, where α, β are complex numbers. The reader can easily check that D_1 is a commutative C^*-subalgebra of $M_2(\mathbb{C})$ and is unitarily equivalent to D_0 via the unitary $\begin{bmatrix} 2^{-1/2} & 2^{-1/2} \\ -2^{-1/2} & 2^{-1/2} \end{bmatrix}$.

With this notation, we are able to state our main result.

Theorem 2.6.1. *There is an isomorphism*

$$\rho_1 : C^*(D_\infty) \to B_1 = \{f \in C([0, 1], M_2(\mathbb{C})) \mid f(0), f(1) \in D_1\}$$

with

$$\begin{aligned} \rho_1(a)(t) &= \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{bmatrix} \\ \rho_1(b)(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Composing ρ_1 with conjugation with the unitary which conjugates D_1 with D_0, we also have an isomorphism

$$\rho_0 : C^*(D_\infty) \to B_0 = \{f \in C([0, 1], M_2(\mathbb{C})) \mid f(0), f(1) \in D_0\}$$

with

$$\begin{aligned} \rho_0(a)(t) &= \begin{bmatrix} \cos(\pi t) & i\sin(\pi t) \\ -i\sin(\pi t) & \cos(\pi t) \end{bmatrix} \\ \rho_0(b)(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Comparing the two different versions of the results, the C^*-algebra B_0 seems a little easier to handle than B_1. On the other hand, in the map ρ_1, we see more clearly the representations of the subgroups generated by a and b, respectively. We will prove only the first statement and we will drop the subscript 1 on ρ. The proof will take the rest of this section.

The idea of the proof idea is fairly simple: in the case that our discrete group is abelian, we had an isomorphism between its C^*-algebra and $C(\hat{G})$. We first realized that $\hat{G}$ was the set of irreducible representations of the group and our map simply sent a group element g to the function whose value at a representation χ was $\chi(g)$. The same principle will work here, with a few minor modifications. The space $[0, 1]$ on the right is (almost) the set of irreducible representations, (almost) all of which are two-dimensional. Our map ρ will take a group element g to the function whose value at π is just $\pi(g)$.

We begin our proof with the easy steps. First, with $\rho(a)$ and $\rho(b)$ defined as in the theorem, it is a simple matter to check that they lie in B_1, are both unitaries and satisfy the same relations as a, b in D_∞. From this fact, it follows that ρ extends to a $*$-homomorphism from $\mathbb{C}D_\infty$ into B_1.

Next, we check that the image of $\mathbb{C}D_\infty$ is dense. Observe that since it contains the unit and $\rho(a) + \rho(a^{-1})$, the closure contains all matrices of the form

$$\begin{bmatrix} f(t) & 0 \\ 0 & f(t) \end{bmatrix},$$

where f is any continuous function on $[0, 1]$. It also contains $-i\rho(a)+i\rho(a^{-1})$ and multiplying this by functions of the type above, we see the closure also contains all matrices of the form

$$\begin{bmatrix} f(t) & 0 \\ 0 & -f(t) \end{bmatrix},$$

where f is a continuous function which vanishes at $0, 1$. By taking sums, we see the closure of the image contains all functions of the form

$$\begin{bmatrix} f(t) & 0 \\ 0 & g(t) \end{bmatrix},$$

where f, g are continuous functions with $f(0) = g(0), f(1) = g(1)$. Finally, by multiplying this by $\rho(b)$ and taking sums, we see the closure of the range contains all of B_1. Our final task is to show that ρ is isometric and we will be done.

If t is any point of $[0,1]$, it is clear that the map sending an element a of the group algebra $\mathbb{C}D_\infty$ to $\rho(a)(t)$ is a $*$-homomorphism to $M_2(\mathbb{C})$. That means it is also a representation of $\mathbb{C}D_\infty$ on a 2-dimensional Hilbert space. We will call this representation π_t. From the definition of the norm on $\mathbb{C}D_\infty$ 2.4.5, we have

$$\|a\| \geq \sup\{\|\rho(a)(t)\| \mid t \in [0,1]\},$$

for every a in $\mathbb{C}D_\infty$. Now, it is not true that all of these representations are irreducible, but we will prove the following.

Lemma 2.6.2. *Let $(\mathcal{H}, \pi)$ be an irreducible representation of D_∞. Then for some t in $[0,1]$ and some subspace $N \subset \mathbb{C}^2$ which is π_t-invariant, π is unitarily equivalent to the restriction of π_t to N.*

If we accept this for the moment, then by using 2.4.8, we have, for any a in $\mathbb{C}D_\infty$,

$$\|a\| = \sup\{\|\pi(a)\| \mid \pi \text{ irreducible }\} = \sup\{\|\pi_t(a)\| \mid t \in [0,1]\} = \|\rho(a)\|$$

and so we see that ρ is isometric.

Let us give a proof of the lemma. The operator $\pi(b)$ is a self-adjoint unitary. The first case to consider is $\pi(b) = 1$. In this case, an operator on $\mathcal{H}$ commutes with the image of π if and only if it commutes with $\pi(a)$. The fact that π is irreducible then means that its restriction to the subgroup generated by a is also irreducible and since this group is abelian, our representation must be 1-dimensional. Suppose that $\pi(a) = z$, for some z in $\mathbb{T}$. The relation $bab = a^{-1}$ and $\pi(b) = 1$ means that $\pi(a) = z$ must also be real. We conclude that $z = \pm 1$. We have exactly two such irreducible representations $\pi(a) = 1 = \pi(b)$ and $\pi(a) = -1, \pi(b) = 1$. Observe that the former is obtained by restricting π_0 to $N = \{(\alpha, \alpha) \mid \alpha \in \mathbb{C}\}$ and the latter by restricting π_1 to N.

Similar arguments deal with the second case when $\pi(b) = -1$. Here there are two irreducible representations: $\pi(a) = 1, \pi(b) = -1$ and $\pi(a) = -1 = \pi(b)$. These are obtained by restricting π_0 and π_1 to $N = \{(\alpha, -\alpha) \mid \alpha \in \mathbb{C}\}$.

Now we turn to the case $\pi(b) \neq \pm 1$. This means that $(\pi(b)+1)/2$ is a non-zero projection. Let ξ be any unit vector in its range. From the fact that $(\pi(b)+1)2^{-1}\xi = \xi$, an easy computation yields $\pi(b)\xi = \xi$ as well.

The first observation is that the element of the group algebra $a + a^{-1}$ clearly commutes with a and it is any easy calculation to see that it commutes with b also. Its image under π is self-adjoint and of norm at most 2 (since $\pi(a)$ and $\pi(a^{-1})$ are both unitaries). It follows that $2 + \pi(a) + \pi(a^{-1})$ is a positive operator commuting with the range of π. As π is irreducible, we conclude that it is a scalar. Then $(\pi(a) + \pi(a^{-1}))/2 = (\pi(a) + \pi(a)^*)/2$ is a real number x in $[-1, 1]$ times the identity operator. Choose t in $[0, 1]$ such that $\cos(\pi t) = x$. That is, we have $\pi(a) + \pi(a)^* = 2x$. Let $y = \sin(\pi t)$ and $z = x + iy$ so that $x^2 + y^2 = 1$.

We observe the following fact that will be used several times

$$\pi(a)^* - z = \pi(a)^* - x - iy = x - \pi(a) - iy = -\pi(a) + \bar{z}.$$

Notice that this also means that $\pi(a) - z = -\pi(a)^* + \bar{z}$. Next, we claim that $(\pi(a) - z)\xi$ and $(\pi(a)^* - z)\xi$ are orthogonal. Observing that since $\pi(a)$ is unitary, $\pi(a)\xi$ is also a unit vector, we have

$$\begin{aligned}
< (\pi(a) - z)\xi, (\pi(a)^* - z)\xi > &= < (\pi(a) - z)\xi, (-\pi(a) + \bar{z})\xi > \\
&= - < \pi(a)\xi, \pi(a)\xi > + < \pi(a)\xi, \bar{z}\xi > \\
&\quad + < z\xi, \pi(a)\xi > - < z\xi, \bar{z}\xi > \\
&= -1 + z < \pi(a)\xi, \xi > \\
&\quad +z < \xi, \pi(a)\xi > -z^2 \\
&= -1 + z < (\pi(a) + \pi(a)^*)\xi, \xi > -z^2 \\
&= -1 + z < 2x\xi, \xi > -z^2 \\
&= -1 + 2zx - z^2 \\
&= -1 + 2x^2 + 2ixy - x^2 + y^2 - 2ixy \\
&= 0.
\end{aligned}$$

We next want to compute the effects of the two operators $\pi(a)$ and $\pi(b)$ on the two vectors $(\pi(a) - z)\xi$ and $(\pi(a)^* - z)\xi$. First, we have

$$\begin{aligned}
\pi(a)(\pi(a) - z)\xi &= \pi(a)(-\pi(a)^* + \bar{z})\xi \\
&= -\pi(a)\pi(a)^*\xi + \bar{z}\pi(a)\xi \\
&= -\xi + \bar{z}\pi(a)\xi \\
&= \bar{z}(\pi(a) - z)\xi.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\pi(a)(\pi(a)^* - z)\xi &= \pi(a)\pi(a)^*\xi - z\pi(a)\xi \\
&= \xi - z\pi(a)\xi \\
&= z(\bar{z} - \pi(a))\xi \\
&= z(\pi(a)^* - z)\xi.
\end{aligned}$$

Now we turn to the action of $\pi(b)$ (which is simpler):

$$\begin{aligned}\pi(b)(\pi(a)-z)\xi &= (\pi(b)\pi(a)-z\pi(b))\xi \\ &= (\pi(a)^*\pi(b)-z\pi(b))\xi \\ &= (\pi(a)^*-z)\pi(b)\xi \\ &= (\pi(a)^*-z)\xi.\end{aligned}$$

A similar computation shows that $\pi(b)(\pi(a)^*-z)\xi = (\pi(a)-z)\xi$.

It is clear from these equations that the subspace spanned by these two vectors is invariant under π. Since the unitary $\pi(b)$ maps one to the other, they must have the same length. If that happens to be zero, then $\pi(a)\xi = z\xi$ and $\pi(a)^*\xi = z\xi$. It follows that $z = \pm 1$ and that the span of ξ itself is invariant under π. It is a simple matter to check that we have listed all the 1-dimensional respresentation above and accounted for them with π_0 and π_1 already. Otherwise, we may re-scale these vectors so that they are unit length and, with respect to that basis, π is the same as π_t. This completes the proof of Theorem 2.6.1.

Leaving all these calculations aside, what have we learned from this example? The set of all irreducible representations of a discrete group G is usually denoted by $\hat{G}$ (which extends the definition we had before from the abelian case). The first thing we see is that a thorough understanding of this set is important to understand $C^*(G)$. Secondly, we can loosely interpret $C^*(G)$ as $C(\hat{G})$, even beyond the abelian case, if we allow the functions to take values in matrix algebras. Underlying all of this is the fact that $\hat{G}$ can be equipped with a natural topology. We did this in the abelian case and we won't discuss the general, but notice here that the space is non-Hausdorff: it's an open interval with two limit points at each end.

Finally, we remark that the short version of what we have done is to realize that our group D_∞ is composed of two abelian subgroups, $\mathbb{Z}$ and $\mathbb{Z}_2$. In short, we are able to get complete information about the irreducible representations of D_∞ from the same information about the two constituents. This is called the Mackey machine after George Mackey, one of the leading figures in the theory of group representations. In brief, if one has a normal subgroup H in a group G, there is a natural construction of irreducible representations of G from those of H. More impressively, this will often be exhaustive.

Exercise 2.6.1. *For $n \geq 3$, let D_n be the dihedral group of order $2n$. That is,*

$$D_n = \{e, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\},$$

where a, b *satisfy* $a^n = e, b^2 = e, bab = a^{-1}$.

1. *List all the irreducible representations of* D_n. *(Hint: each of these also gives an irreducible representation of* D_∞.*)*
2. *Describe* $C^*(D_n)$.

2.7 The group $\mathbb{F}_2$

In this section, we investigate the C^*-algebras associated with the free group on two generators, denoted $\mathbb{F}_2$. Here, the key point is that C^*-algebra is plural: the full and reduced C^*-algebras are different. This is because $\mathbb{F}_2$ is not amenable. Rather than go too far into that, we will mainly concentrate on just how different these C^*-algebras are.

We will begin with a brief review of the group $\mathbb{F}_2$. We denote the generators by a and b. A *word* in a, b, a^{-1}, b^{-1} is simply a finite string of these symbols written sequentially. There is also a special word e which is the empty word consisting of no symbols. Two words may be concatenated by simply writing them side by side. Of course, the order matters. We denote the concatenation of w_1 and w_2 by $w_1 w_2$. Notice that $ew = w = we$ for all w and that concatenation is an associative operation. We define an equivalence relation on the set of all words by first setting

$$w_1 w_2 \sim w_1 a^{-1} a w_2 \sim w_1 a a^{-1} w_2 \sim w_1 b^{-1} b w_2 \sim w_1 b b^{-1} w_2,$$

for all words w_1, w_2 and then letting $\sim$ denote the transitive closure of this relation. We also say that a word is reduced if it contains no pair a, a^{-1} or b, b^{-1} adjacent. Each $\sim$-equivalence class contains a unique reduced word. The collection of $\sim$-equivalence classes of words, with concatenation is our group $\mathbb{F}_2$.

The main feature of the group (in fact, it is really the definition) is that, given any group G with two elements g, h in G, there is a unique group homomorphism $\alpha : \mathbb{F}_2 \to G$ such that $\alpha(a) = g$ and $\alpha(b) = h$.

As we said above, our aim is to show that the C^*-algebras $C^*(\mathbb{F}_2)$ and $C^*_r(\mathbb{F}_2)$ are very different. Compare them with the following two theorems.

Theorem 2.7.1. *If* A *is a unital* C^**-algebra containing unitaries* u, v*, then there exists a unique unital* $*$*-homomorphism* $\rho : C^*(\mathbb{F}_2) \to A$ *such that* $\rho(a) = u, \rho(b) = v$. *In particular, for every* $n \geq 1$*, there is a surjective* $*$*-homomorphism* $\rho : C^*(\mathbb{F}_2) \to M_n(\mathbb{C})$.

Theorem 2.7.2. *The C^*-algebra $C_r^*(\mathbb{F}_2)$ is simple; that is, it has no non-trivial ideals. In particular, it has no finite-dimensional representations.*

Theorem 2.7.3. *The trace on $C_r^*(\mathbb{F}_2)$ is unique.*

Let us start by proving Theorem 2.7.1. Choose $\pi : A \to \mathcal{B}(\mathcal{H})$ which is a faithful, unital representation. Then $\pi(u)$ and $\pi(v)$ are unitaries. That means there is a group homomorphism $\alpha : \mathbb{F}_2 \to \mathcal{U}(\mathcal{H})$ which maps a to $\pi(u)$ and b to $\pi(v)$. This means that α is a unitary representation of $\mathbb{F}_2$ and extends to β, a unital representation of $\mathbb{C}\mathbb{F}_2$ and also a unital representation of $C^*(\mathbb{F}_2)$. It is clear that the image of the former is contained in the $*$-algebra generated by $\pi(u)$ and $\pi(v)$ while the image of the latter is in the closure of this. In any case, both images are contained in $\pi(A)$. We are done when we let $\rho = \pi^{-1} \circ \beta$.

For the second statement, it is a fairly simple matter to see that the smallest C^*-algebra which contains

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/n} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & e^{2\pi i(n-1)/n} \end{bmatrix}$$

is the set of diagonal matrices. It follows that the smallest C^*-algebra which contains this and

$$v = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix}$$

is $M_n(\mathbb{C})$.

Now we turn to the reduced C^*-algebra, which is trickier. We start with a fairly useful result that the trace is faithful.

Theorem 2.7.4. *Let G be a discrete group. If c is in $C_r^*(G)$ and $\tau(c^*c) = 0$, then $c = 0$. That is, the trace on $C_r^*(G)$ is faithful.*

Proof. In this proof, we identify $C_r^*(G)$ with its image under π_λ. Let g be in

G and compute

$$\begin{aligned}
\|c\delta_g\|^2 &= < c\delta_g, c\delta_g > \\
&= < c\lambda_g\delta_e, c\lambda_g\delta_e > \\
&= < \lambda_g^* c^* c\lambda_g\delta_e, \delta_e > \\
&= \tau(\lambda_g^* c^* c\lambda_g) \\
&= \tau(\lambda_g\lambda_g^* c^* c) \\
&= \tau(c^* c) \\
&= 0.
\end{aligned}$$

We see that c is zero on a basis for $\ell^2(G)$ and hence is zero. □

We next turn to some much more technical results. The next is particularly opaque. Of course, we will find it useful. To put the result in some kind of informal way, the expression within the norm on left hand side of the conclusion is a kind of averaging process being applied to the operator c. The result shows that this substantially reduces the norm, under certain hypotheses. The reader might want to try Exercise 2.7.4 as a little warm-up in a different setting.

Lemma 2.7.5. *Let $\mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces. Suppose that c is an operator on $\mathcal{H}_0 \oplus \mathcal{H}_1$ having the form $\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}$, $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$ or $\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}$. Also suppose that $u_1, \ldots, u_N$ are unitaries such that $u_m^* u_n$ has the form $\begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$, for all $m \neq n$. Then we have*

$$\|\frac{1}{N}\sum_{n=1}^{N} u_n c u_n^*\| \leq \frac{2}{\sqrt{N}}\|c\|.$$

Proof. We first consider the case $c = \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}$. If $d = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$, then we have $c^*d = d^*c = 0$ so

$$\|c+d\|^2 = \|(c+d)^*(c+d)\| = \|c^*c + d^*d\| \leq \|c^*c\| + \|d^*d\| = \|c\|^2 + \|d\|^2.$$

Next, it is easy to see that for all $n > 1$, we have

$$d_n = (u_1^* u_n)c(u_1^* u_n)^* = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}\begin{bmatrix} * & * \\ * & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}.$$

Summing over $n > 1$ yields a term of the same form. Combining these two together, we have

$$\begin{aligned} \|\sum_{n=1}^{N} u_n c u_n^*\|^2 &= \|u_1(c + \sum_{n=2}^{N} u_1^* u_n c (u_1^* u_n)^*) u_1^*\|^2 \\ &= \|c + \sum_{n=2}^{N} u_1^* u_n c (u_1^* u_n)^*\|^2. \end{aligned}$$

Now we are in a position to apply our computation above with d being the sum on the right. This yields

$$\begin{aligned} \|\sum_{n=1}^{N} u_n c u_n^*\|^2 &\leq \|c\|^2 + \|\sum_{n=2}^{N} u_1^* u_n c (u_1^* u_n)^*\|^2 \\ &= \|c\|^2 + \|\sum_{n=2}^{N} u_n c u_n^*\|^2. \end{aligned}$$

Continuing by induction, dividing by N^2 and taking square roots yields the answer, without the factor of 2.

