Lecture Notes on Smale Spaces

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Contents

Part I

Smale Spaces

Chapter 1

Dynamical preliminaries

1.1 Preliminaries on dynamical systems

In this section, we will introduce some simple notions in the study of topological dynamical systems: periodic points, non-wandering, irreducibilty and mixing. Each can be regarded as a type of recurrence property.

For much of the time, we will work with a compact metric space (X, d) together with a homeomorphism f of X. For the moment, however, we will give our definitions in somewhat greater generality.

For a metric space (X, d) we employ the following notation for an open ball

$$X(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\},$$

for all x in X, $\epsilon > 0$.

Definition 1.1.1. *Let X be a set and $f : X \to X$ be a bijection. We say that x in X is a* fixed-point *of f if $f(x) = x$. If n is a positive integer and x is in X, we say that x is a* periodic point of period n *if $f^n(x) = x$. The least such positive integer n is called the* period *of x. For any positive integer m, we let $Per_m(X, f)$ denote the set of all periodic points of period m. We also let*

$$Per(X, f) = \cup_{m \geq 1} Per_m(X, f)$$

denote the set of all periodic points.

Next, we consider the notion of non-wandering which requires some topology on X. As we mentioned above, this is a kind of recurrence condition on the points of X. There are a number of these available, but this is the most natural for the systems which we will consider later.

Definition 1.1.2. *Let X be a topological space and let $f : X \to X$ be a homeomorphism. Let x be in X. We say that x is* non-wandering *if, for every non-empty open set, U, containing x, there is a positive integer n such that $f^n(U) \cap U$ is non-empty. Any point which is not non-wandering is called* wandering.

We let $NW(X, f)$ denote the set of non-wandering points of X. We say that (X, f) is non-wandering *if every point of X is non-wandering.*

Let us make a few simple remarks about the definition. A point is non-wandering if and only if, for every non-empty set U containing the point, there is a z in U and positive integer n with $f^n(z)$ also in U. (If y is in $f^n(U) \cap U$, let $z = f^{-n}(y)$.)

We observe that every periodic point is clearly non-wandering. Moreover, if the periodic points of X are dense, then every point is non-wandering.

Proposition 1.1.3. *Let X be a topological space and let f be a homeomorphism of X.*

1. *The set of non-wandering points is f-invariant: that is, x is non-wandering if and only if $f(x)$ is also.*

2. *The set of non-wandering points is closed.*

3. *If X is compact, then the set of non-wandering points is non-empty.*

Next, we turn to the definition of irreducibility.

Definition 1.1.4. *Let X be a topological space and let f be a homeomorphism of X. We say the system (X, f) is* irreducible *if, for every (ordered) pair of non-empty open sets, U, V, there is a positive integer n such that $f^n(U) \cap V$ is non-empty.*

It is clear that every irreducible system is non-wandering. The converse is false; for example the identity map on a set X (having at least two points) is non-wandering but not irreducible.

This condition is often referred to as one-sided topological transitivity and is equivalent (in our context) to the existence of a dense forward orbit (see Theorem 5.9 of [**?**]). We have decided on the term irreducibility for two reasons. There is a difference between the notions of one and two-sided topological transitivity (again see section 5.4 of [**?**]) and this can cause some confusion, especially as our systems are all 2-sided. Secondly, the term is standard in the theory of subshifts, and shifts of finite type are very important examples of the types of systems which we will study later.

Finally, we turn to the definition of mixing.

Definition 1.1.5. *Let X be a topological space and let f be a homeomorphism of X. We say the system (X, f) is* mixing *if, for every (ordered) pair of non-empty open sets, U, V, there is a positive integer N such that $f^n(U) \cap V$ is non-empty for all $n \geq N$.*

We observe that every mixing system is also irreducible and hence non-wandering as well. The converse is false. Consider the case that X consists of two points which the map f exchanges. This is irreducible, but not mixing.

Chapter 2

Definition of a Smale space

2.1 The heuristic definition of a Smale space

In this section, we will provide a heuristic discussion of Smale spaces. This is intended as motivation and will be rather short on rigour. It is important to proceed in this way because the rigourous definition - which we will see in the next section - is really quite opaque without a preliminary discussion to provide some kind of insight.

We will consider a compact metric space (X, d) and a homeomorphism $f : X \to X$. We will require some extra structure. This will take quite some time to describe.

First, for every point x in X, we will have two closed sets E_x and F_x having a number of special properties. We require

P1

$$E_x \cap F_x = \{x\}$$

P2 The cartesian product $E_x \times F_x$ is homeomorphic to a neighbourhood of x in X.

This second item is really too vague. The proper definition in the next section will actually specify this homeomorphism and a number of its properties. But for the moment, this will be enough. That is, locally, X is the product of E_x and F_x.

It is worth noting at this point that the sets E_x and F_x are not unique. For example, if we make both smaller, so long as x is still in the interior of the cartesian product, the result would also satisfy our conditions. They are unique in the sense that any two such choices for E_x will be equal in some neighbourhood of x.

Next, we want to require that these sets be invariant under f. This is a little too much to ask, especially in view of the comments of the last paragraph. Instead, we only require invariance in a local sense.

P3 For all x in X,

$$\begin{aligned} f(E_x) \cap V &= E_{f(x)} \cap V \\ f(F_x) \cap V &= F_{f(x)} \cap V \end{aligned}$$

for some neighbourhood, V, of $f(x)$.

Finally, we come to the crucial conditions: f is contracting on the sets E_x while it is expanding on F_x. For various technical reasons, it is much better to say that f^{-1} is contracting on F_x. Specifically, there is a constant $0 < \lambda < 1$ such that

P4 for all y, z in E_x, we have

$$d(f(y), f(z)) \leq \lambda d(y, z)$$

P5 and for all y, z in F_x, we have

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z).$$

This condition also re-inforces our earlier statement that exact invariance of E_x is not reasonable. We expect that $f(E_x)$ will be smaller than $E_{f(x)}$.

The next section will provide the rigourous definition of Smale space. Alternately, the reader can pass on to the examples in the subsequent section. Most of these should be understandable with the vague notion of Smale space which we have now.

2.2 The rigorous definition of a Smale space

We are now ready to give a precise definition of a Smale space in this section. The definition is rather long. As we go, we will try to provide some comparison with the heuristic version given in the last section.

We begin with a compact metric space (X, d). We let $f : X \to X$ be a homeomorphism of X.

We assume that there is a constant ϵ_X and a map defined on

$$\Delta_{\epsilon_X} = \{(x, y) \mid d(x, y) \leq \epsilon_X\}$$

taking values in X. The map should be continuous in the natural product topology. The image of (x, y) is denoted $[x, y]$. We assume that this satisfies certain axioms.

Before beginning the axioms, let us mention that if one has the heuristic description of Smale space in the last section, then $[x, y]$ should be thought of as the intersection of the sets E_x and F_y. Already we can see that this new version is more rigorous; there is no particularly reason in the last section that the sets E_x and F_y should intersect in a single point, provided x and y are within ϵ_X.

We require $[,]$ to satisfy the following

B1 $[x, x] = x$,

B2 $[x, [y, z]] = [x, z]$, whenever both sides are defined,

B3 $[[x, y], z] = [x, z]$, whenever both sides are defined,

B4 $[f(x), f(y)] = f([x, y])$, whenever both sides are defined.

In terms of the description of the last section and the idea that $[x, y]$ is the intersection of E_x and F_y, B1 is equivalent to P1 of the last section.

Moreover, B4 is analogous to P3, although it is a little stronger because P3 only holds for *some* set V.

The axioms B2 and B3 should be regarded as implying a kind of compatibility between the local product structures of X at nearby points. This is completely missing from our heuristic definition. Let us take a moment to explain it in those terms. We know from P2 that we have some homeomorphism between a neighbourhood of x and $E_x \times F_x$. If y is some point in that neighbourhood, the expectation is that, if we apply this map to the set E_y, or at least the part that also lies in the neighbourhood of x, the result should be of the form $A \times \{y'\}$, for some y' in F_x. Similarly, the image of F_y should be of the form $\{y''\} \times B$. This is a consequence of B2 and B3 given here.

Finally, we require that there is a constant $0 < \lambda < 1$ such that, for all x in X, we have the following two conditions.

C1 For y, z such that $d(x, y), d(x, z) \leq \epsilon_X$ and $[y, x] = x = [z, x]$, we have

$$d(f(y), f(z)) \leq \lambda d(y, z).$$

C2 For y, z such that $d(x, y), d(x, z) \leq \epsilon_X$ and $[x, y] = x = [x, z]$, we have

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z).$$

Notice that if $[y, x]$ is the intersection of E_y and F_x, saying that it is equal to x simply says that x is in E_y. Although it is not stated in the heuristic definition, this should also mean that y is in E_x. So the hypothesis of C1 above is really that y and z are both in E_x. With this in mind, these last two axioms are obviously analogous to P4 and P5 of the last section.

Definition 2.2.1. *A* Smale space *is any quadruple* $(X, d, f, [,])$ *satisfying the axioms B1, B2, B3, B4, C1 and C2.*

A word of warning is in order. The most annoying thing in dealing with Smale spaces is that the bracket map is only defined on points which are close. It is very important to check this at all times, because it is quite easy to be lead to false conclusions if this is ignored. As an example, it is tempting to say that, if x, y are in X and n is a positive integer and $d(x, y), d(f^n(x), f^n(y))$ are both less than ϵ_X, then

$$[f^n(x), f^n(y)] = f^n([x, y]).$$

In fact, this may be false, unless $d(f^k(x), f^k(y))$ is less than ϵ_X for every $1 \leq k \leq n$.

Having just made this remark, it is a good time to note the following simple result.

Theorem 2.2.2. *Let* $(X, d, f, [,])$ *be a Smale space and let* N *be a positive integer. There exists* $\epsilon_X^{(N)}$ *such that if* x, y *are in* X *with* $d(x, y) < \epsilon_X^{(N)}$*, then* $d(f^i(x), f^i(y)) < \epsilon_X$*, for all* $0 \leq i < N$*. Defining* $[x, y]^{(N)} = [x, y]$*, for all* (x, y) *in* $\Delta_{\epsilon_X^{(N)}}$*,* $(X, d, f^n, [,]^{(N)})$ *is also a Smale space.*

Following up on this, it is worth noticing the following result also.

Theorem 2.2.3. *Let $(X, d, f, [,])$ be a Smale space. Defining $[x, y]^{-1} = [y, x]$, for all (x, y) in Δ_{ϵ_X}, $(X, d, f^{-1}, [,]^{-1})$ is also a Smale space.*

Put into words which makes the result seem pretty clear, the local stable sets of $X(X, f)$ are the local unstable sets of $X, f^1)$ and vice verse.

We conclude with one other fairly obvious result: that the product of two Smale spaces is another Smale space.

Theorem 2.2.4. *Let (X_1, d_1, f_1) and (X_2, d_2, f_2) be Smale spaces. Then with the metric*

$$d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2), y_2),$$

$x_1, y_1 \in X_1, x_2, y_2 \in X_2$, $\epsilon = \min\{\epsilon_{X_1}, \epsilon_{X_2}\}$ and bracket

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2), y_2]),$$

$(X_1 \times X_2, d, f_1 \times f_2)$ is a Smale space.

Chapter 3

Examples of Smale spaces

3.1 Shifts of finite type

In this section we will introduce an important class of examples of Smale spaces, called the shifts of finite type. These are, in a certain sense, the most important. An excellent reference for a complete treatment is [**?**].

Let $\mathcal{A}$ denote a finite non-empty set, sometimes called the alphabet. We consider the space

$$\mathcal{A}^{\mathbb{Z}} = \{(a_n)_{n\in\mathbb{Z}} \mid a_n \in \mathcal{A}, \text{ for all } n \in \mathbb{Z}\}.$$

It is given the metric as follows.

Definition 3.1.1. *Let $\mathcal{A}$ be a finite, non-empty set. We define*

$$d_{\mathcal{A}}(a,b) = \inf\{2^{-|n|} \mid n \geq 0, a_i = b_i, \textit{ for all } |i| < n\}.$$

where $a = (a_n)_{n\in\mathbb{Z}}, b = (b_n)_{n\in\mathbb{Z}}$ are in $\mathcal{A}^{\mathbb{Z}}$.

Observe that, $d_{\mathcal{A}}(a,b) = 2^{-n}$, if $n \geq 0$ is the least non-negative integer such that $(a_{-n}, a_n) \neq (b_{-n}, b_n)$. In yet other words, for any $n \geq 0$, $d(a,b) < 2^{-n}$ if and only if $a_i = b_i$, for $|i| \leq n$.

We leave it as an easy exercise that this is indeed a metric. When no confusion will arise, we will often drop the subscript $\mathcal{A}$, but the reader should keep in mind that a specific function is being used. In fact, it is an *ultrametric*: it satisfies the stronger condition

$$d_{\mathcal{A}}(a,b) \leq \max\{d_{\mathcal{A}}(a,c), d_{\mathcal{A}}(c,b)\},$$

for any a, b, c in $\mathcal{A}^{\mathbb{Z}}$. Such spaces have many nice features. We summarize these as follows.

Theorem 3.1.2. *Let $\mathcal{A}$ be a finite, non-empty set. The pair $(\mathcal{A}^{\mathbb{Z}}, d_{\mathcal{A}})$ is an ultrametric space. It is compact and the set of clopen (i.e. closed and open) subsets forms a base for its topology.*

Definition 3.1.3. *We define a* shift map *$\sigma_{\mathcal{A}} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by*

$$(\sigma_{\mathcal{A}}(a))_n = a_{n+1}$$

for any a in $\mathcal{A}^{\mathbb{Z}}$ and n in $\mathbb{Z}$.

Usually no confusion will arise if we drop the subscript $\mathcal{A}$. If one writes out the elements of $\mathcal{A}^{\mathbb{Z}}$ as an infinite string, it is a little hard to make sense of the map; it looks like nothing is happening. The point is that one must keep track of the 0 entry of the sequence. We can do this by inserting a dot between the entries -1 and 0. Then our map looks like

$$\sigma(\ldots a_{-2}a_{-1}a_0.a_1a_2\ldots) = (\ldots a_{-1}a_0a_1.a_2a_3\ldots)$$

so that every entry is moved one place to the left. Sometimes for emphasis, we call $\sigma_{\mathcal{A}}$ the *left* shift.

Theorem 3.1.4. *If $\mathcal{A}$ is a finite set, then $\sigma_{\mathcal{A}}$ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$.*

Proof. It is a trivial matter to see that $\sigma_{\mathcal{A}}$ is a bijection: its inverse is given by shifting the other direction. It is also easy to see from the definition of $d_{\mathcal{A}}$ that, for every a, b in $\mathcal{A}^{\mathbb{Z}}$, we have

$$2^{-1}d_{\mathcal{A}}(a,b) \leq d_{\mathcal{A}}(\sigma_{\mathcal{A}}(a), \sigma_{\mathcal{A}}(b)) \leq 2d_{\mathcal{A}}(a,b)$$

The conclusion follows at once. $\square$

Definition 3.1.5. *If $w = (w_1, \ldots, w_n)$ is a finite sequence in elements of $\mathcal{A}$, we say that w is a* word *in $\mathcal{A}$ and that n is the* length *of the word. Given an element, a, in $\mathcal{A}^{\mathbb{Z}}$, we say that w appears in a if, for some $k \in \mathbb{Z}$,*

$$(a_{k+1}, \ldots, a_{k+n}) = (w_1, \ldots, w_n).$$

If $\mathcal{F}$ is a finite (possibly empty) collection of words in $\mathcal{A}$. We define

$$X_{\mathcal{F}} = \{a \in \mathcal{A}^{\mathbb{Z}} \mid \text{ no word in } \mathcal{F} \text{ appears in } a\}.$$

Theorem 3.1.6. *If $\mathcal{F}$ be a finite (possibly empty) collection of words in $\mathcal{A}$, then $X_{\mathcal{F}}$ is a closed subset of $\mathcal{A}^{\mathbb{Z}}$. Moreover, it is invariant under $\sigma_{\mathcal{A}}$; a is in $X_{\mathcal{F}}$ if and only if $\sigma_{\mathcal{A}}(a)$ is.*

Definition 3.1.7. *Let $\mathcal{F}$ be a finite collection of words in $\mathcal{A}$. The restriction of σ_A to $X_{\mathcal{F}}$ is denoted by $\sigma_{\mathcal{F}}$. Any non-empty system obtained as $(X_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is called a* shift of finite type.

Definition 3.1.8. *Let $\mathcal{F}$ be a finite collection of words in $\mathcal{A}$. Let N be the maximum length of the words in $\mathcal{F}$ (or $N = 1$ if $\mathcal{F}$ is empty). For a, b in $X_{\mathcal{F}}$ with $d_{\mathcal{A}}(a,b) \leq 2^{-N}$, define $[a,b]$ in $\mathcal{A}^{\mathbb{Z}}$ by*

$$([a,b]_{\mathcal{F}})_n = \begin{cases} b_n & n \leq 0 \\ a_n & n \geq 1 \end{cases}$$

Theorem 3.1.9. *If $\mathcal{F}$ be a finite collection of words in $\mathcal{A}$, then $(X_{\mathcal{F}}, \sigma_{\mathcal{F}}, d_{\mathcal{A}}, [,]_{\mathcal{F}})$ is a Smale space.*

Now that we have given the abstract definition, we will not use it again. We will instead produce two classes of examples of shifts of finite type. However, our classes are exhaustive in the sense that every shift of finite type is topologically conjugate to one in each class.

Let G be a (finite) directed graph. That is, G consists of a vertex set, G^0, an edge set, G^1, and two maps $i, t : G^1 \to G^0$. This can be viewed geometrically as follows: each vertex is a point and each edge, e, is drawn as an arrow from $i(e)$ ("i" for "initial") to $t(e)$ ("t" for "terminal").

Definition 3.1.10. *Let G be a finite directed graph. We define*

$$X_G = \{(e_n)_{n=-\infty}^{\infty} \mid e_n \in G^1, t(e_n) = i(e_{n+1}), \text{ for all } n \in \mathbb{Z}\} \subseteq (G^1)^{\mathbb{Z}}.$$

The pair (X_G, σ_{G^1}) is called the edge shift *of the graph.*

The space X_G can be viewed as the space of doubly infinite paths in the graph.

Exercise 3.1.11. *Let G be a finite graph. Prove that X_G is non-empty if and only if G has a cycle.*

Exercise 3.1.12. *Let G be a finite graph.*

1. *A vertex v in G^0 is a* source *if $t^{-1}\{v\}$ is empty. Let H be the graph obtained from G by deleting all sources and all edges e with $i(e)$ a source. Prove that $X_G = X_H$.*

2. *A vertex v in G^0 is a* sink *if $i^{-1}\{v\}$ is empty. Let K be the graph obtained from G by deleting all sources and all edges e with $t(e)$ a sink. Prove that $X_G = X_K$.*

3. *Given any G (with Σ_G non-empty), prove that there is a subgraph H with no sources or sinks such that $X_G = X_H$.*

Remark 3.1.13. *In view of the last part of Exercise 3.1.12, we will often assume a graph G has no sources or sinks.*

To see this is a shift of finite type, we use G^1 as our alphabet $\mathcal{A}$ and set $\mathcal{F} = \{(e, f) \mid t(e) \neq i(f)\}$. It is easy to see that $X_{\mathcal{F}} = X_G$.

In this case, we can give a slightly different definition of the metric and also the bracket.

Theorem 3.1.14. *Let G be a finite directed graph with at least one cycle. The function defined by*

$$d_G(e,f) = \begin{cases} 2 & t(e_0) \neq i(e_1) \\ \inf\{1, 2^{-n} \mid (e_{1-n}, \ldots, e_n) = (f_{1-n}, \ldots, f_n)\} & t(e_0) = t(f_0) \end{cases}$$

is an ultrametric metric on X_G. For any e, f in X_G with $d_G(e,f) \leq 1$, the sequence

$$([e,f]_G)_n = \begin{cases} f_n & n \leq 0 \\ e_n & n \geq 1 \end{cases}$$

for n in $\mathbb{Z}$, is in X_G and $(X_G, d_{G^1}, \sigma_{G^1}, [,]_G)$ is a Smale space.

While the definition looks exactly the same as before, the point of the theorem is that the bracket now has a larger domain.

We now define another class of shifts of finite type and, again, we begin with a finite directed graph, G. We add the hypothesis that G has no multiple edges. That is, if e and f are in G^1 with $i(e) = i(f)$ and $t(e) = t(f)$, then $e = f$.

Definition 3.1.15. *Let G be a finite directed graph with no multiple edges. The* vertex shift *of G is the shift of finite type with alphabet G^0 and*

$$\mathcal{F}_G = \{(v,w) \mid i^{-1}\{v\} \cap t^{-1}\{w\} = \emptyset\}.$$

This means that the vertex shift of G is the set of all $(v_n)_{n\in\mathbb{Z}}$ in $(G^0)^{\mathbb{Z}}$ such that, for every n in $\mathbb{Z}$, there is an edge e_n in G^1 with $i(e_n) = v_n$ and $t(e_n) = v_{n+1}$.

Theorem 3.1.16. *Let G be a finite directed graph with no multiple edges. The map $i_\infty : X_G \to X_{\mathcal{F}_G}$ defined by*

$$i_\infty((e_n)_{n\in\mathbb{Z}}) = ((i(e_n))_{n\in\mathbb{Z}})$$

for $((e_n)_{n\in\mathbb{Z}})$ in X_G is a homeomorphism and satisfies $\imath_\infty \circ \sigma_{G^1} = \sigma_{G^0} \circ i_\infty$.

We will not give a proof of theorem. It is quite simple. The point can be made quite simply by observing that, if the graph has no multiple edges then an infinite sequence of edges is uniquely determined by the vertices through which it passes.

Definition 3.1.17. *Let G be a finite directed graph. For simplicity, assume $G^0 = \{1, 2, \ldots, N\}$, for some positive integer N. The* adjacency matrix *of G, A_G is the $N \times N$ matrix with m, n entry equal to the number of edges from m to n, for $1 \leq m, n \leq N$. That is, we have*

$$A_G(m, n) = \#(i^{-1}\{m\} \cap t^{-1}\{n\}).$$

The first remark is that the adjacency matrix is always square and has non-negative integer entries. More importantly, it is easy to see that any such matrix arises from a graph. Said differently, there is an obvious construction of a finite directed graph from a square matrix with non-negative integer entries.

It is also easy to observe that the graph has no multiple edges if and only if the matrix takes only 0 and 1 as values.

It follows then that for any square matrix with non-negative integer entries, one can construct an associated edge shift and, if the matrix has $0, 1$ entries, an associated vertex shift as well.

Our final result shows the importance of vertex and edge shifts among shifts of finite type: up to conjugacy, there are all of them.

Theorem 3.1.18. *Let $\mathcal{F}$ be a finite collection of words over the alphabet $\mathcal{A}$. Then there exists a finite directed graph G and a homeomorphism $h : X_{\mathcal{F}} \to X_G$ such that $h \circ \sigma_{\mathcal{F}} = \sigma_{G^1} \circ h$. In particular, every shift of finite type is topologically conjugate to an edge shift.*

Proof. Let L be a positive integer which exceeds the length of every word in $\mathcal{F}$. We define G by

$$G^0 = \{(x_1, x_2, \ldots, x_L) \mid x \in X_{\mathcal{F}}\}$$

and

$$G^1 = \{(x_1, x_2, \ldots, x_{L+1}) \mid x \in X_{\mathcal{F}}\}$$

with

$$\begin{aligned} i(x_1, x_2, \ldots, x_{L+1}) &= (x_1, x_2, \ldots, x_L) \\ t(x_1, x_2, \ldots, x_{L+1}) &= (x_2, x_3, \ldots, x_{L+1}), \end{aligned}$$

for any $(x_1, x_2, \ldots, x_{L+1})$ in G^1.

We also define $h : X_{\mathcal{F}} \to X_G$ by

$$h(x)_n = (x_n, x_{n+1}, \ldots, x_{n+L}),$$

for x in $\mathcal{F}$ and n in $\mathbb{Z}$. It is easy to check that h maps into X_G, that it is continuous, injective and satisfies $h \circ \sigma_{\mathcal{F}} = \sigma_{G^1} \circ h$. The only non-trivial item is that h is surjective. To see this, let $(y_n)_{n \in \mathbb{Z}}$ be a sequence in X_G. this means that, for each n, y_n is a word of length $N+1$ in $\mathcal{A}$. We let $y_n = (y_{n,1}, y_{n,2}, \ldots, y_{n,L+1})$. Define $x_n = y_{n,1}$, for all n in $\mathbb{Z}$.

We first claim that for any $0 \leq l \leq L$, we have $x_{n+l} = y_{n,l+1}$, for all n. This is true for $l = 0$ (and all n) by definition. Assuming it holds for $l < L$, we verify it for $l+1$ as follows. Since y_n is a path in G, we know that

$$(y_{n,2}, \ldots, y_{n,L+1}) = t(y_n) = i(y_{n+1}) = (y_{n+1,1}, \ldots, y_{n+1,L}).$$

Simply comparing entry $1 \leq l+1 \leq L$ from both, we see that $y_{n,l+2} = y_{n+1,l+1}$. By induction hypothesis, the latter equals x_{n+1+l}. This completes the induction step.

Now we can check that the sequence x_n is actually in $X_{\mathcal{F}}$. We must see that no word in x lies in $\mathcal{F}$, but to do this is suffices to check words of length less than or equal to L. If $l \leq L$, we have $x_n, x_{n+1}, \ldots, x_{n+l-1}$ is equal to $y_{n,1}, y_{n,2}, \ldots, y_{n,l}$ which is contained in y_n which in turn is contained in an element of $X_{\mathcal{F}}$ and hence is not in $\mathcal{F}$.

Finally, we check that

$$h(x)_n = (x_n, x_{n+1}, \ldots, x_{n+L}) = (y_{n,1}, y_{n,2}, \ldots, y_{n,L} + y_n.$$

□

Exercise 3.1.19. *Let G be a finite graph (without sources or sinks). prove that the following are equivalent.*

1. *(X_G, σ) is irreducible* **??**.
2. *For every ordered pair v_1, v_2 in G^0, there is a path p with $i(p) = v_1$ and $t(p) = v_2$.*
3. *For every pair $1 \leq m, n \leq N$, there exists $k \geq 1$ such that $A_G^k(m,n) > 0$. (Such a matrix is called* irreducible*.)*

Exercise 3.1.20. *Let G be a finite graph (without sources or sinks). prove that the following are equivalent.*

1. *(X_G, σ) is mixing* **??**.
2. *There is $k \geq 1$ such that, for every ordered pair v_1, v_2 in G^0, there is a path p of length k with $i(p) = v_1$ and $t(p) = v_2$.*
3. *There exists $k \geq 1$ such that $A_G^k(m, n) > 0$, for every pair $1 \leq m, n \leq N$. (Such a matrix is called* primitive*.)*

3.2 Hyperbolic toral automorphisms

We will begin with a very specific example and then discuss some generalizations. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Observe that $\det(A) = 1$. We first regard A as a linear map of $\mathbb{R}^2$. As $\det(A) = 1$, $A\mathbb{Z}^2 = \mathbb{Z}^2$ and so A induces a map, f, of the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which is the 2-torus. Let q denote the quotient map from $\mathbb{R}^2$ to $\mathbb{T}^2$. The metric we put on $\mathbb{T}^2$ is the quotient one. This means that for points x, y in $\mathbb{R}^2$ which are sufficiently close, $d(q(x), q(y)) = |x - y|$. We claim that $(\mathbb{T}^2, f)$ is a Smale space. (Actually, $(\mathbb{R}^2, A)$ would have been a Smale space, except for the fact that $\mathbb{R}^2$ is not compact; this is a crucial axiom.)

To see the local product structure, we need a description of the eigenvalues and eigenvectors of A. Let $\gamma = (1 + \sqrt{5})/2$, which satisfies $\gamma^2 = \gamma + 1$ and $\gamma > 1$. The eigenvalues of A are γ^2 and γ^{-2}. The associated eigenvectors are $v_1 = (\gamma, 1)$ and $v_2 = (-1, \gamma)$. For any point x in $\mathbb{R}^2$, we define

$$\begin{aligned} E_{q(x)} &= \{q(x + tv_2) \mid |t| \leq \epsilon\} \\ F_{q(x)} &= \{q(x + tv_1) \mid |t| \leq \epsilon\} \end{aligned}$$

where $\epsilon > 0$ is some sufficiently small fixed parameter. If $y = x + tv_2, z = x + sv_2$, for $|s|, |t| \leq \epsilon$, then we have

$$\begin{aligned} d(f(q(y)), f(q(z))) &= d(q(Ay), q(Az)) \\ &= |Ay - Az| \\ &= |A(x + tv_2) - A(x + sv_2)| \\ &= |(t - s)Av_2| \\ &= |(t - s)\gamma^{-2}v_2| \\ &= \gamma^{-2}|(t - s)v_2| \\ &= \gamma^{-2}d(q(y), q(z)) \end{aligned}$$

This shows the contracting nature of f on $E_{q(x)}$, since $\gamma^{-2} < 1$. The contracting nature of f^{-1} on $F_{q(x)}$ is done in a similar way. The fact that the vectors v_1, v_2 form a basis for $\mathbb{R}^2$ means that the map sending the pair $(q(x+tv_2), q(x+sv_1))$ in $E_{q(x)} \times F_{q(x)}$ to $q(x+tv_2+sv_1)$ is a homeomorphism to a neighbourhood of $q(x)$ in $\mathbb{T}^2$.

