The Orbit Structure of Cantor Minimal $\mathbb{Z}^2$-systems

Thierry Giordano[*]
Department of Mathematics and Statistics,
University of Ottawa,
585 King Edward, Ottawa, Ontario, Canada

Ian F. Putnam[†]
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada

Christian F. Skau[‡]
Norwegian University of Science and Technology (NTNU),
N-7034 Trondheim, Norway

Abstract

The paper considers minimal, free actions of the group $\mathbb{Z}^2$ on a compact, totally disconnected space having no isolated points. Under a hypothesis involving the existence of sufficiently many 'small, positive' cocycles, a procedure is given for finding a nested sequence of compact, open sub-equivalence relations of the orbit relation. It is shown that the union of this sequence, which is an AF-relation, is orbit equivalent to the orbit relation for the $\mathbb{Z}^2$-action. As a consequence of this and earlier results on orbit equivalence for $\mathbb{Z}$-actions

[*]Supported in part by a grant from NSERC, Canada
[†]Supported in part by a grant from NSERC, Canada
[‡]Supported in part by the Norwegian Research Council

and AF-relations, a complete invariant is obtained for orbit equivalence for: minimal AF-relations, minimal $\mathbb{Z}$-actions and minimal, free $\mathbb{Z}^2$-actions satisfying the hypothesis. Two classes of examples are presented and shown to satisfy the hypothesis.

1 Introduction

We continue our investigations of the orbit structure of minimal dynamical systems on Cantor sets. Our main interest is in the case of a free minimal action, φ, of $\mathbb{Z}^2$, the free abelian group on two generators, on a Cantor set, X. However, we will often work more generally and consider G to be a countable discrete abelian group acting on a topological space X. Specifically, this means that for every a in G, we have a homeomorphism $\varphi^a : X \to X$. These satisfy the conditions $\varphi^0(x) = x$, for all x in X and $\varphi^a \circ \varphi^b = \varphi^{a+b}$, for all a, b in G. The action is *free* if, for any x in X and a in G, we have $\varphi^a(x) = x$ only if a is the identity element of G. The *orbit* of a point x in X is the set $\{\varphi^a(x) \mid a \in G\}$, which we denote $\mathcal{O}_\varphi(x)$. The action is minimal if the only closed φ-invariant subsets Z in X(i.e. $\varphi^a(Z) = Z$, for all a in G) are X and the empty set. Equivalently, the action is minimal if the orbit of every point x is dense in X.

Given two free group actions (X, G, φ) and (Y, H, ψ), an *isomorphism* between them is a homeomorphism $h : X \to Y$ and a group isomorphism $\alpha : G \to H$ such that, for all a in G, X, we have

$$h \circ \varphi^a = \psi^{\alpha(a)} \circ h.$$

Recall from [GPS2] that, given two free group actions (X, G, φ) and (Y, H, ψ), an *orbit equivalence* between them is a map $h : X \to Y$ which is a homeomorphism such that, for every x in X, we have

$$h(\mathcal{O}_\varphi(x)) = \mathcal{O}_\varphi(h(x)).$$

It is clear from the definitions that every isomorphism is also an orbit equivalence.

The basic problem under consideration is to determine when two free group actions are orbit equivalent. This is the natural development for topological dynamics analogous to the program in measurable dynamics initiated by Henry Dye [D] and pursued by Orstein-Weiss [OW1, OW2] and Connes-Feldman-Weiss [CFW] and developed in the Borel case by Kechris et. al. [JKL].

It turns out that for connected spaces, any orbit equivalence is also an isomorphism [PPZ]. For this reason, we will restrict our attention to spaces which are totally disconnected. That is, the sets which are both closed and open (henceforth called clopen) generate the topology of X. Most of our results also assume minimality of the action. The case of the group of integers, $\mathbb{Z}$, was treated in [GPS1]. Here, we focus on the case of $\mathbb{Z}^2$.

Leaving out many details and definitions for the moment, let us give some indication of our attack on the problem. Given a free action of a group as above, the real object of interest is the equivalence relation whose equivalence classes are the orbits of the action. In this way, free actions of groups are special cases of an equivalence relation R on a set X, also called principal groupoids. In the topological situation, both the space X and the relation R are given topologies satisfying various conditions; here we are concerned with étale equivalence relations.