The second case for c is obtained simply by taking adjoints of the first. For the third, any such c can be written as $p_0 c + p_1 c$, where p_i is the orthogonal projection of $\mathcal{H}_0 \oplus \mathcal{H}_1$ onto $\mathcal{H}_i$, $i = 0, 1$. We can apply the first and second cases to $p_0 c$ and $p_1 c$ respectively to get

$$\|\frac{1}{N}\sum_{n=1}^{N} u_n c u_n^*\| \leq \frac{1}{\sqrt{N}}\|p_0 c\|^2 + \frac{1}{\sqrt{N}}\|p_1 c\|^2 \leq \frac{2}{\sqrt{N}}\|c\|^2.$$

$\square$

We are going to apply this to the elements of the left regular representation of $\mathbb{F}_2$.

Lemma 2.7.6. *For all w in $\mathbb{F}_2$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \lambda_{a^n}\lambda_w\lambda_{a^{-n}} = \begin{cases} \lambda_w & w = a^k, k \in \mathbb{Z} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. First of all, if w is a power of a, then for each n, $\lambda_{a^n}\lambda_w\lambda_{a^{-n}} = \lambda_w$ and the conclusion is clear. If not, then w contains either a b or b^{-1}. Then we have $w = a^k w_0 a^l$, where w_0 begins and ends in b or b^{-1} (including the case that it is a power of b).

Let $\mathcal{H}_0$ be the span of all $\delta_{w'}$ where w' is empty or begins with $a^{\pm 1}$. Also let $\mathcal{H}_1$ be the span of the remaining elements of the group, those which begin with $b^{\pm 1}$.

We make two simple observations: $\lambda_{w_0}\mathcal{H}_0 \subset \mathcal{H}_1$ and $\lambda_a\mathcal{H}_1 \subset \mathcal{H}_0$. It follows that $b = \lambda_{w_0}$ and $u_i = \lambda_{a^i}, 1 \leq i \leq n$ satisfy the hypotheses of Lemma 2.7.5. Hence, we have

$$\begin{aligned}
\lim_N \|\frac{1}{N}\sum_{n=1}^N \lambda_{a^n}\lambda_w\lambda_{a^{-n}}\| &= \lim_N \|\frac{1}{N}\sum_{n=1}^N \lambda_{a^n}\lambda_{a^k}\lambda_{w_0}\lambda_{a^l}\lambda_{a^{-n}}\| \\
&= \lim_N \|\frac{1}{N}\sum_{n=1}^N \lambda_{a^n}\lambda_{w_0}\lambda_{a^{-n}}\| \\
&\leq \lim_N \frac{1}{\sqrt{N}}\|\lambda_{w_0}\| \\
&= 0.
\end{aligned}$$

□

Lemma 2.7.7. *For any c in $C_r^*(\mathbb{F}_2)$, we have*

$$\lim_{M,N\to\infty} \frac{1}{MN}\sum_{m=1}^M\sum_{n=1}^N \lambda_{b^n a^m} c \lambda_{a^{-m}b^{-n}} = \tau(c).$$

Proof. By some standard estimates, it suffices to check this on elements of the group algebra $\mathbb{C}\mathbb{F}_2$. Then by linearity, it suffices to check it for $c = \lambda_w$, for some w in $\mathbb{F}_2$.

The first case to consider is when w is the identity element and then the conclusion is clear. The second is that $w = a^k$, for some $k \neq 0$. In this case, λ_w commutes with each λ_{a^n} and so we get

$$\lim_{M,N\to\infty} \frac{1}{MN}\sum_{m=1}^M\sum_{n=1}^N \lambda_{b^n a^m} c \lambda_{a^{-m}b^{-n}} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \lambda_{b^n} c \lambda_{b^{-n}}$$

and the conclusion is an application of the last Lemma (with b replacing a).

The final case is for $w \neq a^k$, for any integer k. Then given $\epsilon > 0$, we may find N_0 such that for all $N \geq N_0$, we have

$$\| \frac{1}{N} \sum_{n=1}^{N} \lambda_{a^n} \lambda_w \lambda_{a^{-n}} \| < \epsilon.$$

Then for every m, we also have

$$\| \frac{1}{N} \sum_{n=1}^{N} \lambda_{b^m a^n} \lambda_w \lambda_{a^{-n} b^{-m}} \| < \epsilon.$$

and hence

$$\| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda_{b^n a^m} c \lambda_{a^{-m} b^{-n}} \| < \epsilon,$$

provided $N \geq N_0$. This completes the proof. $\square$

We are now ready to prove Theorem 2.7.2. Suppose that $C_r^*(\mathbb{F}_2)$ has a non-zero ideal I and let c be some non-zero element of that ideal. We know from Theorem 2.7.4 that $\tau(c^*c) > 0$. We also know that c^*c is also in the ideal. Looking at the last Lemma, the expression on the left hand side (using c^*c instead of c) lies in the ideal, for every value of M, N. As the ideal is closed, $\tau(c^*c) > 0$ is in the ideal. But this element is invertible and it follows that $I = C_r^*(\mathbb{F}_2)$.

As for the proof of Theorem 2.7.3, if ϕ is any other trace on $C_r^*(\mathbb{F}_2)$, if we apply it to $\frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \lambda_{b^n a^m} c \lambda_{a^{-m} b^{-n}}$ we clearly get $\phi(c)$ because of the trace property. On the other hand, applying it to the scalar $\tau(c)$ yields $\tau(c)$. The conclusion follows from Lemma 2.7.7.

Exercise 2.7.1. *Let u be the $N \times N$ unitary matrix*

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/N} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & e^{2\pi i(N-1)/N} \end{bmatrix}.$$

Define a map $E : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ by

$$E(a) = \frac{1}{N} \sum_{n=1}^{N} u^n a u^{-n},$$

for a in $M_N(\mathbb{C})$.

1. *Prove that the range of E is contained in the C^*-subalgebra D consisting of diagonal matrices.*

2. *Prove that $E(d) = d$ if d is diagonal. In consequence, $E \circ E = E$.*

3. *Prove that $E(a^*a)$ is positive for every a in $M_N(\mathbb{C})$.*

4. *Prove that E is faithful, meaning that if $E(a^*a) = 0$, for some a in $M_N(\mathbb{C})$, then $a = 0$.*

5. *Prove that if a in $M_N(\mathbb{C})$ and c, d are in D, then $E(cad) = cE(a)d$.*

6. *Prove that $\|E\| = 1$.*

Exercise 2.7.2. *Let u, v be the $N \times N$ unitary matrices*

$$u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{2\pi i/N} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & e^{2\pi i(N-1)/N} \end{bmatrix}, v = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & & & 1 \\ 1 & 0 & & \cdots & 0 \end{bmatrix}.$$

Prove that, for any c in $M_N(\mathbb{C})$, we have

$$\frac{1}{N^2} \sum_{m=1}^{N} \sum_{n=1}^{N} v^n u^m c u^{-m} v^{-n} = \tau(c),$$

where τ is the usual trace on $M_N(\mathbb{C})$.

Exercise 2.7.3. *Show that Theorem 2.7.3 is false for $C^*(G)$.*

2.8 Locally compact groups

In this section, we discuss, informally, a more general construction of the group C^*-algebra. The starting point is the notion of a topological group. By a *topological group* we mean a group G, endowed with a topology in which the product, regarded as a map from $G \times G$ (with the product topology) to G, and the inverse, regarded as a map from G to itself, are both continuous. It is frequently assumed that G is Hausdorff.

There are many examples. Of course, any group at all, given the discrete topology will qualify. The real numbers with the usual operation of addition and the usual topology is also an example. So is the circle group $\mathbb{T}$. A more exotic example is $GL(n,\mathbb{R})$, the set of invertible real $n \times n$ matrices with matrix multiplication as group operation.

The first fundamental new ingredient is a left-invariant measure called the *Haar* measure.

Theorem 2.8.1. *Let G be a locally compact, Hausdorff group. There exists a positive regular Borel measure μ on G with the property that $\mu(gE) = \mu(E)$, for all Borel sets $E \subset G$ and $g \in G$, where $gE = \{gh \mid h \in E\}$. Moreover this measure is unique, up to a scalar multiple.*

In the case that the group G is discrete, Haar measure is simply counting measure.

A complete treatment of this subject would start with a proof of this crucial fact, which is why we aren't going to do that.

Let us take a little detour for a moment to return to the topic of discrete abelian groups which was covered in section 2.5. Recall that we had an isomorphism from $C^*(G)$ to $C(\hat{G})$; let us denote it by α for the moment. We observed earlier that $\hat{G}$ is both a group and a compact, Hausdorff topological space. It is a fairly easy thing to see that it is actually a topological group. Hence, it has a Haar measure, μ. This measure is actually finite in this case, so we assume $\mu(\hat{G}) = 1$. With this measure available, we have a very nice description of the inverse of our isomorphism. Suppose that f is in $C(\hat{G})$. Then we have

$$\alpha^{-1}(f)(g) = \left(\int f(\chi)\overline{\chi(g)} d\mu(\chi) \right)$$

This should be recognizable. When $G = \mathbb{Z}$ and $\hat{G} = \mathbb{T}$, the right hand side is the conventional Fourier series for f. Notice that we are defining $\alpha^{-1}(f)$ as a function on G. In fact, we have to be very careful with this and exactly which f we use. Some care is needed when discussing the map above; the same formula also defines a unitary operator between $L^2(\hat{G},\mu)$ and $\ell^2(G)$.

We now return to the general case of topological groups which are not necessarily abelian. There is an interesting and useful consequence of the uniqueness of the Haar measure as follows. If we fix a group element g in G, and look at the function defined on Borel sets E by $\mu_g(E) = \mu(Eg)$, it is clear that this is also a left-invariant measure. Hence there is a constant $\Delta(g)$ such that $\mu_g = \Delta(g)\mu$. The function $\Delta : G \to \mathbb{R}^+$ is a group homomorphism (with range having multiplication as group operation). The function is identically one if and only if the left-invariant measure happens to be right-invariant as well.

Our first step, when we had a discrete group, was to form the group algebra. Of course, one can still do this, but instead we will consider $C_c(G)$, the continuous complex-valued functions of compact support on G, rather than $\mathbb{C}G$ which are the functions of finite support. If G is discrete, this is just $\mathbb{C}G$ as before. Notice here that the inclusion $G \subset C_c(G)$ fails unless G is discrete, since a function that is one at a single group element and zero elsewhere will not be continuous if the group is not discrete.

We regard $C_c(G)$ as a linear space in the obvious way and define the product by the formula

$$ab(g) = \int_{h \in G} a(h)b(h^{-1}g)d\mu(h),$$

for all a, b in $C_c(G)$ and g in G. We also define an involution by

$$a^*(g) = \Delta(g)^{-1}\overline{a(g^{-1})},$$

for all a in $C_c(G)$ and g in G.

We leave it for the reader to verify that $C_c(G)$ is a $*$-algebra with this product and involution. The computations are not a complete triviality since one sees rather concretely exactly why the measure needs to be left-invariant and also why the function Δ makes its surprise entrance in the involution. Of course, if G is discrete, the measure is simply counting measure, the integral simply becomes a sum and our product is exactly the same as we saw in Theorem 2.4.2. In addition, counting measure is clearly right-invariant as well as left-invariant and this is why Δ did not appear earlier in the Chapter.

The other important thing to notice here is that $C_c(G)$ has no unit unless G is discrete. Essentially, this is the same issue that $C_c(G)$ does not contain G.

We also define a norm on $C_c(G)$ by the formula

$$\|a\|_1 = \int_{g \in G} |a(g)|d\mu(g),$$

for all a in $C_c(G)$. The completion of $C_c(G)$ in this norm is $L^1(G)$. Except for the change of ℓ to L, this is the same as the discrete case. In fact, the formulae we have above also hold for the product and involution on $L^1(G)$ making it a Banach algebra with an isometric involution.

The next item that requires modification is the notion of unitary representation. Usually in the case of a topological group G, a unitary representation is $(u, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space as before, u is a group homomorphism from G to the unitary group on $\mathcal{H}$ *and* we require that u is continuous when G is considered with its given topology and the unitaries are considered with the strong operator topology. The left regular representation is defined exactly as before, replacing $\ell^2(G)$ by $L^2(G, \mu)$, where μ is Haar measure. The fact that the operators $\lambda_g, g \in G$, are unitary is a direct consequence of the fact that the Haar measure is left-invariant. It should also be checked that this is continuous in the sense above.

Why is this the right thing to do? The crucial result which persists from the discrete case here is that there are bijective correspondences between: unitary representations of G, non-degenerate representations of $C_c(G)$ (though not unital since $C_c(G)$ has no unit in general) and non-degenerate representations of $L^1(G)$. The passage from first to second to third is very much as before. If $(u, \mathcal{H})$ is a unitary representation of G, then, for any a in $C_c(G)$, the formula

$$\pi_u(a) = \int_{g \in G} a(g) u_g$$

defines a bounded linear operator. Also, this formula will extend continuously to define π_u on $L^1(G)$. It is in going back that subtleties arise. Of course, $C_c(G) \subset L^1(G)$ is valid, but one cannot restrict to $G \subset C_c(G)$ to get from representations of the $*$-algebras back to unitary representations of the group. Nevertheless, the definitions of the full and reduced norms and the completions of the group algebras into C^*-algebras $C^*(G)$ and $C^*_r(G)$ proceeds as before.

In the special case that G is commutative, but not necessarily discrete, we again know that $C^*(G)$ will be isomorphic to $C_0(X)$, for some locally compact space X. In fact, the space X is again the dual group:

$$\hat{G} = \{\chi : G \to \mathbb{T} \mid \chi \text{ a continuous group homomorphism } \}.$$

The only new item here is the word 'continuous' (which can obviously be omitted with no change if the group happens to be discrete). Again, we have the isomorphism $C^*(G) \cong C_0(\hat{G})$. It is interesting to note that we can still define, for any group element g, the function $\hat{g}(\chi) = \chi(g), \chi \in \hat{G}$. The only problem now with the map $g \to \hat{g}$ is that g is not an element of $C_c(G)$, nor is $\hat{g}$ in $C_0(\hat{G})$, when G is not discrete.

Of course, we have been assuming implicitly here that $\hat{G}$ has a topology. Let us just spell out exactly what that is, particularly since it will be nice to see the parallel with the discrete case before. For each χ in $\hat{G}$, $K \subset G$ which is compact and $\epsilon > 0$, we define

$$U(K, \epsilon) = \{\chi' \in \hat{G} \mid |\chi'(g) - \chi(g)| < \epsilon, \text{ for all } g \in K\}.$$

These sets form a neighbourhood base for a topology on $\hat{G}$. In this topology, $\hat{G}$ is again a topological group.

As an example, we have $\hat{\mathbb{R}} \cong \mathbb{R}$ and the isomorphism, which we find convenient to write here as a pairing, is given by

$$< r, s >= e^{2\pi i r s}, r, s \in \mathbb{R}.$$

That is, if one fixes s, this defines a map $\chi_s(r) =< r, s >$ from $\mathbb{R}$ to the circle which is a continuous group homomorphism. Moreover, $s \to \chi_s$ is an isomorphism between $\mathbb{R}$ and $\hat{\mathbb{R}}$.

It is again true here that the inverse of the isomorphism α from $C^*(G)$ to $C_0(\hat{G})$ is given by Fourier transform

$$\alpha^{-1}(f)(r) = \int_{\mathbb{R}} f(s) e^{-2\pi i r s} ds,$$

for f in $C_c(\mathbb{R})$ and r in $\mathbb{R}$, or for a general abelian topological group G,

$$\alpha^{-1}(f)(g) = \int_{\hat{G}} f(\chi) \overline{< g, \chi >} d\mu(\chi),$$

for f in $C_c(\hat{G})$ and g in G.

Exercise 2.8.1. *Let G be a locally compact Hausdorff group. Prove that if Haar measure is counting measure, then G is discrete.*

Exercise 2.8.2. *Let G be the $ax + b$ group:*

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a > 0, b \in \mathbb{R} \right\}$$

with matrix multiplication as group operation. (There is a little ambiguity here: sometimes one takes $a \neq 0$ instead.)

1. *Prove that $a^{-2}dadb$ is the Haar measure on G. (That is, it is Lebesgue measure on $(0, \infty) \times \mathbb{R}$ times the function $(a, b) \to a^{-2}$.) In other words, if we let E be the set of all matrices with a in $[a_1, a_2]$ and b in $[b_1, b_2]$, then*

$$\mu(E) = (a_1^{-1} - a_2^{-1})(b_2 - b_1).$$

2. *Find Δ.*

Chapter 3

Groupoid C^*-algebras

We have so far seen two very important constructions of C^*-algebras. To any locally compact Hausdorff space X, we have $C_0(X)$. In fact, this construction yields exactly the class of commutative C^*-algebras. The other construction associates a C^*-algebra $C^*(G)$ to any discrete group G. In fact, there is also $C_r^*(G)$ as well, so this particular construction yields (potentially) two (or more) C^*-algebras.

Our next objective is a construction which combines these two, in some sense. In particular, these will both be seen as special cases of our new construction. That sounds impressive, but we should warn readers that, rather than expecting some elegant general structure which subsumes both, what we construct, étale groupoids, really look a little more like some Frankenstein-type monster, built out of parts of both.