To see the bracket operation in this example, we can do no better than our original discussion. The point $[q(x), q(y)]$ is the unique point in $E_{q(x)} \cap F_{q(y)}$.

There are many generalizations of this example possible. First, let A be any $N \times N$ matrix with integer entries and determinant either 1 or -1. In exactly the same fashion as above, we can construct a map f of the N-torus, $\mathbb{T}^N$. If we assume that the matrix has no eigenvalues of absolute value 1, then we construct $E_{q(x)}$ as before, using all eigenvectors whose associated (complex) eigenvalues have absolute value less than 1. Similarly, $F_{q(x)}$ is constructed from all eigenvectors whose eigenvalues are greater than 1 in absolute value. Obviously, some care must be taken in the case of complex eigenvalues and eigenvectors, but we leave this to the reader to sort out.

These are all examples of Anosov diffeomorphisms. Let M be a compact Riemannian manifold and let f be a diffeomorphism of M. We say that (M, f) is an Anosov diffeomorphism if we may find constants $C \geq 0$ and $0 < \lambda < 1$ and a splitting of the tangent space of M

$$TM = E^s \oplus E^U$$

into Tf-invariant sub-bundles such that, for all $n \geq 1$, we have

$$\begin{aligned} \parallel T(f^n)\xi \parallel &\leq C\lambda^n \parallel \xi \parallel \quad \text{for all } \xi \in E^s, \\ \parallel T(f^{-n})\eta \parallel &\leq C\lambda^n \parallel \eta \parallel \quad \text{for all } \eta \in E^u. \end{aligned}$$

The equations above look reminiscent of the definition of Smale space, but slightly different. This can be improved. In Exercises 6.4.1 and 6.4.2 of [**?**], it is shown that the definition given above is equivalent to requiring the existence is a Riemannian metric in which we have

$$\begin{aligned} \parallel (Tf)\xi \parallel &\leq \lambda \parallel \xi \parallel \quad \text{for all } \xi \in E^s, \\ \parallel T(f^{-1})\eta \parallel &\leq \lambda \parallel \eta \parallel \quad \text{for all } \eta \in E^u. \end{aligned}$$

which looks considerably more like the condition we want.

3.3 Basic sets of Axiom A Systems

We will now spend a short time discussing Smale's Axiom A systems. In a certain sense this doesn't belong in a section on examples; particularly since we won't present any explicit ones. However, this class of dynamical systems has been of great interest and, in a certain sense, be regarded as the raison d'etre for Smale spaces. Our treatment will also be quite brief. We refer the reader to [**?**] for more extensive discussions.

Smale's program for differential dynamics begins with a compact manifold, M, with a diffeomorphism $f : M \to M$. We consider the set of non-wandering points, $NW(f)$. The key ingredient in the definition of Axiom A is to suppose that the tangent bundle of M, when restricted to $NW(f)$ has a global splitting

$$T_{NW(f)}M = E^s \oplus E^u$$

and the same conditions hold on these spaces as for Anosov diffeomorphisms: for all $n \geq 1$, we have

$$\begin{aligned} \| T(f^n)\xi \| &\leq C\lambda^n \| \xi \| \quad \text{for all } \xi \in E^s, \\ \| T(f^{-n})\eta \| &\leq C\lambda^n \| \eta \| \quad \text{for all } \eta \in E^u. \end{aligned}$$

To say this another way, an Anosov diffeomorphism is an Axiom A system in which every point is non-wandering.

The other requirement for Axiom A systems is that the periodic points are dense in the non-wandering set. From our point of view here, this is needed to prove that the non-wandering set is actually a Smale space.

Theorem 3.3.1. *If (M, f) is an Axiom A system, then $(NW(f), f \mid NW(f))$ and all the basic sets are Smale spaces.*

We will not give a proof. The essential features of a proof may be found in section 6.4 of [**?**]. (See especially 6.4.9 and 6.4.13.)

Smale proposed the class of Axiom A systems for study for several reasons. First, he believed that they should be generic in a certain sense. Second, they should display structural stability: any sufficiently small perubation of such a map should actually be topologically conjugate to the original map. It seems that they may actually coincide with the class of structurally stable maps. Finally, Smale hoped that they could be classified by relatively simple combinatorial data in the same sort of fashion that Morse-Smale systems could be described. We will not concern ourselves here with all the developments of this program, but [] is an excellent reference.

One of Smale's great insights was that, even though one began with a system which was smooth, the non-wandering set itself would not usually be a sub-manifold. The first example of this was the horse-shoe. It is a diffeomorphism of the two-sphere where the non-wandering set consists of a repelling fix-point, an attracting fix-point and an invariant Cantor set where stable and unstable sub-bundles of the tangent bundle are both one-dimensional. This phenomenon has now become very well known and the non-wandering set is very typically some sort of fractal object. This is our motivation for moving from the smooth category to the topological one.

3.4 n-solenoids

In this section, we will produce on of the simplest examples which goes beyond the cases of shifts of finite type and Anosov diffeomorphisms. These are a certain kind of hybrid between the two; the stable sets are totally disconnected (like a shift of finite type) while the unstable sets are one-dimensional euclidean spaces.

We will also give two descriptions of them, one is rather topological and the other is algebraic. Later we will see generalizations if both of these directions.

Let us start with the topological version. We let S^1 denote the circle, which we regard as $\mathbb{R}/\mathbb{Z}$, both as a group and a topological space. That is, we will write the elements as real numbers $r \in \mathbb{R}$, with the understanding that $r = s$ if and only if $r - s$ is an integer.

Fix a positive integer $n \geq 2$ and let

$$X = \{(x_0, x_1, \ldots) \mid x_k \in \mathbb{R}/\mathbb{Z}, nx_{k+1} = x_k, k \geq 1\}.$$

It is a compact metric space in the metric

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{k=0}^{\infty} n^{-k} \inf\{|x_k - y_k + l| \mid l \in \mathbb{Z}\}.$$

and we define a homeomorphism f of X by

$$f(x_0, x_1, \ldots) = (nx_0, nx_1, \ldots).$$

Observe that it is invertible with

$$f^{-1}(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

Now we turn to the issue of the bracket. We let $\epsilon_X = .5$. If $d(x, y) < \epsilon_X$, it follows that we may assume x_0 and y_0 are chosen so that $|x_0 - y_0| < .5$ For convenience, let $t = x_0 - y_0$. We define

$$[x, y] = (y_0 + t, y_1 + n^{-1}t, y_2 + n^{-2}t, \ldots)$$

or $[x, y]_k = y_k + n^{-k}t, k \geq 0$. Observe that $[x, y]_0 = y_0 + t = x_0$.

First, we should observe that $n[x, y]_{k+1} = [x, y]_k$, so that $[x, y]$ is indeed in X.

We will only check the final properties, leaving the others as an exercise for the reader.

Let us suppose that $[x, y] = y$. Simply comparing the first entries we see that $x_0 = [x, y]_0 = y_0$. It follows that

$$f(y) = (ny_0, ny_1, ny_2, \ldots) = (ny_0, y_0, y_1, \ldots) = (nx_0, x_0, y_1, \ldots)$$

and from this we see that

$$\begin{aligned}
d(f(x), f(y)) &= \sum_{k=2}^{\infty} n^{-k} \inf\{|x_{k-1} - y_{k-1} + l| \mid l \in \mathbb{Z}\} \\
&= 2^{-1} \sum_{k=1}^{\infty} n^{-k} \inf\{|x_k - y_k + l| \mid l \in \mathbb{Z}\} \\
&= n^{-1} d(x, y).
\end{aligned}$$

On the other hand, suppose that $[x, y] = x$. This means that

$$(y_0 + t, y_1 + n^{-1}t, y_2 + n^{-2}t, \ldots) = (x_0, x_1, \ldots)$$

It is then a direct computation that

$$\begin{aligned}
d(f^{-1}(x), f^{-1}(y)) &= \sum_{k=0}^{\infty} n^{-k} \inf\{|x_{k+1} - y_{k+1} + l| \mid l \in \mathbb{Z}\} \\
&= \sum_{k=0}^{\infty} n^{-k-1} n^{-k-1} |t| \\
&= n^{-1} d(x, y).
\end{aligned}$$

Notice that we have also shown that, for any x in X and $K \geq 0$,

$$X^s(x, 2^{-K}) = \{y \in X \mid x_k = y_k, 0 \leq k \leq K\}.$$

In addition, we have also shown that, for any x in X and $\epsilon > 0$,

$$X^u(x, \epsilon) = \{y \in X \mid y_k = x_k + t^{-nk}, |t| < \epsilon\}.$$

Now we take the other approach which is more algebraic.

We begin with some very basic material on the p-adic numbers. We include it for completeness and refer the reader to [**?**] for a complete treatment.

Let p be any prime. We define a kind of absolute value function on the set of rational numbers, $\mathbb{Q}$, by $|0|_p = 0$ and

$$|p^k \frac{r}{s}|_p = p^{-k},$$

where k is any integer and r, s are non-zero integers relatively prime to p. The formula $|a-b|_p$ then defines a metric on $\mathbb{Q}$ and we let $\mathbb{Q}_p$ denote its completion. (In fact, this is actually an ultrametric; that is, it satisfies a stronger condition than the usual triangle inequality where the sum is replaced by the maximum.) It is a field called the *p-adic numbers*. Topologically, $\mathbb{Q}_p$ is a locally compact and totally disconnected ultrametric space. We let $\mathbb{Z}_p$ denote the closure of the usual integers, which is a compact, open subset called the *p-adic integers*. It is a subring and any non-zero integer relatively prime to p has an inverse in $\mathbb{Z}_p$. The most interesting dynamical feature of $\mathbb{Q}_p$ is that multiplication by p is a contraction (by the factor p^{-1}). Multiplication by any non-zero integer relatively prime to p is an isometry.

The constructions above can be extended, replacing the prime p with any positive integer n greater than 2. We first define $|a|_n = \sum_{p|n} |a|_p$, where a is a rational number. We let $\mathbb{Q}_n$ and $\mathbb{Z}_n$ be the completions of $\mathbb{Q}$ and $\mathbb{Z}$ respectively in the associated metric. It is a consequence of the Chinese Remainder Theorem that

$$\mathbb{Q}_n \cong \Pi_{p|n} \mathbb{Q}_p,$$

with the rational numbers embedded diagonally on the right. Both $\mathbb{Q}_n$ and $\mathbb{Z}_n$ are rings and, while the former is not a field, the latter contains an inverse for every non-zero integer relatively prime to m. Again in the natural metric, multiplication by n on $\mathbb{Q}_n$ contracts; specifically, we have

$$|na - nb|_n \leq 2^{-1} |a - b|_n,$$

for all a, b in $\mathbb{Q}_m$. Multiplication by any non-zero integer relatively prime to n is an isometry.

We give the space $\mathbb{Q}_n \times \mathbb{R}$ the metric

$$d((a, r), (b, s)) = |a - b|_n + |r - s|,$$

for all a, b in $\mathbb{Q}_m$ and r, s in $\mathbb{R}$. It is clearly translation invariant.

We leave it as an exercise to show that the subgroup

$$\{(n^{-i} j, n^{-i} j) \mid i \in \mathbb{N}, j \in \mathbb{Z}\}$$

is a discrete subgroup and is invariant under multiplication by (n, n). For simplicity, we denote this by $\mathbb{Z}[1/n]$ (which suppresses the diagonal embedding into $\mathbb{Q}_n \times \mathbb{R}$). We also leave it as an exercise to check that the quotient group $\mathbb{Q}_n \times \mathbb{R}/\mathbb{Z}[1/n]$ is compact and that

$$\begin{aligned} & d((a, r) + \mathbb{Z}[1/n], (b, s) + \mathbb{Z}[1/n]) \\ = \; & \inf\{d((a, r) + (c, c), (b, s) + (c, c)) \mid (c, c) \in \mathbb{Z}[1/n]\} \end{aligned}$$

defines a metric on this quotient space, which we denote X. We let $< a, b >$ denote the equivalence class of (a, b) in the quotient. We define

$$f < a, b >=< na, nb >, a \in \mathbb{Q}_n, b \in \mathbb{R}.$$

This expands in the first factor and contracts in the second.

3.5 Williams' one-dimensional solenoids

We begin this section with a simple class of examples. Let K be a finite directed graph. We regard this as a topological space, with a metric d. Let $f : K \to K$ be a map satisfying the following conditions.

1. f maps vertices to vertices.

2. f is continuous.

3. f is surjective.

4. the restriction of f to each edge is locally expanding; that is, there are constants $\delta > 0$, $\lambda > 1$ such that, if x, y are on the same edge and $d(x, y) \leq \delta$, then $d(f(x), f(y)) \geq \lambda d(x, y)$.

5. f is 'flattening' at the vertices; that is, there is a constant $k \geq 1$ such that each vertex, v, has a neighbourhood V such that $f^k(V)$ is homeomorphic to an open interval with $f^k(v)$ in its interior.

Let us consider an explicit example: suppose K has one vertex v, and two edges a and b. We describe f as $a \to aab$ and $b \to ab$. By this, we mean that a is divided into three equal length subintervals. The first is mapped homeomorphically onto a (and is uniformly stretched by 3), the second is mapped to a also and the third to b. The interval b is divided into two equal subintervals. The first is mapped to a (uniformly stretched by 2) and the second to b.

For the locally expanding axiom, the constant $\delta = \frac{1}{6}$ (notice that there are two distinct points in the interior of the a edge both mapped to v) and $\lambda = 2$. In the flattening axiom, we may use $k = 1$. Notice that the image of a small open ball around v (which looks like a point with four 'legs' is an interval containing the start of the a edge and the end of the b edge.

Now we let X be the inverse limit of the system

$$K \xleftarrow{f} K \xleftarrow{f} K \xleftarrow{f} \cdots .$$

More explicitly, we write

$$X = \{(x_0, x_1, x_2, \ldots) \mid x_n \in K, f(x_{n+1}) = x_n, \text{ for all } n \geq 0\}.$$

We define a metric d on X by

$$d((x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots)) = \sum_{n \geq 0} 2^{-n} d(x_n, y_n),$$

for all $(x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots)$ in X. The map on X, also denoted by f, is defined as

$$f(x_0, x_1, x_2, \ldots) = (f(x_0), f(x_1), f(x_2), \ldots),$$

for all $(x_0, x_1, x_2, \ldots)$ in X. Notice that the inverse is given by

$$f^{-1}(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

In this example, the local contracting sets are totally disconnected, while the local expanding sets are homeomorphic to intervals in the real line.

This example (or a small variation of it) is due to R.F. Williams. He also gave a more general construction where the space K is a branched manifold, of arbitrary dimension. In these examples, the local contracting set is totally disconnected, while the local expanding set is homeomorphic to an open ball in Euclidean space.

3.6 Substitution tiling systems

We work in Euclidean space $\mathbb{R}^d$, $d \geq 1$. In fact, all our examples with be with $d = 1, 2$. For x in $\mathbb{R}^d$ and $r > 0$, we let $B(x, r)$ denote the open ball of radius r centred at x.

We assume to have a finite number of subsets $p_1, \ldots, p_N$ of $\mathbb{R}^d$. These should be homeomorphic to a closed ball, but in fact, we may even assume for the moment that each is a polyhedron. We may also allow that two of them are the same subset, but carry different labels. We call these sets the proto-tiles.

We also have a constant $\lambda > 1$ and, for each $i = 1, \ldots, N$, $\omega(p_i)$, which is a collection of subsets, each of which is a translate of one of the originals, whose interiors are pairwise disjoint and whose union is the set λp_i.

We define a tile to be any translate of of one of the proto-tiles. We extend our definition of ω by setting $\omega(p_i + x) = \omega(p_i) + \lambda x$, for any i and x in $\mathbb{R}^d$.

A partial tiling T is a collection of tiles whose interiors are pairwise disjoint. A tiling is a partial tiling whose union is $\mathbb{R}^d$. We may extend our definition of ω to collections of tiles by $\omega(T) = \cup_{t \in T} \omega(t)$. Note that this is again a partial tiling. This now also allows us to iterate ω; $\omega^k(p_i)$ makes sense for any $k \geq 1$.

If T is a partial tiling, x in is $\mathbb{R}^d$ and $r > 0$, we use a slight abuse of notation by setting

$$T \cap B(x, r) = \{t \in T \mid t \subset B(x, r)\}.$$

We define Ω to be the set of tilings T such that, for any $r > 0$, there is $k \geq 1$, $1 \leq i \leq N$ and x in $\mathbb{R}^d$ such that

$$T \cap B(0, r) \subset \omega^k(p_i) + x.$$

The first basic facts are summarized as follows.

Lemma 3.6.1. *1. Ω is non-empty.*

2. $\omega(\Omega) = \Omega$.

The next step is to introduce a metric on Ω. The idea is that two elements, T, T' are close if, after a small translation, they agree on a large ball around the origin. More precisely, $d(T, T')$ is the infimum of all $\epsilon > 0$ such that, there exist x, x' in $B(0, \epsilon)$ such that

$$(T - x) \cap B(0, \epsilon^{-1}) = (T' - x') \cap B(0, \epsilon^{-1}).$$

(If no such $\epsilon > 0$ exists, we set $d(T, T') = 1$.)

We say that Ω has *finite local complexity* or FLC if, for every $r > 0$, modulo translation, there are only finitely many collections $T \cap B(x, r)$, T in Ω, x in $\mathbb{R}^d$. In the case that our proto-tiles are polyhedra and in the substitution ω, tiles meet full face to full face, this is automatic.

The remaining important properties are summarized below.

Lemma 3.6.2. *1. ω is continuous.*

2. Ω is compact if and only if it has finite local complexity.

3. $\omega : \Omega \to \Omega$ is injective if and only if Ω contains no periodic tilings.

3.7 $\frac{n}{m}$-solenoids

We begin with some very basic material on the p-adic numbers. We include it for completeness and refer the reader to [?] for a complete treatment.

Let p be any prime. We define a kind of absolute value function on the set of rational numbers, $\mathbb{Q}$, by $|0|_p = 0$ and

$$|p^k \frac{r}{s}|_p = p^{-k},$$

where k is any integer and r, s are non-zero integers relatively prime to p. The formula $|a-b|_p$ then defines a metric on $\mathbb{Q}$ and we let $\mathbb{Q}_p$ denote its completion. (In fact, this is actually an ultrametric; that is, it satisfies a stronger comdition than the usual triangle inequality where the sum is replaced by the maximum.) It is a field called the p-adic numbers. Topologically, $\mathbb{Q}_p$ is a locally compact and totally disconnected ultrametric space. We let $\mathbb{Z}_p$ denote the closure of the usual integers, which is a compact, open subset. It is also a subring and any non-zero integer relatively prime to p has an inverse in $\mathbb{Z}_p$. The most interesting dynamical feature of $\mathbb{Q}_p$ is that multiplication by p is a contraction (by the factor p^{-1}). Multiplication by any non-zero integer relatively prime to p is an isometry.

If $p < q$ are prime numbers, we consider $\mathbb{Z}[(pq)^{-1}]$, the subgroup of the rationals generated by all numbers of the form $(pq)^{-k}, k \geq 1$. The map sending r in $\mathbb{Z}[(pq)^{-1}]$ to (r, r, r) in $\mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q$ embeds the former as a lattice: its image is discrete and the quotient by this image is compact. We let X denote this quotient and ρ denote the quotient map.

We also define $\varphi : X \to X$ by

$$\varphi \circ \rho(a, r, b) = \rho\left(p^{-1}qa, p^{-1}qr, p^{-1}qb\right),$$

for a in $\mathbb{Q}_p$, r in $\mathbb{R}$ and b in $\mathbb{Q}_q$. Locally, our space X looks like $\mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q$ and multiplication by $p^{-1}q$ expands in the first two factors and contracts in the third. Without giving a precise definition, this means that (X, φ) is a Smale space with local unstable sets that are homeomorphic to opens subsets of $\mathbb{Q}_p \times \mathbb{R}$ and local stable sets that are homeomorphic to open sets in $\mathbb{Q}_q$.

We can extend the construction above to the case where $2 \leq m < n$ are relatively prime integers. We first define $|a|_m = \sum_{p|m} |a|_p$, where a is a rational number. We let $\mathbb{Q}_m$ and $\mathbb{Z}_m$ be the completions of $\mathbb{Q}$ and $\mathbb{Z}$ respectively in the associated metric. It is a consequence of the Chinese Remainder Theorem that

$$\mathbb{Q}_m \cong \Pi_{p|m} \mathbb{Q}_p,$$

with the rational numbers embedded diagonally on the right. Both $\mathbb{Q}_m$ and $\mathbb{Z}_m$ are rings and, while the former is not a field, the latter contains an inverse for every non-zero integer relatively prime to m. Again in the natural metric, multiplication by m on $\mathbb{Q}_m$ contracts; specifically, we have

$$|ma - mb|_m \leq 2^{-1}|a - b|_m,$$

for all a, b in $\mathbb{Q}_m$. Multiplication by any non-zero integer relatively prime to m is an isometry.

We define X to be the quotient of $\mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n$ by the lattice $\mathbb{Z}[(mn)^{-1}]$ and define φ with the same formula as earlier, replacing $p^{-1}q$ by $m^{-1}n$. The same comments we made in the p, q-case on the stable and unstable sets are valid here. We refer to (X, φ) as the $\frac{n}{m}$-solenoid.

3.8 Wieler's solenoids

In [], Wieler gave a very general construction for Smale spaces. The story begins with Williams' one-dimensional solenoids which we saw in **??** and their generalizations to higher dimensions which we discussed without going into detail.

Basically, by starting with a branched manifold and finite-to-one self-map of this space, Williams showed how to obtain a Smale space by taking an inverse limit. This involved some very subtle properties on the map, even in the one-dimensional case. The resulting Smale space had stable sets which were totally disconnected and unstable sets which were Euclidean. These were for different reasons. The unstable sets being Euclidean was because the starting space was a branched manifold. The stable sets being totally disconnected was due to the inverse limit construction.

Wieler's starting point was to drop the hypothesis that the initial space was a branched manifold. In fact, she gave no real conditions on that space at all. The resulting inverse limit space has stable sets that are totally disconnected, but no restrictions on the unstable sets.

A nice idea, but the immediate problem in doing this is that Williams' subtle conditions are heavily dependent on having a tangent space to the branched manifold and a derivative for the map. So the subtle conditions of **??** needed to be generalized to the setting of metric spaces.

Recall that for a metric space (Y, d), a function $g : Y \to Y$ is *open* if, for every x in Y and $\epsilon > 0$, there is a $\delta > 0$ such that $Y(g(x), \delta) \subseteq g(Y(x, \epsilon))$. In addition, such a map is *locally expanding* if there is a $\beta > 0$ and $0 < \gamma < 1$ such that, for all x, y with $d(x, y) < \beta$, we have $d(x, y) \leq \gamma d(g(x), g(y))$.

We encourage the reader to check that the maps on the wedge of circles from **??** fail to satisfy either of these conditions. The open condition fails because no image of a neighbourhood of the vertex is open and the locally expansive condition fails because $g(x) = g(y)$ will occur for two distinct points near the vertex, x and y, which are arbitrarily close.

Wieler introduced two weaker versions that we will call *eventually open*, or EO, and *eventually locally expanding* or ELE.

Let (Y, d) be a compact metric space and $g : Y \to Y$ be a continuous surjection. For $\beta > 0, 0 < \gamma < 1, K \geq 1$, we say that (Y, d, g) satisfies

1. EO if, for all $0 < \epsilon \leq \beta$ and x in Y, we have

$$g^K(Y(g^K(x), \epsilon)) \subseteq g^{2K}(Y(x, \gamma\epsilon)).$$

2. ELE if, for every x, y in Y with $d(x, y) < \beta$, we have

$$d(g^K(x), g^K(y)) \leq \gamma^K d(g^{2K}(x), g^{2K}(y)).$$

We encourage the reader to try the following exercise: remove copies of g^K from these two conditions to end up with the statements that g^K is open and locally expansive (or at least something close to that).

Back in section **??**, we defined non-wandering for homeomorphisms, but the same definition is valid for continuous maps: a point x is non-wandering if, for every open set $x \in U$, there a positive integer n such that $f^n(U) \cap U$ is non-empty.

Wieler proved two main theorems. The first is the 'construction theorem' which shows that with these conditions, the inverse limit construction leads to a Smale space. The second is a 'realization theorem' that all irreducible Smale spaces with totally disconnected stable sets may be produced in this way.

Theorem 3.8.1 (Wieler)**.** *Let (Y, d) be a compact metric space and $g : Y \to Y$ be a continuous surjection which satisfies EO and ELE for the constants $\beta > 0, 0 < \gamma < 1, K \geq 1$. Let $\hat{Y}$ be the inverse limit of the system*

$$Y \xleftarrow{g} Y \xleftarrow{g} Y \xleftarrow{g} \cdots ,$$

and $\hat{g}$ be the homeomorphism of $\hat{Y}$ defined by

$$\hat{g}(y_0, y_1, \ldots) = (g(y_0), g(y_1), \ldots),$$

for $(y_0, y_1, \ldots)$ in $\hat{Y}$.

1. *$(\hat{Y}, \hat{g})$ is a Smale space (with suitable metric).*

2. *The local stable sets of $(\hat{Y}, \hat{g})$ are totally disconnected.*

3. *If (Y, g) is non-wandering, then so is $(\hat{Y}, \hat{g})$.*

4. *If (Y, g) is non-wandering and has a dense forward orbit, then $(\hat{Y}, \hat{g})$ is irreducible.*

Theorem 3.8.2 (Wieler)**.** *Let (X, f) be an irreducible Smale space with $X^s(x, \epsilon)$ totally disconnected for every x in X and $0 < \epsilon \leq \epsilon_X$. Then there exists (Y, g) satisfying EO, ELE, every point is non-wandering and with a dense forward orbit such that $(\hat{Y}, \hat{g})$ is topologically conjugate to (X, f).*

Example 3.8.3. *Let G be a finite directed graph and let Y_G be the one-sided shift space*

$$Y_G = \{e \in \left(G^1\right)^{\mathbb{N}} \mid t(e_n) = i(e_{n+1}), n \geq 0\}.$$

with the left shift map

$$\sigma(e)_n = e_{n+1}, e \in Y_G, n \geq 0.$$

If we assume for simplicity that $\#i^{-1}\{v\} > 1$, for every vertex v in G^0, then (Y, σ) satisfies EO and ELE with $K = 1$, which is always a convenient simplification. The system $(\hat{Y}, \hat{\sigma})$ is the shift of finite type (X_G, σ).

Chapter 4

The basic structure of Smale spaces

4.1 Local stable and unstable sets

This section is devoted to the definition and study of the local stable and unstable sets in a Smale space. In our informal definition of section **??**, these are the sets E_x and F_x, but here we begin from the rigorous definition of the following section.

Recall then that in a Smale space, we have a compact metric space, (X, d), and a homeomorphism $f : X \to X$.

We also have a constant ϵ_X and a map defined on

$$\Delta_{\epsilon_X} = \{(x, y) \mid d(x, y) \leq \epsilon_X\}$$

taking values in X and satisfying certain axioms. One of these axioms is the existence of a constant $0 < \lambda < 1$ such that

C1 for y, z such that $d(x, y), d(x, z) \leq \epsilon_X$ and $[y, x] = x = [z, x]$, we have

$$d(f(y), f(z)) \leq \lambda d(y, z),$$

C2 for y, z such that $d(x, y), d(x, z) \leq \epsilon_X$ and $[x, y] = x = [x, z]$, we have

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z).$$

We are now ready to give the definition of the local stable and unstable sets.

Definition 4.1.1. *For each x in X and $0 < \epsilon \leq \epsilon_X$, we define*

$$X^s(x, \epsilon) = \{y \mid d(x, y) < \epsilon, [y, x] = x\} \quad (4.1)$$
$$X^u(x, \epsilon) = \{y \mid d(x, y) < \epsilon, [x, y] = x\}. \quad (4.2)$$

The former will be referred to as the local stable sets *and the latter as the* local unstable sets *at x.*

We quickly observe the alternate characterization of these points.

Lemma 4.1.2. *Suppose $d(x, y) \leq \epsilon_X$.*

1. *$[x, y] = x$ if and only if $[y, x] = y$.*
2. *$[x, y] = y$ if and only if $[y, x] = x$.*

Proof. We prove only the "only if" part of the first statement. The others are similar. We calculate

$$\begin{aligned} [y,x] &= [y,[x,y]] && \text{by hypothesis,} \\ &= [y,y] && \text{by axiom B2,} \\ &= y && \text{by axiom B1.} \end{aligned}$$

□

The next result simply states that, under suitable conditions, $[x,y]$ lies in $X^s(x,\epsilon_X)$ and $X^u(y,\epsilon_X)$.

Lemma 4.1.3. *Suppose that x and y are in X and $d(x,y), d(x,[x,y])$ and $d(y,[x,y])$ are all less than ϵ_X. Then we have*

$$\begin{aligned} [x,y] &\in X^s(x,\epsilon_X) \\ [x,y] &\in X^u(y,\epsilon_X) \end{aligned}$$

Proof. For the first, we check

$$\begin{aligned} [[x,y],x] &= [x,x] && \text{by hypothesis B3,} \\ &= x && \text{by hypothesis B1.} \end{aligned}$$

For the second, we check

$$\begin{aligned} [y,[x,y]] &= [y,y] && \text{by hypothesis B2,} \\ &= y && \text{by hypothesis B1.} \end{aligned}$$

□

The sets $X^s(x,\epsilon)$ and $X^u(x,\epsilon)$ are exactly the sets E_x and F_x of the last section, except that we have added a parameter ϵ to allow us to control their size. We have now set things up in such a way that our earlier hypothesis P2 is now a consequence of the other axioms.