An equivalence relation is minimal if every equivalence class is dense. This extends the usual definition for group actions. It is also possible to define an invariant measure for equivalence, which also extends the usual one for actions. It is then possible to define an invariant D_m for an étale equivalence relation on a Cantor set. It has the structure of an ordered abelian group with distinguished positive element and is an invariant of orbit equivalence.

Aside from group actions, there is a class of such étale equivalence relations called AF-relations. The terminology is borrowed from C^*-algebra theory and stands for 'approximately finite'. This is a beautiful class of such relations; it is rich enough to include many interesting examples but also has enough structure to prove many results. The first important result is a classification theorem for AF-relations, due to Elliott and Krieger. Building on this, for minimal AF-relations, D_m is a complete invariant for orbit equivalence in the minimal case [GPS1]. Finally, there is a result in [GPS2] which we refer to as the absorption theorem which states, roughly, that 'small' extensions of a minimal AF-relation are orbit equivalent to the original relation.

Having at our disposal a complete invariant for orbit equivalence for minimal AF-relations, the problem is then to show that the equivalence relation for a free action of a group may be written as 'small' extension of an AF-relation. By the absorption theorem, this orbit relation is then orbit equivalent to the AF-relation and hence comes under the classification theorem for AF-relations. The solution of this extension problem depends on the group and consists of finding some sort of Rohlin approximation for the group. In the analogous theory for measurable dynamics, the notion of 'small' is just differing on a set of measure zero. In the topological case, it is much more subtle.

The program outlined above differs somewhat from what has evolved in earlier work. The focus of [GPS1] was on $\mathbb{Z}$-actions and the classification for AF-relations was obtained a a consequence. With some hindsight, the development outlined above seems more natural. A complete treatment of this point of view is now in preparation [PPZ].

There has been other work along these lines for $\mathbb{Z}^2$-actions on the Cantor set. A. Forrest [F] (see also [Ph] for another treatment) produced large AF-subrelations of the orbit relation of a Cantor minimal $\mathbb{Z}^2$-system. Such subrelations also appear implicitly in the work of Bellissard, Benedetti and Gambaudo [BBG] and also in Benedetti and Gambaudo [BG]. Unfortunately, these methods do not keep track of the difference between the two and, it seems at present, that the absorption theorem cannot be applied. Here, we use a different approach. We use cocycles for the action to create the AF-relation and its extension. The drawback of the new method is that it needs to assume the existence of sufficiently many cocycles with certain properties. Results about the existence are notably missing, although we show that they do exist for several examples of interest.

The paper is organized as follows. The second section deals with basic notions regarding étale equivalence relations and the invariants and recalls earlier results. The third section discusses basic ideas regarding cocycles for $\mathbb{Z}^2$-actions and introduces the notions of positive cocycles, proper cocycles and 'small' cocycles. In section three, we state our main results, a classification of minimal Cantor $\mathbb{Z}^2$-actions satisfying certain hypotheses up to orbit equivalence. The proofs are deferred to sections 4 through 7. In the fourth section, we establish various preliminary results which relate cocycles and subequivalence relations. In section five, we describe an inductive procedure for finding cocycles and their associated equivalence relations. The result is to produce an AF-subequivalence relation R of the orbit relation. In section 6, we describe the 'boundary' of the subrelation R; this is the extension of the relation R which realizes the orbit relation. In fact, the relation R with its boundary do not quite meet the hypotheses of our absorption theorem and we must pass to a subequivalence relation of R, R'. This is described in section 7 and we complete the proofs of the main results. In section 8, we consider a number of examples, showing that they satisfy the hypotheses of the main result.

2 Local actions

Let X denote a compact, metric space and let R be an equivalence relation on X. We denote the two canonical projections from R to X by r and s: $r(x, y) = y, s(x, y) = x$, for all (x, y) in R. We let $\Delta = \{(x, x) \mid x \in X\}$ denote the diagonal in R. We state the following definition and refer the reader to Remark 3.6 of [GPS2].