The goal though is not so unreasonable and let us take a few minutes to explain why, at least for people with some interest in topological dynamical systems. (Ergodic theorists will probably already have realized that they should be looking in the direction of von Neumann algebras.)

What is a topological dynamical system? A reasonable simple answer is that is a continuous self-map, φ, of topological space, X. But that is misleading, even if it is not deliberately so. It would be impossible to do any mathematics that one could legitimately call dynamical systems without *iterating* the map. From this point of view, the dynamical system really consists of a collections of maps, $\varphi^n, n \geq 1$, with the condition that $\varphi^n \circ \varphi^m = \varphi^{n+m}$, for all $n, m \geq 1$. Here, one sees at once that the collection of maps contains some algebraic structure. In this case, it is that of the semigroup of natural numbers. Of course, in any general theory, groups are simpler than semigroups and so we would prefer to assume that φ is actually a homeomorphism and consider the family of maps indexed by $\mathbb{Z}$ rather than $\mathbb{N}$. Even in this simple case, we can see that we would like to consider a single C^*-algebra which is built from $C_0(X)$ and $C^*(\mathbb{Z})$. Of course, the two of these separately aren't very much use since neither remembers on its own what φ is.

3.1 Groupoids

We begin with the definition of a groupoid. As the name would suggest, it is rather like that of a group. The key point is that the product of an arbitrary pair of elements may not be defined. That is, we specify a subset (called G^2) of $G \times G$ and the product gh is only defined for pairs (g, h) in G^2. The first axiom is a fairly natural generalization of associativity. One might reasonably expect the next axiom to concern the existence of an identity. That is actually too much to ask. Instead, the other axiom is that there is an inverse operation $g \to g^{-1}$. The condition is rather subtle, but what it contains is that $g^{-1}g$ and gg^{-1} will serve as an identities for certain elements. In particular, the former is a right identity for g and the latter is a left identity. These may not be equal; indeed G may have a large collection of identities.

Definition 3.1.1. *A* groupoid *is a non-empty set G, together with a subset $G^2 \subset G \times G$, a map from G^2 to G (the image of (g, h) is denoted gh) and an involution, $g \to g^{-1}$, satisfying the following conditions.*

1. *If g, h, k are in G with $(g, h), (h, k)$ both in G^2, then so are (gh, k) and (g, hk) and we have $(gh)k = g(hk)$. (We write the result as ghk.)*
2. *For all g in G, both (g, g^{-1}) and (g^{-1}, g) are in G^2. If (g, h) is in G^2, then $g^{-1}gh = h$. If (h, g) is in G^2, then $hgg^{-1} = h$.*

The following definitions will be useful. First, since there are multiple identity elements, we prefer to call them units. The letters r, s stand for range and source. The usefulness of this terminology will become apparent later.

Definition 3.1.2. *Let G be a groupoid. The set of* units *of G (or identities) is*

$$G^0 = \{g^{-1}g \mid g \in G\}.$$

We define $r, s : G \to G^0$ by $r(g) = gg^{-1}, s(g) = g^{-1}g$.

Lemma 3.1.3. *Let G be a groupoid.*

1. *For any g in G, $(r(g), g), (g, s(g))$ are in G^2 and $r(g)g = gs(g) = g$.*
2. *If g, h are elements of G, then (g, h) is in G^2 if and only if $s(g) = r(h)$.*
3. *If g, h are elements of G with (g, h) in G^2, then (h^{-1}, g^{-1}) is in G^2 and $(gh)^{-1} = h^{-1}g^{-1}$.*
4. *If g, h are elements of G with (g, h) in G^2, then $r(gh) = r(g)$ and $s(gh) = s(h)$.*
5. *If g is in G^0, then $g^{-1} = g$.*
6. *If g is in G^0, then $r(g) = s(g) = g$.*

Proof. The first statement is just a re-writing of the previous definition in the special case $g = h$.

For the second part, if (g, h) is in G^2, we know that $(g^{-1}, g), (h, h^{-1})$ are also and

$$s(g) = g^{-1}g = g^{-1}(ghh^{-1}) = (g^{-1}gh)h^{-1} = hh^{-1} = r(h).$$

Conversely, suppose that $s(g) = r(h)$. We know that (g, g^{-1}) and (g^{-1}, g) are in G^2, so $(g, g^{-1}g)$ is in G^2. Since $g^{-1}g = s(g) = r(h) = hh^{-1}$, we have (g, hh^{-1}) is in G^2. Similar arguments show that (hh^{-1}, h) is in G^2 and hence $(g, h) = (g, hh^{-1}h)$ is in G^2 as desired.

For the third part, from associativity, we know that $(gh)h^{-1}$ is defined and equals g. It follows then that $((gh)h^{-1})g^{-1}$ is also defined and equals gg^{-1}. On the other hand $((gh)^{-1}, gh)$ is in G^2 and it follows that $((gh)^{-1}, ((gh)h^{-1})g^{-1})$ is also and hence $((gh)^{-1}, gg^{-1})$ is in G^2. This in turn means $((gh)^{-1}, g)$ is in G^2 and $(gh)^{-1}gg^{-1} = (gh)^{-1}$ is in G^2. Hence, we have

$$(gh)^{-1} = (gh)^{-1}gg^{-1} = (gh)^{-1}((gh)h^{-1})g^{-1} = h^{-1}g^{-1}.$$

For the fourth part, we make use of the third part and compute

$$r(gh) = gh(gh)^{-1} = (gh)(h^{-1}g^{-1}) = (g(hh^{-1}))g^{-1} = gg^{-1} = r(g).$$

The other part is done similarly.

The fifth part follows at once from the third and the definition of G^0.

For the last statement, if g is in G^0, then $g = h^{-1}h$ and using part 4, we have

$$s(g) = s(h^{-1}h) = s(h) = h^{-1}h = g.$$

The other part is done similarly. □

The result above gives us a nice picture to have in mind of a groupoid. Imagine that we represent the elements of G^0 as vertices and each element g in G as an edge, starting at $s(g)$ and terminating at $r(g)$. The product is then a rule which assigns to each pair g, h with $s(g) = r(h)$, a new edge gh going from $s(h)$ to $r(g)$. There is a small problem with this: is an element of G^0 a vertex or an edge or both? I find it convenient to let $G \setminus G^0$ be the edges. The success of this picture varies, case by case.

It is a good time to look at some examples.

Example 3.1.4. *Any group G is a groupoid. Use $G^2 = G \times G$. Here $G^0 = \{e\}$.*

Following up on this, it can be shown that a groupoid is a group if and only if G^0 contains a single element (Exercise **??**). The picture we had above for this groupoid is particularly unhelpful. We have a single vertex and a set of edges, which are all loops. The only thing you can tell about the group from the picture is the number of elements! Of course, the problem is that the picture doesn't really show the product very clearly.

Example 3.1.5. *Let X be any set and R be an equivalence relation on X. Define*

$$R^2 = \{((x, y), (y, z)) \mid (x, y), (y, z) \in R\}.$$

Then define the product as $(x, y)(y, z) = (x, z)$ and the inverse as $(x, y)^{-1} = (y, x)$, for all $(x, y), (y, z)$ in R. In this way, R is a groupoid with

$$\begin{aligned} r(x, y) &= (x, y)(x, y)^{-1} = (x, y)(y, x) = (x, x), \\ s(x, y) &= (y, y), \end{aligned}$$

for all (x, y) in R and

$$R^0 = \{(x, x) \mid x \in X\}.$$

We denote the set on the right by Δ_X. In particular, for any set X, $X \times X$ (called the trivial equivalence relation) and Δ_X (called the co-trivial equivalence relation) are groupoids.

In contrast to the situation for groups, our picture of an equivalence relation is quite accurate and helpful.

Proposition 3.1.6. *Let G be a groupoid. The set $R = \{(r(g), s(g)) \in G^0 \times G^0 \mid g \in G\}$ is an equivalence relation on G^0. Moreover, if the map $\alpha(g) = (r(g), s(g))$ from G to R is injective then it is an isomorphism between the groupoids G and R. In this case, we say that G is a* principal *groupoid.*

Let us return to a couple more general examples. The next is quite an important one.

Example 3.1.7. *Let G be a group acting (on the right) on a set X. That is, for each x in X and g in G, we have xg which is in X. These are such that, for fixed g in G, the map $x \to xg$ is a bijection and $x(gh) = (xg)h$, for all g, h in G and x in X.*

Consider the set $X \times G$ and define

$$(X \times G)^2 = \{((x, g), (y, h)) \mid xg = y\}.$$

Then the product and inverse are given by $(x, g)(xg, h) = (x, gh)$ and $(x, g)^{-1} = (xg, g^{-1})$, respectively, for x in X, g, h in G. We note that

$$r(x, g) = (x, g)(x, g)^{-1} = (x, g)(xg, g^{-1}) = (x, e), s(x, g) = (xg, e),$$

for x in X, g in G, and $(X \times G)^0 = X \times \{e\}$.

Example 3.1.8. *Let X be a topological space. Consider the set of all paths in X; i.e. continuous functions $\gamma : [0, 1] \to X$. Two paths γ, γ' are homotopic if there is a continuous function $F : [0, 1] \times [0, 1] \to X$ such that $F(s, 0) = \gamma(s), F(s, 1) = \gamma'(s)$, for all $0 \leq s \leq 1$ and $F(0, t) = \gamma(0) = \gamma'(0), F(1, t) = \gamma(1) = \gamma'(1)$, for all $0 \leq t \leq 1$. That is, one path may be continuously deformed to the other, while holding the endpoints fixed. Homotopy is an equivalence relation. Let $[\gamma]$ denote the homotopy class of γ.*

Let $\pi(X)$ denote the set of all homotopy classes of paths in X. We define

$$\pi(X)^2 = \{([\gamma], [\gamma']) \mid \gamma(1) = \gamma'(0)\}.$$

The product is given by $[\gamma][\gamma'] = [\gamma\gamma']$, where

$$(\gamma\gamma')(s) = \begin{cases} \gamma(2s) & 0 \leq s \leq 1/2, \\ \gamma'(2s - 1) & 1/2 \leq s \leq 1, \end{cases}$$

where γ, γ' are paths. The inverse is given by $\gamma^{-1}(s) = \gamma(1 - s), 0 \leq s \leq 1$. This groupoid is called the fundamental groupoid *of X.*

Now we present a couple of simple ways of constructing new groupoids from old ones. The first is fairly familiar from group theory.

Example 3.1.9. *If G and H are groupoids, then so is their product $G \times H$ in an obvious way.*

The next example is very unfamiliar from group theory.

Example 3.1.10. *If $G_\iota, \iota \in I$, are groupoids, then so is their disjoint union $G = \sqcup_{\iota \in I} G_\iota$ as follows. We define $G^2 = \sqcup_{\iota \in I} G_\iota^2$ and the product is then obvious.*

In fact, that's it!

Theorem 3.1.11. *Every groupoid is isomorphic to the disjoint union of a collection of products of groups and equivalence relations. More precisely, if G is a groupoid, then*

$$G \cong \sqcup_{\iota \in I} G_\iota \times (X_\iota \times X_\iota),$$

where G_ι is a group and X_ι is a set, for all ι.

Proof. Let R be the equivalence relation on G^0 from 3.1.6. Choose I to be a set which indexes the equivalence classes of R. That is, for each ι in I, $X_\iota \subset G^0$ is an equivalence class of R, these are pairwise disjoint for different values of ι and the union over all ι in I is G^0. For each value of ι, choose x_ι in X_ι. Also, for each x in X_ι, we know that (x, x_ι) is in R, so we may choose g_x in G such that $r(g_x) = x$ and $s(g_x) = x_\iota$.

For each ι in I, define

$$G_\iota = \{g \in G \mid r(g) = s(g) = x_\iota\}.$$

It is easy to see that G_ι is a group with identity element x_ι.

We define a map $\alpha : \sqcup_{\iota \in I} G_\iota \times (X_\iota \times X_\iota) \to G$ by

$$\alpha(g, (x, y)) = g_x g g_y^{-1},$$

for all g in G_ι, x, y in X_ι and ι in I. First, let us see that α is well-defined. We know that $s(g_x) = x_\iota = r(g) = s(g) = s(g_y) = r(g_y^{-1})$, and so the product is well-defined.

We also define $\beta : G \to \sqcup_{\iota \in I} G_\iota \times (X_\iota \times X_\iota)$ by

$$\beta(g) = (g_{r(g)}^{-1} g g_{s(g)}, (r(g), s(g))), g \in G.$$

It is a simple exercise to see that β is well-defined and is the inverse of α. $\square$

Of course, this result looks like the end of the road for groupoids. In fact, we will revive them in the next section by studying *topological* groupoids.

Exercise 3.1.1. *Suppose that G is a groupoid with a single unit. Prove that G is a group.*

Exercise 3.1.2. *Prove that the groupoid $X \times G$ of Example 3.1.7 is principal if and only if the action is free (that is, the only x, g for which $xg = x$ is $g = e$).*

Exercise 3.1.3. *Suppose that g is an element of a groupoid G. Prove that the right identity of g is unique. That is, if h is any element of G with (g, h) in G^2 and $gh = g$, then $h = s(g)$.*

Exercise 3.1.4. *In example 3.1.8, let γ be any path in X. Find the simplest possible paths which represent $r[\gamma]$ and $s[\gamma]$. Give a complete description of $\pi(X)^0$.*

Exercise 3.1.5. *Find the spaces X_ι and the groups G_ι in Theorem 3.1.11 for the fundamental groupoid of a space X.*

3.2 Topological groupoids

To get some interest back into our groupoids after Theorem 3.1.11, we introduce topology into the picture.

Definition 3.2.1. *A* topological groupoid *is a groupoid G with a topology such that, when G^2 is given the relative topology of $G \times G$, it is closed and the product (as a map from G^2 to G) and the inverse (as a map from G to itself) are both continuous.*

Let us just note an obvious fact.

Lemma 3.2.2. *If G is a topological groupoid, then the maps r and s are continuous. If G is Hausdorff, then G^0 is closed.*

Proof. The map $g \in G \to g^{-1} \in G$ is continuous, hence so is $g \in G \to (g, g^{-1}) \in G^2 \subset G \times G$. The map r is simply the composition of this map with the product and hence is continuous. The situation for s is similar. For the last statement, if g_α is a net in G^0 converging to g, then we know from part 6 of Lemma 3.1.3 that $g_\alpha = s(g_\alpha)$. Taking limits and using the continuity of s we see that $s(g) = g$ and hence g is in G^0. $\square$

Let us give a couple of examples.

Example 3.2.3. *Let X be any topological space. The groupoid $X \times X$ with the product topology is a topological groupoid. Also, the groupoid Δ_X with the relative topology from $X \times X$ (which makes the map sending x in X to (x, x) in Δ_X a homeomorphism) is a topological groupoid.*

Example 3.2.4. *Let X be a compact space and G be a topological group (or a discrete group, if you prefer). Assume that G acts on X continuously, meaning that the map from $X \times G$ to X of Example 3.1.7 is continuous. With the product topology and the groupoid structure of 3.1.7, $X \times G$ is a topological groupoid. We will actually prove this below in Theorem 3.2.11.*

Our next task is to define special classes of topological groupoids called r-discrete groupoids and étale groupoids. The idea in both cases is essentially the same (although the latter is a strictly stronger condition) and we discuss it a little.

When considering the C^*-algebras $C_0(X)$, we would not have got very far if we only considered spaces with the discrete topology. On the other hand, our treatment of group C^*-algebras of Chapter 2 did not suffer a great deal by assuming the group is discrete. What we try to impose with these two new conditions is that, while the groupoid itself may be continuous, the 'group-like' part of it is discrete. This idea comes out nicely in Theorem 3.2.11 where the groupoids $X \times G$ associated with an action of the group G on the space X satisfy both these hypotheses when the group G is discrete.

Definition 3.2.5. *A topological groupoid is r-*discrete *if G^0 is open in G.*

The following result will be useful and helps to explain the terminology.

Lemma 3.2.6. *Let G be an r-discrete groupoid. For every g in G^0, $r^{-1}\{g\}$ and $s^{-1}\{g\}$ are discrete.*

Proof. Suppose that we have a net h_α converging to h, all in $r^{-1}\{g\}$. Then $r(h_\alpha) = r(h) = g$, for all α. We know that $s(h^{-1}) = r(h)$ and so (h^{-1}, h_α) is defined for all α. Moreover, by the continuity of the product, the net $h^{-1}h_\alpha$ converges to $h^{-1}h$, which is in G^0. Since G^0 is open, $h^{-1}h_\alpha$ is in G^0 for all sufficiently large α. Then we know that $h^{-1}h_\alpha = s(h^{-1}h_\alpha) = s(h_\alpha) = h_\alpha^{-1}h_\alpha$. Right multiplying by h_α^{-1} and taking inverse then implies $h_\alpha = h$, for all such α. $\square$

Definition 3.2.7. *A map $f : X \to Y$ between two topological spaces is a* local homeomorphism *if, for ever x in X, there is an open set U containing x such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism.*

Notice the condition that f be open is important. Otherwise the inclusion of a point in any space would satisfy the condition.

Definition 3.2.8. *A topological groupoid is* étale *if the maps r, s are both local homeomorphisms. Any set $U \subset G$ which satisfies the conditions of Definition 3.2.7 for the maps $r, s : G \to G$ is called a G-*set*.*

We note the following; its proof is easy.

Theorem 3.2.9. *Let G be an étale groupoid. Any open subset of a G-set is also a G-set. If Γ is the collection of G-sets, then Γ satisfies the following.*

1. *If γ is in Γ, then so is γ^{-1}.*
2. *If γ_1, γ_2 are in Γ, then so is $\gamma_1\gamma_2$.*
3. *If γ_1, γ_2 are in Γ, then so is $\gamma_1 \cap \gamma_2$.*

Moroever, Γ is a neighbourhood base for the topology on G and $\Gamma_0 = \{\gamma \cap G^0 \mid \gamma \in \Gamma\}$ is a neighbourhood base for the topology of G^0.