Theorem 4.1.4. *There is $0 < \epsilon'_X \leq \epsilon_X/2$ such that, for every $0 < \epsilon \leq \epsilon'_X$, the map*

$$[,] : X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$$

is a homeomorphism to its image, which is an open set containing x.

Proof. First we note that the map is well-defined, since if both y and z are within ϵ of x and $\epsilon \leq \epsilon_X/2$, then $d(y,z) \leq \epsilon_X$ by the triangle inequality. Moreover, since $[,]$ is jointly continuous and $[x,x] = x$, we may find $0 < \delta \leq \epsilon_X$ such that, for all x, y with $d(x,y) \leq \delta$, we have $d(x,[x,y]) \leq \epsilon_X/2$ and $d(x,[y,x]) \leq \epsilon_X/2$. We choose $0 < \epsilon'_X \leq \epsilon_X/2$ so that for all y, z with $d(x,y) \leq \epsilon'_X$ and $d(x,z) \leq \epsilon'_X$, we have $d(x,[y,z]) \leq \delta$. Then we can define a map h on a neighbourhood of x by $h(y) = ([y,x],[x,y])$. By the choice of ϵ'_X this map is defined on the range of $[,]$. It is also clearly continuous.

Next, we verify that h is the inverse of the map in the definition. For y with $d(x,y) < \epsilon'_X$, we have

$$\begin{aligned} [,] \circ h(y) &= [[y,x],[x,y]] \\ &= [y,[x,y]] \\ &= [y,y] \\ &= y, \end{aligned}$$

where we have used Axioms B3, B2 and B1.

Moreover, if we begin with y in $X^u(x,\epsilon)$ and z in $X^s(x,\epsilon)$, then we have

$$\begin{aligned} h([y,z]) &= ([[y,z],x],[x,[y,z]]) \\ &= ([y,x],[x,z]) && \text{by Axioms B2 and B3,} \\ &= (y,z) && \text{by Lemma \textbf{??}.} \end{aligned}$$

Finally, we must check that the image of of our map is open. Let y be in $X^u(x,\epsilon)$ and z be in $X^s(x,\epsilon)$. From the continuity of the bracket, we may choose $\delta' > 0$ such that

$$h(X([y,z],\delta')) \subseteq X(x,\epsilon - d(x,y)) \times X(x,\epsilon - d(x,z)).$$

From this, it follows that $h(X([y,z],\delta'))$ actually lies in the domain of $[,]$ and so $X([y,z],\delta')$ lies in the range of $[,]$. The conclusion follows. □

The next important fact which we want to establish is that the choice of the bracket map is unique (up to the choice of its domain). Once this is established, then we can speak of (X,d,f) being a Smale space when such a bracket exists. We begin with the following lemma.

Lemma 4.1.5. *Suppose that $(X,d,f,[,])$ is a Smale space. Then there is a constant $0 < \epsilon_1$ satisfying the following, for all $0 < \epsilon \leq \epsilon_1$.*

ES *For x and y in X, $d(f^n(x), f^n(y)) < \epsilon$, for all $n \geq 0$, if and only if y is in $X^s(x, \epsilon)$.*

EU *For x and y in X, $d(f^n(x), f^n(y)) < \epsilon$, for all $n \leq 0$, if and only if y is in $X^u(x, \epsilon)$.*

Proof. Choose $0 < \epsilon_1 \leq \epsilon_X$ so that, for all x, y with $d(x, y) < \epsilon_1$, we have $d([y, x], x) < \epsilon_X$. We will prove that conclusion ES holds, only.

First, suppose that y is in $X^s(x, \epsilon)$. It follows from the definition of $X^s(x, \epsilon)$ and hypothesis C1 that

$$d(f(x), f(y)) \leq \lambda d(x, y) < \epsilon.$$

Therefore, we have

$$[f(y), f(x)] = f[y, x] = f(x)$$

or $f(y)$ is in $X^s(f(x), \epsilon)$. The same argument may be repeated inductively to prove the conclusion.

Now suppose that the condition of ES holds. It follows then that $[f^n(y), f^n(x)]$ is defined for all $n \geq 0$. Moreover, by 2.2.3, $[f^n(y), f^n(x)]$ is in $X^u(f^n(x), \epsilon_X)$, for all $n \geq 0$. Now we apply hypothesis B4 to note that

$$f^{-1}[f^n(y), f^n(x)] = [f^{n-1}(y), f^{n-1}(x)]$$

provided n is positive. Then we apply hypothesis C2 to assert

$$\begin{aligned} d(f^{n-1}(x), [f^{n-1}(y), f^{n-1}(x)]) &= d(f^{-1}f^n(x), f^{-1}[f^n(y), f^n(x)]) \\ &\leq \lambda d(f^n(x), [f^n(y), f^n(x)]). \end{aligned}$$

Then an easy induction shows that, for all $n \geq 0$, we have

$$\begin{aligned} d(x, [y, x]) &\leq \lambda^n d(f^n(x), [f^n(y), f^n(x)]) \\ &< \lambda^n \epsilon_X. \end{aligned}$$

Since $\lambda < 1$, we conclude that $x = [y, x]$. □

Theorem 4.1.6. *Let $(X, d, f, [,])$ be a Smale space and let ϵ_1 be as in* **??**. *If x, y are in X with $d(x, y) \leq \epsilon_X$ and $d(x, [x, y]), d(y, [x, y]) < \epsilon_1$, then we have*

$$\begin{aligned} \{[x, y]\} &= X^s(x, \epsilon_1) \cap X^u(y, \epsilon_1) \\ &= \cap_{n \in \mathbb{Z}} \{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\} \end{aligned}$$

Proof. It follows from Lemma **??** that

$$\begin{aligned} & \{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z)) < \epsilon_1, \text{ for all n } \geq 0\} \\ = & \{z \mid d(f^n(x), f^n(z)) < \epsilon_1, \text{ for all n } \geq 0\} \\ & \cap\{z \mid d(f^{-n}(y), f^{-n}(z)) < \epsilon_1, \text{ for all n } \geq 0\} \\ = & X^s(x, \epsilon_1) \cap X^u(y, \epsilon_1) \end{aligned}$$

Now it follows from the hypotheses on x, y and Lemma 2.2.3 that $[x, y]$ is in the final set. If z is any point in this set, we have $[z, x] = x$ and $[y, z] = y$. By Lemma 2.2.2, these imply $[x, z] = z$ and $[z, y] = z$. Thus we have

$$z = [x, z] = [x, [z, y]] = [x, y],$$

by condition B2. □

We note that this theorem says, among other things, that the bracket map is uniquely determined by (X, d, f), provided that it exists.

The last lemma has another immediate consequence: that is, the systems we are considering are expansive. This means that there is a positive constant (here, ϵ_1) so that any two distinct points, no matter how close, may be separated by at least this constant, by applying the map f (or f^{-1}) a number of times to both.

Corollary 4.1.7. *The map f is expansive for the constant ϵ_1; i.e. if x and y are in X and $d(f^n(x), f^n(y)) < \epsilon_1$, for all integers n, then $x = y$.*

Proof. The condition clearly implies that both x and y are in

$$\cap_{n\in\mathbb{Z}}\{z \mid d(f^n(x), f^n(z)) < \epsilon_1, d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\}$$

and hence $x = [x, y] = y$. □

We note that the usual definition of expansive allows for equality in the hypothesis. If one prefers to use that version, then $\epsilon_1/2$ is an expansive constant.

We noted earlier **??** that if (X, f, d) is a Smale space, then so is (X, f^n, d), for any positive integer n, even using the same bracket, but with a smaller domain of definition. The same is also true for $n = -1$ **??**, but that required switching the variables in the definition of the bracket. From these observations, the following is an immediate consequence. We omit the proof. The only troubling part is in the notation because $X^s(x, \epsilon)$ does not make explicit reference to which map is being used. Usually, this is not an issue.

Proposition 4.1.8. *Let (X, f, d) be a Smale space.*

1. *Let $n \geq 1$ and $\epsilon_X^{(n)} > 0$ be as in* **??**. *For all $0 < \epsilon \leq \epsilon_X^{(n)}$, the set $X^s(x, \epsilon)$ (respectively, $X^u(x, \epsilon)$) is the same in the system (X, f^n, d) as for (X, f, d).*

2. *For $0 < \epsilon \leq \epsilon_X$, the set $X^s(x, \epsilon)$ (respectively, $X^u(x, \epsilon)$) in the system (X, f^{-1}, d) is the same as $X^u(x, \epsilon)$ (respectively, $X^s(x, \epsilon)$) for the system (X, f, d).*

4.2 Global stable and unstable sets

In this section, we want to introduce and investigate the notions of global stable and unstable sets for a Smale space. Obviously, there should be some relation with the local stable and unstable sets. Without being very technical, there are two important distinctions. The first is that these sets are actually equivalence classes for equivalence relations which we call stable and unstable equivalence. The second feature is less good: they are much more complicated than their local counterparts.

Before stating the definitions, we establish the following preliminary result.

Lemma 4.2.1. *Let x be in X and $0 < \epsilon < \epsilon_X$. Then we have*

$$f(X^s(x, \epsilon)) \subseteq X^s(f(x), \lambda\epsilon) \subseteq X^s(f(x), \epsilon).$$

Moreover, the set on the left is an open subset of that on the right. Also, we have

$$f^{-1}(X^s(x, \epsilon)) \subseteq X^s(f^{-1}(x), \lambda\epsilon) \subseteq X^s(f^{-1}(x), \epsilon).$$

Moreover, the set on the left is an open subset of that on the right.

Proof. We will only consider the first statement, the other being similar. We first note that the fact that the range is contained in the given set follows from the definitions and the properties of the map f. We must show that the range is open. Begin with y in $X(x, \epsilon)$. As f^{-1} is continuous and hence uniformly continuous, we may find $\epsilon_X > \delta > 0$ such that $d(z, z') < \delta$ implies $d(f^{-1}(z), f^{-1}(z')) < \epsilon_X - d(x, y)$. So if $d(y, z) < \delta$ and, in addition, z is in $X^s(f(x), \epsilon)$, then using $z' = f(y)$, we have

$$d(x, f^{-1}(z)) \leq d(x, y) + d(y, f^{-1}(z)) < eX.$$

In addition, we have

$$[x, f^{-1}(z)] = f^{-1}[f(x), z] = f^{-1}(f(x)) = x,$$

from which we conclude $f^{-1}(z)$ is in $X^s(x, \epsilon)$. This completes the proof. □

Definition 4.2.2. *Let (X, d, f) be a Smale space. We say two points x, y in X are* stably equivalent *and write $x \overset{s}{\sim} y$ if*

$$\lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0.$$

We let $X^s(x)$ be the set of y with $x \stackrel{s}{\sim} y$. We say that x, y are unstably equivalent *and write $x \stackrel{u}{\sim} y$ if*

$$\lim_{n \to -\infty} d(f^n(x), f^n(y)) = 0.$$

We let $X^u(x)$ be the set of y with $x \stackrel{u}{\sim} y$.

It is immediate that each of these is an equivalence relation. It is also fairly clear that $x \stackrel{s}{\sim} y$ (or $x \stackrel{u}{\sim} y$) if and only if $f(x) \stackrel{s}{\sim} f(y)$ (or $f(x) \stackrel{u}{\sim} f(y)$, respectively).

It should also be clear from last lemma that if y is in $X^s(x, \epsilon_X)$, then one can show inductively that, for every positive n, $f^n(y)$ is in $X^s(f^n(x), \epsilon_X)$ and that

$$d(f^n(y), f^n(x)) \leq \lambda^n d(x, y)$$

and since $\lambda < 1$, we have $x \stackrel{s}{\sim} y$. In short, $X^s(x, \epsilon_X) \subseteq X^s(x)$. One might even have anticipated this result from the fact that we called $X^s(x, \epsilon_X)$ the local stable set of x. In a similar way, we have $X^u(x, \epsilon_X) \subseteq X^u(x)$.

We can take these last comments a step further by observing that if $f^n(y)$ is in $X^s(f^n(x), \epsilon_X)$ for any positive integer n, then $f^n(x) \stackrel{s}{\sim} f^n(y)$ and hence, $x \stackrel{s}{\sim} y$. We will show that this is a complete description.

Proposition 4.2.3. *Let x be in X and $0 < \epsilon \leq \epsilon_X$. We have*

$$X^s(x) = \bigcup_{n \geq 0} f^{-n}(X^s(f^n(x), \epsilon))$$

and

$$X^u(x) = \bigcup_{n \geq 0} f^n(X^s(f^{-n}(x), \epsilon)).$$

Proof. We will show only the first statement. The second is obtained in the same way. We have already argued that any point in the set on the left given is in the stable equivalence class of x. Conversely, suppose that $y \stackrel{s}{\sim} x$. Then we may choose $N \geq 0$ sufficiently large so that

$$d(f^n(x), f^n(y)) \leq \epsilon' = \min\{\epsilon, \epsilon_1\}$$

for all $n \geq N$, where ϵ_1 is as given in Lemma **??**. By Lemma **??**, we have $f^N(y)$ is in $X^s(f^N(x), \epsilon)$. This completes the proof. $\square$

Example 4.2.4. *Let G be a graph and X_G be the associated shift of finite type. Recall from the definitions that for any e and f in X_G, we have f is in $X^s(e, \epsilon_X)$ if and only if $e_n = f_n$, for all $n \geq 0$. It follows that $\sigma^N(f)$ is in $X_G^s(\sigma^N(e), \epsilon_X)$ if and only if $e_n = f_n$, for all $n \geq N$. So we see that $e \overset{s}{\sim} f$ if and only if there is some $N \geq 0$ such that $e_n = f_n$, for all $n \geq N$. This is usually referred to as* right tail equivalence*. Analogous statements are available for unstable equivalence, including the notion of left tail equivalence.*

Example 4.2.5. *Let $(\mathbb{T}^2, f)$ be the hyperbolic toral automorphism of section* **??**. *If x is any point in $\mathbb{R}^2$, we have seen the local stable set*

$$(\mathbb{T}^2)^s(q(x), \epsilon) = \{q(x + tv_2) \mid |t| \leq \epsilon\}.$$

We leave it is an exercise to check (from Proposition **??***) that*

$$(\mathbb{T}^2)^s(q(x)) = \{q(x + tv_2) \mid t \in \mathbb{R}\}.$$

(It helps quite a bit to try $x = (0, 0)$ first.)

Observe that this set is dense in $\mathbb{T}^2$, justifying our earlier comment that $(\mathbb{T}^2)^s(q(x))$ is significantly more complicated than $(\mathbb{T}^2)^s(q(x), \epsilon)$.

We note the following fairly easy result for future reference.

Theorem 4.2.6. *Let (X, f, d) be a Smale space.*

1. *The equivalence relations of stable and unstable equivalence are invariant under f; that is, for any points x and y in X, $x \sim_s y$ (or $x \sim_u y$) if and only if $f(x) \sim_s f(y)$ ($f(x) \sim_u f(y)$, respectively).*

2. *The stable (or unstable) equivalence relation in the Smale space (X, f, d) is the same as the stable (or unstable, respectively) equivalence relation in the Smale space (X, f^N, d), for any $N \geq 1$.*

3. *Stable (or unstable) equivalence in the Smale space (X, f, d) is the same as unstable (or stable, respectively) equivalence equivalence in the Smale space (X, f^{-1}, d).*

Proof. The first and thirds parts are obvious. For the second, stable equivalence in (X, f, d) obviously implies stable equivalence in (X, f^N, d), for any $N \geq 1$. Conversely, by the uniform continuity of the maps $f, f^2, \ldots, f^{N-1}$, for any $\epsilon > 0$, we may find $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f^i(x), f^i(y)) < \epsilon$, for all $1 \leq i < N$. It follows that if $d(f^{Nn}(x), f^{Nn}(y))$ tends to 0 as n tends to $+\infty$, then so does $d(f^n(x), f^n(y))$. This means that stable equivalence for f^N implies stable equivalence for f. $\square$

The next result is an easy consequence of **??**. The point is that, now having the notions of global stable and unstable sets, it can be stated in a simpler form.

Proposition 4.2.7. *Let (X, f, d) be a Smale space and x, y be in X with $d(x, y) < \epsilon_X$. Then we have $[x, y]$ is in $X^s(x) \cap X^u(y)$.*

Recall the phenomenon observed in the last examples, namely that the global stable and unstable equivalence classes are dense. In fact, this is always the case in mixing Smale spaces.

Theorem 4.2.8. *Let (X, d, f) be a mixing Smale space and let x be in X. Then $X^s(x)$ and $X^u(x)$ are dense in X. In fact, for any x, y in X, $X^s(x) \cap X^u(y)$ is dense in X.*

Proof. Let δ be positive and let y be in X. We will show that $X^s(x)$ meets $X(y, \delta)$. First, we choose $\epsilon_X > \delta' > 0$ such that if $d(x', y') < \delta'$, then $d([x', y'], y') < \delta/2$. Considering $f^n(x), n \geq 0$, find a subsequence $f^{n_i}(x)$ which converges to some point x_0 in X. Let $U = X(y, \delta/2)$ and $V = X(x_0, \delta'/2)$ and apply the definition of mixing to find a positive integer N such that $f^n(U) \cap V$ is non-empty for all $n \geq N$. Then find $n_i \geq N$ such that $f^{n_i}(x)$ is in $X(x_0, \delta'/2)$. From the choice of N, there is z in U with $f^{n_i}(z)$ in V. This means that $f^{n_i}(z)$ and $f^{n_i}(x)$ are in $X(x_0, \delta'/2)$. It follows that $w = f^{-n_i}[f^{n_i}(x), f^{n_i}(z)]$ is well-defined. More over, $f^{n_i}(w)$ is in $X^u(f^{n_i}(z), \delta/2)$. It follows that $d(w, z) \leq \lambda^{n_i}\delta/2$ and from this that w is in $X(y, \delta)$. On the other hand, $f^{n_i}(w)$ is in $X^s(f^{n_i}(x), \delta/2)$ and from this it follows that w is in $X^s(x)$.

For the last statement. Let U be any open set; we will prove that U meets $X^s(x) \cap X^u(y)$. First, by continuity of the bracket, we may find an open subset $V \subseteq U$ of diameter less than ϵ_X such that $[V, V] \subseteq U$. As $X^s(x)$ and $X^u(y)$ are both dense in X, we may find x' in the former and y' in the latter such that both are in V. The point $[x', y']$ is well-defined, is in U and is in $X^s(x) \cap X^u(y)$. This completes the proof.

$\square$

For a given x in X, the set $X^s(x)$ is a subset of X and has a relative topology from X. This is terrible, in particular it is not locally compact. However, there is another much more natural topology on it. To see this, we will use the description of $X^s(x)$ given in Proposition **??**.

Now each set, $f^{-n}(X^s(f^n(x), \epsilon))$ is given the relative topology of X and the set $X^s(x)$ is given the inductive limit topology. A subset, U, is open if and only if its intersection with $f^{-n}(X^s(f^n(x), \epsilon))$ is open, for all but finitely many n. The unstable set is given a topology in a similar fashion.

In Example **??** above, we leave it as an exercise for the reader to verify that $(\mathbb{T}^2)^s(q(x))$, with this inductive limit topology, is homeomorphic to $\mathbb{R}$. The following will be useful in dealing with this new topology in many cases. The fourth and fifth conditions are particularly helpful because they provide a nice link between the local and global stable/unstable sets.

Theorem 4.2.9. *Let x be a point in the Smale space (X, d, f).*

1. *The sets $X^s(x)$ and $X^u(x)$, endowed with the inductive limit topology above, are locally compact and Hausdorff.*

2. *A sequence y_n in $X^s(x)$ converges to y in $X^s(x)$ if and only if it converges in the usual topology of X and $[y_n, y] = y$, for all n sufficiently large.*

3. *A sequence y_n in $X^u(x)$ converges to y in $X^u(x)$ if and only if it converges in the usual topology of X and $[y, y_n] = y$, for all n sufficiently large.*

4. *Sets of the form $X^s(y, \epsilon)$, where y is in $X^s(x)$ and $0 < \epsilon \leq \epsilon_X$, form a neighbourhood base for the inductive limit topology on $X^s(x)$.*

5. *Sets of the form $X^u(y, \epsilon)$, where y is in $X^u(x)$ and $0 < \epsilon \leq \epsilon_X$, form a neighbourhood base for the inductive limit topology on $X^u(x)$.*

6. *The map $f : X^s(x) \to X^s(f(x))$ is a homeomorphism when these sets are given the inductive limit topologies.*

7. *The map $f : X^u(x) \to X^u(f(x))$ is a homeomorphism when these sets are given the inductive limit topologies.*

Proof. That $X^s(x)$ is Hausdorff follows from the fact that each set in the inductive limit is Hausdorff. We consider the fact that it is locally compact. Let y be a point of $X^s(x)$, which means that $f^N(y)$ is in $X^s(f^N(x), \epsilon_X)$, for some $N \geq 0$. By replacing N by $N + 1$ if necessary, we may assume that $d(f^N(y), f^N(x)) < \epsilon_X$. Choose $0 < \delta < \epsilon_X - d(f^N(y), f^N(x))$ and let $U = f^{-N}(X^s(f^N(x), \delta))$. This set is equal to

$$f^{-N}(X(f^N(x), \delta)) \cap f^{-N}(X^s(f^N(x), \epsilon_X))$$

and, hence, open in the relative topology of $f^{-N}(X^s(f^N(x), \epsilon_X))$. We want to show the same is true for each of the sets $U \subset f^{-n}(X^s(f^n(x), \epsilon_X))$, for every $n \geq N$. For each such n, $f^n(U)$ is open in the relative topology of $X^s(f^n(x), \epsilon_X)$, from Lemma **??**. The desired conclusion follows since f is a homeomorphism. So the set U is open in the inductive limit topology of $X^s(x)$. Its closure is compact in $f^{-N}(X^s(f^N(x), \epsilon_X))$ and hence in $X^s(x)$. This completes the proof of the first part.

For the second statement, suppose that y_n converges to y in the relative topology. Repeating the argument in the first part, for some $N \geq 0$, y is in $f^{-N}(X^s(f^N(x), \epsilon_X))$ and for some $0 < \delta$, the set

$$U = f^{-N}(X^s(f^N(x), \delta))$$

is a neighbourhood of y in $X^s(x)$. So for all n sufficiently large, y_n is in U. Also for all n sufficiently large, y_n is sufficiently close to y so that $d(f^{-k}(y_n), f^{-k}(y)) < \epsilon_X$, for all $0 \leq k \leq N$. Then we have

$$\begin{aligned} [y_n, y] &= f^{-N}[f^N(y_n), f^N(y)] \\ &= f^{-N}(f^N(y)) \\ &= y \end{aligned}$$

because both $f^N(y_n)$ and $f^N(y)$ are in $X^s(f^N(x), \epsilon_X)$.

Now suppose that y_n converges to y and $[y_n, y] = y$, for all n sufficiently large. Then for some N, $f^N(y)$ is in $X^s(f^N(x), \epsilon_X)$ and, again replacing N by $N+1$ if necessary, we may assume that $d(f^N(y), f^N(x)) < \epsilon_X$. Then for n sufficiently large, we have $d(f^N(y_n), f^N(x)) \leq \epsilon_X$ and we can compute

$$\begin{aligned} [f^N(y_n), f^N(x)] &= [[f^N(y_n), f^N(y)], f^N(x)] \\ &= [f^N[y_n, y], f^N(x)] \\ &= [f^N(y), f^N(x)] \\ &= f^N(x) \end{aligned}$$

This means that $f^N(y_n)$ is converging to $f^N(y)$ in the relative topology of $X^s(f^N(x), \epsilon_X)$ and, hence, y_n is converging to y in the relative topology of $f^{-N}(X^s(f^N(x), \epsilon_X))$. Now we note that the inclusion of this set in the inductive limit is continuous and the desired conclusion follows. $\square$

There is one more equivalence relation on the points of X which is quite important. It is just the intersection of stable and unstable equivalence, but it has a number of very nice features which we will exploit.

Definition 4.2.10. *Two points x and y in X are* homoclinic *if they are both stably and unstably equivalent. That is, we have*

$$\lim_{|n| \to \infty} d(f^n(x), f^n(y)) = 0.$$

In this case, we write $x \stackrel{h}{\sim} y$. Here, we denote the equivalence class of x by $X^h(x)$.

It is worth considering an example at this point. If we once again think about our shift of finite type X_G, we see that $e \stackrel{h}{\sim} f$ if and only if $e_n = f_n$ for all but finitely many n.

4.3 Shadowing

In this section we discuss a critical property of Smale spaces called shadowing. The section will be somewhat technical and we will not use the results until later sections, but the concept is crucial.

First of all, if a is in $\mathbb{Z} \cup \{-\infty\}$ and b is in $\mathbb{Z} \cup \{\infty\}$, then we say that $I = (a, b) = \{n \in \mathbb{Z} \mid a < n < b\}$ is an interval in $\mathbb{Z}$.

We begin with a pair of definitions.

Definition 4.3.1. *Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism. For any $\epsilon > 0$, an* ϵ-pseudo-orbit *over a non-empty interval, I, is a collection of points x_n, for each n in I, such that*

$$d(f(x_n), x_{n+1}) \leq \epsilon$$

provided n and $n + 1$ are in I.

Observe first that if x is in X, then for any I, the points $x_n = f^n(x)$ are an ϵ-pseudo-orbit, for any positive ϵ. That is, orbits are pseudo-orbits.

Definition 4.3.2. *Let $\epsilon > 0$ and $\delta > 0$. If x_n and y_n are ϵ-pseudo-orbits over the same interval I, then we say that one* δ-shadows *the other if*

$$d(x_n, y_n) \leq \delta$$

for all n in I. If x is in X, we also say that x_n is δ-shadowed *by (the orbit of) x, if*

$$d(x_n, f^n(x)) \leq \delta$$

for all n in I.

Our objective is to prove the following result.

Theorem 4.3.3. *Let (X, d, f) be a Smale space. For any $\delta > 0$, there is an $\epsilon > 0$ such that every ϵ-pseudo-orbit in X is δ-shadowed by an orbit of X.*

We will need the following result in the proof.

Lemma 4.3.4. *Suppose that $0 < \delta_1 \leq \epsilon_X$. Then there is $\epsilon > 0$ such that, if $d(f(x), x') < \epsilon$, then for all z in $X^s(x, \delta_1)$, $d(x', f(z)) \leq \epsilon_X$ and $[x', f(z)]$ is in $X^s(x', \delta_1)$.*

Proof. First, it is clear that $[x', f(z)]$ is well-defined and in the local stable set of x'. We must find ϵ so that $d(x', [x', f(z)]) \leq \delta_1$.

Consider the set

$$A = \{(x, y, z) \mid d(x, y), d(y, z) \leq \epsilon_X/2, [y, z] = z\}$$

which is compact in $X \times X \times X$. Consider also the function h defined on A by

$$h(x, y, z) = d(x, [x, z]) - d(y, z)$$

which is clearly continuous and hence uniformly continuous. On the set,

$$B = \{(x, y, z) \in A \mid x = y\}$$

which is compact, we have

$$\begin{aligned} h(x, y, z) &= h(y, y, z) \\ &= d(y, [y, z]) - d(y, z) \\ &= d(y, z) - d(y, z) \\ &= 0. \end{aligned}$$

Therefore, there is $\epsilon > 0$ such that, if $d(x, y) < \epsilon$ and $(x, y, z) \in A$, then

$$|h(x, y, z)| < \delta_1(1 - \lambda).$$

Also choose ϵ sufficiently small so that $\epsilon < (1 - \lambda)\delta_1$.

Now consider x, x', z as in the statement. First we have $d(x, z) < \epsilon$ and hence

$$\begin{aligned} d(x', f(z)) &\leq d(x', f(x)) + d(f(x), f(z)) \\ &< \epsilon + \lambda d(x, z) \\ &< (1 - \lambda)\delta_1 + \lambda\delta_1 \\ &\leq \delta_1 \\ &\leq \epsilon_X \end{aligned}$$

Also, we have $(x', f(x), f(z))$ is in the set A and $d(x', f(x)) < \epsilon$ and so we conclude that

$$\begin{aligned} d(x', [f(x), f(z)]) &\leq h(x', f(x), f(z)) + d(f(x), f(z)) \\ &< \delta_1(1 - \lambda) + \lambda d(x, z) \\ &< \delta_1(1 - \lambda) + \lambda\delta_1 \\ &= \delta_1 \end{aligned}$$

This completes the proof. □

Proof of Theorem 4.3.3. First we choose $0 < \delta_1 \leq \epsilon_X/2$ such that

$$[X^u(x,\delta_1), X^s(x,\delta_1)] \subset X(x,\delta),$$

for all x in X. Next, we choose $\epsilon > 0$ as in Lemma **??** which holds for both (X,d,f) and (X,d,f^{-1}).

We will first show the conclusion holds in the case where $I = (a,b)$ is finite and $a < 0$ and $b > 0$. For each $a < i < b-1$, we define a map $g_i : X^s(x_i,\delta_1) \to X^s(x_{i+1},\delta_1)$ by $g_i(z) = [x_{i+1}, f(z)]$. The fact that g_i is well-defined follows from the conclusions of the last Lemma.