Definition 2.1. *Let X be a compact metric space and R be an equivalence relation on X. A topology $\mathcal{T}$ on R is said to be* étale *if the following hold:*

1. *R is σ-compact,*
2. *Δ is both open and closed in R,*
3. *the maps $r, s : R \to X$ are local homeomorphisms; that is, there is a neighbourhood base $\mathcal{N}$ for the topology $\mathcal{T}$ consisting of sets U such that $r(U)$ and $s(U)$ are open and*
$$r : U \to r(U), s : U \to s(U)$$
are homeomorphisms,
4. *for any open sets U and V in R, the set*
$$UV = \{(x, z) \mid \text{ there exists } y \in X, \text{ such that } (x, y) \in U, (y, z) \in V\}$$
is open,
5. *for any open set U in R, the set*
$$U^{-1} = \{(x, y) \mid (y, x) \in U\}$$
is open.

If $\mathcal{T}$ is such a topology, we say that $(X, R, \mathcal{T})$ is a local action.

The reason for the terminology is as follows. Let U be any set satisfying the third condition of the definition. It follows that the map

$$r \circ (s\,|_U)^{-1} : s(U) \to r(U)$$

is a homeomorphism between two open subsets of X. Its graph is simply the set U and its relative topology from $\mathcal{T}$ is the same as that from $X \times X$. We can imagine these maps as a group acting on X. Condition 1 is the analogue of the group being countable, condition 2 is the analogue of freeness and conditions 4 and 5 are closure of this set under composition and inverses.

Consider a free action φ of the countable group G on the compact metric space X. For such an action, we define the equivalence relation

$$R_\varphi = \{(x, \varphi^a(x)) \mid x \in X, a \in G\}.$$

The map sending (x, a) to $(x, \varphi^a(x))$ is a bijection from $X \times G$ to R_φ. We endow R_φ with a topology by transferring the product topology on $X \times G$ by this map. In this way, it is an étale equivalence relation. Here the neighbourhood base $\mathcal{N}$ is the set of all restrictions $\varphi^a\,|_U$, where a is in G and U is an open subset of X.

A word of warning about our notation is in order. We noted above that there is a natural bijection between $X \times G$ and R_φ. We will move freely between these notations. Frequently, it will be simpler to write (x, a) rather than $(x, \varphi^a(x))$.

The second class of examples is the AF-relations.

Definition 2.2. *A local action, $(X, R, \mathcal{T})$, is an AF-relation if*

1. *X is totally disconnected, and*

2. *there is a sequence*

$$R_0 = \Delta \subset R_1 \subset R_2 \subset \cdots$$

   where, for each $n = 0, 1, \ldots$, each R_n is a compact open subequivalence relation of R and

$$\cup_{n=0}^{\infty} R_n = R.$$

As mentioned earlier there is a complete structure theorem for such relations; see Theorem 3.9 of [GPS2].

Several simple notions from dynamics may be extended to equivalence relations as follows

Definition 2.3. *Let X be a compact metric space and let R be an equivalence relation on X. A subset Z of X is* R-invariant *if, for any (x, y) in R with x in Z, it follows that y is in Z. The relation R is* minimal *if the only closed R-invariant subsets of X are X and the empty set. Equivalently, R is minimal if every equivalence class in R is dense in X.*

Definition 2.4. *([GPS2]) Let $(X, R, \mathcal{T})$ be a local action. A measure μ on X is* R-invariant *if, for every set U in the neighbourhood base $\mathcal{N}$, we have $\mu(s(U)) = \mu(r(U))$. We let $M_1(X, R)$ denote the set of all R-invariant probability measures on X.*

We can now extend our notion of orbit equivalence to equivalence relations.

Definition 2.5. *For $i = 1, 2$, let X_i be a compact metric space and R_i be an equivalence relation on X_i. An* orbit equivalence *between them is a homeomorphism*

$$h : X_1 \to X_2$$

such that $h \times h(R_1) = R_2$. If such a map exists, we say that (X_1, R_1) and (X_2, R_2) are orbit equivalent *and write*

$$(X_1, R_1) \sim (X_2, R_2), \text{ or simply } R_1 \sim R_2.$$

If h is an orbit equivalence as above, then h^* induces a bijection between $M_1(X_2, R_2)$ and $M_1(X_1, R_1)$ (see the proof of Theorem 2.2 of [GPS1] or [PPZ]).