While the definitions of r-discrete and étale are rather different looking, let us show they are at least related.

Theorem 3.2.10. *Every étale groupoid is r-discrete.*

Proof. If g is in G, let U be any G-set containing g. Then $r(U)$ is an open set containing $r(g)$ and is contained itself in G^0. In this way, we see that G^0 can be covered by open sets and hence is open. □

Roughly speaking, the condition of r-discrete means that r is locally injective at any point h. The condition that G is étale means it is also onto an open set containing $r(h)$.

Theorem 3.2.11. *Let G be a topological group and X a topological space. Assume that G acts on X (on the right) and that the map $(x, g) \in X \times G \to xg \in X$ is continuous. Then $X \times G$ as a groupoid (3.1.7) with the product topology is a topological groupoid. If G is discrete, then $X \times G$ is both r-discrete and étale.*

Proof. We first show that $(X \times G)^2$ is closed. If $((x_\alpha, g_\alpha), (x_\alpha g_\alpha, h_\alpha))$ is a convergent net, then clearly x_α converges to some x in X, g_α converges to some g in G and h_α converges to some h in G. It follows from the continuity of the action that $x_\alpha g_\alpha$ converges to xg. Thus, our net converges to $((x, g), (xg, h))$, which is in $X \times G$ and the limit of the products is the product of the limits.

From the fact that the inverse map on G is continuous and the hypothesis on the continuity of the action, we see that $(x, g) \to (xg, g^{-1})$ is also continuous.

Now we assume that G is discrete. We compute

$$r(x, g) = (x, g)(x, g)^{-1} = (x, g)(xg, g^{-1}) = (x, e).$$

It follows that, for any g in G, the set $\{g\}$ is open in G and the restriction of r to the open set $X \times \{g\}$ is a homeomorphism to $X \times \{e\} = (X \times G)^0$. The case for the map s is similar. It follows that $X \times G$ is étale and hence r-discrete as well. □

We note the following useful but easy result.

Theorem 3.2.12. *Let G be a topological groupoid and let $H \subset G$ be a subgroupoid. Then H is also a topological groupoid (with the relative topology). If G is r-discrete, then H is also. If H is open in G and G is étale, then H is also étale.*

Exercise 3.2.1. *Let $R = \{(x, x), (-1, 1), (1, -1) \mid -1 \leq x \leq 1\}$ which is considered as an equivalence relation of $[-1, 1]$ and with the relative topology of $[-1, 1]^2$. First, show that R is a topological groupoid. Is it r-discrete? Is it étale?*

Exercise 3.2.2. *Let $R = \{(x, x), (x, -x) \mid -1 \leq x \leq 1\}$ which is considered as an equivalence relation of $[-1, 1]$ and with the relative topology of $[-1, 1]^2$. Is R r-discrete? Is it étale?*

Exercise 3.2.3. *Let $G = \{1, -1\}$ be the cyclic group of order two and consider its action on $X = [-1, 1]$ defined by $x \cdot i = ix$, for $x \in [-1, 1]$ and $i = \pm 1$. Then $X \times G$ is étale by Theorem 3.2.11. Does this fact and Proposition 3.1.6 imply that the equivalence relation of Exercise 3.2.2 is étale?*

Exercise 3.2.4. *Consider the smallest topology on R of Exercise 3.2.2 which includes the relative topology of $[-1, 1]^2$ and the set $\Delta_{[-1,1]}$. With this topology, is R r-discrete? Is it étale? Give two explanations of the last part: one direct and one using the action of Exercise 3.2.3 and Theorem 3.2.12.*

Exercise 3.2.5. *Let $X = [0,1] \cup [2,3]$. Let A be a subset of $[0,1]$ and define*

$$R = \Delta_X \cup \{(x, x+2), (x+2, x) \mid x \in A\},$$

with the relative topology of $X \times X$.

1. *Prove that R is r-discrete.*

2. *Give a necessary and sufficient condition on the set A for R to be étale.*

3.3 The C^*-algebra of an étale groupoid

This section is devoted to the construction of the C^*-algebra of an étale groupoid. For convenience, and so that it parallels with the construction of the C^*-algebra of a discrete group, we have divided this into a sequence of subsections.

3.3.1 The fundamental lemma

Lemma 3.3.1. *Let G be an étale groupoid. Let p denote the product map from G^2 to G. If U and V are G-sets, then $p(U \times V \cap G^2)$ is open in G^2 and the restriction of p to $U \times V \cap G^2$ is a homeomorphism to its image. Moreover, $p(U \times V \cap G^2)$ is a G-set.*

Proof. Define $W = s(U) \cap r(V)$, which is an open subset of G^0, and let $U' = (s|_U)^{-1}(W) \subset U$ and $V' = (r|_V)^{-1}(W) \subset V$. It follows from the fact that U and V are G-sets, that U' and V' are also. We have $s(U') = r(V') = W$. In addition, it is an easy consequence of part 2 of Lemma 3.1.3 that $U \times V \cap G^2 = U' \times V' \cap G^2$. Hence, by replacing U, V by U', V' we may assume that our G-sets also satisfy $s(U) = r(V)$.

By definition, $U \times V \cap G^2$ is an open subset of G^2. The map $(r|_V)^{-1} \circ s|_U : U \to V$ is a homeomorphism, since it is the composition of two homeomorphisms. Moreover, the map f from U to $U \times V$ defined by $f(g') = (g', (r|_V)^{-1} \circ s|_U(g'))$ has range in G^2 since $r((r|_V)^{-1} \circ s|_U(g'))) = s(g')$. We claim that its range is exactly $U \times V \cap G^2$. We have already shown the range is contained in this set. For the reverse inclusion, suppose (g, h) is in $U \times V \cap G^2$. Then g is in U, h is in V and $s(g) = r(h)$ since (g, h) is in G^2. It follows that $h = (r|_V^{-1}) \circ (s|_U)(g)$ and we are done. It is clear that f is injective. It follows at once from the facts that $r|_V$ and $s|_U$ are homeomorphisms that the map f is continuous. Consider π_1, the natural projection onto U. It is clear that $\pi_1 \circ f$ is the identity on U. From this and the fact that f is onto, we deduce that π_1 is injective as well. Both are clearly continuous, hence they are both homeomorphisms.

Let p denote the product map restricted to the domain $U \times V \cap G^2$ and let its range be W. We first claim that $r(W)$ is open. We showed in Lemma 3.1.3 that $r(gh) = r(g)$, for any (g, h) in G^2. This fact can be re-written here as $r|_W \circ p = r|_U \circ \pi_1$ as functions on $U \times V \cap G^2$. The fact that p is onto is simply the definition of W. We know that the range of the right hand side is $r(U)$. Hence $r(W) = r(U)$ is open.

Next we claim that $r|_W$ and p are homeomorphisms. The right hand side of $r|_W \circ p = r|_U \circ \pi_1$ is a bijection and p is onto, hence p and $r|_W$ are injective. Both p and $r|_W$ are continuous and bijective. The fact that their composition is a homeomorphism implies that both are homeomorphisms.

Now we claim that $r|_W$ is a homeomorphism to its image. But this follows from the equation $r|_W \circ p = r|_U \circ \pi_1$ and the fact that we have established that the other three maps are homeomorphisms.

The proof that $s(W)$ is open and that $s|_W$ is a homeomorphism is done in a similar way. This completes the proof.

$\square$

3.3.2 The $*$-algebra $C_c(G)$

Definition 3.3.2. *Let G be an étale groupoid which is locally compact and Hausdorff. We denote by $C_c(G)$ the set of compactly supported continuous complex-valued functions on G. We also denote by $\mathcal{C}(G)$ those functions in $C_c(G)$ which are supported in some G-set.*

Lemma 3.3.3. *Let G be an étale groupoid which is locally compact and Hausdorff. Every element of $C_c(G)$ is a sum of functions in $\mathcal{C}(G)$.*

Proof. Let a be in $C_c(G)$. Using the fact that the support of a is compact and that the G-sets form a neighbourhood base for the topology, we may find a finite cover, $\mathcal{U}$, of the support of a by G-sets. We may find a partition of unity α of G which is subordinate to the cover $\mathcal{U} \cup \{G \setminus supp(a)\}$. Then we have

$$a = \sum_{U \in \mathcal{U}} \alpha_U a.$$

$\square$

Theorem 3.3.4. *Let G be an étale groupoid which is locally compact and Hausdorff. Then $C_c(G)$, with the obvious linear structure, and multiplication and involution given by*

$$a \cdot b(g) = \sum_{r(h)=r(g)} a(h)b(h^{-1}g),$$

for a, b in $C_c(G)$ and g in G,

$$a^*(g) = \overline{a(g^{-1})},$$

for a in $C_c(G)$ and g in G, is a $*$-algebra.

Proof. Our first observation is that, since the inverse map is a homeomorphism of G, the function a^* is clearly again in $C_c(G)$. Our next task is to see that the product is well-defined. First, as a has compact support and we know from Lemma 3.2.6 and Theorem 3.2.10 that $r^{-1}\{r(g)\}$ is discrete, the term $a(h)$ is non-zero for only finitely many values of h. Hence, the formula for the product yields a well-defined function on G. Moreover, this multiplication clearly distributes over addition. Hence, in view of Lemma 3.3.3, to see that the product ab is in $C_c(G)$, it suffices for us to check that this holds for a, b in $\mathcal{C}(G)$. Suppose that a is supported in the G-set U, while b is supported in the G-set V. Consider the function c sending (h_1, h_2) in G^2 to $a(h_1)b(h_2)$. This is supported in $U \times V \cap G^2$. Moreover, if ab is non-zero on g, then so is c on some point of $p^{-1}\{g\}$ (p is the product map). It follows that ab is supported on $p(U \times V \cap G^2)$, which is a G-set and that $ab \circ p|_{U \times V \cap G^2} = c$. It follows from Lemma 3.3.1 that ab is in $\mathcal{C}(G)$.

Of course, there are a number of issues still to be addressed: associativity, $(ab)^* = b^*a^*$, etc. We leave these an as exercise for the reader. □

We take a moment to observe that, in the case our groupoid is an equivalence relation, the product and adjoint have a particularly simple form.

Theorem 3.3.5. *If R is an equivalence relation on X and is equipped with an étale topology, then for a, b in $C_c(R)$, the product is given by*

$$a \cdot b(x, y) = \sum_{z \in [x]_R} a(x, z)b(z, y),$$

and the adjoint by

$$a^*(x, y) = \overline{a(y, x)},$$

for all (x, y) in R.

Proof. Looking at the formula in 3.3.4 for the product, we see that for (x, y) in R,

$$a \cdot b(x, y) = \sum_{r(x,y)=r(w,z)} a(w, z)b((w, z)^{-1}(x, y)).$$

As $r(x, y) = (x, x)$ and $r(w, z) = (w, w)$, the condition $r(x, y) = r(w, z)$ is simply that $w = x$. Then the condition that $w, z)$ is in R simply means z is in $[x]_R$. Finally, in this case, $(w, z)^{-1}(x, y) = (z, w)(x, y) = (z, y)$ and we are done.

The formula for the adjoint is immediate from the definitions. □

Example 3.3.6. *Let X be a compact Hausdorff space and consider Δ_X as a groupoid. Here, of course, $r^{-1}\{(x, x)\} = (x, x)$, for every unit (x, x). It follows that the product on $C_c(\Delta_X) = C(\Delta_X)$ is simply pointwise product of functions. In this case, $C_c(\Delta_X)$ is commutative and isomorphic to $C(X)$.*

Example 3.3.7. *We consider the equivalence relation $X \times X$ on the set X. In order for this to be r-discrete, X must have the discrete topology and for X to be compact it must also be finite. So we assume X is a finite set with the discrete topology and consider $X \times X$. The formula for the product becomes*

$$ab(x, y) = \sum_{z \in X} a(x, z)b(z, y),$$

for all (x, y) in $X \times X$. Moreover, we also have

$$a^*(x, y) = \overline{a(y, x)},$$

for all (x, y) in $X \times X$. These formulae should look familiar; $C_c(X \times X)$ is isomorphic to $M_N(\mathbb{C})$, where N is the cardinality of X.

Example 3.3.8. *If G is a discrete group, then $C_c(G) = \mathbb{C}G$, and the structure as a $*$-algebra is the same as earlier.*

Our next observation is fairly easy, but it is extremely helpful in understanding the construction. In example 3.3.7 above, we saw that the total equivalence on a space with N points yields the $*$-algebra of $N \times N$ matrices. But more is true; the algebra of continuous functions on this space (which is just $\mathbb{C}^N$) occurs in a natural way as the subalgebra of diagonal matrices. In fact, this is a general phenomenon for $*$-algebras of étale groupoids.

Theorem 3.3.9. *Let G be an étale groupoid which is locally compact and Hausdorff. Assume that the unit space G^0 is compact. Then the map $\Delta : C(G^0) \to C_c(G)$ defined by*

$$\Delta(f)(g) = \begin{cases} f(g) & g \in G^0 \\ 0 & g \notin G^0 \end{cases}$$

is a unital, injective $*$-homomorphism.

Proof. The only non-trivial aspect of the proof is the observation that, since G^0 is both open and compact, the function $\Delta(f)$ is continuous and compactly supported. The remaining details are rather straightforward and we omit them. $\square$

In particular, it is worth noting that $C_c(G)$ is unital, when G^0 is compact.

Also, the result in the more general case when the unit space is only locally compact also holds, but the domain of Δ is $C_c(G^0)$. Of course, it would be more satisfying with what we have seen to this point to have $C_0(G^0)$ instead.

Before passing on to more analytic matters, we want to consider the special case of particular interest: group actions.

Theorem 3.3.10. *Let G be a discrete group acting continuously on the compact Hausdorff space X. We regard $X \times G$ as a groupoid 3.1.7 and consider the $*$-algebra $C_c(X \times G)$.*

1. *For each g in G, let u_g denote the characteristic function of $X \times \{g\}$. Then u_g is a unitary in $C_c(G)$. Moreover, $u_g u_h = u_{gh}$, for all g, h in G.*

2. *For any f in $C(X)$ and g in G, we have*

$$u_g \Delta(f) u_g^* = \Delta(f^g),$$

where $f^g(x) = f(xg)$, for all x in X. In particular,

$$u_g \Delta(C(G^0)) u_g^* = \Delta(C(G^0)).$$

3. *For any f in $C_c(X \times G)$, we may write*

$$f = \sum_{g \in G} \Delta(f_g) u_g,$$

where f_g is in $C(X)$, for each g in G and only finitely many are non-zero.

Proof. We begin by making the simple observation that in the product topology $X \times \{g\}$ is compact and open and so u_g is indeed in $C_c(X \times G)$, for any g in G. Next, we compute

$$u_g^*(x,h) = \overline{u_g((x,h)^{-1})} = \overline{u_g(xh,h^{-1})} = u_g(xh,h^{-1}).$$

This has value one when $h^{-1} = g$ and zero otherwise. In other words, it equals $u_{g^{-1}}$.

Next, we compute $u_g u_h$ as

$$u_g u_h(x,k) = \sum_{l \in G} u_g(x,l) u_h((x,l)^{-1}(x,k)) = \sum_{l \in G} u_g(x,l) u_h(xl, l^{-1}k).$$

The first term in the sum is zero except for $l = g$, when it is one, so the sum reduces to a single term $u_h(xg, g^{-1}k)$. This is zero unless $g^{-1}k = h$ or $k = gh$, in which case it is one. We conclude that $u_g u_h = u_{gh}$.

That u_g is unitary follows from the first two parts above.

For the second part, we will compute $u_g \Delta(f)$ and $\Delta(f^g) u_g$ and see that they are equal. The conclusion then follows from this and the first statement. We have

$$\begin{aligned} (u_g \Delta(f))(x,h) &= \sum_{r(x,k)=r(x,h)} u_g(x,k) \Delta(f)((x,k)^{-1}(x,h)) \\ &= \sum_{xk=xh} u_g(x,k) \Delta(f)((xk, k^{-1}h)). \end{aligned}$$

From the definition of $\Delta(f)$, the sum reduces to a single term, when $k = h$. Then from the definition of the first term, we get zero unless $k = h = g$, in which case, the value is $f(xg)$. On the other hand, we have

$$\begin{aligned} (\Delta(f^g) u_g)(x,h) &= \sum_{r(k)=r(h)} \Delta(f^g)(x,k) u_g((x,k)^{-1}(x,h)) \\ &= \sum_{r(k)=r(h)} \Delta(f^g)(x,k) u_g((xk, k^{-1}h) \\ &= \Delta(f^g)(x,e) u_g((xe, e^{-1}h) \\ &= f^g(x) u_g(x,h) \\ &= f(xg) u_g(x,h). \end{aligned}$$

Again the result is zero unless $h = g$ and in this case it is $f(xg)$.

For the last part, for each g in G, let $f_g(x) = f(x, g)$. It follows from the compactness of the support of f that this is non-zero for only finitely many g. Moreover, $\Delta(f_g)u_g$ is the function that at (x, h) takes the value $f(x, g)$ when $h = g$ and is zero otherwise. In fact, this was proved in the second calculation of the second statement above. The conclusion follows at once. This completes the proof. $\square$

Continuing our analogy of $C_c(G)$ with the $N \times N$ matrices and $C(G^0)$ being the diagonal matrices, we see here that the elements $u_g, g \in G$ should be regarded as permutation matrices.

3.3.3 The left regular representation

We now want to define an analogue of the left regular representation of a group for an étale groupoid G. As we are going to avoid the notion of a representation of a groupoid itself, this will be at the level of the algebra $C_c(G)$. In fact, we will define one representation for each element of the unit space. Our left regular representation will be the direct sum of all of these.