In an analogous fashion, we may define a map $h_i : X^u(x_i,\delta_1) \to X^u(x_{i-1},\delta_1)$ by $h_i(z) = [f^{-1}(z), x_{i-1}]$, for $a+1 < i < b$. We let g and h be the union of the functions g_i, $a < i < b-1$, and h_i, $a+1 < i < b$, respectively.

We define sets

$$S_i = \left[h^{b-1+i}(X^u(x_{b-1},\delta_1), g^{1-a+i}(X^s(x_{a+1},\delta)\right],$$

for every $a < i < b$, and we claim that any point x in S_0 will shadow the pseudo-orbit. It is clear that

$$\begin{array}{rcl} h^{b-1+i}(X^u(x_{b-1},\delta_1) & \subset & X^u(x_i,\delta_1) \\ g^{1-a+i}(X^s(x_{a+1},\delta_1) & \subset & X^s(x_i,\delta_1) \end{array}$$

and then from the choice of δ_1, we have $S_i \subset X(x_i,\delta)$. Then it suffices for us to show that $f(S_i) = S_{i+1}$.

Choose y in $X^s(x_{a+1},\delta_1)$ and z in $X^u(x_{b-1},\delta_1)$. Let $y' = g^{i-a-1}(y)$ and $z' = h^{i-b}(z)$ so that we have $[h(z'),y'] \in S_i$ and $[z', g(y')] \in S_{i+1}$. Any element of S_i can be obtained in this way. We will show that f carries the former to the latter. In the following computation, one must verify that all bracket operations are defined. We leave this tedious aspect of the proof to the reader. We have

$$\begin{array}{rcl} f([h(z'),y']) & = & f(\left[\left[f^{-1}(z'),x_i\right],y'\right]) \\ & = & \left[f(\left[f^{-1}(z'),x_i\right]), f(y')\right] \\ & = & [[z', f(x_i)]), f(y')] \\ & = & [z', f(y')] \\ & = & [z', [x_{i+1}, f(y')]] \\ & = & [z', g(y')] \end{array}$$

This completes the proof.

We now address the problem when the interval is infinite. In fact, we consider the case only for $I = \mathbb{Z}$ and leave the half-open cases for the reader.

We begin by considering $I_n = (-n, n)$, for any positive integer n. Notice that the choice of δ_1 is independent of n. This also means that every point of

$$S^{(n)} = \left[h^{n-1}(X^u(x_{n-1}, \delta_1)), g^{n-1}(X^s(x_{-n+1}, \delta_1))\right]$$

will δ-shadow the pseudo-orbit over the interval I_n. The same is true of the closure of $S^{(n)} = \overline{S^{(n)}}$. It follows directly from the definitions that $\overline{S^{(n)}} \supseteq \overline{S^{(n+1)}}$ for all n. The intersection of all $\overline{S^{(n)}}$ is non-empty and any point in this intersection will δ-shadow the pseudo-orbit over all of $\mathbb{Z}$. This completes the proof. $\square$

As we indicated earlier, most important applications of this result will be seen later.

4.4 Periodic points in Smale spaces

As our first important application of shadowing, we prove the following result: the set of non-wandering points in a Smale space is the closure of the set of periodic points. One containment is quite obvious; every periodic point is non-wandering and the set of non-wandering points is closed, so the closure of the periodic points is contained in the non-wandering set.

Theorem 4.4.1. *Let (X, d, f) be a Smale space. Then the set of periodic points for f, $Per(X, f)$, is dense in $NW(X, f)$. In particular, if X is non-wandering, then $Per(X, f)$ is dense in X.*

Proof. Let x_0 be a non-wandering point of X and let ϵ_0 be positive. Let $\delta = \epsilon_1/2$, where ϵ_1 is the expansiveness constant for (X, d, f) **??**. Choose $\epsilon_2 > 0$ so that every ϵ_2-pseudo-orbit is δ-shadowed by an orbit. Let ϵ be the minimum of ϵ_0, δ and ϵ_2 and let $V = X(x_0, \epsilon)$ Since x_0 is non-wandering, there is a positive integer n and a point x in V, with $f^n(x)$ also in V. Now we define $x_{in+j} = f^j(x)$, for any i in $\mathbb{Z}$ and $0 \leq j < n$. It is easy to verify that this is a ϵ-pseudo-orbit over $\mathbb{Z}$. Then we may find a point y whose orbit δ-shadows x_n. In particular, y is in $X(x, \delta)$ and hence in $X(x_0, \epsilon_0)$. We claim that y is periodic. To see this we note that, for any integer i, $x_{n+i} = x_i$ and so we have

$$\begin{aligned} d(f^i(y), f^{i+n}(y)) &\leq d(f^i(y), x_i) + d(x_{i+n}, f^{i+n}(y)) \\ &\leq \delta + \delta \\ &\leq \epsilon_1. \end{aligned}$$

Applying the expansiveness condition **??** to the points y and $f^n(y)$, we see that are equal and hence, y is periodic. □

This last result has a nice consequence which is the converse of 4.2.8. In fact, we need only

Theorem 4.4.2. *Let (X, f) be a non-wandering Smale space. If (X, f) is mixing, then $X^s(x)$ and $X^u(x)$ are dense in X, for any x in X. Conversely, if $X^s(x)$ and $X^u(x)$ are dense in X, for any periodic point,x in X, then (X, f) is mixing.*

Proof. We have already shown the only if direction in 4.2.8. As for the converse, let U and V be non-empty open sets. As (X, f) is non-wandering, we may apply Theorem 4.4.1 to find a periodic point x in U. Let p be its period.

Find $\epsilon > 0$ such that $X^u(x, \epsilon)$ is contained in U. It follows from **??** that the sets $f^{mp}(X^u(x, \epsilon)), m \geq 1$ are increasing and their union is dense in X. The same is true of we replace x with $f^i(x), 1 \leq i < p$. So we may find $M \geq 1$ such that

$$f^{mp}(X^u(f^i(x), \epsilon)) \cap V \neq \emptyset$$

for all $m \geq M$ and $0 \leq i < p$. Let $N = Mp$. We claim that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$. For $n \geq N$, write $n = mp + i$, for some $m \geq M$ and $0 \leq i < p$. Then we have

$$\begin{aligned} f^n(U) \cap V &\supseteq f^n(X^u(x, \epsilon)) \cap V \\ &= f^{mp} \circ f^i(X^u(x, \epsilon)) \cap V \\ &\supseteq f^{mp}(X^u(f^i(x), \epsilon)) \cap V \\ &\neq \emptyset. \end{aligned}$$

This completes the proof. □

There is a natural question: if (X, f) is a Smale space, then is $NW(X, f)$ also? Of course, $NW(X, f)$ is closed and invariant under f, but we need to see it is closed under the bracket operation as well.

Lemma 4.4.3. *Let (X, d, f) be a Smale space. If x and y are periodic and $d(x, y) \leq \epsilon_X$, then $[x, y]$ is non-wandering.*

Proof. Let p and q be the respective periods of x and y. Let U be any open set containing $[x, y]$ and choose $\delta > 0$ such that $X([x, y], 2\delta) \subseteq U$.

By the shadowing property of **??**, we may find $\epsilon > 0$ such that every ϵ-pseudo-orbit is δ-shadowed by an orbit. Since $[x, y]$ is in $X^s(x)$ and $f^p(x) = x$, we may find $i > 0$ with $d(f^{ip}([x, y]), x) < \epsilon/2$. In a similar way, we may find $j, k, l > 0$ with

$$d(f^{-jp}([y, x]), x), d(f^{kq}([y, x]), y), d(f^{-lq}([x, y]), y) < \epsilon/2.$$

It is then a simple matter to see that the sequence

$$\begin{array}{lll} [x, y], & f([x, y]), \ldots, f^{ip-1}([x, y]) & \\ & f^{-jp}([y, x]), \ldots, f^{-1}([y, x]), & [y, x] \\ & f([y, x]), \ldots, f^{kq-1}([y, x]) & \\ & f^{-lq}([x, y]), \ldots, f^{-1}([x, y]) & [x, y] \end{array}$$

is an ϵ-pseudo-orbit. Approximating this by a δ-orbit proves that $f^{ip+jp+kq+lq}(U) \cap U$ is non-empty. This completes the proof. □

Theorem 4.4.4. *Let (X, d, f) be a Smale space. If x and y are non-wandering and $d(x, y) < \epsilon_X$, then $[x, y]$ is also non-wandering. In particular, $(NW(X, f), d, f)$ is also a Smale space.*

Proof. We know from Theorem 4.4.1 that there exist sequences of periodic points $x_n, y_n, n \geq 1$ converging to x, y, respectively. For sufficiently large n, we also have $d(x_n, y_n) < \epsilon_X$ and so by Lemma 4.4.3, $[x_n, y_n]$ is non-wandering. Since the bracket is continuous, this sequence converges to $[x, y]$ and since the set of non-wandering points is closed, we conclude that $[x, y]$ is non-wandering.

It follows at once that $(NW(X, f), d, f)$ is also a Smale space, although one needs to reduce the constant ϵ_X to something strictly smaller. □

4.5 Decomposition of Smale spaces

We have introduced the notions of non-wandering, irreducibility and mixing and we have noticed that mixing implies irreducibity, which implies non-wandering. These implications hold in generality and the converses do not. We now want to restrict our consideration to the case of Smale spaces.

The converse directions are still false. For example, the finite disjoint union of at least two irreducible Smale spaces is still non-wandering but no longer irreducible. The remarkable fact which we will prove is that every non-wandering Smale space arises in exactly this way; that is, it may be decomposed into a finite number of irreducible components. The precise statement is given in Corollary 4.5.6

There is a similar situation for mixing. If (X, f) is a mixing Smale space and we let $Y = X \times \{0, 1, \ldots, n-1\}$ with $g(x, i) = (f(x), i+1)$, where $i+1$ is understood modulo n, then (Y, g) is also a Smale space (one needs to set the metric and bracket so that $[(x, i), (y, j)]$ is only defined when $i = j$). It is a fairly simple matter to see that (Y, g) is still irreducible, but not mixing. For example, if U is a non-empty subset of $X \times \{0\}$, then $f^k(U) \cap U$ is empty unless k is a multiple of n. Again, it turns out that any irreducible Smale space looks something like this. The precise statement is given in Corollary 4.5.7.

The two results we have mentioned above are the main points of this section. It is possible to simply read and understand them and then go on without looking at the rest of the section. The techniques used in proving them are rather interesting and we do obtain some other interesting results along the way.

Our analysis of this part of the structure of Smale spaces requires the following definition of a relation on the set of periodic points.

Definition 4.5.1. *Let (X, f) be a Smale space.*

If x and y are periodic points, we define $x \preceq y$ if $X^u(x) \cap X^s(y)$ is non-empty.

The next result establishes some basic properties of this relation.

Proposition 4.5.2. *Let (X, f) be a Smale space.*

1. *The relation $\preceq$ on $Per(X, f)$ is reflexive and transitive.*

2. *For periodic points x and y, $x \preceq y$ if and only if $f^i(x) \preceq f^i(y)$, for all integers i.*

3. *If x and y are periodic points with $d(x,y) \leq \epsilon_X$, then $x \preceq y$ (and $y \preceq x$).*

4. *For a periodic point x and integer i, if $x \preceq f^i(x)$ then $f^i(x) \preceq x$ also.*

Proof. For the first part, x is in $X^u(x) \cap X^s(x)$, so the relation is reflexive.

We suppose that $x \preceq y$ and $y \preceq z$. We will prove that $X^u(x) \cap X^s(z)$ is non-empty. Let p be the product of the periods of x, y and z. Let u be any point in $X^u(x) \cap X^s(y)$ and let w be any point of $X^u(y) \cap X^s(z)$. Then as n tends to plus infinity, the sequences $f^{np}(u)$ and $f^{-np}(w)$ both tend to y. Choose n sufficiently large so that $d(f^{np}(u), f^{-np}(w))$ is less than ϵ_X. Let $v = [f^{-np}(w), f^{np}(u)]$. Now the point v is stably equivalent to $f^{-np}(w)$ which is stably equivalent to $f^{-np}(z)$ which is just z, by the choice of p. Similarly, v is unstably equivalent to x and this completes the proof.

The second part follows at once from the observation that

$$X^u(f^i(x)) \cap X^s(f^i(y)) = f^i(X^u(x)) \cap f^i(X^s(y)) = f^i(X^u(x) \cap X^s(y)).$$

The third statement follows from the simple observation **??** that when $d(x,y) \leq \epsilon_X$, we have $[y,x]$ is in $X^u(x) \cap X^s(y)$.

For the fourth part, we let p be the period of x and simply make repeated use of the second part :

$$f^i(x) \preceq f^{2i}(x) \preceq \cdots \preceq f^{pi}(x) = x.$$

$\square$

The first part of the next definition is a standard one for any relation that is reflexive and transitive.

Definition 4.5.3. *Let (X,d,f) be a Smale space and let x, y be periodic points of X.*

1. *We define $x \sim y$ if $x \preceq y$ and $y \preceq x$.*

2. *We define $x \approx y$ if $f^i(x) \sim f^j(y)$, for some integers i, j.*

We observe the following easy result.

Proposition 4.5.4. 1. *On $Per(X,f)$, $\sim$ is an equivalence relation.*

2. *On $Per(X, f)$, $\approx$ is an equivalence relation. Moreover, $\preceq$ induces a partial order of the $\approx$-equivalence classes by defining one class to be less than another if they have respective elements x and y with $x \preceq y$.*

3. *If x, y are periodic points with $d(x, y) \leq \epsilon_X$, then $x \sim y$.*

Proof. All items here are quite trivial, except the assertion about the partial order on the $\approx$-classes. For this, we must check anti-symmetry and transitivity. Suppose we can find periodic points x, x', y, y' with $x \approx x', y \approx y', x \preceq y, y' \preceq x'$. Then we may find integers i, j, i', j' with $f^i(x) \sim f^{i'}(x'), f^j(y) \sim f^{j'}(y')$. Then using part 2 of Proposition 4.5.2, we have

$$x \preceq y \preceq f^{j'-j}(y') \preceq f^{j'-j}(x') \preceq f^{i-i'+j'-j}(x).$$

Now we can apply part 4 of 4.5.2 to add to the end $\preceq x$. Hence each $\preceq$ may be replaced by $\sim$. In particular, $x \approx y$ and we are done.

For transitivity, we begin with periodic points x, y, y', z with $x \preceq y, y \approx y', y' \preceq z$. We find integers i, j with $f^i(y) \sim f^j(y')$. Then we have

$$x \approx f^{i-j}(x) \preceq f^{i-j}(y) \preceq y' \preceq z$$

and we are done. □

Theorem 4.5.5. *Let (X, f) be a Smale space.*

1. *There are a finite number of $\sim$-equivalence classes in $Per(X, f)$, their closures are pairwise disjoint and clopen in $NW(X, f)$ and the union of their closures is $NW(X, f)$.*

2. *The are a finite number of $\approx$-equivalence classes in $Per(X, f)$, their closures are pairwise disjoint, clopen, f-invariant and each is an irreducible Smale space. Their union is $NW(X, f)$.*

3. *If $Y \subseteq NW(X, f)$ is an f-invariant, clopen subset and (Y, f) is irreducible, then $Y \cap Per(X, f) = Per(Y, f)$ is exactly one $\approx$-equivalence class in $Per(X, f)$. That is, the decomposition of $NW(X, f)$ into sets satisfying these conditions is unique. These subsets are called the* irreducible components *of (X, f).*

4. *There is a natural order on the irreducible components of (X, f) induced by $\preceq$. More specifically, if Y and Y' are irreducible components of X, we say that $Y \preceq Y'$ if for any periodic points y in Y and y' in Y', we have $y \preceq y'$.*

Proof. We know that $Per(X, f)$ is contained in $NW(X, f)$ and is dense by Theorem 4.4.1. If x is in $NW(X, f)$, the open set $X(x, \epsilon_X/2)$ meets $Per(X, f)$, but by part 3 of Proposition 4.5.4, it meets only one $\sim$-equivalence class in $Per(X, f)$. This shows at once that x lies in the closure of exactly one $\sim$-equivalence class. It also follows that for any y in $X(x, \epsilon_X/4)$, $X(y, \epsilon_X/4)$ also meets only the same $\sim$-equivalence class. It follows then that the closure of a $\sim$-equivalence class is open, as well as closed, and their union is all of $NW(X, f)$. The fact there are only a finite number follows from compactness of $NW(X, f)$.

Since each $\sim$-equivalence class is contained in a single $\approx$-equivalence class, it follows at once that there are only finitely many $\approx$-equivalence class and the closure of each is clopen in $NW(X, f)$ and their union is $NW(X, f)$. The fact that these closures are f-invariant follows from the simple fact that $x \approx f(x)$, for any periodic point, so the $\approx$-equivalence classes are also f-invariant. Finally, we must prove that if Y is the closure of a single $\approx$-equivalence class then (Y, f) is irreducible. Let U, V be non-empty open sets in Y. Let x be a periodic point in U and let y be a periodic point in V. Let p be the product of their periods. Since $x \approx y$, we may find integers i and j such that there is z in $X^u(f^i(x)) \cap X^s(f^j(y))$. Then as m, n tends to infinity, the sequence $f^{-i-mp}(z)$ converges to x and $f^{-j+np}(z)$ converges to y. Thus we can find m, n such that the former is in U and the latter is in V and $-i - mp < -j + np$. It follows that $f^{i-j+(m-n)p}(U) \cap V$ is non-empty.

Suppose Y, Z are both clopen, f-invariant subsets of $NW(X, f)$ and (Y, f) and (Z, f) are irreducible. The sets $U = Y \cap Z, V = Y - Z$ are clopen, disjoint f-invariants of Y. If both are non-empty, these sets would contradict the fact that (Y, f) is irreducible. Hence, either Y and Z are disjoint or $Y \subseteq Z$. The same argument applied to $U = Y \cap Z, V = Z - Y$ shows that either Y and Z are disjoint or equal. Thus the Y of the hypothesis must be exactly one of the irreducible components found in part 2.

The statement is clear.

□

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We restate part of the last result in a simple form.

Corollary 4.5.6. *Let (X, d, f) be a non-wandering Smale space. Then there are open, closed, pairwise disjoint, f-invariant subsets $X_1, \ldots, X_n$ of X, whose union is X, and so that $(X_i, d, f|X_i)$ is irreducible, for each $1 \leq i \leq n$. Moreover, these sets are unique up to relabelling.*

Corollary 4.5.7. *Let (X, d, f) be an irreducible Smale space. Then there are open, closed, pairwise disjoint sets $X_1, X_2, \ldots, X_N$ whose union is X. These sets are cyclicly permuted by f and $f^N|X_i$ is mixing for every $1 \leq i \leq N$. Moreover, N is the number of $\sim$-equivalence classes in $Per(X, f)$.*

Proof. As (X, f) is irreducible, $Per(X, f)$ has a single $\approx$-equivalence class and this in turns means that the $\sim$-equivalence classes must be cyclicly permuted by f. So letting X_i be the closures of the $\sim$-equivalence classes yields the decomposition as claimed. It remains for us to show that $f^N|X_i$ is mixing for every $1 \leq i \leq N$. Since stable and unstable equivalence classes in X are the same for f^n as for f (**??**), the relation $\sim$ is the same for f^n as for f. Therefore, it suffices for us to prove that if (X, f) is a non-wandering Smale space with a single $\sim$ equivalence class in $Per(X, f)$, then (X, f) is mixing. We will do so by appealing to Theorem 4.4.2; that is, it suffices to prove that for any periodic point x in X, $X^s(x)$ and $X^u(x)$ are dense in X. We will show only the former.

Let x be periodic and let U be open. As (X, f) is non-wandering, we may find a periodic point y in U. Let p be the products of their periods. Our hypothesis is that $x \sim y$, so we may find z in $X^u(y) \cap X^s(x)$. The sequence $f^{-np}(z), n \geq 1$, remains in $X^s(x)$ and converges to y which lies in the open set U. So for some n we have $f^{-np}(z)$ is in $X^s(x) \cap U$ and we are done. □

There is another nice interpretation of the number of $\sim$-equivalence classes in an irreducible Smale space, as follows. Observe that the greatest common divisor of a subset of the naturals exists for infinite sets as well.

Theorem 4.5.8.

If (X, f) is an irreducible Smale space, then the number of $\sim$-equivalence class in $Per(X, f)$ is equal to

$$gcd\{n \in \mathbb{N} \mid Per_n(X, f) \neq \emptyset\}.$$

Proof. Let N be the number of $\sim$-equivalence class in $Per(X, f)$. It is immediate from Corollary 4.5.7 that $Per_n(X, f)$ is empty if n is not a multiple of N.

Conversely, suppose that x is a periodic point of period p. As remarked above, this must be a multiple of N, say $p = mN$. If $p = N$, then N appears in the set on the right and the greatest common divisor is N as desired. We now assume $m > 1$.

As x and $f^N(x)$ lie in the same $\sim$-equivalence class, we may find y in $X^u(x) \cap X^s(f^N(x))$. This means that the sequence $f^{-jp}(y), j \geq 1$, converges to x while $f^{jp}(y), j \geq 1$, converges to $f^N(x)$.

Let ϵ_1 be the expansiveness constant for (X, f). Let $\delta = \epsilon_1/2$ and choose $\epsilon_2 > 0$ so that every ϵ_2-pseudo-orbit is δ-shadowed by an orbit.

Find j, k such that

$$d(x, f^{-jp}(z)), d(f^N(x), f^{kp}(z)) < \epsilon_2.$$

Consider the following sequence

$$f^{-1}(x)f^{-jp}(y)f^{-jp+1}(y)\cdots f^{kp-1}(y)f^N(x)f^{N+1}(x)\cdots f^{p-2}(x)$$

repeated cyclicly. This is a ϵ_2-pseudo-orbit and so it may be δ-shadowed by the orbit of a point z. Also observe that the pseudo-orbit above is periodic of period $(j+k+1)p - N$. It follows then that the orbit of $f^{(j+k+1)p-N}(z)$ 2δ-shadows that of z and hence by expansiveness they are equal. Thus, we have a point of period $(j+k+1)p - N$ and the greatest common divisor of this and p is evidently N, as desired. □

Proposition 4.5.9. *Let x be a point in a Smale space (X, f, d).*

1. *The accumulation points of the sequence $f^n(x), n \geq 0$ all lie in a single irreducible component of (X, f, d), which we denote by $Irr^+(x)$. Similarly, the accumulation points of the sequence $f^{-n}(x), n \geq 0$ all lie in a single irreducible component if (X, f, d), which we denote by $Irr^-(x)$.*

2. *We have $Irr^-(x) \preceq Irr^+(x)$.*

3. *The point x is non-wandering if and only if $Irr^+(x) = Irr^-(x)$.*

Proof. First, we show that the accumulation points of $f^n(x), n \geq 1$ are non-wandering. If y is such a point, then there is an increasing sequence $n_k, k \geq 1$ such that $\lim_k f^{n_k}(x) = y$. If U is any neighbourhood of y, we may find k such that $f^{n_k}(x)$ and $f^{n_{k+1}}(x)$ are both in U. This completes the proof.

If y, z are periodic points which are both accumulation points of $f^n(x), n \geq 1$, then we can find $m \geq 1$ such that $d(f^m(x), y) \leq \epsilon_X$ and also so that $d([f^m(x), y], f^m) \leq \epsilon_X$. Then we can find $n > m$ such that $d(f^m(x), y) \leq \epsilon_X/2$ and $\lambda^{n-m} < 1/2$. It follows then that $d(f^{n-m}[f^m(x), y]), f^n(x)) < \epsilon_X/2$ and $[z, f^{n-m}[f^m(x), y]]$ is defined. This point is clearly stably equivalent to z

and unstably equivalent to $f^{n-m}[f^m(x), y]$, which in turn is unstably equivalent to $f^m(y)$. We conclude that $f^m(y) \preceq z$. It follows that the irreducible component containing y is less than that containing z. Interchanging the roles of y and z, we obtain the reverse inequality and so by **??**, the two irreducible classes are equal.

The second statement is proved in much the same way. Let y be an accumulation point of $f^{-n}(x), n \geq 1$ and z be an accumulation point of $f^n(z), n \geq 1$. We proceed as above, except with $m < 0$ and $n > 0$. We cannot reverse the role of y and z, but we don't need to to arrive at the desired conclusion.

We sketch a proof of the last statement only. First, let y be an accumulation point of $f^{-n}(x), n \geq 1$ and z be an accumulation point of $f^n(z), n \geq 1$. Find periodic points y' close to y and z' close to z. As these lie in the same irreducible component, we know that $z' \preceq f^l(y')$, for some l. We may find w in the unstable set of z' and in the stable set of $f^l(y')$. We then construct an ϵ-pseudo-orbit as follows. The first part is simply $x, f(x), \ldots, f^n(x)$, where $f^m(x)$ is close to z'. The next part is $f^{-i+1}(w), \ldots, f^j(w)$, where $f^{-i}(w)$ is close to z' and $f^j(w)$ is close to $f^l(y')$ The final part is $f^{-n}(x), \ldots, f^{-1}(x), x$, Approximating this periodic pseudo-orbit by a periodic point shows that x can be approximated by periodic points and hence is non-wandering. $\square$

Part II

Maps

Chapter 5

Properties of maps between Smale spaces

5.1 Preliminaries on maps

In this section, we give some basic definitions of maps and factor maps between dynamical systems and establish basic properties of them in the case that both domain and range are Smale spaces.

Definition 5.1.1. *Let (X, f) and (Y, g) be dynamical systems. A* map $\pi : (Y, g) \to (X, f)$, *is a continuous function $\pi : Y \to X$ such that $\pi \circ g = f \circ \pi$. A* factor map *from (Y, g) to (X, f) is a map for which $\pi : Y \to X$ is also surjective.*

Definition 5.1.2. *A map $\pi : (Y, g) \to (X, f)$ is* finite-to-one *if there is a constant $M \geq 1$ such that $\#\pi^{-1}\{x\} \leq M$, for all x in X.*

A very easy observation for finite-to-one maps is that they preserve periodic points in a strong sense.

Proposition 5.1.3. *Let $\pi : (Y, g) \to (X, f)$ be a finite-to-one map. For any y in Y, y is periodic if and only if $\pi(y)$ is.*

Proof. Notice that y is periodic if and only if $\{g^n(y) \mid n \in \mathbb{Z}\}$ is finite. An analogous statement holds for $\pi(y)$. It follows easily from the definition that

$$\pi\{g^n(y) \mid n \in \mathbb{Z}\} = \{f^n(\pi(x)) \mid n \in \mathbb{Z}\}.$$

As π is finite-to-one, $\{g^n(y) \mid n \in \mathbb{Z}\}$ is finite if and only if its image under π is. □

Now we consider the situation where one or both of our dynamical systems are Smale spaces. The first important result is basically that a map between two Smale spaces is necessarily compatible with the two bracket operations.

Theorem 5.1.4. *Let (Y, g) and (X, f) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be a map. There exists $\epsilon_\pi > 0$ such that, for all y_1, y_2 in Y with $d(y_1, y_2) \leq \epsilon_\pi$, then both $[y_1, y_2], [\pi(y_1), \pi(y_2)]$ are defined and

$$\pi([y_1, y_2]) = [\pi(y_1), \pi(y_2)].$$

Proof. Let ϵ_X, ϵ_Y be the Smale space constants for X and Y, respectively. As Y is compact and π is continuous, we may find a constant $\epsilon > 0$ such that, for all y_1, y_2 is in Y with $d(y_1, y_2) < \epsilon$, we have $d(\pi(y_1), \pi(y_2)) < \epsilon_X/2$. From the continuity of the bracket map, we may choose ϵ_π such that $0 < \epsilon_\pi < \epsilon_Y$ and for all y_1, y_2 in X with $d(y_1, y_2) \leq \epsilon_\pi$, we have

$$d(y_1, [y_1, y_2]), d(y_2, [y_1, y_2]) < \epsilon.$$

Now assume y_1, y_2 are in Y with $d(y_1, y_2) \leq \epsilon_\pi$. It follows that, $[y_1, y_2]$ is defined and we have the estimates above. Then, inductively for all $n \geq 0$, we have

$$d(g^n(y_1), g^n[y_1, y_2]) \leq \lambda^n d(y_1, [y_1, y_2]) \leq \epsilon$$

and also

$$d(g^{-n}(y_2), g^{-n}[y_1, y_2]) \leq \lambda^n d(y_2, [y_1, y_2]) \leq \epsilon$$

It follows from the choice of ϵ that, for all $n \geq 0$, we have

$$\begin{aligned} d(\pi(g^n(y_1)), \pi(g^n[y_1, y_2])) &\leq \epsilon_X/2 \\ d(fn(\pi(y_1)), f^n(\pi[y_1, y_2])) &\leq \epsilon_X/2 \end{aligned}$$

and similarly

$$d(f^{-n}(\pi(y_2)), f^{-n}(\pi[y_1, y_2])) \leq \epsilon_X/2.$$

On the other hand, these two estimates are also satisfied replacing $f^n(\pi[y_1, y_2])$ by $f^n[\pi(y_1), \pi(y_2)]$ and so, by expansiveness of (X, f), we have the desired conclusion. □

In the case where the domain is a shift of finite type with a specific presentation, we introduce the notion of a map being regular. It is really just an analogue of the conclusion of the last Theorem appropriate for shifts of finite type.