Definition 2.6. *For $i = 1, 2$, let $(X_i, R_i, \mathcal{T}_i)$ be local actions. An* isomorphism *between them is given by an orbit equivalence, $h : X_1 \to X_2$, such that*

$$h \times h : R_1 \to R_2$$

is a homeomorphism. If such a map exists, we say that the local actions are isomorphic *and we write*

$$(X_1, R_1) \cong (X_2, R_2), \text{ or simply } R_1 \cong R_2.$$

We now recall our main invariant (although using new notation) for a local action, $(X, R, \mathcal{T})$. We let $C(X, \mathbb{Z})$ denote the set of continuous functions from

X to the integers $\mathbb{Z}$. It is a group with operation point-wise addition. It also has a positive cone consisting of the non-negative functions. We let

$$B_m(X,R) = \{f \in C(X,\mathbb{Z}) \mid \int_X f\,d\mu = 0, \text{ for all } \mu \in M_1(X,R)\}.$$

which is clearly a subgroup of $C(X,\mathbb{Z})$ and

$$D_m(X,R) = C(X,\mathbb{Z})/B_m(X,R),$$

be the quotient group. For any f in $C(X,\mathbb{Z})$, we let $[f]$ denote its class in the quotient group. We also define a cone

$$D_m(X,R)^+ = \{[f] \mid f \geq 0\},$$

and a distinguished positive element $u_m = [1]$. The triple $(D_m(X,R), D_m(X,R)^+, [1])$ is an invariant of orbit equivalence [PPZ]. In the case of AF-relations and $\mathbb{Z}$-actions, it is a simple dimension group with no infinitesimal elements. Moreover, any such ordered group which is acyclic arises in this way from a minimal AF-relation and also from a minimal $\mathbb{Z}$-action. In [GPS1], it is shown that it is a complete invariant for the class of local actions which include minimal AF-relations and minimal $\mathbb{Z}$-actions on Cantor sets. Our aim here is to include in this class all minimal, free $\mathbb{Z}^2$-actions on Cantor sets. We achieve this only for actions having sufficiently many cocycles, as described in the next section.

To establish that two local actions are orbit equivalent, we have a fundamental result from [GPS2]. We state it here for completeness.

Theorem 2.7. *Let $(X, R, \mathcal{T})$ be a minimal AF-relation. Suppose that Y_1, Y_2 are closed subsets of X and $\alpha : Y_1 \to Y_2$ be a homeomorphism such that the following conditions hold.*

1. $(Y_1 \times Y_2) \cap R = \emptyset$,
2. $\mu(Y_1) = \mu(Y_2) = 0$, *for all R-invariant probability measures μ on X,*
3. *the relation $(Y_i \times Y_i) \cap R$, with the relative topology from R, is an étale equivalence relation on Y_i, for $i = 1, 2$,*
4. *the map α induces an isomorphism between $(Y_1 \times Y_1) \cap R$ and $(Y_2 \times Y_2) \cap R$.*

   The equivalence relation generated by R and graph(α) is orbit equivalent to R and, in particular, is affable.

If A and B are two sets and $f : A \to B$ is a function, we are using the notation $graph(f) = \{(a, f(a)) \mid a \in A\} \subset A \times B$. Some would probably like to point out that a function f is *defined* to be a subset of $A \times B$ and our use of the word 'graph' is redundant. We use it to emphasize that it is a subset of the Cartesian product.

3 Cocycles and Positive Cocycles

We will use the L^∞-norm on $\mathbb{Z}^2$; that is, we set $|(i,j)| = \max\{|i|, |j|\}$. For any n in $\mathbb{Z}^2$ and positive integer m, we let $B(n,m) = \{n' \in \mathbb{Z}^2 \mid |n'-n| \leq m\}$.

We begin with some basic definitions for cohomology. The basic references are [FM, R].

Definition 3.1. *Let φ be a free action of $\mathbb{Z}^2$ on the compact space X. A* one-cocycle taking values in $\mathbb{Z}$ *or just* cocycle *for φ is a continuous function*

$$\theta : X \times \mathbb{Z}^2 \to \mathbb{Z}$$

such that, for all x in X and m, n in $\mathbb{Z}^2$, we have

$$\theta(x, m+n) = \theta(x, m) + \theta(\varphi^m(x), n).$$

We let $Z^1(X, \varphi)$ denote the set of all cocycles, which is a group under addition. If f is in $C(X, \mathbb{Z})$, then the function $bf(x,n) = f(\varphi^n(x)) - f(x)$ is called a coboundary. *We let $B^1(X, \varphi)$ denote the set of coboundaries. It is easily seen to be a subgroup of $Z^1(X, \varphi)$. We let*

$$H^1(X, \varphi) = Z^1(X, \varphi)/B^1(X, \varphi)$$

denote the quotient group.