Theorem 3.3.11. *Let u be a unit in the locally compact, Hausdorff étale groupoid G. For each a in $C_c(G)$ and ξ in $\ell^2(s^{-1}\{u\})$, the equation*

$$(\pi^u_\lambda(a)\xi)(g) = \sum_{r(h)=r(g)} a(h)\xi(h^{-1}g),$$

for any g in $s^{-1}\{u\}$, defines an element of $\ell^2(s^{-1}\{u\})$. Moreover, $\pi^u_\lambda(a)$ is a bounded linear operator on $\ell^2(s^{-1}\{u\})$ whose norm is bounded by a constant depending on a, but not on u. The function $\pi^u_\lambda : C_c(G) \to \mathcal{B}(\ell^2(s^{-1}\{u\})$ is a representation of $C_c(G)$.

Proof. First of all, observe that if $r(g) = r(h)$ then $h^{-1}g$ is defined and $s(h^{-1}g) = s(g) = u$, so the terms in the sum are all defined. Secondly, as a is compactly supported and $r^{-1}\{r(g)\}$ is discrete, the sum in the formula has only finitely many non-zero terms. Hence the function $(\pi^u_\lambda(a)\xi)$ is well-defined for any g in G, in particular, on $s^{-1}\{u\}$. It is also clearly linear in a.

Now let us assume that a is in $\mathcal{C}(G)$ and is supported in the G-set U. In this case, for a fixed g, $r^{-1}\{r(g)\} \cap supp(a)$ is at most a single point, $h_g = (r|_U)^{-1}(r(g))$. Therefore we see that $(\pi^u_\lambda(a)\xi)(g)$ is either zero or

$a(h_g)\xi(h_g^{-1}g)$. Considered as a function of g, it is simply a permutation of the original function ξ times another sequence which is bounded above by $\|a\|_\infty$. Indeed, the ℓ^2-norm of this vector is bounded above by the ℓ^2-norm of ξ times $\|a\|_\infty$. Hence we see the result is in $\ell^2(s^{-1}\{u\})$. As any function in $C_c(G)$ is a sum of elements of $\mathcal{C}(G)$, we see that $\pi_\lambda^u(a)$ indeed maps $\ell^2(s^{-1}\{u\})$ to itself and is a bounded linear operator.

We now verify that π_λ^u is a representation. For any a, b in $C_c(G)$, we show that $\pi_\lambda^u(a)\pi_\lambda^u(b) = \pi_\lambda^u(ab)$. To verify this, it suffices to let each side act on a vector in $\ell^2(s^{-1}\{u\})$. It suffices to check $\xi = \delta_{g_0}$, for some fixed g_0 in $s^{-1}\{u\}$, since these form an orthonormal basis. Here, we again use the notation δ_{g_0} to mean the function that is one at g_0 and zero elsewhere. In this case, we have

$$(\pi_\lambda^u(a)\delta_{g_0})(g) = \sum_{r(h)=r(g)} a(h)\delta_{g_0}(h^{-1}g) = a(gg_0^{-1}).$$

From this we see that

$$\pi_\lambda^u(a)\delta_{g_0} = \sum_{s(g)=u} a(gg_0^{-1})\delta_g.$$

The map taking g with $s(g) = u$ to gg_0^{-1} is well-defined and $s(gg_0^{-1}) = r(g_0)$. The map taking h with $s(h) = r(g_0)$ to hg_0 is also well-defined and $s(hg_0) = s(g_0) = u$. It is easily seen that these maps are inverses, so we may also write

$$\pi_\lambda^u(a)\delta_{g_0} = \sum_{s(g)=r(g_0)} a(g)\delta_{gg_0}.$$

We first compute

$$\begin{aligned}
\pi_\lambda^u(a)\pi_\lambda^u(b)\delta_{g_0} &= \pi_\lambda^u(a)\left[\sum_{s(g)=u} b(gg_0^{-1})\delta_g\right] \\
&= \sum_{s(h)=u}\sum_{s(g)=u} a(hg^{-1})b(gg_0^{-1})\delta_h.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
\pi_\lambda^u(ab)\delta_{g_0} &= \sum_{s(h)=u} (ab)(hg_0^{-1})\delta_h \\
&= \sum_{s(h)=u}\left[\sum_{r(k)=r(hg_0^{-1})} a(k)b(k^{-1}hg_0^{-1})\right]\delta_h.
\end{aligned}$$

We observe that $r(hg_0^{-1}) = r(h)$. Considering h fixed for the moment, the map that sends g in $s^{-1}\{u\}$ to hg^{-1} has range in $r^{-1}\{r(h)\}$. The map that sends k in $r^{-1}\{r(h)\}$ to $k^{-1}h$ has range in $s^{-1}\{u\}$. Moreover, the composition of these two maps in either order is the identity. It follows that each is a bijection. We conclude that

$$\sum_{s(g)=u} a(hg^{-1})b(gg_0^{-1}) = \sum_{r(k)=r(hg_0^{-1})} a(k)b(k^{-1}hg_0^{-1}).$$

This completes the proof that $\pi_\lambda^u(a)\pi_\lambda^u(b) = \pi_\lambda^u(ab)$.

Finally, let us check that $\pi_\lambda^u(a^*) = \pi_\lambda^u(a)^*$. Let g_0, g_1 be in $s^{-1}\{u\}$. We compute

$$\begin{aligned}
< \pi_\lambda^u(a^*)\delta_{g_0}, \delta_{g_1} > &= < \sum_{s(g)=u} a^*(gg_0^{-1})\delta_g, \delta_{g_1} > \\
&= \sum_{s(g)=u} \overline{a(g_0 g^{-1})} < \delta_g, \delta_{g_1} > \\
&= \overline{a(g_0 g_1^{-1})} \\
&= \sum_{s(g)=u} < \delta_{g_0}, a(gg_1^{-1})\delta_g > \\
&= < \delta_{g_0}, \pi_\lambda^u(a)\delta_{g_1} > .
\end{aligned}$$

□

Theorem 3.3.12. *Let G be a locally compact Hausdorff étale groupoid. If g is any element of G, the representations $\pi_\lambda^{r(g)}$ and $\pi_\lambda^{s(g)}$ are unitarily equivalent.*

Proof. The map sending h to hg is a bijection from $s^{-1}\{r(g)\}$ to in $s^{-1}\{s(g)\}$ and therefore implements a unitary operator between their respective ℓ^2 spaces. We leave it as an exercise to check that this unitary intertwines the two representations as claimed. □

Definition 3.3.13. *Let G be a locally compact Hausdorff étale groupoid. The left regular representation of G is the direct sum $\pi_\lambda = \oplus_{u \in G^0} \pi_\lambda^u$.*

It is not hard to see that the fact that G may be partitioned into pairwise disjoint sets as

$$G = \cup_{u \in G^0} s^{-1}\{u\}$$

means that

$$\ell^2(G) = \oplus_{u \in G^0} \ell^2(s^{-1}\{u\}).$$

In fact, the same formula that defines π_λ^u for each u in G^0 may also be used to define π_λ^u on $\ell^2(G)$. So in fact, we could also have defined the left regular representation in this way on $\ell^2(G)$ and then observed that each subspace $\ell^2(s^{-1}\{u\})$ is invariant.

Notice in the last proof that we have shown

$$\pi_\lambda^u(a)\delta_{g_0} = \sum_{s(g)=r(g_0)} a(g)\delta_{gg_0},$$

for any a in $C_c(G)$ and $s(g_0) = u$. Let h be any element of G and apply this in the case $u = s(h)$ and $g_0 = u$, and then take the inner products with δ_h and we obtain the following.

Theorem 3.3.14. *Let G be a locally compact Hausdorff étale groupoid. For any a in $C_c(G)$ and h in G, we have*

$$a(h) = < \pi_\lambda^{s(h)}(a)\delta_{s(h)}, \delta_h > .$$

The following is a trivial consequence of the formula above.

Corollary 3.3.15. *Let G be a locally compact Hausdorff étale groupoid. The left regular representation of $C_c(G)$ is faithful.*

3.3.4 $C^*(G)$ and $C_r^*(G)$

We have almost everything to define the reduced C^*-algebra of an étale groupoid, namely the left regular representation and the knowledge that it is faithful. It also means we have almost everything we need to define the full C^*-algebra. The only missing ingredient is the knowledge that taking a single element of $C_c(G)$ and the supremum of its norms in all possible representations is finite. We start with the elements of $\mathcal{C}(G)$.

Lemma 3.3.16. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space and let a be any function in $\mathcal{C}(G)$.*

1. *a^*a and aa^* both lie in $\Delta(C(G)^0)$.*
2. *$\|a\|_\infty^2 = \|aa^*\|_\infty = \|a^*a\|_\infty$. (N.B. The product being used is* not *the pointwise product, but the product from $C_c(G)$.)*

3. *If π is any representation of the $*$-algebra $C_c(G)$ on the Hilbert space $\mathcal{H}$, then $\|\pi(a)\| \leq \|a\|_\infty$.*

Proof. Let U be a G-set which contains the support of a. Then for any g in G, we compute

$$aa^*(g) = \sum_{r(h)=r(g)} a(h)a^*(h^{-1}g) = \sum_{r(h)=r(g)} a(h)\overline{a(g^{-1}h)}.$$

For the term $a(h)\overline{a(g^{-1}h)}$ to be non-zero, it is necessary that both h and $g^{-1}h$ lie in U. As $s(h) = s(g^{-1}h)$ and s is injective on U, we conclude that $g^{-1}h = h$ is necessary for the term to be non-zero. This implies g is a unit. We have shown that a necessary condition for $aa^*(g)$ to be non-zero is that g is in G^0. This means that $aa^* = \Delta((aa^*)|_{G^0})$. Continuing with the computation above for the case that g is a unit (so $r(g) = g$), we see

$$aa^*(g) = \sum_{r(h)=g} a(h)\overline{a(h)} = |a|^2((r|_U)^{-1}(g)).$$

Since $r|_U$ is a homeomorphism, the conclusion of part two follows.

The first two parts also hold since a^* is also in $\mathcal{C}(G)$.

For the third part, we use the first part and the fact that $\pi \circ \Delta$ is a $*$-representation of the C^*-algebra $C(G^0)$ and hence is contractive. Thus, we have

$$\begin{aligned} \|\pi(a)\|^2 &= \|\pi(a)\pi(a)^*\| \\ &= \|\pi(aa^*)\| \\ &= \|\pi(\Delta(aa^*|_{G^0}))\| \\ &\leq \|aa^*|_{G^0}\|_\infty \\ &= \|aa^*\|_\infty \\ &= \|a\|_\infty^2. \end{aligned}$$

□

Now we want to move on from the special elements of $\mathcal{C}(G)$ to all of $C_c(G)$.

Theorem 3.3.17. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space and let a be any function in $C_c(G)$. There is a constant A such that*

$$\|\pi(a)\| \leq A,$$

for all representations π of $C_c(G)$.

Proof. We know we can find functions $a_1, \ldots, a_K$ in $\mathcal{C}(G)$ such that $a = \sum_{k=1}^{K} a_k$. Then, for any π, we have

$$\|\pi(a)\| = \|\pi(\sum_k a_k)\| \leq \sum_k \|\pi(a_k)\| \leq \sum_k \|a_k\|_\infty.$$

$\square$

We are now ready to define full and reduced norms on $C_c(G)$ and then the associated full and reduced C^*-algebras. It is worth remarking that both norms in the next two definitions are valid norms because of Corollary 3.3.11.

Definition 3.3.18. *Let G be a locally compact, Hausdorff étale groupoid. We define its C^*-algebra to be the completion of $C_c(G)$ in the norm*

$$\|a\| = \sup\{\|\pi(a)\| \mid \pi : C_c(G) \to \mathcal{B}(\mathcal{H})\},$$

for any a in $C_c(G)$, and we denote it by $C^*(G)$.

Definition 3.3.19. *Let G be a locally compact Hausdorff étale groupoid. Its* reduced C^*-algebra *is the completion of $C_c(G)$ in the norm*

$$\|a\|_r = \|\pi_\lambda(a)\| = \sup\{\|\pi_\lambda^u(a)\| \mid u \in G^0\}.$$

Exercise 3.3.1. *Show that Theorem 3.3.4 is false for r-discrete groupoids. (Hint: consider Example 3.2.1.)*

Exercise 3.3.2. *Let G be a groupoid with the discrete topology. Prove that for any two elements g_1, g_2 in G, we have*

$$\delta_{g_1}\delta_{g_2} = \delta_{g_1 g_2}$$

if (g_1, g_2) is in G^2 and is zero otherwise.

Exercise 3.3.3. *Let G be a finite group and let it act on itself by translation. That is, consider $G \times G$ with the action $(g, h) \in G \times G \to gh \in G$. In other words, $G \times G$ is a groupoid with*

$$(G \times G)^2 = \{((g, h), (gh, k)) \mid g, h, k \in G\}$$

and groupoid product $((g, h), (gh, k)) \to (g, hk)$. Prove that $C_c(G \times G)$ is isomorphic to $M_N(\mathbb{C})$, where N is the cardinality of G. (Hint: there are two proofs. One involves showing that this groupoid is principal.)

Exercise 3.3.4. *Let $X = \{1, 2, 3, \cdots, N\}$ and let S_N be the full permutation group of X. Let $p_n, 1 \leq n \leq N$ be the projection in $C(X)$ which is 1 at n and 0 elsewhere. Prove that*

$$\Delta(p_N)C_c(X \times G)\Delta(p_N) \cong C^*(S_{N-1}).$$

If you want to be ambitious, try to prove that

$$C_c(X \times G) \cong M_N(C^*(S_{N-1})).$$

Exercise 3.3.5. *Let Γ be a discrete group and let Γ_0 be a subgroup of index $N > 1$. Let $\Gamma_0\backslash\Gamma$ denote the set of all right cosets, $\Gamma_0\gamma, \gamma \in \Gamma$. Choose $\gamma_1, \gamma_2, \ldots, \gamma_N$ to be representatives of the N right cosets. Assume $\gamma_1 = e$. (If you like, let $\Gamma = \mathbb{Z}$, $\Gamma_0 = N\mathbb{Z}$ and $\gamma_i = i - 1$.)*

We let G be the transformation groupoid associated with the action of Γ on $\Gamma_0\backslash\Gamma$: $(\Gamma_0\gamma_i)\gamma = \Gamma_0(\gamma_i\gamma)$. Of course, everything gets the discrete topology.

1. *Find a set of matrix units $e_{i,j}, 1 \leq i, j \leq N$ in $C_c(G)$. (Hint: Try letting $e_{1,1}$ be the function which is one at (Γ_0, e) and zero elsewhere.)*

2. *A typical element a in $C_c(G)$ can be written as a finite linear combination of functions on G which are one at a single element and zero elsewhere. Write such a function out and compute $e_{1,1}ae_{1,1}$.*

3. *The set $e_{1,1}C_c(G)e_{1,1}$ is $*$-subalgebra of $C_c(G)$ with unit, $e_{1,1}$. Describe it as a more familiar $*$-algebra.*

3.4 The structure of groupoid C^*-algebras

In this section, we want to prove some basic results about the structure of $C^*_r(G)$ and $C^*(G)$, when G is a locally compact, Hausdorff étale groupoid. For technical simplicity, we will usually also assume that the unit space G^0 is compact.

3.4.1 The expectation onto $C(G^0)$

We have already seen that there is a natural $*$-homomorphism $\Delta : C(G^0) \to C_c(G)$. We have done this under the added hypothesis that G^0 is compact. That was mainly for convenience; the same thing will work in the general case, replacing $C(G^0)$ by $C_c(G^0)$. But we will continue to work in the case G^0 is compact.

What we would like to do now is find a kind of inverse to this map. Specifically, we will have a map $E : C_c(G) \to C(G^0)$. It is as simple to describe as Δ: simply restrict the function to G^0. Of course, the weak point about this map is that it is *not* a homomorphism. But it does have a number of useful features. First, it extends continuously to both full and reduced groupoid C^*-algebras. Secondly, the extension to the reduced C^*-algebra is faithful in a certain sense. That is very useful because it is very difficult in general to write down elements in the reduced C^*-algebra, but $C(G^0)$ is quite explicit.

Theorem 3.4.1. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space. The map E from $C_c(G)$ to $C(G^0)$ defined by $E(a) = a|_{G^0}$ is a contraction for both full and reduced norms. That is, for all a in $C_c(G)$, we have*

$$\|E(a)\|_\infty \leq \|a\|_r \leq \|a\|.$$

The map extends to a contraction on both $C_r^*(G)$ and $C^*(G)$, both denoted by E.

For any f, g in $C(G^0)$ and a in $C_r^*(G)$, we have

$$E(\Delta(f)a\Delta(g)) = fE(a)g.$$

In particular, we have $E \circ \Delta(f) = f$, for all f in $C(G^0)$.

On $C_r^*(G)$, E is faithful in the sense that if a is in $C_r^*(G)$ and $E(a^*a) = 0$, then $a = 0$.

Proof. First of all, we noted above in Theorem 3.3.14 that for any a in $C_c(G)$, we have

$$a(h) =< \pi_\lambda^{s(h)}(a)\delta_{s(h)}, \delta_h >,$$

for any h in G. It follows then that

$$\begin{aligned}\|E(a)\|_\infty &\leq \|a\|_\infty \\ &\leq \sup\{| < \pi_\lambda^{s(h)}(a)\delta_{s(h)}, \delta_h > | \mid h \in G\} \\ &\leq \|a\|_r,\end{aligned}$$

and we are done.

The second part is done by first considering a in $C_c(G)$. Here the conclusion is a direct computation, which we omit. The general case is obtained by taking limits.

For the last statement, we first consider the case when a is in $C_c(G)$. Then for any g in G, we have

$$\begin{aligned}
\|\pi_\lambda^{s(g)}(a)\delta_g\|^2 &= < \pi_\lambda^{s(g)}(a)\delta_g, \pi_\lambda^{s(g)}(a)\delta_g > \\
&= \sum_{s(h)=u} \sum_{s(k)=u} < a(hg^{-1})\delta_h, a(kg^{-1})\delta_k > \\
&= \sum_{s(h)=u} a(hg^{-1})\overline{a(hg^{-1})} \\
&= \sum_{r(l)=u} \overline{a(l^{-1})}a(l^{-1}) \\
&= \sum_{r(l)=u} a^*(l)a(l^{-1}u) \\
&= (a^*a)(u) \\
&= E(a^*a)(u).
\end{aligned}$$

Now let a be any element of $C_r^*(G)$ with $E(a^*a) = 0$. We choose a sequence a_k in $C_c(G)$ which converges to a. Using the equation we have above for the elements a_k, we have

$$\begin{aligned}
\|\pi_\lambda^{s(g)}(a)\delta_g\|^2 &= \lim_k \|\pi_\lambda^{s(g)}(a_k)\delta_g\|^2 \\
&= \lim_k E(a_k^*a_k)(u) \\
&= E(a^*a)(u) \\
&= 0.
\end{aligned}$$

Thus we see that $\pi_\lambda^{s(g)}(a)$ is zero on a basis for the Hilbert space and hence is zero. $\square$

3.4.2 Traces on groupoid C^*-algebras

The title of this section more or less says it all: we want to have a method of producing traces on the C^*-algebras.