Definition 5.1.5. *Let G be a graph, (X_G, σ) be the associated shift of finite type and (X, f) be a Smale space. We say that a map $\pi : (X_G, \sigma) \to (X, f)$ is* regular *if, for all x, y in X_G with $t(x^0) = t(y^0)$, we have $d(\pi(x), \pi(y)) \leq \epsilon_X$ and*

$$\pi[x, y] = [\pi(x), \pi(y)].$$

In fact, after replacing the domain by a higher block presentation, we may assume any map from a shift of finite type is regular.

Theorem 5.1.6. *Let (Y, ψ) be a shift of finite type, (X, φ) be a Smale space and $\pi : (Y, \psi) \to (X, f)$ be a map. Then there exists a graph G and a conjugacy $h : (X_G, \sigma) \to (Y, \psi)$ such that $\pi \circ h$ is regular.*

When G satisfies the conclusion of this Theorem, we will say (rather imprecisely) that G is a presentation of π.

One thing which has been missing from our discussion so far is any kind of example of a map. We correct that now, showing a very simple way to obtain maps between shifts of finite type; if the shifts are given by finite graphs, then maps arise naturally from from graph homomorphisms. We define these, although the definition must be fairly obvious.

Definition 5.1.7. *Let G and H be finite graphs. A* graph homomorphism *$\theta : H \to G$ is a pair of maps (both denoted θ) $\theta : H^0 \to G^0$ and $\theta : H^1 \to G^1$ such that $\theta \circ i_H = i_G \circ \theta$ and $\theta \circ t_H = t_G \circ \theta$.*

The following is an easy consequence of the definitions and we omit the proof.

Proposition 5.1.8. *Let G and H be finite directed graphs and let $\theta : H \to G$ be a graph homomorphism. The associated map $\theta : (X_H, \sigma) \to (X_G, \sigma)$ defined by $\theta(y)_n = \theta(y_n), n \in \mathbb{Z}, y \in X_H$ is regular.*

We will often be concerned with factor maps, that is maps which are surjective, so we would like to have a simple criterion which implies this for maps arising from graph homomorphisms.

Proposition 5.1.9. *Let G and H be finite directed graphs and let $\theta : H \to G$ be a graph homomorphism. Suppose that*

1. *$\theta : H^0 \to G^0$ is surjective and,*

2. *for every vertex v in H^0, the map*
$$\theta : t^{-1}\{v\} \to t^{-1}\{\theta(v)\}$$
is surjective.

Then the associated map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is a factor map. The same conclusion holds replacing t by i.

Proof. We know that the map θ is continuous and so $\theta(\Sigma_H)$ is a closed subset of Σ_G. It suffices then for us to prove that it is dense. To do so, it suffices to consider x in Σ_G and $n \geq 1$. Using the first hypothesis, we may find v in H^0 such that $\theta(v) = t(x_n)$. The second hypothesis implies that we may find y_n in H^1 with $ty_n) = v$ and $\theta(y_n) = x_n$. It follows then that $\theta(i(y_n)) = i(x_n) = t(x_{n-1})$. Continuing in this way we may construct a path $y_{-n}, \ldots, y_n$ in H such that $t(y_i) = x_i, -n \leq i \leq n$. Now extend this finite path to any element of Σ_H which we call y and we see that $d(\theta(y), x) \leq 2^{-1}$. As x and n were arbitrary, $\theta(\Sigma_H)$ is dense in Σ_G. $\square$

The following result is also rather technical, but it is somewhat interesting on its own and will be useful later.

Proposition 5.1.10. *Let G be a finite directed graph such that (X_G, σ) is irreducible, (X, f) be a Smale space and $\pi : (X_G, \sigma) \to (X, f)$ be a regular, finite-to-one map. If y, y' are in X_G such that $\pi(y) = \pi(y')$ and $y_n = y'_n$ for all but finitely many integers n, then $y = y'$.*

Proof. First, we can replace y and y' by $\sigma^k(y)$ and $\sigma^k(y')$, for any integer k. So let us assume that $y_n = y'_n$, for all $n \leq 0$ and $n \geq n_0$, for some fixed positive n_0.

Since X_G is irreducible, there is a path from $t(y_{n_0}) = t(y'_{n_0})$ to $i(y_0) = i(y'_0)$. Let $x_0, \ldots, x_p$ be the sequence $y_0, \ldots, y_{n_0}$ concatenated with this path. That is, this is a cycle of length $p > n_0$ and which agrees with y on its first n_0 entries.

Let x in X_G be the unique periodic point of period p which extends this path.

Define $w = [y, x]$ and $w' = [y', x]$. These are clearly well-defined and satisfy the same two hypotheses are the pair y, y'. In addition, $w_n = w'_n = x_n$, for all $n \leq 0$. Then let $z = \sigma^{-n_0}[\sigma^{n_0}(x), \sigma^{n_0}(w)]$ and $z' = \sigma^{-n_0}[\sigma^{n_0}(x), \sigma^{n_0}(w')]$. Again, these are clearly well-defined and satisfy the same two hypotheses as the pair y, y'. In addition, for $n \leq 0$, we have

$$z'_n = [\sigma^{n_0}(x), \sigma^{n_0}(w')]_{n-n_0} = \sigma^{n_0}(w')_{n-n_0} = w'_n = x_n.$$

Similarly, $z_n = x_n$ for $n \leq 0$.

For $n > n_0$, we have

$$z'_n = [\sigma^{n_0}(x), \sigma^{n_0}(w')]_{n-n_0} = \sigma^{n_0}(x)_{n-n_0} = x_n.$$

and also $z_n = x_n$.

Finally, if $0 \leq n \leq n_0$, we have

$$z_n = [\sigma^{n_0}(x), \sigma^{n_0}(w)]_{n-n_0} = \sigma^{n_0}(w)_{n-n_0} = w_n = [y, x]_n = y_n = x_n.$$

and if $0 \leq n \leq n_0$, we have

$$z'_n = [\sigma^{n_0}(x), \sigma^{n_0}(w')]_{n-n_0} = \sigma^{n_0}(w')_{n-n_0} = w'_n = [y', x]_n = y'_n.$$

We conclude that $z = x$ and is periodic. As $\pi(z') = \pi(z)$ and π is finite-to-one, by 5.1.3, z' is also periodic. If the period is q, then $\sigma^{pq}(z') = z'$ and it follows that $y'_n = z'_n = z_n = y_n$ for $1 \leq n \leq n_0$ and hence $y' = y$. □

We return to the general situation of maps between Smale spaces. The next three lemmas are rather technical and will not be used until later, but we find it convenient to note them here.

Lemma 5.1.11. *Let $\pi : Y \to X$ be a continuous map and let x_0 be in X with $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$ finite. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\pi^{-1}(X(x_0, \delta)) \subset \cup_{n=1}^{N} Y(y_n, \epsilon)$.*

Proof. If there is no such δ, we may construct a sequence $x^k, k \geq 1$ in X converging to x_0 and a sequence $y^k, k \geq 1$ with $\pi(y^k) = x^k$ and y^k not in $\cup_{n=1}^{N} Y(y_n, \epsilon)$. Passing to a convergent subsequence of the y^k, let y be the limit point. Then y is not in $\cup_{n=1}^{N} Y(y_n, \epsilon)$, since that set is open, while $\pi(y) = \lim_k \pi(y^k) = \lim_k x^k = x_0$. This is a contradiction to $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$. □

Lemma 5.1.12.
1. *Let (Y, g) and (X, f) be Smale spaces and let $\pi : (Y, g) \to (X, f)$ be a finite-to-one factor map. If y, y' are periodic points in Y with $d(y, y') < \epsilon_\pi$ and $\pi(y) = \pi(y')$, then $y = y'$.*
2. *Let G be finite directed graph, , (X, f) be a Smale space and $\pi : (X_G, \sigma) \to (X, f)$ be a regular, finite-to-one factor map. If y, y' is are periodic points in X_G with $t(y_0) = t(y'_0)$ and $\pi(y) = \pi(y')$, then $y = y'$.*

Proof. From either hypothesis, we know that $[y, y']$ is defined and $\pi[y, y'] = [\pi(y), \pi(y')] = \pi(y) = \pi(y')$. By Proposition 5.1.3, $[y, y']$ is periodic. As it is stably equivalent to y which is also periodic, they are equal. Similarly, $[y, y']$ is equal to y' and we are done. □

Lemma 5.1.13. *Let $\pi : (Y, g) \to (X, f)$ be a finite-to-one factor map between Smale spaces and suppose that (X, f) is non-wandering. If x_0 is any point in X, then there exists $\delta > 0$ such that, for every periodic point x in $X(x, \delta)$,*

$$\#\pi^{-1}\{x\} \leq \#\pi^{-1}\{x_0\}$$

Proof. Let $\pi^{-1}\{x_0\} = \{y_1, \ldots, y_N\}$. Choose $\epsilon_\pi > \epsilon > 0$ so that the sets $Y(y_n, \epsilon), 1 \leq n \leq N$ are pairwise disjoint. Apply the Lemma 5.1.11 to find δ satisfying the conclusion there for this ϵ.

Choose a periodic point x in $X(x, \delta)$. It follows that $\pi^{-1}\{x\} \subseteq \cup_{n=1}^{N} Y(y_n, \epsilon)$. It follows from Lemma 5.1.12 that, for any $1 \leq n \leq N$, the set $\pi^{-1}\{x\} \cap Y(y_n, \epsilon)$ contains at most one point. It follows at once that $\#\pi^{-1}\{x\} \leq N = \#\pi^{-1}\{x_0\}$ as desired. □

For the last part of this section, we turn to a fairly basic construction, which is usually called either the pullback of the fibred product.

Definition 5.1.14. *Suppose that $(X, f), (Y_1, \psi_1)$ and (Y_2, ψ_2) are dynamical systems and that $\pi_1 : (Y_1, g_1) \to (X, f)$ and $\pi_2 : (Y_2, g_2) \to (X, f)$ are maps. Their* fibred product *is the space*

$$Z = \{(y_1, y_2) \mid y_1 \in Y_1, y_2 \in Y_2, \pi_1(y_1) = \pi_2(y_2)\}$$

equipped with the relative topology from $Y_1 \times Y_2$, together with the map $h = g_1 \times g_2$, which is clearly a homeomorphism of this space. There are canonical maps from this system to (Y_1, g_1) and (Y_2, g_2) defined by $\rho_1(y_1, y_2) = y_1$ and $\rho_2(y_1, y_2) = y_2$ which satisfy $\pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$.

One of the most basic features of the fibred product is that properties of the map π_1 (or π_2) tend to be inherited by ρ_2 (or ρ_1, respectively). Here is a useful example.

Theorem 5.1.15. *Suppose that $(X, f), (Y_1, g_1)$ and (Y_2, g_2) are dynamical systems and that $\pi_1 : (Y_1, g_1) \to (X, f)$ and $\pi_2 : (Y_2, g_2) \to (X, f)$ are maps. Let (Z, h) be their fibred product with maps ρ_1 and ρ_2 as above. If π_1 (or π_2) is a factor map, then so is ρ_2 (or ρ_1, respectively).*

Proof. Let y_2 be in Y_2. Since π_1 is surjective, we may find y_1 in Y_1 such that $\pi_1(y_1) = \pi_2(y_2)$. Thus (y_1, y_2) is in Z and $\rho_2(y_1, y_2) = y_2$. □

We finish by observing that the fibred product of two Smale spaces is again a Smale space.

Theorem 5.1.16. *Suppose that $(X, f), (Y_1, g_1)$ and (Y_2, g_2) are Smale spaces and that $\pi_1 : (Y_1, g_1) \to (X, f)$ and $\pi_2 : (Y_2, g_2) \to (X, f)$ are maps. The fibred product is also a Smale space with the metric*

$$d((y_1, y_2), (y_1', y_2')) = \max\{d(y_1, y_1'), d(y_2, y_2')\},$$

constant $\epsilon_Z = \min\{\epsilon_{\pi_1}, \epsilon_{\pi_2}\}$ and bracket

$$[(y_1, y_2), (y_1', y_2')] = ([y_1, y_1'], [y_2, y_2']),$$

provided $d(y_1, y_1'), d(y_2, y_2') \leq \epsilon_Z$.

5.2 s/u-resolving and s/u-bijective maps

In this section, we discuss special classes of maps called s-resolving, u-resolving, s-bijective and u-bijective maps. These maps possess many nice properties but the most important is that our invariants will behave in a functorial way with respect to them.

It is an easy consequence of the definitions that if (Y, ψ) and (X, φ) are Smale spaces and

$$\pi : (Y, \psi) \to (X, \varphi)$$

is a map, then $\pi(Y^s(y)) \subset X^s(\pi(y))$ and $\pi(Y^u(y)) \subset X^u(\pi(y))$. We recall the following definition due to David Fried [**?**].

Definition 5.2.1. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a map. We say that π is s-resolving *(or* u-resolving*) if, for any y in Y, its restriction to $Y^s(y)$ (or $Y^u(y)$, respectively) is injective.*

It is probably nice to give a simple example of such a map. It is interesting to compare this result (especially the hypotheses) with Proposition 5.1.9.

Proposition 5.2.2. *Let G and H be finite graphs and let $\theta : H \to G$ be a graph homomorphism. Suppose that, for every v in H^0, $\theta|t^{-1}\{v\}$ is injective. Then the map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is s-resolving. Similarly, if for every v in H^0, $\theta|i^{-1}\{v\}$ is injective, then θ is u-resolving.*

The proof is quite trivial and we omit it.

The following is a useful technical preliminary result.

Proposition 5.2.3. *Let (X, f) and (Y, g) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be an s-resolving (or u-resolving) map. With ϵ_π as in Theorem **??**, *if y_1, y_2 are in Y with $\pi(y_1)$ in $X^u(\pi(y_2), \epsilon_X)$ (or $\pi(y_1)$ in $X^u(\pi(y_2), \epsilon_X)$, respectively) and $d(y_1, y_2) \leq \epsilon_\pi$, then $y_2 \in Y^u(y_1, \epsilon_\pi)$ ($y_2 \in Y^s(y_1, \epsilon_\pi)$, respectively).*

Proof. It follows at once the from hypotheses that

$$\pi[y_1, y_2] = [\pi(y_1), \pi(y_2)] = \pi(y_1).$$

On the other hand $[y_1, y_2]$ is stably equivalent to y_1 and, since π is s-resolving, $[y_1, y_2] = y_1$. $\square$

Resolving maps have many nice properties, the first being that they are finite-to-one. We establish this, and a slight variant of it, as follows.

Theorem 5.2.4. *Let (X, f) and (Y, g) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be an s-resolving map. There is a constant $M \geq 1$ such that

1. *for any x in X, there exist $y_1, \ldots, y_K$ in Y with $K \leq M$ such that*

$$\pi^{-1}(X^u(x)) = \cup_{k=1}^{K} Y^u(y_k),$$

   and

2. *for any x in X, we have $\#\pi^{-1}\{x\} \leq M$.*

In particular, π is finite-to-one.

Proof. Cover Y with balls of radius $\epsilon_\pi/2$, then extract a finite subcover, whose elements we list as $B_m, 1 \leq m \leq M$. We claim this M satisfies the desired conclusions.

For the first statement, given x in X and y in $\pi^{-1}(X^u(x))$, it is clear that $Y^u(y) \subset X^u(x)$. We must show that there exist at most M unstable equivalence classes in $\pi^{-1}(X^u(x))$. For this, it suffices to show that if $y_i, 1 \leq i \leq M+1$, are in Y with $\pi(y_i)$ and $\pi(y_j)$ unstably equivalent, for all i, j, then y_i and y_j are unstably equivalent for some $i \neq j$. Choose $n \leq 0$ such that $f^n(\pi(y_i))$ is in $X^u(f^n(\pi(y_j)), \epsilon_X)$, for all $1 \leq i, j \leq M+1$. From the pigeon hole principle, there exists distinct i and j such that $g^n(y_i)$ and $g^n(y_j)$ lie in the same B_m, for some $1 \leq m \leq M$. These points satisfy the hypotheses of **??** and it follows that they are unstably equivalent. Then y_i are y_j are also unstably equivalent.

For the second statement, suppose $\pi^{-1}\{x\}$ contains distinct points $y_1, \ldots, y_{M+1}$. Let δ denote the minimum distance, $d(y_i, y_j)$, over all $i \neq j$. Choose $n \geq 1$ such that $\lambda^n \epsilon_\pi < \delta$. Consider the points $g^n(y_i), 1 \leq i \leq M+1$. By the pigeon-hole principle, there exists g$i \neq j$ with $g^n(y_i)$ and $\psi^n(y_j)$ in the same set B_m. We have

$$\pi(g^n(y_i)) = f^n(\pi(y_i)) = f^n(x) = f^n(\pi(y_j)) = \pi(g^n(y_j)).$$

From Proposition **??**, $g^n(y_i)$ is in $Y^u(g^n(y_j), \epsilon_\pi)$. This implies that y_i is in $Y^u(y_j, \lambda^n \epsilon_\pi)$. As $\lambda^n \epsilon_\pi < \delta$, this is a contradiction. □

Although the definition of s-resolving is given purely at the level of the stable sets as sets, various nice continuity properties follow.

Theorem 5.2.5. *Let (X, f) and (Y, g) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be either an s-resolving or a u-resolving map. For each y in Y, the maps

$$\pi : Y^s(y) \to X^s(\pi(y)), \pi : Y^u(y) \to X^u(\pi(y))$$

*are continuous and proper, where the sets above are given the topologies of Proposition **??**.*

Proof. From the symmetry of the statement, it suffices to consider the case that π is s-resolving.

We use the characterization of limits in $Y^s(y)$ and $X^s(\pi(y))$ given in Proposition **??**. From this, and Theorem **??**, it is easy to see that π is continuous on $Y^s(y)$. The same argument covers the case of π on $Y^u(y)$.

To see the map π on $Y^s(y)$ is proper, it suffices to consider a sequence y_n in $Y^s(y)$ such that $\pi(y_n)$ is convergent in the topology of $X^s(\pi(y))$, say with limit x, and show that it has a convergent subsequence. As Y is compact in its usual topology, we may find y' which is a limit point of a convergent subsequence $y_{n_k}, k \geq 1$. It follows that

$$\pi(y') = \pi(\lim_k y_{n_k}) = \lim_k \pi(y_{n_k}) = \lim_n \pi(y_n) = x.$$

We also have, for k sufficiently large,

$$\pi[y_{n_k}, y'] = [\pi(y_{n_k}), \pi(y')] = [\pi(y_{n_k}), x] = x,$$

since $\pi(y_{n_k})$ is converging to x in the topology on $X^s(\pi(y))$ and using Proposition **??**. We know that $\pi^{-1}\{x\}$ is finite and contains y' and $[y_{n_k}, y']$, for all k sufficiently large. Moreover, y' is the limit of the sequence $[y_{n_k}, y']$. It follows that there is K such that $[y_{n_k}, y'] = y'$, for all $k \geq K$. From this, we see that y' is in $Y^s(y_{n_k}) = Y^s(y)$ and that the subsequence y_{n_k} converges to y' in $Y^s(y)$.

To see the map π on $Y^u(y)$ is proper, we begin in the same way with a sequence y_n such that $\pi(y_n)$ has limit x in the topology of $X^u(\pi(y))$. Again we obtain a subsequence y_{n_k} with limit y' in Y. Then we have, for k sufficiently large,

$$\pi[y_{n_k}, y'] = [\pi(y_{n_k}), \pi(y')] = [\pi(y_{n_k}), x] = \pi(y_{n_k}),$$

since $\pi(y_{n_k})$ is converging to x in the topology on $X^u(\pi(y))$ and using Proposition **??**. On the other hand, $[y_{n_k}, y']$ and y_{n_k} are stably equivalent and since π is s-resolving, this implies they are equal. It follows that y' is in $Y^u(y_{n_K}) = Y^u(y)$ and y_{n_k} is converging to y' in the topology of $Y^u(y)$. □

There has been extensive interest in s/u-resolving maps. We will need a slightly stronger condition, which we refer to as s/u-bijective maps.

Definition 5.2.6. *Let (X, f) and (Y, g) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be a map. We say that π is s-bijective *(or* u-bijective*) if, for any y in Y, its restriction to $Y^s(y)$ (or $Y^u(y)$, respectively) is a bijection to $X^s(\pi(y))$ (or $X^u(\pi(y))$, respectively).*

It is relatively easy to find an example of a map which is s-resolving, but not s-bijective and we will give one in a moment. However, one important distinction between the two cases should be pointed out at once. The image of a Smale space under an s-resolving map is not necessarily a Smale space. The most prominent case is where the domain and range are both shifts of finite type and the image is a sofic shift, which is a much broader class of systems. (See [**?**].) This is not the case for s-bijective maps (or u-bijective maps).

Theorem 5.2.7. *Let (Y, g) and (X, f) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be either an s-bijective map or a u-bijective map. Then $(\pi(Y), f|_{\pi(Y)})$ is a Smale space.

Proof. The only property which is not clear is the existence of the bracket: if y_1 and y_2 are in Y and $d(\pi(y_1), \pi(y_2)) < \epsilon_X$, then it is clear that $[\pi(y_1), \pi(y_2)]$ is defined, but we must see that it is in $\pi(Y)$. If π is s-bijective, then $[\pi(y_1), \pi(y_2)]$ is stably equivalent to $\pi(y_1)$ and hence in the set $\pi(Y^s(y_1))$ and hence in $\pi(Y)$. A similar argument deals with the case π is u-bijective. □

If $\pi : (Y, g) \to (X, f)$ is a factor map and every point in the system (Y, g) is non-wandering (including the case that (Y, g) is irreducible), then it follows that the same is true of (X, f) and in this case, any s-resolving factor map is also s-bijective, as we will show. The distinction is important for us; although we are mainly interested in our homology theory for irreducible systems, the various self-products we constructed earlier, and which will be used in the later definitions, will almost never be irreducible.

Example 5.2.8. *Consider (Y, g) to be the shift of finite type associated with the following graph:*

$$\begin{tikzcd} v_1 \arrow[loop left] \arrow[r] & v_2 \arrow[loop right] \end{tikzcd}$$

$$\begin{tikzcd} v_3 \arrow[loop left] \arrow[r] & v_4 \arrow[loop right] \end{tikzcd}$$

and (X, f) to be the shift of finite type associated with the following graph:

$$\begin{tikzcd} w_1 \arrow[loop left] \arrow[r, bend left] \arrow[r, bend right] & w_2 \arrow[loop right] \end{tikzcd}$$

It is clear that there is a factor map from (Y, g) to (X, f) obtained by mapping the loops in the first graph to those in the second 2-to-1, while mapping the other two edges injectively. The resulting factor map is s-resolving and u-resolving but not s-bijective or u-bijective.

Theorem 5.2.9. *Let (X, f) and (Y, g) be Smale spaces and let*

$$\pi : (Y, g) \to (X, f)$$

be an s-resolving factor map. Suppose that each point of (Y, g) is non-wandering. Then π is s-bijective.

We now prove a version of Lemma **??** for local stable sets.

Lemma 5.2.10. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces and suppose x_0 in X is periodic and $\pi^{-1}\{x_0\} = \{y_1, y_2, \ldots, y_N\}$. Given $\epsilon_0 > 0$, there exist $\epsilon_0 > \epsilon > 0$ and $\delta > 0$ such that*

$$\pi^{-1}(X^s(x_0, \delta)) \subset \cup_{n=1}^{N} Y^s(y_n, \epsilon).$$

Proof. First, since x_0 is periodic, so is each y_n. Choose $p \geq 1$ such that $\psi^p(y_n) = y_n$, for all $1 \leq n \leq N$, and hence $\varphi^p(x_0) = x_0$. The system (Y, ψ^p)

is also a Smale space. Choose $\epsilon_0 > \epsilon > 0$ to be less than the Smale space constant for this. Also, choose ϵ sufficiently small so that the sets $Y(y_n, \epsilon), 1 \leq n \leq N$ are pairwise disjoint and so that $\psi^p(Y(y_n, \epsilon)) \cap B(y_m, \epsilon) = \emptyset$, for $m \neq n$. Use the Lemma **??** to find δ such that $\pi^{-1}(X(x, \delta)) \subset \cup_{n=1}^{N} Y(y_n, \epsilon)$.

Now suppose that x is in $X^s(x_0, \delta)$ and $\pi(y) = x$. It follows that y is in $Y(y_m, \epsilon)$, for some m. Now consider $k \geq 1$. We have $\pi(\psi^{kp}(y)) = \varphi^{kp}(x)$ which is in $X^s(x_0, \lambda^{kp}\delta) \subset X(x, \delta)$. It follows that $\psi^{kp}(y)$ is in $\cup_{n=1}^{N} Y^s(y_n, \epsilon)$ for all $k \geq 1$. It then follows from the choice of ϵ and induction that $\psi^{kp}(y)$ is in $Y^s(y_m, \epsilon)$ for all $k \geq 1$. This means that y is in $Y^s(y_m, \epsilon)$. $\square$

We are now prepared to give a proof of Theorem 5.2.9.

Proof. In view of the structure Theorem **??**, it suffices for us to consider the case that (X, φ) is irreducible. First, choose a periodic point x_0 satisfying the conclusion of Lemma **??**. Let $\pi^{-1}\{x_0\} = \{y_1, \ldots, y_N\}$. We will first show that, for each $1 \leq n \leq N$, $\pi : Y^s(y_n) \to X^s(x_0)$ is open and onto. We choose $\epsilon_0 > 0$ so that the sets $Y(y_n, \epsilon_0), 1 \leq n \leq N$, are pairwise disjoint. We then choose $\epsilon_0 > \epsilon > 0$ and $\delta > 0$ as in Lemma **??**. Let x be any point in $B(x_0, \delta)$. We know that $\pi^{-1}(\{x\})$ is contained in $\cup_{n=1}^{N} Y^s(y_n, \epsilon_0)$. As the map π is s-resolving, it is injective when restricted to each of the sets $Y^s(y_n, \epsilon)$. This means that $\pi^{-1}\{x\}$ contains at most one point in each of these sets. On the other hand, it follows from our choice of x_0 that $\pi^{-1}\{x\}$ contains at least N points. We conclude that, for each n, $\pi^{-1}\{x\} \cap Y^s(y_n, \epsilon)$ contains exactly one point. Let $W_n = \pi^{-1}(X(x, \delta)) \cap Y^s(y_n, \epsilon)$. The argument above shows that π is a bijection from W_n to $X^s(x_0, \delta)$, for each n. It is clearly continuous and we claim that is actually a homeomorphism. To see this, it suffices to show that, for any sequence y^k in W_n such that $\pi(y^k)$ converges to some x in $X^s(x_0, \delta)$, it follows that y^k converges to some y in W_n. As $\epsilon < \epsilon_0$, the closure of W_n is a compact subset of $Y^s(y_n, \epsilon_0)$. So the sequence y^k has limit points; let y be one of them. By continuity, $\pi(y) = x$. On the other hand, there is a unique point y' in W_n such that $\pi(y') = x$. Thus, y and y' are both in $Y^s(y_n, \epsilon_0)$ and have image x under π. As π is s-resolving, $y = y'$ and so y is in W_n. So the only limit point of the sequence y^k is y' and this completes the proof that π is a homeomorphism.

Since W_n is an open subset of $Y^s(y_n, \epsilon)$, we know that $Y^s(y_n) = \cup_{l \geq 0} \psi^{-l}(W_n)$ and the topology is the inductive limit topology. Similarly, $X^s(x_0) = \cup_{l \geq 0} \varphi^{-l}(X^s(x_0, \delta))$ and the topology is the inductive limit topology. It follows at once that π is a homeomorphism from the former to the latter.

Now we turn to arbitrary point y in Y and $x = \pi(y)$ in X and show that $\pi : Y^s(y) \to X^s(x)$ is onto. We choose x_0 and $\{y_1, \ldots, y_N\}$ to be periodic points as above so that $\pi : Y^s(y_n) \to X^s(x_0)$ are homeomorphisms. By replacing x_0 by another point in its orbit (which will satisfy the same condition), we may assume that x is in the closure of $X^s(x_0)$. Then, we may choose y_n such that y is in the closure of $Y^s(y_n)$. There exists a point y' is $Y^s(y_n)$ in $Y^u(y, \epsilon_Y/2)$ and so that $x' = \pi(y')$ is in $X^u(x, \epsilon_X/2)$. The map π may be written as the composition of three maps. The first from $Y^s(y, \epsilon_Y/2)$ to $Y^s(y', \epsilon_Y)$ sends z to $[y', z]$. The second from $Y^s(y', \epsilon_Y)$ to $X^s(x')$ is simply π. The third is the map from $X^s(x', \epsilon_X/2)$ to $X^s(x, \epsilon_X)$ sends z to $[x, z]$. Each is defined on an open set containing y, y' and x', respectively and is an open map. The conclusion is that there exists some $\epsilon' > 0$ such that $\pi(Y^s(y, \epsilon')) = U$ is an open set in $X^s(x)$ containing x. It then follows that

$$X^s(x) = \cup_{l \geq 0} \varphi^{-l}(U) = \cup_{l \geq 0} \pi(\psi^{-l}(Y^s(y, \epsilon))) \subset \pi(Y^s(y)).$$

□

Now we want to observe that although the property of a map being s-bijective is defined purely at the level of stable sets, continuity properties follow as a consequence.

Theorem 5.2.11. *Let (X, φ) and (Y, ψ) be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be an s-bijective (or u-bijective) map. Then for each y in Y, the map $\pi : Y^s(y) \to X^s(\pi(y))$ (or $\pi : Y^u(y) \to X^u(\pi(y))$, respectively) is a homeomorphism.