As we noted in the introduction, there is a natural bijection between $X \times \mathbb{Z}^2$ and R_φ sending (x, n) to $(x, \varphi^n(x))$. There will be times when it will be easier to consider our cocycles as defined on R_φ. It this case, the cocycle condition of the definition becomes

$$\theta(x, z) = \theta(x, y) + \theta(y, z),$$

for all $(x, y), (y, z)$ in R_φ. Moreover, if f is in $C(X, \mathbb{Z})$, then

$$bf(x, y) = f(y) - f(x)$$

for all (x, y) in R_φ.

Definition 3.2 (defn:basic-cocycles). *We let ξ_0 and η_0 denote the maps*

$$\xi_0(x,(i,j)) = i, \eta_0(x,(i,j)) = j,$$

for all x in X and (i,j) in $\mathbb{Z}^2$.

It is easy to see that ξ_0 and η_0 are both cocycles.

We want to introduce the notion of positivity for cocycles.

Definition 3.3. *Let φ be a free action of $\mathbb{Z}^2$ on X. Let C be a subset of $\mathbb{Z}^2$.*

1. *A cocycle θ is* positive with respect to C *if*

$$\theta(X \times C) \geq 0.$$

2. *A cocycle θ is* proper with respect to C *if the map*

$$\theta : X \times C \to \mathbb{Z}$$

is proper (i.e.,the pre-image of any finite set is compact).

3. *A cocycle is* strictly positive with respect to C *if it is proper and positive with respect to C.*

Lemma 3.4. *Let θ be a cocycle for (X,φ), let C be a subset of $\mathbb{Z}^2$ and let h be in $C(X,\mathbb{Z})$.*

1. *If θ is positive with respect to C, then $\theta + bh : X \times C \to \mathbb{Z}$ is bounded below.*

2. *If θ is proper with respect to C, then so is $\theta + bh$.*

Proof. It is easy to see that, since h is bounded, the cocycle bh is a bounded function on $X \times \mathbb{Z}^2$. The result follows immediately. □

If θ is a cocycle, then we have $\theta(x,-n) = -\theta(\varphi^{-n}(x), n)$, for all x in X and n in $\mathbb{Z}^2$. From this fact, the following result follows easily. We omit the proof.

Lemma 3.5. 1. *If the cocycle θ is positive with respect to $C \subset \mathbb{Z}^2$, then $-\theta$ is positive with respect to $-C$.*

2. *If θ is proper with respect to C then it is also proper with respect to $-C$.*

In fact, the sets C we will often use are of a very special form. We define the following.

Definition 3.6. *Let r, r' be positive real numbers. We define*

$$C(r, r') = \{(i, j) \in \mathbb{Z}^2 \mid j \leq ri, j \leq r'i\}.$$

The definition of both of these can be extended to include the cases $r = +\infty$ (using the convention $+\infty \cdot 0 = 0$) and $r = 0$.

Let a, b be elements of $\mathbb{Z}^2$ which generate $\mathbb{Z}^2$ as a group. We define

$$< a, b >= \{ia + jb \mid i, j \geq 0\}.$$

In addition to the notion of positive cocycle, we will also use the notion of a small cocycle as follows.

Definition 3.7. *Let θ be a cocycle for (X, φ) and let M be a positive integer. We say that $\theta \leq M^{-1}$ if $|\theta(x, n)| \leq 1$ for all x in X and n in $B(0, M)$. We say that θ is* small *if $\theta \leq 2^{-1}$.*

If ξ and η are two cocycles on $X \times \mathbb{Z}^2$, then we define $\xi \times \eta : X \times \mathbb{Z}^2 \to \mathbb{Z}^2$ by

$$\xi \times \eta(x, n) = (\xi(x, n), \eta(x, n))$$

for all x in X and n in $\mathbb{Z}^2$.