Definition 3.4.2. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space. A regular Borel measure μ on G^0 is said to be G*-invariant *if, for every G-set U, $\mu(r(U)) = \mu(s(U))$.*

Lemma 3.4.3. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space. A regular Borel measure μ on G^0 is G-invariant if and only if, for every G-set U, and function $a \geq 0$ in $C_c(r(U))$, we have*

$$\int_{r(U)} a(u) d\mu(u) = \int_{s(U)} a \circ r \circ (s|_U)^{-1}(v) d\mu(v).$$

Proof. First suppose that μ is G-invariant and let U and a be as given above. So $r \circ (s|_U)^{-1}$ is a homeomorphism from $s(U)$ to $r(U)$. We claim that, for any Borel subset $E \subset r(U)$, we have $\mu(E) = \mu(r \circ (s|_U)^{-1}(E))$. The conclusion follows from these two facts.

First, if E is an open subset of $s(U)$, then $V = (s|_U)^{-1}(E)$ is again a G-set. Moreover, $s(V) = E$ while $r(V) = r \circ (s|_U)^{-1}(E)$. Our claim follows from an application of the definition to the G-set V.

Conversely, suppose the condition stated holds and let U be a G-set. It suffices then to note again that $r \circ (s|_U)^{-1}$ is a homeomorphism from $s(U)$ to $r(U)$

$$\begin{aligned}
\mu(r(U)) &= \sup\{\int_{r(U)} a(u) d\mu(u) \mid 0 \leq a \leq 1, a \in C_c(r(U))\} \\
&= \sup\{\int_{s(U)} a \circ r \circ (s|_U)^{-1}(v) d\mu(v) \mid 0 \leq a \leq 1, a \in C_c(r(U))\} \\
&= \sup\{\int_{s(U)} b(u) d\mu(u) \mid 0 \leq b \leq 1, a \in C_c(s(U))\} \\
&= \mu(s(U)).
\end{aligned}$$

□

Theorem 3.4.4. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space and suppose that μ is a regular, G-invariant, Borel probability measure on G^0. The formula*

$$\tau(a) = \int_{G^0} a(u) d\mu(u)$$

for a in $C_c(G)$ extends to a continuous trace on $C_r^*(G)$ and also on $C^*(G)$, both denoted τ.

Proof. For the first statement, we observe that the map τ is simply the composition of E of Theorem 3.4.1 with the linear functional defined by μ in $C(G^0)$. Hence it is bounded and extends continuously.

Let us now verify the trace condition. It suffices (in view of Lemma 3.3.3) to consider functions a and b that are supported on G-sets U and V, respectively.

In this case, for any unit u,

$$ab(u) = \sum_{r(h)=u} a(h)b(h^{-1}u) = \sum_{r(h)=u} a(h)b(h^{-1}).$$

For a given u, $a(h)$ is zero unless h is in U and hence $u = r(h)$ is in $r(U)$. Similarly, the term $b(h^{-1})$ is zero unless h^{-1} is in V and $u = r(h)$. We conclude that the sum $ab(u)$ is zero unless u is in $r(U \cap V^{-1})$ and in this case, its value is $a((r|_{U\cap V^{-1}})^{-1}(u))b((r|_{U\cap V^{-1}})^{-1}(u)^{-1})$. Notice that if W is a G-set and $s(h) = u$, then h^{-1} is the unique element of W^{-1} (also a G-set) with $s(h^{-1}) = u$. This may be summarized as $((r|_W)^{-1}(u))^{-1} = (s|_{W^{-1}})^{-1}(u)$. We may conclude

$$ab(u) = \begin{cases} a((r|_{U\cap V^{-1}})^{-1}(u))b((s|_{U^{-1}\cap V})^{-1}(u)) & u \in r(U \cap V^{-1}) \\ 0 & \text{otherwise.} \end{cases}$$

By simply reversing the roles of a and b (and also U and V), we see that

$$ba(v) = \begin{cases} b((r|_{U^{-1}\cap V})^{-1}(v))a((s|_{U\cap V^{-1}})^{-1}(v)) & v \in r(U^{-1} \cap V) \\ 0 & \text{otherwise.} \end{cases}$$

We now observe that $r(U^{-1} \cap V) = s(U \cap V^{-1})$ and for v in this set, we claim that

$$ab(r \circ s(U \cap V^{-1})^{-1}(v)) = ba(v).$$

In fact, it is clear that the terms involving a are the same for both. As for the b terms, it suffices to see that if g is in a G-set W^{-1}, then $(s|_W)^{-1} \circ r(g) = g^{-1}$, so that

$$(s|_{U^{-1}\cap V})^{-1} \circ r \circ s(U \cap V^{-1})^{-1}(v) = (s(U \cap V^{-1})^{-1}(v))^{-1} = (r|_{U^{-1}\cap V})^{-1}(v).$$

This completes the proof. □

In fact, if the groupoid G is also principal, then the construction above accounts for *every* trace on the reduced C^*-algebra.

Theorem 3.4.5. *Let G be a locally compact, Hausdorff étale groupoid with compact unit space. If G is principal, then every trace on $C^*_r(G)$ arises from a G-invariant regular Borel probability measure as in Theorem 3.4.4.*

Proof. Let τ be a trace on $G_r^*(G)$. First, we prove that for any a in $C_c(G)$ with $a(u) = 0$, for all u in G^0, we have $\tau(a) = 0$. Let g is any element in the support of a. From the hypothesis above, and the fact that G^0 is open, g is not in G^0. As G is principal, this means that $r(g) \neq s(g)$. Then we may choose a G-set U_g which is a neighbourhood of g with $r(U_g) \cap s(U_g) = \emptyset$.

The sets $U_g, g \in \mathit{supp}(a)$, form an open cover of $\mathit{supp}(a)$ from which we extract a finite subcover. Together with $G \setminus \mathit{supp}(a)$, this forms a finite open cover of G. We find a partition of unity subordinate to this cover and multiply this pointwise against a. In this way, we have $a = a_1 + a_2 + \ldots + a_N$, where each a_n is supported in a set U_n with $r(U_n) \cap s(U_n) = \emptyset$. It now suffices to prove that if a is supported on a G-set U with $r(U) \cap s(U) = \emptyset$, then $\tau(a) = 0$.

Define functions b, c by

$$b(u) = \begin{cases} |a(g)|^{1/2} & g \in U, r(g) = u \\ 0 & \text{otherwise} \end{cases}$$

$$c(g) = \begin{cases} |a(g)|^{-1/2} a(g) & g \in U, a(g) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is a simple exercise to check that b, c are in $C_c(G)$. We compute

$$bc(g) = \sum_{r(h)=r(g)} b(h)c(h^{-1}g).$$

The b is zero unless $h = r(g)$, some some g in U and the a term is zero unless $a(g) \neq 0$. In this case, $bc(g) = b(r(g))c(g) = a(g)$. We conclude in any case that $bc = a$.

Next, we compute

$$cb(g) = \sum_{r(h)=r(g)} c(h)b(h^{-1}g).$$

The c term is non-zero only when h is in U. Also the b term is non-zero only on the units, so in order to get something other than zero, we must have g in U and then $cb(g) = c(g)b(g^{-1}g)$. But for any g in U, $g^{-1}g = s(g)$ is in $s(U)$ and b is zero on this set since b is supported in $r(U)$. We conclude that $cb = 0$. Then we have

$$\tau(a) = \tau(bc) = \tau(cb) = \tau(0) = 0.$$

We now know that $\tau(a) = \tau(E(a))$, for every a in $C_c(G)$. The functional $\tau \circ \Delta$ is a positive linear functional on $C(G^0)$ and so

$$\tau(\Delta(f)) = \int_{G^0} f(u) d\mu(u),$$

for some probability measure μ on G^0. It remains for us to prove that μ is G-invariant. Let U be a G-set and suppose $a \geq 0$ is in $C_c(r(U))$. Define b in $C_c(G)$ by

$$b(g) = \begin{cases} a(r(g))^{1/2} & g \in U, \\ 0 & \text{otherwise.} \end{cases}$$

As b is supported in U, both b^*b and bb^* are in $C(G^0)$. We compute

$$bb^*(u) = \sum_{r(g)=u} b(g)b^*(g^{-1}u) = \sum_{r(g)=u} b(g)b^*(g^{-1}) = \sum_{r(g)=u} b(g)^2.$$

Clearly for $bb^*(u)$ to be non-zero, we must have a g in U with $r(g) = u$. That is, $bb^*(u)$ is zero unless u is in $r(U)$ and in this case, with g in U, $r(g) = u$, $bb^*(u) = b(g)^2 = a(r(g)) = a(u)$. On the other hand, we compute

$$\begin{aligned} b^*b(u) &= \sum_{r(g)=u} b^*(g)b(g^{-1}u) \\ &= \sum_{r(g)=u} b(g^{-1})b(g^{-1}u) \\ &= \sum_{r(g)=u} b(g^{-1})^2 \\ &= \sum_{s(g)=u} b(g)^2. \end{aligned}$$

For b^*b to be non-zero, we need $u = s(g)$ for some g in U. So u is in $s(U)$ and in this case, letting $g = (s|_U)^{-1}(u)$ be the unique element of U with $s(g) = u$,

$$b^*b(u) = b(g)^2 = a(r(g)) = a(r \circ (s|_U)^{-1}(u)).$$

From the trace property, we know that $\tau(bb^*) = \tau(b^*b)$. It follows that the condition of Lemma 3.4.3 is satisfied, so μ is G-invariant. $\square$

3.4.3 Ideals in groupoid C^*-algebras

We now turn to the issue of describing the ideals in the C^*-algebras of an étale groupoid. The results are remarkably similar to the situation with traces. In stead of G-invariant measures on G^0, we introduce the notion of G-invariant open sets in G^0. We see how such a set gives rise to an ideal and, in the case that G is principal, all ideals arise in this way.

Definition 3.4.6. *Let G be a locally compact, Hausdorff étale groupoid. A subset $X \subset G^0$ is said to be G-invariant if, for any g in G, $r(g)$ is in X if and only if $s(g)$ is in X.*

Of course, another way to say this is that X is the union of equivalence classes for the equivalence relation of Proposition 3.1.6. Notice also that, for such an X, $r^{-1}(X) = s^{-1}(X)$.

Lemma 3.4.7. *Let G be a locally compact, Hausdorff étale groupoid and suppose that U is an open G-invariant subset of G^0. If b is any element of $C_c(G)$ with support in $r^{-1}(U)$, then there exists a continuous function f on G^0 with support in U such that $\Delta(f)b = b = b\Delta(f)$.*

Proof. Let $K \subseteq r^{-1}(U)$ be a compact set such that $b = 0$ on $G - K$. Then $r(K) \cup s(K)$ is a compact subset of U. We may find a continuous function of compact support on U which is identically 1 on $r(K) \cup s(K)$. We extend this function to be zero on $G^0 - U$. For any g in G, we compute

$$\Delta(f)b(g) = \sum_{r(h)=r(g)} \Delta(f)(h)b(h^{-1}g).$$

Since $\Delta(f)$ is supported on G^0, the only non-zero term in the sum is for $h = r(g)$ in which case we get $f(r(g))b(g)$. If $b(g)$ is non-zero, it follows that g is in K and hence $f(r(g)) = 1$. This proves that $\Delta(f)b = b$. The other computation is done in a similar manner. □

Theorem 3.4.8. *Let G be a locally compact, Hausdorff étale groupoid and suppose that U is an open subset of G^0 and X is its closed complement in G^0. Then U is G-invariant if and only if X is. Moreover, in this case, the following hold.*

1. *$G_U = r^{-1}(U)$ is open in G and $G_X = r^{-1}(X)$ is closed in G.*

2. *With the relative topologies of G, both G_U and G_X are étale groupoids.*

3. *By extending functions to be zero on $G - G_U$, $C_c(G_U) \subseteq C_c(G)$ is a $*$-closed, two-sided ideal. Moreover, this inclusion extends to $C_r^*(G_U) \subseteq C_r^*(G)$ and $C^*(G_U) \subseteq C^*(G)$ and in each, the subset is a $*$-closed, two-sided ideal.*

4. *The restriction map $\rho : C_c(G) \to C_c(G_X)$ is a surjective $*$-homomorphism which extends to surjective $*$-homomorphisms $\rho : C_r^*(G) \to C_r^*(G_X)$ and $\rho : C^*(G) \to C^*(G_X)$.*

5. *The kernel of $\rho : C^*(G) \to C^*(G_X)$ is $C^*(G_U)$.*

Proof. The first thing to say is that the proof is long! Mainly because the statement should probably be broken down into several theorems, some of which need little lemmas.

For the first statement, assume that U is G-invariant. Let g be in G with $r(g)$ in X. This means that $r(g)$ is not in U and hence $s(g)$ is also not in U, as U is G-invariant. Hence $s(g)$ is in X. As this argument did not use the fact that U was open, the same argument shows that if X is G-invariant, then so is U.

Part 1 follows from the continuity of r. For part 2, it is clear from the G-invariance condition that both G_U and G_X are groupoids. The fact that G_U is étale follows from Theorem 3.2.12. As for G_X, let g be an element of G_X. It has a neighbourhood V in G such that $r(v)$ is open in G^0 and $r : V \to r(V)$ is a homeomorphism. Then $V \cap G_X$ is a neighbourhood of g in the relative topology of G_X. From the G-invariance of X, we have $r(V \cap G_X) = r(V) \cap G_X$, which is an open set in G_X. Moreover, it follows immediately from the definition of the relative topology that $r : V \cap G_X \to r(V \cap G_X)$ is a homeomorphism. A similar argument proves the analogous statement for s.

In part 3, it is a general fact in topology, that if A is a topological space and $B \subseteq A$ is open, by extending functions to be zero, we have $C_c(B) \subseteq C_c(A)$ and is a linear subspace. We know that the map sending g to g^{-1} is a homeomorphism of G; let us verify G_U is mapped to itself. If g is in G_U, then by definition, $r(g)$ is in U, so $s(g)$ is in U by G-invariance, and so $r(g^{-1}) = s(g)$ is in U and so g^{-1} is in G_U. It follows from this that if a is in $C_c(G_U)$, then so is a^*. It remains for us to verify that $C_c(G_U)$ is an ideal in $C_c(G)$. To this end, let a, b be two functions in $C_c(G)$ and assume the latter is compactly supported in G_U. We may find f as in Lemma 3.4.7 and we have $ab = a\Delta(f)b$ and it suffices for us to prove that $a\Delta(f)$ is in $C_c(G_U)$. For g in G, we compute

$$a\Delta(f)(g) - \sum_{r(h)=r(g)} a(h)\Delta(f)(h^{-1}g).$$

As $\Delta(f)$ is supported in G^0, the only non-zero term in the sum is for $h = g$ and we have $a\Delta(f)(g) = a(g)f(s(g))$. This is non-zero only if g is in the support of a and also in the support of f. We claim this set is a compact subset of G_U. Any such g has $s(g)$ in the support of f which is contained in U, so this is a subset of U. To see it is compact, suppose that g_n is a sequence in this set. As the support of a is compact, it has a subsequence which converges to a point, g, in G. Then applying s to this subsequence, we have a sequence in the support of f which converges. As f has compact support in U, we know $s(g)$ is in U. Hence g is in G_U as desired.

To prove the inclusion $C^*(G_U) \subseteq C^*(G)$, we must show that the norms we obtain on $C_c(G_U)$ obtained by considering all representations and the norm we obtain by considering all representations of $C_c(G)$ and restricting to $C_c(G_U)$ are the same. More precisely, for each b in $C_c(G_U)$, we must prove that

$$\begin{aligned} & \sup\{\|\pi(b)\| \mid \pi \text{ a representation of } C_c(G_U)\} \\ = \quad & \sup\{\|\pi(b)\| \mid \pi \text{ a representation of } C_c(G)\}. \end{aligned}$$

Since every representation of $C_c(G)$ restricts to one of $C_c(G_U)$, the inequality $\geq$ is immediate.

We will now show that every representation $(\pi, \mathcal{H})$ of $C_c(G_U)$ extends to one, $(\tilde{\pi}, \mathcal{H})$, of $C_c(G)$. In fact, to obtain the norm on $C_c(G_U)$, we need only consider those representations which extend to irreducible representations of $C^*(G_U)$. Let ξ be any non-zero vector in $\mathcal{H}$. It follows that $\pi(C^*(G_U))\xi$ is a dense subspace of $\mathcal{H}$. As $C_c(G_U)$ is norm dense in $C^*(G_U)$, $\pi(C_c(G_U))\xi$ is also dense in $\mathcal{H}$. Let a be in $C_c(G)$. We define an operator $\tilde{\pi}(a)$ on $\pi(C_c(G_U))\xi$ by setting

$$\tilde{\pi}(a)\pi(b)\xi = \pi(ab)\xi,$$

for any b in $C_c(G_U)$. To see this is well-defined we must check that $\pi(b)\xi = 0$ implies $\pi(ab)\xi = 0$. Let f be as in Lemma 3.4.7 for the element b. Then we have

$$\pi(ab)\xi = \pi(a\Delta(f)b)\xi = \pi(a\Delta(f))\pi(b)\xi = 0,$$

since $a\Delta(f)$ is in $C_c(G_U)$.