Proof. The proof is the general fact that if A, B are locally compact Hausdorff spaces and $f : A \to B$ is a continuous, proper bijection, then f is a homeomorphism. This can be seen as as follows. Let A^+ and B^+ denote the one-point compactifications of A and B, respectively. That the obvious extension of f to a map between these spaces is continuous follows from the fact that f is proper. Since this extension is a continuous bijection between compact Hausdorff spaces, it is a homeomorphism. The result follows from this argument and Theorem **??**. □

We have established a number of properties of s/u-bijective maps. We now want to consider the constructions from the last section of fibred products and self-products as they pertain to s/u-resolving maps and s/u-bijective maps. The first basic result is the following.

Let us consider for a moment the special case of maps between two shifts of finite type. We begin with the following easy result. We describe a simple condition on the underlying graphs which is related.

Definition 5.2.12. *Let G and H be graphs. A graph homomorphism $\theta : H \to G$ is* left-covering *if it is surjective and, for every v in H^0, the map $\theta : t^{-1}\{v\} \to t^{-1}\{\theta(v)\}$ is a bijection. Similarly, π is* right-covering *if it is surjective and, for every v in H^0, the map $\theta : i^{-1}\{v\} \to i^{-1}\{\theta(v)\}$ is a bijection.*

The following result is obvious and we omit the proof.

Theorem 5.2.13. *If G and H are graphs and $\theta : H \to G$ is a left-covering (or right-covering) graph homomorphism, then the associated map $\theta : (\Sigma_H, \sigma) \to (\Sigma_G, \sigma)$ is an s-bijective (or u-bijective, respectively) factor map.*

The following is a result due to Kitchens - see Proposition 1 of [**?**] and the discussion just preceding it.

Theorem 5.2.14. *Let $\pi : (\Sigma, \sigma) \to (\Sigma_G, \sigma)$ be an s-bijective (or u-bijective) factor map. There exists a graph H, a left-covering graph homomorphism (or right-covering graph homomorphism, respectively) $\theta : H \to G$ and a conjugacy $h : (\Sigma_H, \sigma) \to (\Sigma, \sigma)$ such that $\pi \circ h = \theta$.*

We close with a result which is analogous to Theorem **??**.

Theorem 5.2.15. *Suppose that $(X, \varphi), (Y_1, \psi_1)$ and (Y_2, ψ_2) are dynamical systems and that $\pi_1 : (Y_1, \psi_1) \to (X, \varphi)$ and $\pi_2 : (Y_2, \psi_2) \to (X, \varphi)$ are maps. Let (Z, ζ) be their fibred product with maps ρ_1 and ρ_2 as in Definition 5.1.11. If π_1 (or π_2) is s-bijective, then so is ρ_2 (or ρ_1, respectively). If π_1 (or π_2) is u-bijective, then so is ρ_2 (or ρ_1, respectively).*

5.3 The degree of a factor map

The main topic of this section is the notion of the degree of a factor map. We will start in quite a general setting and later go on to investigate the situation for s-bijective and u-bijective factor maps.

Definition 5.3.1. *Let (Y, g) and (X, f) be irreducible Smale spaces and let $\pi : (Y, g) \to (X, f)$ be a finite-to-one factor map. We define the* degree *of π as*

$$deg(\pi) = \inf\{\#\pi^{-1}\{x\} \mid x \in X\}.$$

The following result is an immediate consequence of Lemma 5.1.13. We omit the proof.

Theorem 5.3.2. *Let (Y, g) and (X, f) be irreducible Smale spaces and let $\pi : (Y, g) \to (X, f)$ be a finite-to-one factor map. We have*

$$deg(\pi) = \inf\{\#\pi^{-1}\{x\} \mid x \in Per(X, f)\}.$$

Theorem 5.3.3. *Let (Y, g) and (X, f) be irreducible Smale spaces and let $\pi : (Y, g) \to (X, f)$ be a finite-to-one factor map. The set of all points x in X such that there exists $\delta > 0$ with $\#\pi^{-1}\{x'\} = deg(\pi)$ for every periodic point x' in $X(x, \delta)$ is open and dense.*

Proof. The fact that the set is open is clear. Let us now show it is dense. Let x be in X and $\delta > 0$. First, choose x_0 so that $\#\pi^{-1}\{x_0\} = deg(\pi)$. By Lemma 5.1.13, there is δ_0 such that every periodic point in $X(x_0, \delta_0)$ has exactly $deg(\pi)$ pre-images under π. By the irreducibility of (X, f), there are positive integers m, n such that

$$f^{-m}(X(x_0, \delta_0)) \cap X(x, \delta), f^{-n}(X(x, \delta)) \cap X(x_0, \delta_0)$$

are both non-empty. In fact, we may construct an ϵ-pseudo-orbit which is periodic and meets both $X(x_0, \delta_0)$ and $X(x, \delta)$. Then by shadowing, we may find a periodic point y whose orbit meets the same two sets. Of course, the value of $\#\pi^{-1}\{f^k(y)\}$ is constant in k. It follows that there is a periodic point z in $X(x, \delta)$ with $\#\pi^{-1}\{z\} = deg(\pi)$ and the conclusion follows from Lemma 5.1.13. $\square$

Our next goal is to get some more precise information on the size of a pre-image of a point, especially under s-bijective or u-bijective maps.

Basically, a Smale space (or any topological dynamical system) is irreducible if and only if it has a point with a dense forward orbit.

But we can do a little better than that; the first result below is stronger than the 'only if' part and the second is stronger than the 'if' part.

The first result is quite standard and proof can be found in many dynamical systems texts. We will provide a sketch for completeness.

Proposition 5.3.4. *Let (X, f) be a dynamical system with X compact, metric. If it is irreducible, then the set of all points with dense forward orbit is a dense G_δ subset of X. Also, the set of points whose forward orbit and backward orbit are both dense is a dense G_δ in X.*

Proof. Fix $n \geq 1$ and cover X with all n^{-1}-balls, then extract a finite subcover, which we denote $\{B_1, B_2, \ldots, B_M\}$. Define Y_n be the set of all points x such that, for $1 \leq m \leq M$, there is $k \geq 1$ with $f^k(x) \in B_m$. It is easy to check that Y_n is open. Next, we claim that Y_n is dense. Let x be any point in X and let $\epsilon > 0$. By irreducibility, we may find $k_1 \geq 1$ such that $f^{k_1}(B(x, \epsilon)) \cap B_1$ is non-empty. Again by irreducibility, find $k_2 \geq 1$ such that $f^{k_2}(B(x, \epsilon) \cap f^{-k_1}(B_1)) \cap B_2$ is non-empty. Continuing in this way, we find positive integers $k_1, \ldots, k_M$ such that

$$B(x, \epsilon) \cap f^{-k_1}(B_1)) \cap \ldots \cap f^{-k_M}(B_M))$$

is non-empty and this set is clearly in Y_n.

It follows from the Baire category theorem that $\cap_{n=1}^{\infty} Y_n$ is dense in X and we claim that any point in this intersection has a dense forward orbit. Let y be in $\cap_{n=1}^{\infty} Y_n$, x be in X and $\epsilon > 0$. Find n with $2n^{-1} < \epsilon$ and the B in the selected collection of n^{-1}-balls which contains x. So $B \subseteq B(x, \epsilon)$. We have $y \in \cap_{n'=1}^{\infty} Y_{n'} \subseteq Y_n$ and by definition, there is a positive integer k with $f^k(y) \in B \subseteq B(x, \epsilon)$.

The same argument also shows that the set of points whose backward orbit is dense is also a dense G_δ and the final statement follows since the intersection of two dense G_δ's is also a dense G_δ. □

Now we turn to the converse.

Proposition 5.3.5. *Let (X, f) be a Smale space. If there is a point whose forward orbit (or backward orbit) limits on every periodic point of X, then (X, f) is irreducible.*

Proof. Let x be the point described by the hypothesis. Let y be an accumulation point of the backward orbit of x. It is clearly non-wandering and so there are periodic points arbitrarily close. It follows that y is also a limit point of the forward orbit of x. By patching the forward orbit of x that gets close to y with part of the backward orbit of x that begins close to y we can form pseudo-orbits from x to itself. It follows then that x is in the non-wandering set and lies in one irreducible component. The orbit of x will remain in the same irreducible component of the non-wandering set and for this forward orbit to limit on every periodic point, X has only a single irreducible component. □

We now bring factor maps into the picture. This is analogous to results in section 9.1 of Lind and Marcus [].

Lemma 5.3.6. *Let (Y, g) and (X, f) be Smale spaces and let $\pi : (Y, g) \to (X, f)$ be an s-bijective factor map. Assume that x, x' are in X and x has a dense forward orbit. Then we have*

$$\#\pi^{-1}\{x\} \leq \#\pi^{-1}\{x'\}.$$

The same conclusion holds if π is u-bijective and x has a dense backward orbit.

Proof. List $\#\pi^{-1}\{x\} = \{y_1, \ldots, y_I\}$. Since the orbit of x is dense, we may find an increasing sequence of positive integers n_k such that $f^{n_k}(x)$ converges to x. Passing to a subsequence, we may assume that for each $1 \leq i \leq I$, the sequence $g^{n_k}(y_i)$ converges to some point of y and by continuity these points must all lie in $\pi^{-1}\{x'\}$. We claim that no two sequences can have the same limit. This will complete the proof. If they do, then for some i, j we have $d(g^{n_k}(y_i), g^{n_k}(y_j))$ tends to zero as k goes to infinity. Notice that

$$\pi(g^{n_k}(y_i)) = f^{n_k}(\pi(y_i)) = f^{n_k}(x) = f^{n_k}(\pi(y_j)) = \pi(g^{n_k}(y_i)).$$

By **??**, for k sufficiently large, we have

$$g^{n_k}(y_i) \in Y^u(g^{n_k}(y_j), \epsilon_\pi).$$

and this implies that

$$y_i \in Y^u(y_j, \lambda^{n_k}\epsilon_\pi).$$

Since this is true for all k, $y_i = y_j$ and we are done. □

The following result is in exact analogy with 9.1.2 []:

Theorem 5.3.7. *Let (X, φ) be an irreducible Smale space. Let $\pi : (Y, g) \to (X, f)$ be either an s-bijective or a u-bijective factor map. If x is any point of X with a dense forward orbit and a dense backward orbit then*

$$deg(\pi) = \#\pi^{-1}\{x\}.$$

In fact, this and 5.3.4, actually give us four different definitions of the degree for s-bijective or u-bijective factor maps:

1. the minimum number of points in a pre-image under π,

2. the number of pre-images under π of a point with dense forward orbit and backward orbit,

3. the minimum number of points in a pre-image under π of a periodic point,

4. the 'typical' number of points in a pre-image under π.

Chapter 6

Existence of maps between Smale spaces

6.1 Maps and partitions

The last Chapter discussed properties of maps between Smale spaces and some of the consequences of these properties. Here, we turn to a different problem: the existence of such maps. There are some special situations, especially for shifts of finite type, when it is rather easy to construct maps between systems. In general, it is not. The first really deep examples were given by Adler and Weiss and their construction was both elegant and surprising.

Before we get into the technical details, we will discuss some of the ideas rather informally. To do so, let us look at a very simple and familiar notion: decimal expansion. Here we will see all the principles which are used throughout the rest of this section. (This approach is similar to that given by Adler in [].)

The real numbers are defined by various axioms. On the other hand, we are all used to writing real numbers in decimal form, but the two are really distinct. For a moment, let us be really pedantic and ask whether $\frac{1}{2}$ is the same as .5? Of course, they are *equal*, but are they *the same*? The first is the real number obtained as the multiplicative inverse of $1+1$, the addition of the multiplicative identity to itself. The second is defined to be $1+1+1+1+1$, divided by 10, where 10 is the number of fingers possessed by the average person. (Already there is something a little artificial about the latter.) Let us ask the same question about $\frac{1}{3}$ and $.3333\cdots$. Now the latter involves summing an infinite series.

Let us repeat the question one more time with 1, $1.0000\cdots$ and $.99999\cdots$. The first is a real number (the multiplicative identity), the second is the sum of a rather trivial infinite series and the third is more complicated. Here we are getting into deeper waters: the decimal expansion is not unique. Of course, we are all used to this and have stopped worrying about it, but we would like to put this idea in a more rigorous way.

To simplify things, let us just work with real numbers between 0 and 1. With all of our discussion of shifts of finite type, we hope the reader will be happy to define the decimal expansion as a (one-sided) infinite sequence of digits $0,1,2\ldots,9$, or an element of $\{0,1,2,\ldots,9\}^{\mathbb{N}}$. To every decimal expansion $i=(i_k)_{k=1}^{\infty}$, we can assign a real number

$$\rho(i)=\sum_{k=1}^{\infty} i_k 10^{-k}.$$

This ρ is a continuous function from $\{0, 1, 2, \ldots, 9\}^{\mathbb{N}}$ to $[0, 1]$. Moreover, if i, j are in $\{0, 1, 2, \ldots, 9\}^{\mathbb{N}}$ and $\rho(i) = \rho(j)$, then either $i = j$ or $i_k = 9, j_k = 0$, for all k sufficiently large (or the other way around). In short, ρ is very close to being a bijection. But let us consider its domain and range as topological spaces. In the former, the only connected subsets are the points, while the latter is connected. If we were just given those two topological space to begin, it might be rather surprising that there is a continuous function between them with this property.

Next, we want to look a little closer at the definition of our function ρ. It is clear that it depends of the additive structure of the real numbers and we would like a different definition which is more purely topological. Consider that if r is a real number in $[0, 1]$, then the first digit of its decimal expansion simply tells us which of the intervals $[\frac{i}{10}, \frac{i+1}{10}], 0 \leq i \leq 9$ contains x. To get the second digit of the decimal expansion, we consider $10x$, remove the integer part and ask the same question. Already, this is looking more dynamical: the map $x \to 10x$ is doing something important. The key feature of the intervals is that f maps them bijectively onto the whole space. The part about ignoring the integer part of $10x$ can be achieved by simply replacing $[0, 1]$ by $\mathbb{R}/\mathbb{Z}$. So define $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $f(x) = 10x$. To get the decimal expansion for x, we simply look at its trajectory: $x, f(x), f^2(x), \ldots$ and observe in which interval as above each point lies.

Of course, this approach does not quite work, because a point such as .1 does not lie in a unique interval. But this is exactly what our function ρ does: rather than having a function which takes a real number to its decimal expansion, it goes the other way and we can say $\rho(i_1, i_2, \ldots) = x$ if $f^{k-1}(x)$ lies in $[\frac{i_k}{10}, \frac{i_k+1}{10}]$, for every $k \geq 1$. Or equivalently, x lies in $f^{1-k}[\frac{i_k}{10}, \frac{i_k+1}{10}]$, for every $k \geq 1$. Written yet one more way, we have

$$\{\rho(i)\} = \cap_{k=1}^{\infty} f^{1-k} \left[\frac{i_k}{10}, \frac{i_k + 1}{10}\right].$$

About the only part of this which is not quite suitable for Smale spaces is the fact that the dynamical system $(\mathbb{R}/\mathbb{Z}, f)$ is not invertible. But let us summarize what we have seen: if we have a dynamical system (X, f) we can use partitions of X to try to construct factor maps from symbolic systems, such as shifts of finite type, onto X. The remainder of this section is devoted to making these ideas rigorous.

One of the first ambiguities is whether it is easier to use open sets or closed sets in our partitions. In either case, let us set out the notion of regular open and regular closed sets. We will not use it extensively, but it does appear.

Definition 6.1.1. *In a topological space, a* regular closed set *is a set which is the closure of its interior. A a* regular open set *is a set which is the interior of its closure.*

Our first definition is quite a general one.

Definition 6.1.2. *Let X be a topological space and f be a homeomorphism of X. An* open generating partition *for (X, f) is a finite collection of open, non-empty sets, $\mathcal{P}$, satisfying the following.*

1. *The sets are pairwise disjoint; that is, $P \cap P'$ is empty if $P \neq P'$ in $\mathcal{P}$.*
2. *Their union is dense: $\cup_{P \in \mathcal{P}} \overline{P} = \overline{\cup_{P \in \mathcal{P}} P} = X$.*
3. *If $P : \mathbb{Z} \to \mathcal{P}$ is any function, then*
$$\cap_{K=1}^{\infty} \overline{\cap_{k=-K}^{K} f^{-k}(P(k))}$$
is either empty or a single point.

Example 6.1.3. *Let us consider the n-solenoid of 3.4 with $n = 5$. We use the topological notation from 3.4: points in X are infinite sequences $(x_0, x_1, \ldots)$ where each x_0 is a real number, but represents a class in $\mathbb{R}/\mathbb{Z}$. These satisfy $5x_i = x_{i-1}$, for all $i > 0$. Our map is*

$$\begin{aligned} f(x_0, x_1, \ldots) &= (5x_0, 5x_1, \ldots), \\ f^{-1}(x_0, x_1, \ldots) &= (x_1, x_2, \ldots). \end{aligned}$$

For $1 \leq k \leq 5$, define

$$P_k = \{x \in X \mid x_1 \in \left(\frac{k-1}{5}, \frac{k}{5}\right)\}.$$

It is clear that

$$\overline{P_i} = \{x \in X \mid x_1 \in \left[\frac{k-1}{5}, \frac{k}{5}\right]\}.$$

The first two properties for an open generating partitions are clear. The third is also satisfied, but we will not prove this for the moment.

The one remark we would like to make at this point is just to compare

$$\cap_{K=1}^{\infty} \overline{\cap_{k=-K}^{K} f^{-k}(P(k))}$$

with the simpler expression

$$\cap_{K=1}^{\infty} \cap_{k=-K}^{K} \overline{f^{-k}(P(k))} = \cap_{k=-\infty}^{\infty} f^{-k}\left(\overline{P(k)}\right).$$

We will show later in this section that the former is always exactly one point. But if we let P be the function that is constantly one, it is a simple matter to check that the second intersection contains at least two points:

$$(0,0,0,\ldots), (0,5^{-1},5^{-2},5^{-3},\ldots).$$

That is, if we had used the simpler condition above in the last part of the definition, this particular example would not be a generating partition. This justifies our choice of the more subtle formulation.

In view of the final condition in this definition, let us observe the following definition and easy topological fact.

Definition 6.1.4. *Let X be a compact metric space and A be any subset of X. We define the* diameter *of A, denoted $diam(A)$, to be*

$$diam(A) = \sup\{d(x,y) \mid x,y \in A\}.$$

Lemma 6.1.5. *Let X be a compact metric space and $C_1 \supseteq C_2 \supseteq \cdots$ be a decreasing sequence of closed, non-empty subsets. The we have $\cap_{K=1}^{\infty} C_K$ is a single point if and only if*

$$\lim_{K\to\infty} diam(C_K) = 0.$$

Proof. First, notice that the containment of the sets easily implies that the sequence is decreasing and non-negative, so it has a limit. Secondly, as the sets are closed and X is compact, the intersection is always non-empty. In view of these facts, the statement is equivalent to showing that $\cap_{K=1}^{\infty} C_K$ contains at least two points if and only if $\lim_{K\to\infty} diam(C_K) > 0$.

First suppose $\cap_{K=1}^{\infty} C_K$ contains at least two points, say $x \neq y$. As x,y are in C_K for every K, $diam(C_K) \geq d(x,y) > 0$. Hence, we have $\lim_{K\to\infty} diam(C_K) \geq d(x,y) > 0$.

Next suppose that $\lim_{K\to\infty} diam(C_K) = d > 0$. Then for every $K \geq 1$, we have $diam(C_K) = d/2$ and so we may find x_K, y_K in C_K with $d(x_K, y_K) > d/2$. As X is compact, we may pass to subsequences which are converging to x and y respectively. Fix K and we claim that x is in C_K. For $k \geq K$, x_k is in $C_k \subseteq C_K$ as desired. As C_K is closed, x is in C_K. The same argument shows that y is in C_K and so both x and y are in $\cap_{K=1}^{\infty} C_K$. On the other hand $d(x, y)$ is the limit of a subsequence of $d(x_K, y_K)$ which is bounded below by $d/2$ and we conclude that $x \neq y$. $\square$

Theorem 6.1.6. *Let X be a compact metric space and f be a homeomorphism of X. Let $\mathcal{P}$ be an open generating partition for (X, f). Define*

$$X_{\mathcal{P}} = \{P \in \mathcal{P}^{\mathbb{Z}} \mid \cap_{K=1}^{\infty} \overline{\cap_{k=-K}^{K} f^{-k}(P(k))} \neq \emptyset\}.$$

For any P in $X_{\mathcal{P}}$, define $\pi_{\mathcal{P}}(P)$ to be the unique point in

$$\cap_{K=1}^{\infty} \overline{\cap_{k=-K}^{K} f^{-k}(P(k))}.$$

1. *$X_{\mathcal{P}}$ is a closed subset of $\mathcal{P}^{\mathbb{Z}}$ and is invariant under σ.*
2. *The map*
$$\pi_{\mathcal{P}} : (X_{\mathcal{P}}, \sigma) \to (X, f)$$
is a factor map.
3. *The set*
$$\{x \in X \mid \#\pi_{\mathcal{P}}^{-1}\{x\} = 1\}$$
is a dense G_δ.

Proof. We have already observed that, for any P in $\mathcal{P}^{\mathbb{Z}}$, the sequence of sets $\overline{\cap_{k=-K}^{K} f^{-k}(P(k))}, K \geq 1$ is decreasing and each is closed. By hypothesis, the intersection is either a single point or is empty. In the latter case, there must exist a $K \geq 1$ such that $\cap_{k=-K}^{K} f^{-k}(P(k))$ is empty. This set is open and entirely contained in the complement of $X_{\mathcal{P}}$. Hence, $X_{\mathcal{P}}$ is closed.

If P is in $X_{\mathcal{P}}$, then it is clear that for any $K > 1$, we have

$$\begin{aligned} f\left(\overline{\cap_{k=-K}^{K} f^{-k}(P(k))}\right) &= \overline{\cap_{k=-K}^{K} f^{-k+1}(P_(k))} \\ &= \overline{\cap_{l=-K+1}^{K+1} f^{-l}(P(l+1)} \\ &= \overline{\cap_{l=-K+1}^{K+1} f^{-l}(\sigma(P)(l))} \end{aligned}$$

and also that

$$\overline{\cap_{l=-K-1}^{K+1} f^{-l}(\sigma(P)(l))} \subseteq \overline{\cap_{l=-K+1}^{K+1} f^{-l}(\sigma(P)(l))} \subseteq \overline{\cap_{l=-K+1}^{K-1} f^{-l}(\sigma(P)(l))}.$$

It follows easily that $\cap_{K \geq 1} \overline{\cap_{l=-K}^{K} f^{-l}(\sigma(P)(l))}$ is also a single point, so $\sigma(P)$ is in $X_{\mathcal{P}}$ and that $\pi_{\mathcal{P}}(\sigma(P)) = f(\pi_{\mathcal{P}}(P))$.

Next, we check that $\pi_{\mathcal{P}}$ is surjective. Let x be in X. Fix $K \geq 1$. We claim that there is at least one $P : \{-K, \ldots, K\} \to \mathcal{P}$ such that x is in the closure of $\cap_{k=-K}^{K} f^{-k}(P(k))$. We know that $\cup_{P \in \mathcal{P}} P$ is open and dense in X. It follows that

$$\cap_{k=-K}^{K} f^{-k} \left(\cup_{P \in \mathcal{P}} P \right)$$

is also open and dense. Moreover, as the elements of $\mathcal{P}$ are pairwise disjoint, we also have

$$\cap_{k=-K}^{K} f^{-k} \left(\cup_{P \in \mathcal{P}} P \right) = \cup_{P : \{-K, \ldots, K\} \to \mathcal{P}} \cap_{k=-K}^{K} f^{-k}(P(k)).$$

The claim follows from this.

We can then define a graph whose vertices at level $K \geq 1$ are those P where the closure of the intersection contains x. We obtain an edge from P at level K to $P|\{-K+1, \ldots, K-1\}$. This is an infinite tree with non-empty vertex sets at each level and hence contains an infinite path. This path then determines P in $X_{\mathcal{P}}$ with $\pi_{\mathcal{P}}(P) = x$.

Let us prove the last statement using the same framework. Consider

$$A = \cap_{k \in \mathbb{Z}} f^{-k} \left(\cup_{P \in \mathcal{P}} P \right).$$

This is a countable intersection of open dense sets in X, hence it is a dense G_δ. If x is any point in A, then for each $K \geq 1$, the function $P : \{-K, \ldots, K\} \to \mathcal{P}$ with x in $\overline{\cap_{k=-K}^{K} f^{-k}(P(k))}$ is unique. It follows that $\pi_{\mathcal{P}}^{-1}\{x\}$ is a single point. $\square$

Remark 6.1.7. *One important thing to note in this last result is that the system (X, f) is* not *assumed to be a Smale space, nor does the conclusion assert that $(X_{\mathcal{P}}, \sigma)$ is necessarily a shift of finite type, even if we assume (X, f) is a Smale space.*

In the case that our system (X, f) is a Smale space, we would like to do somewhat better with our partitions. The first reasonable requirement is that the elements should be nice in the local product structure.

Definition 6.1.8. *Let (X, d, f) be Smale space. A subset R of X is a* rectangle *if*

1. *$diam(R) \leq \epsilon_X$, and*
2. *$[R, R] = R$.*

Let us note the following simple properties of rectangles.

Lemma 6.1.9. *Let (X, d, f) be a Smale space.*

1. *Let R be a subset of X with $diam(R) \leq \epsilon_X$. R is a rectangle if and only if $[R, R] \subseteq R$.*
2. *If R and R' are rectangles, then so is $R \cap R'$.*
3. *If R is a rectangle and $diam(f^{-1}(R)) \leq \epsilon_X$ (or $diam(f(R)) \leq \epsilon_X$) , then $f^{-1}(R)$ ($f(R)$, respectively) is also a rectangle.*

Proof. Notice that $[x, x] = x$ implies that $R \subseteq [R, R]$ always holds and the first statement follows from this.

For the second statement, we have

$$[R \cap R', R \cap R'] \subseteq [R, R] = R.$$

An analogous argument shows $[R \cap R', R \cap R'] \subseteq R'$ and we are done.

The third statement follows from the invariance of the bracket under f. $\square$

Now we come to one of the key definitions in the subject, that of Markov partition.

Definition 6.1.10. *Let (X, d, f) be Smale space. An* open Markov partition *for (X, d, f) is a finite collection of open, non-empty sets $\mathcal{R}$ satisfying the following.*

1. *For each R in $\mathcal{R}$, R and $f^{-1}(R)$ are rectangles.*
2. *The sets are pairwise disjoint; that is, $R \cap R'$ is empty if $R \neq R'$ are in $\mathcal{R}$.*
3. *Their union is dense: $\cup_{R \in \mathcal{R}} \overline{R} = \overline{\cup_{R \in \mathcal{R}} R} = X$.*

4. *If for any* R, R' *in* $\mathcal{R}$ *with* $R \cap f^{-1}(R')$ *non-empty, then for all* x *in* R *and* y *in* $f^{-1}(R')$, $[y, x]$ *is defined and we have*

$$[f^{-1}(R'), R] = R \cap f^{-1}(R').$$

(This is usually called the Markov *property.)*

Remark 6.1.11. *Usually the elements of a Markov partition are defined to be regular closed sets. This is why we have chosen the term* open *Markov partition. In the usual definition, the third condition and fifth conditions must use the interiors of the sets.*

Let us also note that one containment in condition 4 is trivial:

$$[f^{-1}(R'), R] \supseteq [R \cap f^{-1}(R'), R \cap f^{-1}(R')] \supseteq R \cap f^{-1}(R').$$

It is a good idea to give an example at this point. The following is more than an example, it is one of the prime motivations for this definition.

Proposition 6.1.12. *Let* G *be a finite directed graph and* (X_G, σ) *be the associated shift of finite type. For each edge* e *in* G^1, *define*

$$C_e = \{y \in X_G \mid y_0 = e\}.$$

1. *For each* e *in* G^1, C_e *is open and closed and both* C_e *and* $\sigma^{-1}(C_e)$ *are rectangles.*

2. *The sets* $C_e, e \in G^1$, *are pairwise disjoint and union to* X_G.

3. *For* e, e' *in* G^1, $C_e \cap \sigma^{-1}(C_{e'})$ *is non-empty if and only* $t(e) = i(e')$.

4. $C_e, e \in G^1$ *is an open Markov partition for* (X_G, σ).

Proof. It is an easy matter to check that, for any e in G^1,

$$\sigma^{-1}(C_e) = \{y \in X_G \mid y_1 = e\}.$$

The rest of the proof is an easy consequence and we omit the details. □

We use the notation C_e as a reference to cylinder sets.

As we indicated just before this result, it is important, not because it is hard, obviously, but because it provides motivation for the concept.

In some sense, the Markov partition is capturing all of the data that is in the graph. More precisely, if we define a graph whose vertices are the elements of the Markov partition and with an edge from C_e to $C_{e'}$ precisely when $Ce \cap \sigma^{-1}(C_{e'})$ is non-empty, then the graph we construct is exactly G^1.

If our original graph had no multiple edges, then one could define a partition parameterized by the vertices of G and this would recover G in the same way. There are simply some problems here with multiple edges.

We have stated the result above mostly for motivational purposes, but it has a vast generalization which we now consider.