Definition 3.8. *Let ξ and η be two cocycles for (X, φ). We say that $\xi \times \eta$ is* surjective *if, for every x in X, the map*

$$\xi \times \eta : \{x\} \times \mathbb{Z}^2 \to \mathbb{Z}^2$$

is surjective. We also say that $\xi \times \eta$ is proper *if it is proper as a map from $X \times \mathbb{Z}^2$ to $\mathbb{Z}^2$.*

Finding small, positive cocycles reduces to finding clopen sets with certain properties which we outline below. For practical purposes, this is the result which we will use in examples to find cocycles.

Theorem 3.9. *Let (X, φ) be a minimal, free Cantor $\mathbb{Z}^2$-system. Let a, b be generators for $\mathbb{Z}^2$. Suppose that for any $N \geq 1$, we may find clopen sets A and B such that*

1. *A and $\varphi^{-a}(B)$ and disjoint, $\varphi^{-b}(A)$ and B are disjoint,*

2. *$A \cup \varphi^{-a}(B) = \varphi^{-b}(A) \cup B$,*

3. *the sets $\varphi^{i(a+b)}(A \cup \varphi^{-a}(B))$, are disjoint for $0 \leq i \leq N$.*

Then for any $M \geq 1$, we may find a cocycle θ which is strictly positive on $C =< a, b >= \{ia + jb \mid i, j \geq 0\}$ and $\theta \leq M^{-1}$.

Proof. Given $M \geq 1$, choose $N \geq 1$ such that $\{ia + jb \mid -N \leq i, j \leq N\}$ contains $B(0, M)$ in $\mathbb{Z}^2$. Define $\theta(x, 0) = 0$, for all x in X. Then define $\theta(x, ia + jb)$ inductively on $|i| + |j|$. If $i > 0$, define

$$\theta(x, ia + jb) = \theta(x, (i-1)a + jb) + \chi_{\varphi^{-(i-1)a-jb}(A)}(x),$$

for x in X. If $i < 0$, define

$$\theta(x, ia + jb) = \theta(x, (i+1)a + jb) - \chi_{\varphi^{-ia-jb}(A)}(x),$$

for x in X. If $j > 0$, define

$$\theta(x, ia + jb) = \theta(x, ia + (j-1)b) + \chi_{\varphi^{-ia-(j-1)b}(B)}(x),$$

for x in X. If $j < 0$, define

$$\theta(x, ia + jb) = \theta(x, ia + (j+1)b) - \chi_{\varphi^{-ia-jb}(B)}(x),$$

for x in X. One must check, inductively on $|i| + |j|$, that θ is well-defined. This follows from the hypothesis. It is easy to check that θ is a cocycle. Note that $\theta(x, a) = \chi_A(x), \theta(x, b) = \chi_B(x)$. It also follows inductively that θ is positive on $< a, b >$. It is also an easy induction argument to show that, for any $i \geq 0$,

$$\theta(x, ia + ib) = \sum_{0 \leq k < i} \chi_{\varphi^{-ka-kb}(A \cup \varphi^{-a}(B))}(x),$$

and, for any $i \leq 0$,

$$\theta(x, ia + ib) = \sum_{i \leq k < 0} -\chi_{\varphi^{-ka-kb}(A \cup \varphi^{-a}(B))}(x),$$

for any x in X. In particular for the case, $i = N$ or $i = -N$, we may use the last hypothesis to conclude that the values of the first expression above is either 0 or 1 and the second is either 0 or -1. To check the smallness of θ, we consider c in $\mathbb{Z}^2$, with $|c| \leq M$. By the choice of N, we may $c = ia + jb$, for some $|i|, |j| \leq N$. It follows from the positivity of θ that

$$-1 \leq \theta(x, -Na - Nb) \leq \theta(x, ia + jb) \leq \theta(x, Na + Nb) \leq 1,$$

for any x in X.