Next, it is an easy estimate which we leave to the reader to check that $\tilde{\pi}(a)$ is bounded and therefore extends to all of $\mathcal{H}$. The computations that $\tilde{\pi}$ is a representation and restricts to π on $C_c(G_U)$ are routine and we omit them.

To prove the inclusion $C_r^*(G_U) \subseteq C_r^*(G)$, we proceed in much the same way, but considering only the left regular representations. Let us add a little notation. If u is in G^0, we let π_λ^u be the representation of $C_c(G)$ as before. If u is also in U, we let $\pi_{\lambda,U}^u$ be the representation of $C_c(G_U)$. It u is unit of G, then we consider the Hilbert space $\ell^2(s^{-1}\{u\})$. It is clear from the fact the U is G-invariant that $s^{-1}\{u\}$ is contained in U if u is in U and is contained in X otherwise. From this observation, it follows almost at once that the representation π_λ^u of $C_c(G)$, when restricted to $C_c(G_U)$, is exactly $\pi_{\lambda,U}^u$ if u is in U and is zero otherwise. We may write this as

$$\oplus_{u\in G^0}\pi_\lambda^u | C_c(G_U) = \left(\oplus_{u\in U}\pi_{\lambda,U}^u\right) \oplus \left(\oplus_{u\in X} 0\right).$$

The desired conclusion follows.

This brings us to the proof of part 4. The fact that the map is linear is obvious. To see that it is surjective, let f be in $C_c(G_X)$ with support in the compact set K. Since K is compact in G_X, which is closed in G, it is also compact in G. Each point of K is contained in an open set in G with compact closure. In this way, we may form an open cover of K in G and extract a finite subcover. The union of this finite subcover, which we denote by V, is then an open set in G containing K and with compact closure. The sets G_X and $G-V$ are both closed and while they may intersect, f is zero on this intersection. It follows that we can extend f to be a continuous function on $G_X \cup (G-V)$ by setting f to be zero on $G-V$. By the Tietze extension Theorem we may find a continuous function $\tilde{f}$ on G which equals f on G_X and is zero on $G-V$. It follows that $\tilde{f}$ is in $C_c(G)$ and $\rho(\tilde{f}) = f$.

We indicate briefly why ρ is a homomorphism. Looking at the formula we have for the product in $C_c(G)$, for a fixed g in G_X, we need to sum over all h with $r(h) = r(g)$. Since G_X is G-invariant, it is irrelevant whether we specify h in G or in G_X; they are equivalent. From this fact, it is easy to see the restriction map is a homomorphism. It is also clear that ρ is $*$-preserving.

Let u be a unit in X. The same reasoning as above shows that the left regular representation for u, of $C_c(G)$ or $C_c(G_X)$ are on exactly the same Hilbert space. Moreover, for f in $C_c(G)$, its image under the left regular representation depends only on its values on G_X. In other words, we have $\pi^u_{\lambda,U}(\rho(f)) = \pi^u_\lambda(f)$. It follows at once that ρ extends to a surjective $*$-homomorphism from $C^*_r(G)$ to $C^*_r(G_X)$.

Extending ρ to the full C^*-algebras is somewhat easier: if π is any representation of $C_c(G_X)$, then $\pi \circ \rho$ is also a representation of $C_c(G)$ and it follows that ρ is contractive for the full C^*-norms.

We next consider part 5. It is clear that $C_c(G_U)$ is contained the kernel of $\rho : C_c(G) \to C_c(G_X)$. We claim that the kernel of $\rho : C_c(G) \to C_c(G_X)$ is contained in $C^*(G_U)$. Suppose that a is in $C_c(G)$ and is zero on G_X. We write $a = \sum_V \alpha_V a$ exactly as in the proof of Lemma 3.3.3 (using V for the elements of the cover of $supp(a)$ instead of U). Let M be the number of sets V in the cover. Let $\epsilon > 0$. For each V in the cover, $\alpha_V a$ is a function in $\mathcal{C}(G)$ and vanishes on G_X. Therefore, we may find β_V in $C_c(G_U)$ with $\|\alpha_V a - \beta_V\|_\infty < \frac{\epsilon}{M}$. Then we have $b = \sum_V \beta_V$ is in $C_c(G_U)$ and

$$
\begin{aligned}
\|a - b\| &= \|\sum_V \alpha_V a - \sum_V \beta_V\| \\
&\leq \sum_V \|\alpha_V a - \beta_V\| \\
&\leq \sum_V \|\alpha_V a - \beta_V\|_\infty \\
&< \sum_V \frac{\epsilon}{M} \\
&= \epsilon,
\end{aligned}
$$

by part 3 of Lemma 3.3.16. Since ϵ was arbitrary, we conclude that a is in the closure of $C_c(G_U)$ as desired.

We know that $C^*(G_U)$ is in the kernel of $\rho : C^*(G) \to C^*(G_X)$ and is an ideal in $C^*(G)$. So we have a well-defined $*$-homomorphism $\bar{\rho} : C^*(G)/C^*(G_U) \to C^*(G_X)$. We claim this map is an isometry. It suffices to verify that on the cosets of elements of $C_c(G)$. By Theorem 1.12.8, there is a representation π of $C^*(G)/C^*(G_U)$ which is an isometry. We can regard π as a representation of $C^*(G)$ by simply first applying the quotient map and then π. This map is zero on $C^*(G_U)$. We claim that there exists a representation π' of $C_c(G_X)$ such that $\pi(a) = \pi'(\rho(a))$, for all a in $C_c(G)$. To see this, let b be in $C_c(G_X)$. Find a in $C_c(G)$ with $\rho(a) = b$ and set $\pi'(b) = \pi(a)$. This is well-defined since any two choices for a differ by an element of $C^*(G_U)$ which is in the kernel of π. Checking that π' is a representation is a trivial matter. It follows now that for any a in $C_c(G)$

$$\|a + C^*(G_U)\| = \|\pi(a)\| = \|\pi'(\rho(a))\| \leq \|\bar{\rho}(a + C^*(G_U))\|.$$

This completes the proof that $\bar{\rho}$ is an isometry. It follows that the kernel of ρ is simply $C^*(G_U)$. □

It is an interesting question to ask whether Theorem 3.4.8 accounts for all ideals in $C^*(G)$. The short answer is 'no'. Suppose $G = \mathbb{F}_2$, the free group on two generators. Here, as a groupoid, the unit space is a single element. On the other hand, $C^*(\mathbb{F}_2)$ has a non-trivial ideal: the kernel of the map to $C^*_r(\mathbb{F}_2)$. In fact, since there are $*$-homomorphism onto every matrix algebra and their kernels are a lot of other ideals as well.

We could also ask whether Theorem 3.4.8 accounts for all ideals in $C^*_r(G)$. The answer is still 'no'. Suppose that $G = \mathbb{Z}$. Again, as a groupoid, the unit space is a single element. On the other hand, $C^*(\mathbb{Z}) = C^*_r(\mathbb{Z}) \cong C(S^1)$ has a great many ideals.

We can finally get a positive answer by looking at ideals in $C^*_r(G)$ with the added hypothesis that G is principal.

We begin with two simple lemmas.

Lemma 3.4.9. *Let A be a C^*-algebra and $e_1, \ldots, e_I$ be positive elements of A with $\| \sum_{i=1}^I e_i^2 \| \leq 1$. Then for any a in A, we have*

$$\| \sum_{i=1}^{I} e_i a e_i \| \leq \|a\|.$$

Proof. Without loss of generality, we may assume that A is acting on a Hilbert space $\mathcal{H}$. Define operators on $\oplus_{i=1}^I \mathcal{H}$

$$\begin{aligned} e(\xi_1, \ldots, \xi_I) &= (e_1\xi_1, e_2\xi_1, \ldots, e_I\xi_1) \\ \tilde{a}(\xi_1, \ldots, \xi_I) &= (a\xi_1, \ldots, a\xi_I). \end{aligned}$$

It is a simple matter to verify that

$$\begin{aligned} e * \tilde{a} e(\xi_1, \ldots, \xi_I) &= (\sum_{i=1}^{I} e_i a e_i \xi_1, 0, \ldots, 0), \\ e * e(\xi_1, \ldots, \xi_I) &= (\sum_{i=1}^{I} e_i^2 \xi_1, 0, \ldots, 0). \end{aligned}$$

From these facts, it follows that

$$\|\sum_{i=1}^{I} e_i a e_i\| = \|e * \tilde{a} e\| \leq \|e^* e\| \|a\| \leq \|a\|.$$

□

The next result is where the groupoid being principal appears in a crucial way.

Lemma 3.4.10. *Let G be a principal, étale groupoid which is locally compact and Hausdorff.*

1. *If a is in $C_c(G)$, then there exist $f_1, f_2, \ldots, f_N$ in $C_c(G^0)$ such that*

$$\sum_{n=1}^{N} \Delta(f_n) a \Delta(f_n) = E(a).$$

2. *If I is an ideal in $C_r^*(G)$, then $E(I) \subset I$.*

Proof. Suppose that a is supported in the compact set $K \subseteq G$. For each g in $K - G^0$, $r(g) \neq s(g)$ since G is principal. Therefore we may find a G-set V_g such that $\overline{r(V_g)}$ and $\overline{s(V_g)}$ are disjoint. These sets form an open cover of $K - G^0$ and we extract a finite subcover $V_1, \ldots, V_M$. For each u in $K \cap G^0$, we may choose an open neighbourhood W_u with compact closure as follows. For any fixed m, u cannot be in both $\overline{r(V_m)}$ and $\overline{s(V_m)}$, so choose W_u so that it is disjoint from one of them. That is, for each m, W_u is disjoint from at least one of $r(V_m)$ and $s(V_m)$. These sets form an open cover of $K \cap G^0$ and we extract a finite subcover $W_1, \ldots, W_N$.

We find functions $f_1, f_2, \ldots, f_N$ in $C_c(G^0)$ with $0 \leq f_n \leq 1$, $f_n = 0$ off of W_{u_n}, and $\sum_{n=1}^{N} f_i^2(u) = 1$, for u in $K \cap G^0$.

It is an immediate consequence of the definition of the product that, for any $1 \leq n \leq N$ and g in G,

$$(\Delta(f_n) a \Delta(f_n))(g) = f_n(r(g)) a(g) f_n(s(g)).$$

We consider two cases. First, suppose that g is in G^0. If $a(g)$ is zero, then this function is also zero at g. If not, then after summing over all n, we get $a(g)$.

If g is not in G^0, we claim $(\Delta(f_n)a\Delta(f_n))(g) = 0$. If g is not in K, then $a(g) = 0$ and the claim holds. If $r(g)$ is not in W_n or if $s(g)$ is not in W_n, then $f(r(g)) = 0$ and the claim holds. It remains for us to consider the case g is in K, $r(g)$ and $s(g)$ are in W_n. Then g is in V_m, for some m, and $r(g)$ is in $r(V_m)$ while $s(g)$ is in $s(V_m)$. But we know that W_n must be disjoint from one of these two, so this case cannot occur. We have now proved that $\sum_{n=1}^{N} \Delta(f_n)a\Delta(f_n) = E(a)$.

For the second part, let b be in I. Let $\epsilon > 0$ and choose a in $C_c(G)$ with $\|b-a\| < \epsilon$. Choose $f_1, \ldots, f_N$ as above for a. Then clearly $\sum_{n=1}^{N} \Delta(f_n)b\Delta(f_n)$ is in I and we have

$$\begin{aligned}
\|E(b) - \sum_{n=1}^{N} \Delta(f_n)b\Delta(f_n)\| &\leq \|E(b) - E(a)\| \\
&\quad + \|\sum_{n=1}^{N} \Delta(f_n)a\Delta(f_n) - \sum_{n=1}^{N} \Delta(f_n)b\Delta(f_n)\| \\
&\leq \|b-a\| + \|\sum_{n=1}^{N} \Delta(f_n)(a-b)\Delta(f_n)\| \\
&\leq \|b-a\| + \|b-a\| \\
&< 2\epsilon,
\end{aligned}$$

having used Lemma 3.4.9 in the penultimate step. As ϵ was arbitrary and I is closed, we conclude that $E(b)$ is in I. □

Theorem 3.4.11. *Let G be a locally compact, Hausdorff étale principal groupoid. If I is a closed two-sided ideal in $C_r^*(G)$. then there is an open G-invariant set $U \subset G^0$ such that*

$$I = C_r^*(G_U).$$

Proof. It is clear that $\Delta^{-1}(I)$ is a closed two-sided ideal in $C(G^0)$ and so by the second part of Exercise 1.8.2, it equals $C_0(U)$, for some open set $U \subseteq G^0$.

First, we prove that U is G-invariant. Suppose that g is in G with $r(g)$ in U. Choose an open G-set V containing g with $r(V) \subseteq U$. Let a be any continuous function supported in V with $a(g) = 1$. We apply Lemma 3.3.16. In fact, the proof there shows that aa^* is not only in $\Delta(C(G^0))$, but is supported in $r(V) \subseteq U$ and so aa^* is in I. It follows then that $(a^*a)^2 = a^*(aa^*)a$ is also in I. But the proof of Lemma 3.3.16 also shows that a^*a is in $\Delta(C(G^0))$ and $(a^*a)(s(g)) = 1$. This implies that $s(g)$ is also in U.

Next, we will show that $C_r^*(G_U) \subseteq I$. Since I is closed, it suffices to check that $C_c(G_U) \subseteq I$. If a is in $C_c(G_U)$, then by Lemma 3.4.7 we may find f in $C_c(U)$ with $\Delta(f)a = a$. Since $\Delta(f)$ is clearly in I and I is an ideal, so is a.

We now turn to the reverse inclusion: $C_r^*(G_U) \supseteq I$. Let a be in I. It follows that a^*a and hence $E(a^*a)$ (by 3.4.10) are in I. By definition of U, $E(a^*a)$ is in $\Delta(C_0(U))$. Next, choose $\epsilon > 0$. As $C_c(G)$ is dense in $C_r(G)$, we may find f in $C_c(G)$ such that

$$\|a - f\|_r, \|a^*a - f^*f\|_r < \epsilon.$$

It follows that $\|E(a^*a) - E(f^*f)\| < \epsilon$ and hence $\Delta^{-1}(E(f^*f))$ is an element of $C(G^0)$ which is less than ϵ in absolute value on $G^0 - U$. Hence, we can find g in $C_c(U)$ such that

$$\|\Delta(g) - E(f^*f)\|_r = \|g - \Delta^{-1}(E(f^*f))\|_\infty < 2\epsilon.$$

It follows that $\Delta(g)$ and hence $f\Delta(g)$ are in $C_c(G_U)$. Moreover, we have

$$\|a - f\Delta(g)\|_r \quad \leq \quad \|a - f\|_r + \|f - f\Delta(g)\|_r$$

□

Exercise 3.4.1. *Explain why the kernel of $\pi : C_c(G) \to C_c(G_X)$ is* not *$C_c(G_U)$ in Theorem 3.4.8. (Hint: you don't have to look for very complicated groupoids. Try the ones in Example 3.3.6.)*

Exercise 3.4.2. *Let $N > 1$ and let $\mathbb{Z}_N$ act on the circle, $\mathbb{T}$, by $z \cdot n = e^{\frac{2\pi i n}{N}} z$. That is, n rotates the circle by angle $\frac{2\pi n}{N}$. Let G be the associated transformation groupoid.*

1. *Describe all the traces on $C_r^*(G)$.*

2. *Describe all ideals in $C_r^*(G)$. Among these, identify the maximal proper ideals. (Observe that, for two open G-invariant sets U, V, $U \subseteq V$ if and only if $C_r^*(G_U) \subseteq C_r^*(G_V)$.)*

3. *For each maximal ideal I in $C_r^*(G)$, describe the quotient $C_r^*(G)/I$.*

Exercise 3.4.3. *For $N > 1$, let G be the groupoid associated with the action of S_N on $\{1, 2, \ldots, N\}$ as in Exercise* **??**.

1. *Explain why Theorem 3.4.8 applies, while 3.4.11 does not.*

2. *Find all open G-invariant subsets X of G^0.*

3. *For $N = 3$ or $N = 4$, prove that there is an ideal which is not of the form $C_r^*(G_U)$, where U is an open G-invariant subset of G^0.*

Exercise 3.4.4. *Let $0 < \theta < 1$ be irrational and let $\mathbb{Z}$ act on the circle, $\mathbb{T}$, by $z \cdot n = e^{2\pi i n\theta} z$. That is, n rotates the circle by angle $2\pi n\theta$. Let G be the associated transformation groupoid.*

1. *Describe a trace $C_r^*(G)$. How many others are there? (This last part might be a little beyond your background.)*

2. *Describe all ideals in $C_r^*(G)$. (This might be a little beyond your background, too.)*

3. *Assume that $0 < \theta < \frac{1}{2}$. Define $f : \mathbb{T} \to [0,1]$ by*

$$f(e^{2\pi i t}) = \begin{cases} \theta^{-1} t & 0 \leq t \leq \theta, \\ \theta^{-1}(2\theta - t) & \theta \leq t \leq 2\theta, \\ 0 & 2\theta \leq t < 1. \end{cases}$$

Verify that f is a continuous function and draw its graph. Define a in $C_c(G)$ by

$$a(e^{2\pi i t}, n) = \begin{cases} \sqrt{f(e^{2\pi i t})(1 - f(e^{2\pi i t}))} & n = -1, \theta \leq t \leq 2\theta, \\ f(e^{2\pi i t}) & n = 0, \\ \sqrt{f(e^{2\pi i t})(1 - f(e^{2\pi i t}))} & n = 1, 0 \leq t \leq \theta, \\ 0 & \text{otherwise}. \end{cases}$$

Compute a^*, a^2 and $\tau(a)$, for each trace τ from the first part.

3.5 AF-algebras

In this chapter, we will construct a particular class of examples of étale groupoids and thoroughly describe their associated C^*-algebras. The class of C^*-algebras which we produce are called AF-algebras, 'AF' standing for 'approximately finite-dimensional'.

We need some combinatorial data to get started. The key notion is that of a Bratteli diagram. It is an infinite graph, consisting of vertices and edges, but these are organized into layers indexed by the natural numbers.