Lemma 6.1.13. *Let (X, d, f) be a Smale space and G be a finite directed graph with (X_G, σ) irreducible. Let $\pi : (X_G, \sigma) \to (X, f)$ be a regular, finite-to-one factor map of degree one. For each e in G^1 and using $C_e \subseteq X_G$ as in 6.1.12, let*

$$R_e = \{x \in X \mid \pi^{-1}\{x\} \subseteq C_e\}.$$

The following hold.

1. *The sets $R_e, e \in G^1$ are pairwise disjoint.*

2. *R_e is open, for e in G^1.*

3. *$\cup_{e \in G^1} R_e$ is dense in X.*

4. *$\pi(C_e)$ is the closure of R_e, for e in G^1.*

5. *R_e is the interior of $\pi(C_e)$, for e in G^1.*

6. *R_e is non-empty, for e in G^1.*

Proof. The first statement follows immediately from the definition and the fact that the sets $C_e, e \in G^1$, are pairwise disjoint.

Next, we show that R_e is open. Let x be in R_e, so $\pi^{-1}\{x\} \subseteq C_e$. As C_e is open, we may find a positive ϵ such that $Y(y, \epsilon) \subseteq C_e$, for every y in $\pi^{-1}\{x\}$. It then follows from **??** that there is a $\delta > 0$ such that $\pi^{-1}(X(x, \delta)) \subseteq C_e$ and hence $X(x, \delta)$ is contained in R_e.

Since the map π is surjective, $\pi^{-1}\{x\}$ is non-empty, for every x in X, and it follows that $R_e \subseteq \pi(C_e)$ for every e in G^1.

If x is in X with a unique y with $\pi(y) = x$, then x is in R_{y_0}. It follows that $\{x \mid \#\pi^{-1}\{x\} = 1\} \subseteq \cup_{e \in G^1} R_e$. The former is dense in X since π is degree one and by **??**, hence the latter is as well.

Next, we prove that, for any e in G^1, R_e is dense in $\pi(C_e)$. Let y be in C_e and let $\epsilon > 0$. We will find a point in $R_e \cap X(\pi(y), \epsilon)$. First, find $J \geq 1$ such that if $y'_j = y_j, -J \leq j \leq J$, then $\pi(y')$ is in $X(\pi(y), \epsilon)$. Let x be any point in X with a unique z in X_G with $\pi(z) = x$. As noted above, x is in R_{z_0}, which is an open set. We can find $K \geq 1$ such that any point z' in X_G with $z'_k = z_k, -K \leq k \leq K$, then $\pi(z')$ also lies in R_{z_0}. We use the fact that X_G is irreducible along with **??** to find a path $u_1, u_2, \ldots, u_L$ from $t(z_K$ to $i(y_{-J})$ and a second path $v_1, v_2, \ldots, v_M$ from $t(y_J)$ to $i(z_{-K})$. Let w be the periodic point in X_G obtained by repeating the sequence

$$z_{-K}, \ldots, z_K, u_1, u_2, \ldots, u_L, y_{-J}, \ldots, y_J, v_1, v_2, \ldots, v_M.$$

First, observe that $\pi(w)$ is in R_{z_0}. Since it periodic Lemma **??** implies that it has a unique pre-image under π. The same holds for any point in its orbit under f. In addition, $\sigma^{J+K+L+1}(w)$ clearly lies in C_e and $\pi(\sigma^{J+K+L+1}(w)) = f^{J+K+L+1}(\pi(w))$ lies in $X(\pi(y), \epsilon)$. It follows that $\pi(\sigma^{J+K+L+1}(w))$ is in $R_e \cap X(\pi(y), \epsilon)$.

We know know that R_e is a dense open subset of $\pi(C_e)$. We claim next that it is actually the interior of $\pi(C_e)$. The containment of R_e in the interior of $\pi(C_e)$ is clear. Now let x be in X_G, $\epsilon > 0$ with $X(x, \epsilon) \subseteq \pi(C_e)$. We claim that x is in R_e. If not, then there is y in X_G with $\pi(y) = x$ and y not in C_e. Then, y is in $\pi(C_{y_0})$ where $y_0 \neq e$. We know that R_{y_0} is dense in $\pi(C_{y_0})$ and so we may find z in $R_{y_0} \cap X(x, \epsilon)$. This means $\pi^{-1}\{z\} \subseteq C_{z_0}$, contradicting our assumption that $X(x, \epsilon) \subseteq \pi(C_e)$. Hence, we have shown that $X(x, \epsilon) \subseteq R_e$.

As we assume our graphs have no sources or sinks, C_e is non-empty, for every e. The same holds for R_e from the second condition. □

Theorem 6.1.14. *Let (X, d, f) be a Smale space and G be a finite directed graph with (X_G, σ) irreducible. Let $\pi : (X_G, \sigma) \to (X, f)$ be a regular, finite-to-one factor map of degree one. The sets $R_e, e \in G^1$, of 6.1.13 form an open Markov partition for (X, f).*

Proof. Some of the properties have already been established in 6.1.13. It remains to see that our sets are rectangles and that they satisfy the Markov property.

First we observe that, for e in G^1, the set $\pi(C_e)$ is a rectangle. We know that for every x, y in C_e, $d(x, y) \leq \epsilon_{X_G}$ and as π is regular, $d(\pi(x), \pi(y)) \leq \epsilon_X$ and $[\pi(x), \pi(y)] = \pi[x, y]$, and $[x, y]_0 = y_o = e$, so $[x, y]$ is in C_e.

If, in addition, $\pi(x)$ and $\pi(y)$ are both in the interior of $\pi(C_e)$, then we may find open sets U, U' around them also contained in $\pi(C_e)$. But from the basic properties of the bracket $[U, U']$ is then also an open set in $\pi(C_e)$ containing $[\pi(x), \pi(y)]$. It follows that R_e is a rectangle.

Exactly the same argument can be used to show that $f^{-1}(R_e)$ is a rectangle and also the fourth property in the definition of an open Markov partition. $\square$

Notice that Proposition 6.1.12 is simply 6.1.14 in the special case that π is the identity map.

This result is really the second in a chain: in 6.1.6 we showed how to get from an open generating partition to a factor map from a subshift. Here, with the additional hypothesis of having Smale spaces, we see how to get from a factor map to an open Markov partition. Our next step will be to close up this chain by showing that an open Markov partition is also an open generating partition. Most of the work is contained in the following.

Lemma 6.1.15. *Let $\mathcal{R}$ be an open Markov partition for the Smale space (X, f). Suppose $0 \leq K, L$ and let $R : \{-K, -K+1, \ldots, L\} \to \mathcal{R}$ be any function.*

1. *If $R(l) \cap f^{-1}(R(l+1))$ is non-empty, for all $0 \leq l < L$, then*
$$[\cap_{l=1}^{L} f^{-l}(R(l)), R(0)] = \cap_{l=0}^{L} f^{-l}(R(l)).$$
2. *If $R(l) \cap f^{-1}(R(l+1))$ is non-empty, for all $0 \leq l < L$, then $\cap_{l=0}^{L} f^{-l}(R(l))$ is also non-empty.*
3. *If x is in $R(l) \cap f^{-1}(R(l+1))$, for all $0 \leq l < L$, then*
$$diam[\cap_{l=0}^{L} f^{-l}(R(l)), x] \leq \lambda^{L} diam(X).$$
4. *If $R(k) \cap f^{-1}(R(k+1))$ is non-empty, for all $-K \leq k < 0$, then*
$$[R(0), \cap_{k=-K}^{-1} f^{-k}(R(k))] = \cap_{k=-K}^{0} f^{-k}(R(k)).$$
5. *If $R(k) \cap f^{-1}(R(k+1))$ is non-empty, for all $-K \leq k < 0$, then $\cap_{k=-K}^{0} f^{-k}(R(k))$ is also non-empty.*

6. *If x is in $R(k) \cap f^{-1}(R(k+1))$, for all $-K \leq k < 0$, then*

$$diam[x, \cap_{k=-K}^{0} f^{-k}(R(k))] \leq \lambda^K diam(X).$$

7. *The intersection*

$$\cap_{k=-K}^{L} f^{-k}(R(k))$$

is non-empty if and only if $R(j) \cap f^{-1}(R(j+1))$ is non-empty, for all $-K \leq j < L$.

8. *If x is in $R(k) \cap f^{-1}(R(k+1))$, for all $-K \leq k < L$, then we have*

$$\lim_{K \to \infty} diam(\cap_{k=-K}^{K} f^{-k}\,(R(k)) = 0.$$

Proof. For the first statement, one containment is relatively easy:

$$\begin{aligned}
[\cap_{l=1}^{L} f^{-l}(R(l)), R(0)] &\supseteq [\cap_{l=0}^{L} f^{-l}(R(l)), \cap_{l=0}^{L} f^{-l}(R(l))] \\
&\supseteq \cap_{l=0}^{L} f^{-l}(R(l)).
\end{aligned}$$

We will prove the reverse inclusion by induction on L, with the case $L = 1$ being trivially true from the definition of a Markov partition. Assume that the statement is true for L. Consider x in $\cap_{l=1}^{L+1} f^{-l}(R(l))$ and y in $R(0)$. In particular, x is in $f^{-1}(R(1))$ and so $[x, y]$ is in $[f^{-1}(R(1)), R(0)] = f^{-1}(R(1)) \cap R(0)$. We also have $[x, y] = f^{-1}[f(x), f[x, y]]$. Here, $f[x, y]$ is in $R(1)$ while $f(x)$ is in $\cap_{l=1}^{L} f^{-l}(R(l+1))$. It follows from the induction hypothesis for the function $R'(l) = R(l+1)$ that $[f(x), f[x, y]]$ is in $\cap_{l=0}^{L} f^{-l}(R(l+1))$ and hence $[x, y]$ is in $\cap_{l=1}^{L+1} f^{-l}(R(l))$. This completes the proof of the first part.

The second part follows from a simple induction argument using the first part.

The third part is also by induction on L, with the case $L = 0$ being obvious. Let us assume this holds for L (and all R, x) and consider the case $L + 1$. We write

$$\begin{aligned}
[\cap_{l=0}^{L+1} f^{-l}(R(l)), x] &= [[\cap_{l=1}^{L+1} f^{-l}(R(l)), R(0)], x] \\
&= [\cap_{l=1}^{L+1} f^{-l}(R(l)), x] \\
&= f^{-1}[\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)]
\end{aligned}$$

The set $[\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)]$ has diameter less than or equal to $\lambda^L diam(X)$, by induction. In addition, we have

$$\big[[\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)], f(x)]\big] = [\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)]$$

and it follows from Axiom C2 for Smale spaces that

$$\begin{aligned} diam(f^{-1}[\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)]) &\leq \lambda diam([\cap_{l=0}^{L} f^{-l}(R(l+1)), f(x)]) \\ &\leq \lambda^{L+1} diam(X). \end{aligned}$$

Parts 4, 5 and 6 are proved in an analogous way to parts 1, 2 and 3.For part 7, the 'only if' part is clear and for the 'if' part we use parts 1, 2, 3 and 4:

$$\begin{aligned} \cap_{k=-K}^{L} f^{-k}(R(k)) &= \left(\cap_{k=-K}^{0} f^{-k}(R(k))\right) \cap \left(\cap_{k=0}^{L} f^{-k}(R(k))\right) \\ &= [R(0), \cap_{k=-K}^{0} f^{-k}(R(k))] \cap [\cap_{k=0}^{l} f^{-k}(R(k)), R(0)] \\ &\supseteq [\cap_{k=0}^{l} f^{-k}(R(k)), \cap_{k=-K}^{0} f^{-k}(R(k))] \end{aligned}$$

which is clearly non-empty.

For the last statement we know from **??** that, for any x in $\cap_{k=-K}^{L} f^{-k}(R(k))$, the map sending y in $\cap_{k=-K}^{L} f^{-k}(R(k))$ to the pair $([x, y], [y, x])$ in $X \times X$ is continuous and injective. It follows from parts 3 and 6 that the diameter of the range of this function tends to zero as K, L tend to infinity. The conclusion follows. □

With these technical results established, the following theorem is more or less immediate and we omit the details for the proof.

Theorem 6.1.16. *Let (X, f) be a Smale space and suppose that $\mathcal{R}$ is an open Markov partition for (X, f). Let G be the graph whose vertex set is $\mathcal{R}$, edge set is all ordered pairs (R, R') where R, R' are in $\mathcal{R}$ and satisfy $R \cap f^{-1}(R')$ is non-empty with $i(R, R') = R$ and $t(R, R') = R'$.*

1. *$\mathcal{R}$ is also an open generating partition for (X, f).*

2. *$X_{\mathcal{R}} = X_G$. In particular, $(X_{\mathcal{R}}, \sigma)$ is a shift of finite type.*

Remark 6.1.17. *In practical terms, the usefulness is in finding Markov partitions and then obtaining a factor map. Intrinsically, the conditions for a Markov partition are simpler to verify than those for a generating partition because it suffices to consider only intersections between $f^{-k}(\mathcal{P})$ when $k = 0$ and $k = 1$. This will be illustrated in the next example. In addition, the Markov partition yields the extra conclusion that the domain of the factor map is a shift of finite type.*

One slightly annoying condition on the elements of a Markov partition is that on their diameter. We will see in a moment an example where we essentially have a Markov partition except that the sets are too big. We present the following result to help deal with this situation.

Theorem 6.1.18. *Let (X, f) be a Smale space and suppose that $\mathcal{R}$ is a finite collection of open, non-empty sets that are pairwise disjoint and whose union is dense in X.*

Let G be the graph whose vertex set is $\mathcal{R}$, edge set is all ordered pairs (R, R') where R, R' are in $\mathcal{R}$ and satisfy $R \cap f^{-1}(R')$ is non-empty with $i(R, R') = R$ and $t(R, R') = R'$. For $K \geq 1$, we identify the functions $R : \{0, \ldots, K\} \to \mathcal{R}$ with G^K.

Suppose that there is $K \geq 1$ satisfying the following.

1. *For R in G^K, if $R(k) \cap f^{-1}(R(k+1))$ is non-empty for $0 \leq k < K+1$, then $\cap_{k=0}^{K+1} f^{-k}(R(k))$ is also non-empty.*

2. *The collection of sets*

$$\{\cap_{k=0}^{K} f^{-k}(R(k)) \mid R \in G^K\}$$

is a Markov partition for (X, f).

The the following hold.

1. *$\mathcal{R}$ is also an open generating partition for (X, f).*

2. *$X_{\mathcal{R}} = X_G$. In particular, $(X_{\mathcal{R}}, \sigma)$ is a shift of finite type.*

Proof. Let R be in G^K. The first hypothesis is that the sets $\cap_{k=0}^{K} f^{-k}(R(k))$ is non empty. If R' is also in G^K, then as the elements of the partition are pairwise disjoint

$$\begin{aligned} & \left(\cap_{k=0}^{K} f^{-k}(R(k))\right) \cap f^{-1}\left(\cap_{k=0}^{K} f^{-k}(R'(k))\right) \\ = \quad & R(0) \cap \left(\cap_{k=1}^{K} f^{-k}(R(k) \cap R'(k-1))\right) \cap f^{-k-1}(R'(K)) \end{aligned}$$

is empty unless $R(k) = R'(k-1)$ for all $1 \leq k \leq K$. In the case that $R(k) = R'(k-1)$ for all $1 \leq k \leq K$, the intersection above is non-empty by the hypothesis.

This proves that the graph associated with the Markov partition $\{\cap_{k=0}^{K} f^{-k}(R(k)) \mid R \in G^K\}$ is precisely G^K. It is a trivial matter to check that $\mathcal{R}$ is a generating partition and that $X_{G^K} = X_G$. The conclusion follows from 6.1.16. $\square$

Example 6.1.19. *Let us return to the example we considered in 6.1.3. We will show that the partition given there is actually an open Markov partition. It is a simple matter to check that for $1 \leq i \leq 5$, we have*

$$\begin{aligned} f^{-1}(P_j) &= \left\{ x \in X \mid x_0 \in \left(\frac{j-1}{5}, \frac{j}{5} \right) \right\} \\ &= \left\{ x \in X \mid x_1 \in \cup_{i=0}^{4} \frac{i}{5} + \left(\frac{j-1}{25}, \frac{j}{25} \right) \right\}. \end{aligned}$$

It follows immediately that, for all $1 \leq i, j, k \leq 5$, we have

$$P_k \cap f^{-1}(P_j) = \left\{ x \in X \mid x_1 \in \frac{k-1}{5} + \left(\frac{j-1}{25}, \frac{j}{25} \right) \right\}.$$

and

$$f^{-1}(P_j \cap f^{-1}(P_i)) = \left\{ x \in X \mid x_0 \in \frac{j-1}{5} + \left(\frac{i-1}{25}, \frac{i}{25} \right) \right\}.$$

We leave it for the reader to check that both sets have diameter less than .5.

Let us take x in $f^{-1}(P_j \cap f^{-1}(P_i))$ and y in $P_k \cap f^{-1}(P_j)$ and we compute, from the formula given in **??**,

$$[x, y] = (x_0, y_1 + \frac{x_0}{5} - \frac{y_1}{5}, \ldots).$$

It follows from the first entry x_0 that this is in $f^{-1}(P_j \cap f^{-1}(P_i))$. On the other hand, from the first entry, since $5y_1 = y_0$, we have $y_1 - \frac{y_0}{5} = \frac{k-1}{5}$. It follows that

$$y_1 + \frac{x_0}{5} - \frac{y_1}{5} \in \frac{k-1}{5} + \left(\frac{j-1}{25}, \frac{j}{25} \right)$$

and so $[x, y]$ is in $P_k \cap f^{-1}(P_j)$.

Part III

Measures

Chapter 7

Measures on Smale spaces

7.1 The Parry measure

Part IV

C^*-algebras

Chapter 8

Constructing C^*-algebras from Smale spaces

8.1 C^*-algebras from equivalence relations

8.2 Étale equivalence relations

As we indicated in the last section our general goal is to create a C^*-algebra from an equivalence relation on a topological space. The space itself must be compact and Hausdorff but the more subtle technical point here is that the equivalence relation must also be endowed with a topology which is compatible in some precise sense with that of X. The prime examples for Smale spaces will be stable and unstable equivalence.

We begin with a a fairly standard notion.

Definition 8.2.1. *Let X be a topological space. A* local homeomorphism *of X is a pair of open subsets $U, V \subseteq X$ and a homeomorhism $h : U \to V$.*

Now we test our readers' memories with a basic question: what is a function? If you will remember, a function from a set X to a set Y is a subset of the cartesian product $X \times Y$ satisfying certain conditions, which we will not recall. For example, the function $f(x) = x^3$ on $\mathbb{R}$ is really

$$\{(x, x^3) \mid x \in \mathbb{R}\}.$$

While this is something mathematicians all agree upon, it is almost never how we think about functions. We are going to do so now, however, and the reason is fairly simple. An equivalence relation on the set X *is* commonly thought of as a set of ordered pairs, satisfying some rather different conditions from functions. The key idea is that we want to see our equivalence relations as made up of functions, or more precisely, local homeomorphisms of the space X.

We let $r, s : X \times X$ be the two canonical projection maps: $s(x, y) = x, r(x, y) = y$. (Caution: this choice is not uniform.) In this way, a local homeomorphism of X, $h : U \to V$ is then a subset $h \subseteq U \times V$. But as both are subsets of X, we will regard h as a subset of $X \times X$. Its domain is $s(h)$ and its range is $r(h)$, so in the definition above, $U = s(h), V = r(h)$ can actually be recovered from h.

Regarding local homeomorphisms as subsets of $X \times X$, we define composition by

$$g \circ h = \{(x, z) \mid \text{ there exists } y, (x, y) \in g, (y, z) \in h\}.$$

Unfortunately, this definition does not quite agree with the usual notion. Instead, we have $g \circ h(x) = h(g(x))$, with our definition on the left and the usual one on the right. One learns to live with this. (The correct solution to this problem would have been to write $(x)f = x^3$ for the cubing function on $\mathbb{R}$, but it is too late to implement that.)

Also note in our definition that we do not assume the range of one function coincides with the domain of the other. Composition is simply defined where it makes sense. Fortunately, the inverse of a functions is nicer:

$$h^{-1} = \{(x, y) \mid (y, x) \in h\}.$$

Definition 8.2.2. *Let X be a topological space. Suppose that Γ is a collection of subsets of $X \times X$ such that*

1. *Each element of Γ is a local homeomorphism of X.*

2. *The collection of sets $U \subseteq X$ such that $id_U = \{(x, x) \mid x \in U\}$ is a basis for the topology on X.*

3. *If γ is in Γ, then so is γ^{-1}.*

4. *If γ_1, γ_2 are in Γ and $(x, y) \in \gamma_1 \cap \gamma_2$, then there is a γ in Γ such that*

$$(x, y) \in \gamma \subseteq \gamma_1 \cap \gamma_2.$$

5. *If γ_1, γ_2 are in Γ and $(x, y) \in \gamma_1, (y, z) \in \gamma_2$, then there is a γ in Γ such that*

$$(x, z) \in \gamma \subseteq \gamma_1 \circ \gamma_2.$$

We call Γ a local action *on X.*

Remark 8.2.3. *A stronger version of the definition is to require $\gamma_1 \cap \gamma_2$ to be in Γ in the third condition and $\gamma_1 \circ \gamma_2$ to be in Γ in the fourth. That works in a number of cases, but is not quite general enough for what we want to do with Smale spaces in the next section.*

Theorem 8.2.4. *Let Γ be a local action on the space X. Then the union of the elements of Γ*

$$R_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma \subseteq X \times X$$

is an equivalence relation on X and Γ is a basis for a topology on R_Γ.

Proof. This sounds impressive, but is more or less a tautology. The fact that R_Γ is reflexive follows from condition 2 of the definition of a local action. That it is symmetric follows from condition 3 and that is it transitive follows from condition 5.

It is obvious the elements of Γ cover R_Γ and condition 4 is the only other property needed to have a basis for a topology. □

Theorem 8.2.5. *Let Γ be a local action on the space X and let R_Γ be the associated étale equivalence relation.*

1. *If U is any element of Γ, then $r(U)$ and $s(U)$ are open sets in X and the maps r, s are homeomorphisms (with U given the relative topology from Γ to $r(U)$ and $s(U)$, respectively.*

2. *If X is locally compact and Hausdorff, then so is R_Γ.*

3. *If $K \subset R_\Gamma$ is compact, then for every x in X, $r^{-1}\{x\} \cap K$ and $s^{-1}\{x\} \cap K$ are finite.*

Lemma 8.2.6. *Let Γ be a local action on the space X and let R_Γ be the associated étale equivalence relation. Any continuous function of compact support on R_Γ, may be written as a finite sum of functions, each supported on a single element of Γ.*

8.3 Étale equivalence relations for Smale spaces

Having given some motivation in the first section and supplied some rigour with the definition of an étale equivalence relation in the second, we are now ready to put these ideas to work on Smale spaces. More specifically, we want to define étale equivalence relations from stable and unstable equivalence.

Let us remark at the beginning that we make no assumptions here about non-wandering, irreducibility or maxing.

Definition 8.3.1. *Let (X, d, f) be a Smale space. Suppose that P is a finite subset of X with $f(P) = P$.*

1. *We define*

$$X^s(P) = \cup_{x \in P} X^s(x), X^u(P) = \cup_{x \in P} X^u(x).$$

   Moreover, each set in the union is given the topology of **??** *while the union is given the disjoint union topology.*

2. *We define*

$$\begin{aligned} R^s(X, f, P) &= \{(x, y) \mid x, y \in X^u(P), y \in X^s(x)\}, \\ R^u(X, f, P) &= \{(x, y) \mid x, y \in X^s(P), y \in X^u(x)\} \end{aligned}$$

Caution is needed: in the definition of $R^s(X, f, P)$, the important condition is that x and y are *stably* equivalent (which is the reason for the notation). The dependence on P is that these points are required to come from $X^u(P)$.

The most obvious problem with the definition we have just given is that the first part contains a nice definition for a topology on $X^u(P)$ and $X^s(P)$ (in fact, we had this topology some time ago), while the sets in the second part are just that, sets. No topology is given. The reason is that this is a subtle point and we need to work to provide one.

Definition 8.3.2. *Let (X, d, f) be a Smale space and P be a finite subset of X with $f(P) = P$.*

1. *We let $\mathcal{N}^s(X, f, P)$ (or simply $\mathcal{N}^s$) denote the set of all (x, y, n, U) satisfying the following conditions.*

(a) (x, y) *is in* $R^s(X, f, P)$,

(b) $n \geq 0$ *is such that* $f^n(y)$ *is in* $X^s(f^n(x), \epsilon_X/2)$ *(see* **??***),*

(c) $U \subseteq X^u(x, \epsilon_X/2)$ *is relatively open and such that* $f^n(U) \subseteq X^u(f^n(x), \epsilon_X/2)$,

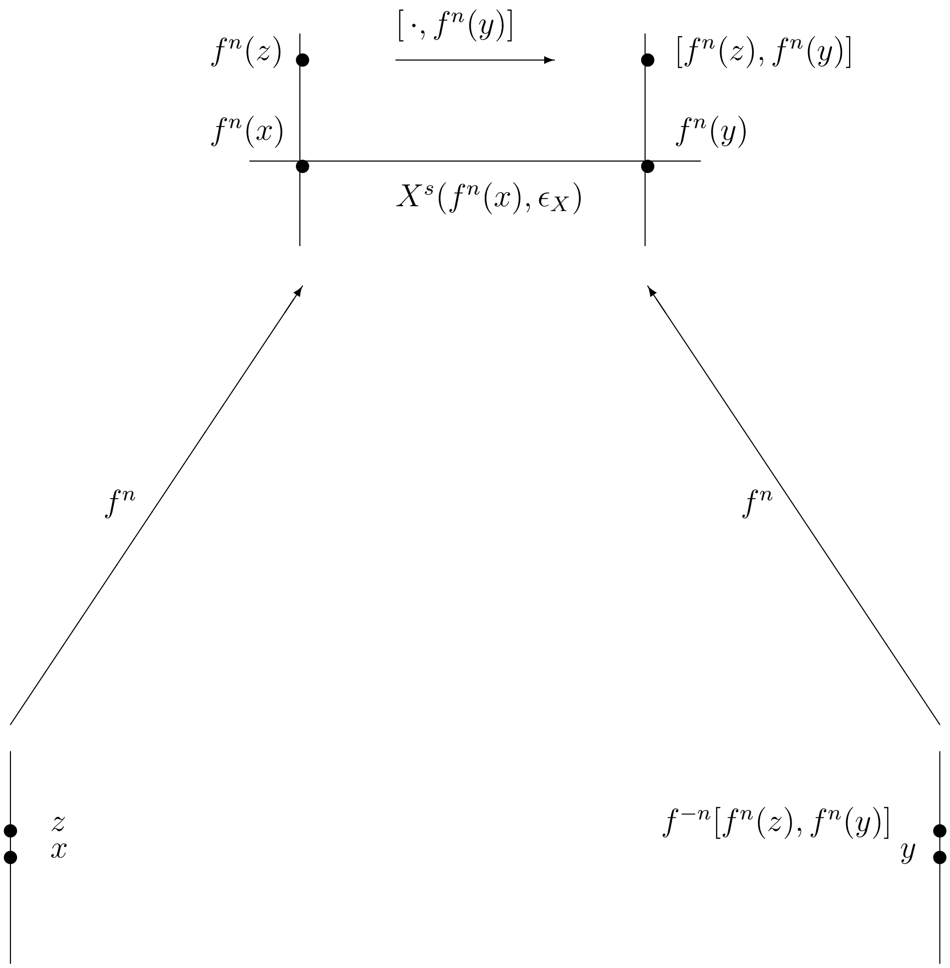
(d) for all z *in* U, *we have* $[f^n(z), f^n(y)]$ *(which we note is well-defined) lies in* $X^u(f^n(y), \epsilon_X/2)$,

(e) for all z *in* U, *we have* $f^{-n}[f^n(z), f^n(y)]$ *lies in* $X^u(y, \epsilon_X/2)$,

2. *For* (x, y, n, U) *in* $\mathcal{N}^s$, *we define*

$$h^s(x, y, n, U) = \{(z, f^{-n}[f^n(z), f^n(y)]) \mid z \in U\} \subset R^s(X, f, P).$$

It is often observed that a picture is worth a thousand words and that is certainly true of this definition, which is fairly opaque in symbols. But let us take a look at the following picture.



The idea is simply this: as x and y are stably equivalent, we can get one into the local stable set of the other, by finding a suitable $n \geq 0$ and applying f^n to both. Next, we can choose an open subset of $X^u(x, \epsilon_X)$ containing x such that $f^n(U)$ lies within $X^u(f^n(x), \epsilon_X/2)$. This set U will need to be very small since f^n will expand the set $X^u(x, \epsilon_X)$, but it exists by continuity. Finally, the map $[\cdot, f^n(y)]$ moves points in $X^u(f^n(x), \epsilon_X/2)$ into the local unstable set of $f^n(y)$ and takes $f^n(x)$ to $f^n(y)$. Again by continuity, we may choose U sufficiently small so that the image of $f^n(U)$ under this bracket operation is contained in $X^u(f^n(y), \epsilon_X/2)$. Our $h^s(x, y, n, U)$ is simply the composition of f^n with $[\cdot, f^n(y)]$, with f^{-n}.

We have now given a proof of the first two part of the following technical result.