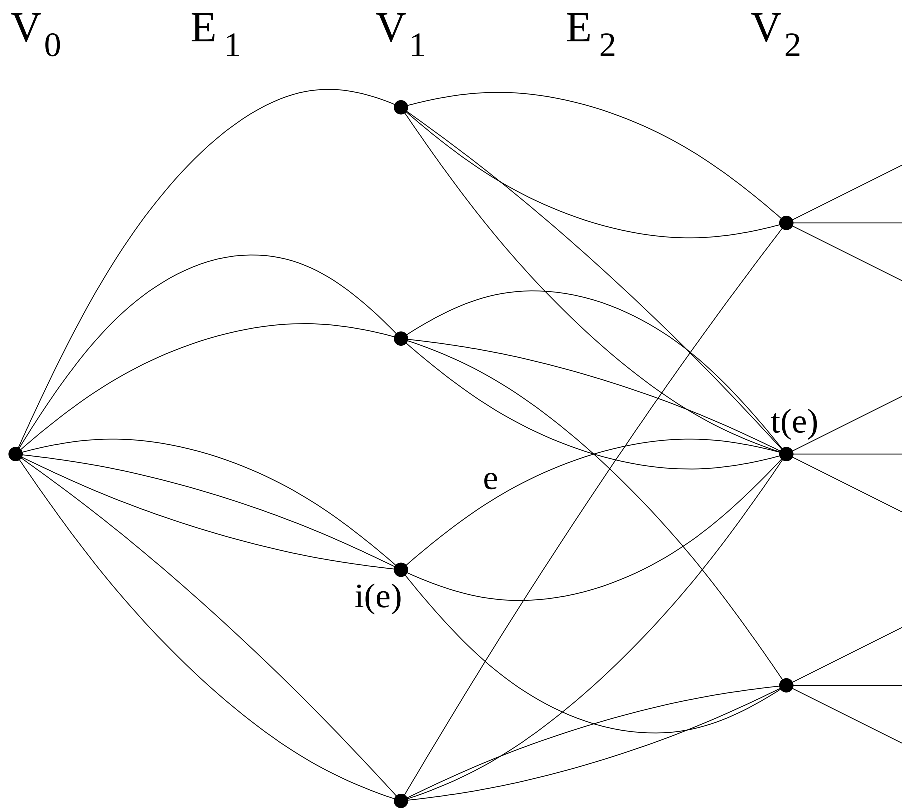

Figure 3.1: A Bratteli diagram.

Definition 3.5.1. *A* Bratteli diagram*, consists of a sequence of finite, pairwise disjoint, non-empty sets, $\{V_n\}_{n=0}^{\infty}$, called the vertices, a sequence of finite non-empty sets $\{E_n\}_{n=1}^{\infty}$ called the edges and maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$, called the initial and terminal maps. We let V and E denote the union of these sets and denote the diagram by (V, E). We will assume that V_0 has exactly one element, denoted v_0, that $i^{-1}\{v\}$ is non-empty for every v in V, and that $t^{-1}\{v\}$ is non-empty for every $v \neq v_0$ in V. (That is, the diagram has no sinks and no sources other than v_0.)*

We may draw the diagram as in Figure 3.1. We say a Bratteli diagram (V, E) has *full edge connections* if for every $n \geq 1$, v in V_{n-1} and w in V_n, there exists e in E_n with $i(e) = v$ and $t(e) = w$.

For $M < N$, a *path from V_M to V_N*, $p = (p_{M+1}, p_{M+2}, \ldots, p_N)$, is a sequence of edges with p_n in E_n, for $M < n \leq N$ and $t(p_n) = i(p_{n+1})$ for

$M < n < N$. We define $i(p) = i(p_{M+1}) \in V_M$ and $t(p) = t(p_N) \in V_N$ for such a path. We let $E_{M,N}$ denote the set of all such paths. If p is in $E_{L,M}$ and q is in $E_{M,N}$ and $t(p) = i(q)$, then p and q may be concatenated and we denote the result by pq in $E_{L,N}$.

Definition 3.5.2. *An* infinite path *in the Bratteli diagram* (V, E) *is a sequence of edges* $(x_1, x_2, \ldots)$ *such that* x_n *is in* E_n *and* $t(x_n) = i(x_{n+1})$ *for* $n \geq 1$. *We let* X_E *denote the set of all infinite paths. If* $N \geq 1$ *and* p *is in* $E_{0,N}$, *we let*

$$C(p) = \{x \in X_E \mid (x_1, x_2, \ldots, x_N) = p\}.$$

We define d *on* $X_E \times X_E$ *by*

$$d(x, y) = \inf\{2^{-N} \mid N \geq 0, x_n = y_n, \textit{ for all } 0 < n \leq N\}.$$

Proposition 3.5.3. *Let* (V, E) *be a Bratteli diagram and let* $N \geq 1$.

1. *d is a metric on X_E.*

2. *For x in X_E, we have*

$$B(x, 2^{1-N}) = C(x_1, \ldots, x_N).$$

3. *For p, q in $E_{0,N}$, $C(p)$ and $C(q)$ are equal when $p = q$ and are disjoint otherwise.*

4. *For p in $E_{0,N}$, $C(p)$ is clopen. (That is, $C(p)$ is both closed and open.)*

5. *The collection $\mathcal{P}_N = \{C(p) \mid p \in E_{0,N}\}$ is a partition of X_E into pairwise disjoint clopen sets.*

6. *For p in $E_{0,N}$, we have*

$$C(p) = \cup_{i(e)=t(p)} C(pe).$$

7. *Each element of $\mathcal{P}_{N+1}$ is contained in a single element of $\mathcal{P}_N$.*

8. *The collection $\cup_{M \geq 1} \mathcal{P}_M$ is a base for the topology of (X_E, d).*

9. *The space X_E is compact.*

We leave the proof as an exercise for the interested reader. Almost all of this is quite easy, with the possible exception of the last part.

Having defined our space X_E, we are now ready to provide an equivalence relation on it. Of course, an equivalence relation is an example of a groupoid. More than that, this will come equipped with a topology in which it is étale.

Definition 3.5.4. *Let (V, E) be a Bratteli diagram.*

1. *For each $N \geq 1$, v in V_N and pair p, q in $E_{0,N}$ with $t(p) = t(q) = v$, we define*

$$\gamma(p, q) = \{(x, y) \in X_E \times X_E \mid x \in C(p), y \in C(q), x_n = y_n, n > N\}.$$

2. *We let $\Gamma_{E,v}$ denote the collection of all such $\gamma(p, q)$ with $t(p) = t(q) = v$. We also let $\Gamma_{E,N}$ denote the union of these over all v in V_N and, finally, we let Γ_E denote the union of these over all $N \geq 1$. (Notice that $\Gamma_{E,v}, \Gamma_{E,N}$ and Γ_E are all collections of subsets of $X_E \times X_E$.)*

3. *We define*

$$\begin{aligned} R_{E,N} &= \cup \Gamma_{E,N} \\ R_E &= \cup \Gamma_E. \end{aligned}$$

(Notice that $R_{E,N}$ and R_E are subsets of $X_E \times X_E$.)

We regard $X_E \times X_E$ as a groupoid with the usual product and involution: $(x, y) \cdot (x', y')$ is defined when $y = x'$ and the result is (x, y') and $(x, y)^{-1} = (y, x)$. We note that $r(x, y) = (x, x)$ and $s(x, y) = (y, y)$.

We summarize the properties of these sets with the following result. All parts follow easily from the definition and we omit the proof.

Lemma 3.5.5. *Let (V, E) be a Bratteli diagram.*

1. *If p, q, p', q' are in $E_{0,N}$ with $t(p) = t(q), t(p') = t(q')$, then $\gamma(p, q) = \gamma(p', q')$ if and only if $(p, q) = (p', q')$ and they are disjoint otherwise.*

2. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then $\gamma(p, q) \subseteq \Delta_{X_E}$ if $p = q$ and $\gamma(p, q) \cap \Delta_{X_E} = \emptyset$ if $p \neq q$.*

3. *If p, q, p', q' are in $E_{0,N}$ with $t(p) = t(q), t(p') = t(q')$, then $\gamma(p, q) \cdot \gamma(p', q') = \gamma(p, q')$ if $q = p'$ and is empty otherwise. In the first case, the product is a bijection*

4. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then $\gamma(p, q)^{-1} = \gamma(q, p)$.*

5. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then $r(\gamma(p, q)) = \gamma(p, p)$ and r is a bijection between these two sets in $X_E \times X_E$. Similarly, $s(\gamma(p, q)) = \gamma(q, q)$ and r is a bijection between these two sets in $X_E \times X_E$.*

6. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then for any $M > N$,*

$$\gamma(p, q) = \cup\gamma(pp', qp'),$$

where the union is over all p' in $E_{N,M}$ with $i(p') = t(p)$.

Again, we omit the proof since all parts are quite easy. The importance though is summarized in the following result.

Theorem 3.5.6. *Let (V, E) be a Bratteli diagram.*

1. *The set $R_E \subseteq X_E \times X_E$ is an equivalence relation. In other words, with the usual product from $X_E \times X_E$, the set R_E is a principal subgroupoid.*

2. *The collection of subsets Γ_E forms a neighbourhood base for a topology on R_E in which it is an étale groupoid.*

3. *On each element of Γ_E, the relative topology from the last part agrees with the relative topology as a subset of $X_E \times X_E$.*

4. *Each element of Γ_E is a compact, open R_E-set.*

Proof. It is clear that for any fixed N, the union of all sets $\gamma(p, p)$, with p in $E_{0,N}$ is exactly Δ_{X_E} and it follows that R_E is reflexive. R_E is symmetric by part 4 of Lemma 3.5.5 and transitive from part 3 of Lemma 3.5.5. Thus, R_E is an equivalence relation.

The collection of sets $\Gamma_{E,N}$ is closed under intersections by part 1 of 3.5.5. The same is true of Γ_E using this fact and part 6 of 3.5.5. This immediately implies that Γ_E is a neighbourhood base for a topology.

The fact that the product and inverse are continuous in this topology follows from parts 3 and 4 of Lemma 3.5.5. The facts that both r and s are open and local homeomorphisms follows from part 5. Thus, with the topology induced by Γ_E, R_E is an étale groupoid.

If we fix p, q in $E_{0,N}$ with $t(p) = t(q)$, then the relative topology on $\gamma(p, q)$, from our base is generated by sets $\gamma(p', q')$ with p', q' in $E_{0,N'}$ with $N' > N$,

$(p'_1, \ldots, p'_N) = p$, $(q'_1, \ldots, q'_N) = q$, and $(p'_{N+1}, \ldots, p'_{N'}) = (q'_{N+1}, \ldots, q'_{N'})$. It is a simple matter to see that

$$\gamma(p', q') = \gamma(p, q) \cap C(p') \times C(q').$$

This implies that the relative topology on $\gamma(p, q)$ induced by Γ_E is the same as the product topology.

We have already seen that $\gamma(p, q)$ is an open R_E-set. The fact it is compact follows from the last statement and the fact that $C(p)$ and $C(q)$ are both closed and hence compact in X_E. □

Definition 3.5.7. *The étale groupoid R_E arising from a Bratteli diagram (V, E) as above is called an AF-groupoid.*

We now move on to investigate the associated groupoid C^*-algebras.

Definition 3.5.8. *For each pair p, q in $E_{0,N}$ with $t(p) = t(q)$, we define $e(p, q)$ to be the characteristic function of $\gamma(p, q)$, which lies in $C_c(R_E)$.*

Lemma 3.5.9.

1. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then $e(p, q)^* = e(q, p)$.*
2. *If p, q, p', q' are in $E_{0,N}$ with $t(p) = t(q) \neq t(p') = t(q')$, then $e(p, q)e(p', q') = 0$.*
3. *If p, q, p', q' are in $E_{0,N}$ with $t(p) = t(q) = t(p') = t(q')$, then $e(p, q)e(p', q') = e(p, q')$ if $q = p'$ and is zero if $q \neq p'$.*
4. *If p, q are in $E_{0,N}$ with $t(p) = t(q)$, then*

$$e(p, q) = \sum_{i(e)=t(p)} e(pe, qe).$$

Proof. The first part is a general fact that for a compact open G-set U, $\chi_U^* = \chi_{U^{-1}}$ combined with part 4 of 3.5.5.

Now consider the case $t(p) = t(q), t(p') = t(q')$ and $q \neq p'$. We compute

$$\left(e(p, q)e(p', q')\right)(x, y) = \sum_{(x,z) \in R_E} e(p, q)(x, z)e(p', q')(z, y).$$

To obtain something other than zero, we would need to find a z with $(z_1, \ldots, z_N) = q$ and $(z_1, \ldots, z_N) = p'$. As $q \neq p'$ there are no such z. This proves part 2 and the second statement of part 3.

For the other statement of part 3, we use the same formula above. For a fixed x, y, to obtain something other than zero, we must find $z = (p'_1, \ldots, p'_N, x_{N+1}, x_{N+2}, \ldots)$ for the first term to be non-zero and $z = (q_1, \ldots, q_N, y_{N+1}, y_{N+2}, \ldots)$. This requires $x_n = y_n$ for all $n > N$ and in this case, there is a unique such z and the value of $e(p,q)e(p',q')(x,y)$ is one. This means that $e(p,q)e(p',q')$ is the characteristic function of $\gamma(p,q')$ as claimed.

The last part follows from the last part of 3.5.5. □

Theorem 3.5.10. *Let (V, E) be a Bratteli diagram.*

1. *For a fixed vertex v in V_N,*

$$A_v = span\{e(p,q) \mid p, q \in E_{0,N}, t(p) = t(q) = v\}$$

is a C^*-subalgebra of $C^*(R_E)$ and is isomorphic to $M_{n(v)}(\mathbb{C})$, where

$$n(v) = \#\{p \in E_{0,N} \mid t(p) = v\},$$

the number of paths from v_0 to v.

2. *For a fixed positive integer N,*

$$A_N = span\{e(p,q) \mid p, q \in E_{0,N}, t(p) = t(q)\}$$

is a unital C^*-subalgebra of $C^*(R_E)$ and equals $\oplus_{v \in V_N} A_v$.

3. *For a fixed positive integer N, A_N is a unital C^*-subalgebra of A_{N+1}.*

4. *$\cup_{N \geq 1} A_N$ is a dense unital $*$-subalgebra of $C^*(R_E)$.*

5. *The full and reduced groupoid C^*-algebras of R_E are the same.*

Proof. The first statement follows from parts 1 and 3 of Lemma 3.5.9 and exercise 1.7.1. If v, w are distinct vertices in V_N, then $A_v \cdot A_w = 0$ follows from part 2 of Lemma 3.5.9 and the second part follows at once.

It is easy to see that, for any N, $\sum_{p \in E_{0,N}} e(p,p) = \chi_{\Delta_{X_E}} = \Delta(1)$ is the unit for $C_c(R_E)$. The fact that A_N is a subalgebra of A_{N+1} follows from part 4 of Lemma 3.5.9.

The proof of part 4 is as follows. First, we have a neighbourhood base for R_E consisting of compact open sets which is also closed under intersections.

It follows that every continuous function of compact support of R_E can be uniformly approximated by a linear combination of such functions. The linear span of these functions is exactly $\cup_{N\geq 1} A_N$. This implies that these two subsets will have the same completion in the C^*-algebra norm.

As each $*$-subalgebra A_N in $C_c(R_E)$ is finite dimensional, it has a unique norm in which it is a C^*-algebra. Thus the norm on $C_c(R_E)$ in which its completion is a C^*-algebra is unique on the dense set $\cup_N A_N$ and hence is unique. It follows that $\| \cdot \|$ and $\| \cdot \|_r$ agree and $C^*(R_E) = C^*_r(R_E)$. □

Exercise 3.5.1. *Let (V, E) be a Bratteli diagram. Let $(x^k, y^k), k \geq 1$ and (x, y) be in R_E. (That is, each x^k and y^k is an infinite sequence in X_E.) Prove that the sequence $(x^k, y^k), k \geq 1$ converges to (x, y) in R if and only it converges in $X \times X$ and there exists $N \geq 1$ such that $R_{E,N}$ contains the sequence and its limit point.*

Exercise 3.5.2. *Consider the Bratteli diagram (V, E) with*

1. *$V_n = \{v_n, w_n\}$, $n \geq 1$,*
2. *there is one edge from v_0 to v_1 and one edge from v_0 to w_1,*
3. *for $n \geq 1$, there is one edge from v_n to v_{n+1}, one edge from v_n to w_{n+1}, and one edge from w_n to w_{n+1}.*

Give a simple concrete description of X (as a familiar space) and R. Also, find a set that is open in R and not open in the relative topology on R from $X \times X$.

Exercise 3.5.3. *Suppose that (V, E) has full edge connections. Prove that every equivalence class in R is dense in X. Describe all ideals in $C^*(R_E)$.*

Exercise 3.5.4. *Find a necessary and sufficient condition on a Bratteli diagram (V, E) such that $C^*(R_E)$ is commutative.*

Exercise 3.5.5. *Let (V, E) be a Bratteli diagram. Use the following two facts: any finitely additive, probability measure on the clopen subsets of a totally disconnected compact space will extend to a probability measure, and that a probability measure on such a space is uniquely determined by its restriction to the clopen sets.*

Find a bijection between the set of R_E-invariant probability measures on X_E and the set of all functions $\nu : V \to [0, 1]$ such that

1. $\nu(v_0) = 1$,

2. $$\nu(v) = \sum_{e \in E_{n+1}, i(e)=v} \nu(t(e)),$$

   for every $n \geq 0$, v *in* V_n.

Exercise 3.5.6. *Consider the following three Bratteli diagrams.*

1. *The diagram described in Exercise 3.5.2.*

2. *The diagram with exactly one vertex and exactly two edges at each level.*

3. *With the same vertex set as Exercise 3.5.2, have one edge from* v_n *to* v_{n+1}*, one from* v_n *to* w_{n+1} *and one from* w_n *to* v_{n+1}.

In each case, find all traces on the C^**-algebras of the Bratteli diagrams.*