Lemma 8.3.3. *1. For any (x, y) in $R^s(X, f, P)$, there is n and U such that (x, y, n, U) is in $\mathcal{N}^s$.*

2. For any (x, y, n, U) in $\mathcal{N}^s$, (x, y) is in $h^s(x, y, n, U)$.

3. If $U \subseteq X^u(x, \epsilon_X/2)$ is open, then $(x, x, 0, U)$ is in $\mathcal{N}^s$ and $h^s(x, x, 0, U)$ is the identity map on U.

4. For any (x, y, n, U) in $\mathcal{N}^s$, the sets $r(h^s(x, y, n, U))$ and $U = s(h^s(x, y, n, U))$ are open is $X^u(P)$.

5. For any (x, y, n, U) in $\mathcal{N}^s$, if we let $V = r(h^s(x, y, n, U))$, then (y, x, n, V) is also in $\mathcal{N}^s$ and $h^s(y, x, n, V) = h^s(x, y, n, U)^{-1}$.

Proof. As we mentioned above, we have already proved the first two statements. The third follows at once from the definition. For the fourth, the statement $U = s(h^s(x, y, n, U))$ is obvious. Also, we observe that the map sending w in $X^u(f^n(x), \epsilon_X/2)$ to $[w, f^n(y)]$ has an inverse on the intersection of its range with $X^u(f^n(y), \epsilon_X/2)$ and both are continuous. It follows that this is an open map. Then $h^s(x, y, n, U)$ is a composition of open maps. The last part follows from the fact that V is open, given in part four. □

Of course, we have verified most of the conditions for an étale topology as in **??**. The last one is slightly more technical.

Lemma 8.3.4. *1. Let (x, y, n, U) be in $\mathcal{N}^s$ and let $n' > n$. Then there exists $U' \subseteq U$ such that (x, y, n', U') is in $\mathcal{N}^s$ and*

$$h^s(x, y, n', U') \subseteq h^s(x, y, n, U).$$

2. Let (x, y, m, U) and (y, z, n, V) be in $\mathcal{N}^s$. There exist $l \geq 0$, $W \subseteq U$ such that (x, z, l, W) is in $\mathcal{N}^s$ and

$$h^s(x, z, l, W) \subseteq h^s(x, y, m, U) \circ h^s(y, z, n, V).$$

Proof. Let us consider the first part. Let $V = r(h^s(x, y, n, U))$. Let $U_0 = \{z \in U \mid f^{n'}(z) \in X^u(f^{n'}(x), \epsilon_X/2)\}$ and $V_0 = \{w \in V \mid f^{n'}(w) \in$

$X^u(f^{n'}(y), \epsilon_X/2)\}$. Clearly, $U_0 \subseteq U$ and $V_0 \subseteq V$ are open in the relative topology of $X^u(P)$. Finally, define $U' = s(U_0 \times V_0 \cap h^s(x, y, n, U)$.

We must show that $h^s(x, y, n', U') \subseteq h^s(x, y, n, U))$. Let (z, w) be in the former. This means that $f^{n'}(z)$ is in $X^u(f^{n'}(x), \epsilon_X/2)$, $f^{n'}(w)$ is in $X^u(f^{n'}(y), \epsilon_X/2)$ and $w = f^{-n'}[f^{n'}(z), f^{n'}(y)]$. We know from the choice of n that $f^n(y)$ is in $X^s(f^n(x), \epsilon_X/2)$. It follows that for every $n \leq k \leq n'$, we have $f^k(z) \in X^u(f^k(x), \epsilon_X/2)$ and $f^k(y)$ is in $X^s(f^k(x), \epsilon_X/2)$. We conclude that for all k in this range $f^{-k}[f^k(z), f^k(y)]$ is defined and is independent of k. We conclude that $w = f^{-n}[f^n(z), f^n(y)]$ and so (z, w) is in $h^s(x, y, n, U)$.

We now consider the second part. First, choose $l \geq m, n$ such that $f^l(y)$ is in $X^s(f^l(x), \epsilon_X/4)$ and $f^l(z)$ is in $X^s(f^l(y), \epsilon_X/4)$. Next, from the first part of the Lemma, we know we can find $U' \subseteq U$ and $V' \subseteq V$ such that

$$\begin{aligned} h^s(x, y, l, U') &\subseteq h^s(x, y, m, U) \\ h^s(y, z, l, V') &\subseteq h^s(y, z, n, V) \end{aligned}$$

We define

$$\begin{array}{rcl} W = & s(h^s(x, y, l, U') \cap (U' \times V')) & \subseteq U', \\ W' = & r(h^s(x, y, l, U')) \cap V' & \subseteq V'. \end{array}$$

We claim that

$$h^s(x, y, l, W) \circ h^s(y, z, l, W') = h^s(x, z, l, W).$$

Let $(x', y') \in h^s(x, y, l, W), (y', z') \in h^s(y, z, l, W')$. This means that $f^l(y') = [f^l(x'), f^l(y)]$ and that $f^l(z') = [f^l(y'), f^l(z)]$. Then, observing that all brackets are well-defined, we have

$$f^l(z') = [f^l(y'), f^l(z)] = [[f^l(x'), f^l(y)], f^l(z)] = [f^l(x'), f^l(z)],$$

which means that (x', z') is in $h^s(x, z, l, W)$. For the reverse inclusion (which is the only one we will actually need), let (x', z') be in $h^s(x, z, l, W)$. As x' is in W, there exists y' with (x', y') in $h^s(x, y, l, U') \cap (U' \times V')$. In particular, y' is in V' and also in $r(h^s(x, y, l, U'))$. Hence y' is in W'. Thus there is a z'' with (y', z'') in $h^s(y, z, l, W')$. That is, we have $f^l(z'') = [f^l(y'), f^l(z)]$. It follows that

$$f^l(z'') = [f^l(y'), f^l(z)] = [[f^l(x'), f^l(y)], f^l(z)] = [f^l(x'), f^l(z)] = f^l(z'),$$

and hence $z'' = z'$. This completes the proof of the reverse inclusion.

To complete the proof we observe

$$\begin{aligned} h^s(x,z,l,W) &= h^s(x,y,l,W) \circ h^s(y,z,l,W') \\ &\subseteq h^s(x,l,m,U) \circ h^s(y,z,n,V). \end{aligned}$$

□

The last two Lemmas together provide a proof of the following.

Theorem 8.3.5. *Let (X,d,f) be a Smale space and let P be a finite f-invariant subset of X.*

1. *The collection of sets $\{h^s(x,y,n,U) \mid (x,y,n,U) \in \mathcal{N}^s(X,f,P)\}$ is a basis for an étale topology on the equivalence relation $R^s(X,f,P)$.*

2. *The collection of sets $\{h^u(x,y,n,U) \mid (x,y,n,U) \in \mathcal{N}^u(X,f,P)\}$ is a basis for an étale topology on the equivalence relation $R^u(X,f,P)$.*

Definition 8.3.6. *Let (X,d,f) be a Smale space and let P be a finite f-invariant subset of X.*

1. *We define $S(X,f,P)$ to be the reduced C^*-algebra of the étale equivalence relation $R^s(X,f,P)$; that is,*

$$S(X,f,P) = C^*_r(R^s(X,f,P)).$$

2. *We define $U(X,f,P)$ to be the reduced C^*-algebra of the étale equivalence relation $R^u(X,f,P)$; that is,*

$$U(X,f,P) = C^*_r(R^u(X,f,P)).$$

8.4 Ideals in the C^*-algebras

In this section, we want to study the ideal structure in our C^*-algebras $S(X, f, P)$ and $U(X, f, P)$ associated to a Smale space (X, f). One important consequence is that we will see exactly how these C^*-algebras depend on the choice of set of periodic points P.

8.4.1 General results

The fundamental results underlying all of this section are the following. The first is very classical and we provide the statement here for readers with little background in C^*-algebra theory, since it provides a nice context for the second result. (In fact, it can be seen as a special case.) We will not give proofs, but refer interested readers to [].

Theorem 8.4.1. *Let X be a locally compact Hausdorff space. Suppose that U is an open subset of X and let Z be its closed complement. Define*

$$I_U = \{f \in C(X) \mid f|Z = 0\}.$$

1. *I_U is a closed, $*$-closed, two-sided ideal in $C_0(X)$ and the restriction of functions to U is an isomorphism from I_U to $C_0(U)$.*

2. *The quotient of $C_0(X)$ by I_U is $*$-isomorphic to $C_0(Z)$.*

3. *Every closed, $*$-closed, two-sided ideal in $C_0(X)$ is of the form I_U, for some open subset U of X.*

Definition 8.4.2. *Let R be an equivalence relation on the set X. We say that a subset $Z \subseteq X$ is R-invariant if, for any x in Z and (x, y) in R, we have y is in Z. In this case, we define $R_Z = R \cap (Z \times Z)$.*

Theorem 8.4.3. *Let R be an étale equivalence relation on the locally compact space X. Suppose that U is an open, R-invariant subset of X and let Z be its closed complement, which is also R-invariant. We let $R_U = R \cap U \times U$ and $R_Z = R \cap Z \times Z$.*

1. *Then R_U is an étale equivalence relation on U (using the relative topologies for each).*

2. *By extending functions to be zero, $C_c(R_U)$ is a $*$-closed, two-sided ideal in $C_c(R)$.*

3. *The inclusion of $C_c(R_U)$ as a $*$-closed, two-sided ideal in $C_c(R)$ extends to an isometric inclusion of $C_r^*(R_U)$ in $C_r^*(R)$) and the image is a closed, $*$-closed, two sided ideal.*

4. *R_Z is an étale equivalence relation on Z (using the relative topologies for each). Moreover, the quotient C^*-algebra $C_r^*(R))/C_r^*(R_U)$ is isomorphic to $C_r^*(R_Z)$.*

5. *Every closed, $*$-closed, two-sided ideal in $C_r^*(R)$ is of the form $C_r^*(R_U)$, for some open R-invariant subset U of X.*

8.4.2 Irreducible and mixing cases

Theorem 8.4.4. *If (X, f) is a mixing Smale space and P is a finite, non-empty f-invariant set, then $S(X, f, P)$ and $U(X, f, P)$ are both simple. That is, neither has a closed, two-sided ideal other than 0 and the entire C^*-algebra.*

Proof. We consider the Case of $S(X, f, P)$ only. Let U be a non-empty open X^s-invariant subset of $X^u(P)$. We will show that $U = X^u(P)$ and the conclusion follows from part 5 of Theorem 8.4.3. Let y be any point in U. As U is open in $X^u(P)$, we may find $\delta > 0$ such that $X^u(y, \delta) \subseteq U$. Then, there is an open set $y \in V \subseteq X(y, \epsilon_X)$ such that $[V, y] \subseteq X^u(y, \delta)$. Let x be any point in $X^u(P)$. It follows from Theorem 4.2.8 that $X^s(x) \cap V$ is non-empty. Let z be any point in the intersection. It follows that $[z, y]$ is still in $X^s(x)$ and also in U. As U was assumed to be X^s-invariant, x is also in U. This completes the proof. □

Theorem 8.4.5. *Let (X, f) be an irreducible Smale space and P be a finite, non-empty f-invariant set. Let $X_1, \ldots, X_N$ be as given in Corollary 4.5.7. Then we have*

$$\begin{aligned} S(X, f, P) &= \oplus_{n=1}^{N} S(X_n, f^N, P \cap X_n), \\ U(X, f, P) &= \oplus_{n=1}^{N} U(X_n, f^N, P \cap X_n). \end{aligned}$$

Each summand is simple. Moreover, the summands are cyclicly permuted by the automorphisms α_s and α_u.

Proof. It follows from Corollary 4.5.7 that each $(X_i, f|_{X_i}, d)$ is invariant under f^N. Moreover, f and f^N have the same stable and unstable equivalence relations by Theorem 4.2.6. The conclusions follows easily from these facts and we omit the details. □

8.4.3 General case

In this last subsection, we will consider the ideal structure of $S(X, f, P)$ for an arbitrary Smale space (X, f, d). On the other hand, we will restrict our attention to $S(X, f, P)$ and leave the reader to work out the obvious analogous statements for $U(X, f, P)$.

We have seen in section **??** that in a general Smale space (X, d, f), the set of irreducible components, which we now denote by $Irr(X, d, f)$, is finite and partially ordered by $\preceq$. We say a subset $\mathcal{I}$ of $Irr(X, d, f)$ is up-hereditary if whenever Y is in $\mathcal{I}$ and $Y \preceq Y'$, then Y' is also $\mathcal{I}$.

Definition 8.4.6. *Let (X, d, f) be a Smale space and let P be a finite, f-invariant subset of X. We define*

$$Irr(P)^+ = \{Y \in Irr(X, d, f) \mid p \preceq Y, \text{ for some } p \in P\}.$$

Lemma 8.4.7. *Let Y be an irreducible component of the Smale space (X, d, f). We have*

$$\{Y' \in Irr(X, d, f) \mid Y \preceq Y'\} = \{Irr^+(x) \mid d(x, Y) \leq \epsilon_X\}.$$

Proof. We begin with the containment $\subseteq$: let Y' be an irreducible component with $Y \preceq Y'$. This means that there is a point x in X with $Irr^+(x) = Y'$ and $Irr^-(x) = Y$. For some integer n, we have $d(f^n(x), Y) < \epsilon_X$, while $Irr^+(f^n(x)) = Irr^+(x) = Y'$.

Conversely, if $d(x, Y) \leq \epsilon_X$, then we may find y in Y with $d(x, y) \leq \epsilon_X$. It then follows that $Irr^-([x, y]) = Irr^-(y) = Y$, while $Irr^+([x, y]) = Irr^+(x) = Y'$ and so $Y \preceq Y'$. □

Lemma 8.4.8. *Let (X, d, f) be a Smale space and let P be a finite, f-invariant subset of X.*

1. *If $\mathcal{A}$ be an up-hereditary subset of $Irr(P)^+$, then*

$$U_{\mathcal{A}} = \{x \in X^u(P) \mid Irr^+(x) \in \mathcal{A}\}$$

is an open subset of $X^u(P)$ which is invariant under $R^s(X, f, P)$.

2. *If U is an open subset of $X^u(P)$ which is invariant under $R^s(X, f, P)$, then*

$$\mathcal{A}_U = \{Irr^+(x) \mid x \in U\}$$

is an up-hereditary subset of $Irr(P)^+$.

3. *The maps above are a containment-preserving bijection between the up-hereditary subset of $Irr(P)^+$, and the $R^s(X,f,P)$-invariant, open subsets of $X^u(P)$.*

Proof. First, we show that $U_{\mathcal{A}}$ is open: if x is in $U_{\mathcal{A}}$, so $Irr^+(x)$ is in $\mathcal{A}$. Then we may find a positive integer n such that $d(f^n(x), Irr^+(x)) < \epsilon_X$. By continuity, we may find an open set V in X such that $d(f^n(y), Irr^+(x)) < \epsilon_X$. It follows from Lemma 8.4.7 that $Irr^+(x) \preceq Irr^+(f^n(y)) = Irr^+(y)$. As $\mathcal{A}$ is up-hereditary, we see that $Irr^+(y)$ is in $\mathcal{A}$, for every y in V.

Next, we observe that if x and y are stably equivalent, we have $Irr^+(x) = Irr^+(y)$ and it follows at once that $U_{\mathcal{A}}$ is invariant under stable equivalence.

If $\mathcal{A}$ and $\mathcal{A}'$ are both up-hereditary and $\mathcal{A} \subseteq \mathcal{A}'$, then it is clear that $U_{\mathcal{A}} \subseteq U_{\mathcal{A}'}$.

Now we consider U, an $R^s(X,f,P)$-invariant, open subsets of $X^u(P)$. First, we show that $\mathcal{A}_U$ is contained in $Irr(P)^+$. Let x be in $X^u(P)$. This means that $\omega^-(x)$ contains some point, p, of P. It follows that $p \preceq Irr^+(x)$, so $Irr^+(x)$ is in $Irr(P)^+$.

Next, we show that $\mathcal{A}_U$ is up-hereditary. Let x be in U and $Irr^+(x) \preceq Y$, for some Y in $Irr(X,f,d)$.

We may find a positive integer n and point y in $Irr^+(x)$ such that $d(f^n(x), y) < \epsilon_X/3$. Observe that $z = [f^n(x), y]$ has $Irr^+(z) = Irr^+(x)$ and $Irr^-(z) = Irr^-(y) = Irr^+(x)$ and so belongs to $Irr^+(x)$. Next, find an open set $x \in V \subseteq X^u(x, \epsilon_X) \cap U$ such that $f^n(V) \subseteq X^u(f^n(x), \epsilon_X/3)$. As the periodic points in $Irr^+(x)$ are dense, we may find one, say p, sufficiently close to z such that $[p, f^n(x)]$ is sufficiently close to $[z, f^n(x)] = [[f^n(x), y], f^n(x)] = f^n(x)$ so it lies in $f^n(V)$.

As $Irr^+(x) \preceq Y$, we may find a point w with p in $\omega^-(w)$ and $Irr^+(w) = Y$. Then we may choose a negative integer m so that $u = [f^m(w), f^n(x)]$ is also in $f^n(V)$. Hence $f^{-n}(u)$ is in $V \subseteq U$ and

$$Irr^+(f^{-n}(u)) = Irr^+(u) = Irr^+(f^m(w)) = Irr^+(w) = Y.$$

This proves that Y is also in $\mathcal{A}$ as desired.

It is clear that the two constructions are inverse to each other. This completes the proof. □

Theorem 8.4.9. *Let (X,d,f) be a Smale space and let P be a finite, f-invariant subset of X. If $\mathcal{A}$ be an up-hereditary subset of $Irr(P)^+$ then $U_{\mathcal{A}}$ gives rise to a closed two-sided ideal in $S(X,f,P)$, which we denote by $I_{\mathcal{A}}$.*

The correspondence sending A of $Irr(P)^+$ to I_A in $S(X, f, P)$ is a containment-preserving bijection between the up-hereditary subsets of $Irr(P)^+$ and the closed, two-sided ideals of $S(X, f, P)$.

Proof. This is an immediate consequence of the last Lemma and Theorem 8.4.3. □

Th next result allows us to compare C^*-algebras $S(X, f, P)$ and $S(X, f, Q)$, at least in the special case that P is a subset of Q. Recall that if A is a C^*-subalgebra of a C^*-algebra B if whenever a is in A an d b is in B with $0 \leq b \leq a$, then b is also in A. The simplest example is to consider $A = \overline{hBh}$, where h is any positive element of B. Also recall that the C^*-subalgebra A is called *full* if the closed two-sided ideal it generates is all of B.

Theorem 8.4.10. *Let (X, d, f) be a Smale space and let $P \subseteq Q$ be finite, f-invariant subsets of X.*

1. *The characteristic function of $X^u(P)$, which we denote by χ_P for simplicity, is a continuous, bounded on $X^u(Q)$ and hence lies in the multiplier algebra of $S(X, f, Q)$. Moreover, we have*
$$\chi_P S(X, f, Q)\chi_P = S(X, f, P).$$
In particular, $S(X, f, P)$ is an hereditary C^*-subalgebra of $S(X, f, Q)$.

2. *The closed two-sided ideal in $S(X, f, Q)$ generated by $S(X, f, P)$ corresponds to the up-hereditary set $Irr(P)^+ \subseteq Irr(Q)^+$ under Theorem 8.4.9.*

3. *$S(X, f, P)$ is a full hereditary C^*-subalgebra of $S(X, f, Q)$ if and only if $Irr(P)^+ = Irr(Q)^+$. Equivalently, for every q in Q, there exists p in P with $p \preceq q$.*

Proof. That χ_P is continuous follows from the definition of the topology on $X^u(P)$ as the disjoint union topologies on the subsets $X^u(p), p \in P$. The rest is a simple consequence.

The second and third parts are immediate consequences and we omit the details. □

If A is a full, hereditary C^*-subalgebra of B, then A and B are *strongly Morita equivalent*. Indeed, the set $\overline{AB}$ is a left-A, right-B equivalence bimodule. We refer the reader to [] for more information on Morita equivalence.

The following result allows us to describe our C^*-algebras up to Morita equivalence.

Corollary 8.4.11. *Let (X, d, f) be a Smale space and let P and Q be finite, f-invariant subsets of X. If $Irr(P)^+ = Irr(Q)^+$, then the C^*-algebras $S(X, f, P)$ and $S(X, f, Q)$ are strongly Morita equivalent.*

Proof. We apply Theorem 8.4.10 to $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. It is immediate from the definition that the hypothesis $Irr(P)^+ = Irr(Q)^+$ implies that $Irr(P)^+ = Irr(P \cup Q)^+$. Therefore, $S(X, f, P)$ is a full hereditary subalgebra of $S(X, f, P \cup Q)$ and hence they are Morita equivalent. Similarly, $S(X, f, Q)$ and $S(X, f, P \cup Q)$ are Morita equivalent. The conclusion follows. $\square$

The following is an obvious consequence, but since it is perhaps the most useful one, we state it for completeness.

Corollary 8.4.12. *If (X, f, d) is an irreducible Smale space and P and Q are any finite, f-invariant subsets of X, then $S(X, f, P)$ and $S(X, f, Q)$ are strongly Morita equivalent.*

8.5 C^*-algebras associated with shifts of finite type

In this section, we give quite a concrete description of the C^*-algebras which arise from a shift of finite type.

Let (X, d, f) be a shift of finite type and suppose that $P \subseteq X$ is a finite, f-invariant set. We note the following fact.

Proposition 8.5.1. *Let (X, f) be a shift of finite type and let $P \subseteq X$ be a finite, f-invariant set. Then there exists a finite directed graph G and a conjugacy $g : (X, f) \to (X_G, \sigma)$ such that the map from P to G^0 defined by*

$$p \to t(g(p)_0), p \in P,$$

is injective.

Proof. By definition $(X, f) = (X_{\mathcal{F}}, \sigma_{\mathcal{F}})$, for some finite alphabet $\mathcal{A}$ and some finite set of words $\mathcal{F}$. Let N be any positive integer which is longer than the lengths of all words in $\mathcal{F}$ and so that the least period of any element of P divides N. Notice that this means that, for any x in P, x is uniquely determined by $x_1, x_2, \ldots, x_N$. Define

$$G^0 = \{(x_1, x_2, \ldots, x_N) \mid x \in X\}$$

and

$$G^1 = \{(x_1, x_2, \ldots, x_{N+1}) \mid x \in X\}$$

with

$$\begin{aligned} i(x_1, x_2, \ldots, x_{N+1}) &= (x_1, x_2, \ldots, x_N) \\ t(x_1, x_2, \ldots, x_{N+1}) &= (x_2, x_3, \ldots, x_{N+1}), \end{aligned}$$

for any $(x_1, x_2, \ldots, x_{N+1})$ in G^1.

We claim that $(X_{\mathcal{F}}, \sigma_{\mathcal{F}})$ is conjugate to (X_G, σ). In fact the proof is exactly the same as was done in **??** and we omit the details. The fact that this conjugacy satisfies the desired condition is immediate from that proof and our choice for N. $\square$

In fact, the statement of the last result is slightly misleading. It gives the impression that one would typically start with a shift of finite type and a finite set of periodic points P and then go about finding a presentation of the shift by a finite directed graph.

In fact, the choice of P is not terribly important for the construction of the C^*-algebras. In practice, one would typically begin with a finite directed graph G, and want to consider the shift (X_G, σ) and its C^*-algebra. Then one would look for a P satisfying the conclusion of the Proposition. This is not so difficult: find a cycle in G of minimum length. This cycle, repeated over and over, would then form a periodic point or even a finite collection of periodic points, which could be used for P. This would satisfy the conclusion of the last Proposition. As we will see if the next section, it is possible that we would have more requirements from P.

Let us begin then with a finite directed graph G and finite set of periodic points P in X_G and assume that the conclusion of Proposition 8.5.1 holds.

We make a number of definitions beginning with

$$P^- = \{(\ldots, p_{-2}, p_{-1}, p_0) \mid p \in P\},$$

and for a sequence $(\ldots, p_{-2}, p_{-1}, p_0)$ in P^-, we define $t(p) = t(p_0)$. Notice that our condition from 8.5.1 on P means that p is uniquely determined by $t(p)$. Also let $P^0 = t(P^-) \subseteq G^0$.

For each $n \geq 1$, let G^n denote the paths in G of length n. The terminal and initial maps extend to G^n in an obvious way:

$$\begin{aligned} t(\xi_1, \xi_2, \ldots, \xi_n) &= t(\xi_n), \\ i(\xi_1, \xi_2, \ldots, \xi_n) &= i(\xi_1), \end{aligned}$$

for ξ in G^n.

We define

$$R_n = \{(\xi, \eta) \mid \xi, \eta \in G^{2n}, i(\xi), i(\eta) \in P^0, t(\xi) = t(\eta)\}.$$

For (ξ, η) in R_n, we define

$$\begin{aligned} E(\xi, \eta) = \{(x, y) \mid\ & (\ldots, x_{-n-1}, x_{-n}), (\ldots, y_{-n-1}, y_{-n}) \in P^-, \\ & (x_{1-n}, \ldots, x_n) = \xi, (y_{1-n}, \ldots, y_n) = \eta, \\ & x_m = y_m, \text{ for all } m > n\}. \end{aligned}$$

Lemma 8.5.2. *1. For each $n \geq 1$ and (ξ, η) in R_n, the set $E(\xi, \eta)$ is a compact, open subset of $R^s(X_G, \sigma, P)$.*

2. *For each $n \geq 1$ and (ξ, η) in R_n, the maps r and s are local homeomorphisms to $X^u(P)$ and*

$$\begin{aligned} s(E(\xi,\eta)) &= \{x \mid && (\ldots, x_{-n-1}, x_{-n}) \in P^-, \\ & && x_{1-n}, \ldots, x_n = \xi\} \\ r(E(\xi,\eta)) &= \{y \mid && (\ldots, y_{-n-1}, y_{-n}) \in P^-, \\ & && y_{1-n}, \ldots, y_n = \eta\} \end{aligned}$$

3. *The collection of all sets $E(\xi, \eta)$, with $n \geq 1$, $(\xi, \eta) \in R_n$ are a basis for the topology on $R^s(X_G, \sigma, P)$.*

In view of the first part of this last Lemma, for any $n \geq 1$ and (ξ, η) in R_n, we define $e(\xi, \eta)$ to be the characteristic function of $E(\xi, \eta) \subseteq R^s(X_G, \sigma, P)$ so that it lies in our $*$-algebra $C_c(R^s(X_G, \sigma, P))$.

Lemma 8.5.3. *Let $n \geq 1$ be fixed and $(\xi, \eta), (\xi', \eta')$ be in R_n.*

1. *If $t(\xi) \neq t(\xi')$, then $e(\xi, \eta)e(\xi', \eta') = 0$.*
2. *If $\eta \neq \xi'$, then $e(\xi, \eta)e(\xi', \eta') = 0$.*
3. *If $\eta = \xi'$ and $t(\xi) \neq t(\xi')$, then $e(\xi, \eta)e(\xi', \eta') = e(\xi, \eta')$.*
4. *We have $e(\xi, \eta)^* = e(\eta, \xi)$.*

Proposition 8.5.4. *Let $n \geq 1$ be fixed.*

1. *For v a vertex of G^0, the set*

$$S_n(X_G, \sigma, P, v) = span\{e(\xi, \eta) \mid (\xi, \eta) \in R_n, t(\xi) = v\}$$

is a finite-dimensional C^*-subalgebra of $C_c(R^s(X_G, \sigma))$ and is isomorphic to $M_{k(P,n,v)}$, where $k(P, n, v)$ is the number of paths ξ in G^{2n} with $i(\xi) \in P^0$ and $t(\xi) = v$.

2. *The set*

$$S_n(X_G, \sigma, P) = span\{e(\xi, \eta) \mid (\xi, \eta) \in R_n\}$$

is a finite-dimensional C^*-subalgebra of $C_c(R^s(X_G, \sigma))$ and

$$S_n(X_G, \sigma, P) = \oplus_{v \in G^0} S_n(X_G, \sigma, P, v).$$

Lemma 8.5.5. *Let $n \geq 1$ and (ξ, η) be in R_n. Let p and q be the unique elements of P^- with $t(p) = i(\xi), t(q) = i(\eta)$.*

We have

$$e(\xi, \eta) = \sum_{e \in G^1, i(e) = t(\xi)} e(p_0 \xi e, q_0 \eta e),$$

where $p_0 \xi e = (p_0, \xi_{1-n}, \ldots, \xi_n, e)$ and $q_0 \eta e = (q_0, \eta_{1-n}, \ldots, \eta_n, e)$ are paths of length $2n + 2$.

In particular, we have

$$S_n(X_G, \sigma, P) \subseteq S_{n+1}(X_G, \sigma, P).$$

Proof. It suffices for us to prove that

$$E(\xi, \eta) = \cup_{e \in G^1, i(e) = t(\xi)} E(p_0 \xi e, q_0 \eta e),$$

and that the sets in the union are pairwise disjoint.

□

Theorem 8.5.6. *Let (X_G, σ) be the edge shift on the graph G and assume that P is a finite set of periodic points satisfying the conclusion of 8.5.1. Then we have*

$$\cup_{n \geq 1} S_n(X_G, \sigma, P) \subseteq S(X_G, \sigma, P)$$

is dense. In particular, $S(X_G, \sigma, P)$ is an approximately finite-dimensional C^*-algebra, or an AF-algebra.

8.6 Properties of the C^*-algebras

8.7 Maps and homomorphisms