The positivity of θ on C follows from the definitions. To see that θ is strictly positive on C, we claim that it suffices to show that there is a positive integer K such that

$$\cup_{k=0}^{K} \varphi^{ka}(A) = \cup_{k=0}^{K} \varphi^{kb}(B) = X.$$

To see this, suppose that i, j are positive integers with at least one them greater than lK. Suppose that $i \geq lK$. It follows that, for any x in X,

$$\begin{aligned}
\theta(x, ia + jb) &\geq \theta(x, ia) \\
&= \sum_{i'=0}^{i-1} \theta(\varphi^{i'a}(x), a) \\
&\geq \sum_{l'=0}^{l-1} \sum_{k=0}^{K-1} \chi_A(\varphi^{l'K+k}(x)) \\
&\geq \sum_{l'=0}^{l-1} 1 \\
&= l.
\end{aligned}$$

It follows that θ is proper on C.

To see that such K exists, we proceed as follows. First, we claim that $A \cap B$ is non-empty. Assume the contrary. It follows from the cocycle condition that B is contained in $\varphi^{-b}(B)$. Let μ be any invariant probability measure. Its support is invariant and hence all of X. Therefore it is positive on any non-empty open set. As it is invariant, the measure of $\varphi^{-b}(B) - B$ is zero. It follows that B is invariant under φ^{-b}. Similarly, $\varphi^{-a}(A) = A$. Let x be any point of A. We may find i, j such that $\varphi^{ia+jb}(x)$ is in B. But then, $\varphi^{ia}(x)$ is in $\varphi^{ia}(A) = A$ and in $\varphi^{-jb}(B) = B$. This is a contradiction.

Before continuing, let us make two important observations which both follow from the hypothesis. If x is $A - B$, then it is in $\varphi^{-b}(A)$ and hence $\varphi^{b}(x)$ is in A. Secondly, if x is in B then $\varphi^{-a}(x)$ is in $\varphi^{-a}(B)$, and hence

$\varphi^{-a}(x)$ is in B or $\varphi^{-b}(A)$, hence $\varphi^{-a+b}(x)$ is in A. Also, these possibilities are mutually exclusive.

Next, we claim that, for any x in A, there is a $j \geq 1$ such that $\varphi^{jb}(x)$ is in B. Suppose the contrary. It follows then $\varphi^{jb}(x)$ is in A, for all $j \geq 1$. Find (i_1, j_1) in $\mathbb{Z}^2$ with $\varphi^{i_1 a + j_1 b}(x)$ in A, $i_1, j_1 \geq 1$ and so that i_1 chosen as small as possible. We claim that for $0 \leq i \leq i_1, j_1 \leq j$, the point $\varphi^{ia+jb}(x)$ is not in B. This is true for $i = 0$ by our assumption above. Suppose that there is some i, j in the given range with $\varphi^{ia+jb}(x)$ in B. Consider first the case $i \geq 2$. As $0 < i \leq i_1$, and from our choice of i_1, $\varphi^{(i-1)a+(j+1)b}(x)$ is not in A and so we must have $\varphi^{(i-1)a+jb}(x)$ is in B. Continuing in this way, we are left to consider the case $i = 1$. But then $\varphi^{jb}(x)$ is in A and in $\varphi^{-a}(B)$, which contradicts these sets being disjoint. Therefore the claim is established. We continue inductively to define $i_n > i_{n-1}, j_n > j_{n-1}$ such that $\varphi^{ia+jb}(x)$ is not in B for $0 \leq i \leq i_n$ and $j_n \leq j$. But the union of these sets over all j contains arbitrarily large rectangles. This contradicts the minimality of the action (see the proof of 5.7 for details).

In an analogous fashion, for any x in B, there is $i \geq 1$ such that $\varphi^{ia}(x)$ is in A. To show the existence of K as above, we let x be in X. We claim that for some $i > 0$, $\varphi^{ia}(x)$ is in A. As $A \cap B$ is non-empty, we may select $i, j \geq 0$ such that $\varphi^{ia+jb}(x)$ is in $A \cap B$. Select such i, j with j as small as possible. If $j = 0$, then we are done. If $j > 0$, consider $\varphi^{ia+(j-1)b}(x)$, which is in $\varphi^{-b}(A)$. If this is not in A, then it is in $\varphi^{-a}(B)$, in which case $\varphi^{(i+1)a+(j-1)b}(x)$ is in B. It then follows from the claim above that, for some $i' > i$, $\varphi^{i'a+(j-1)b}(x)$ is in $A \cap B$. But this contradicts the minimality of the choice of j. Therefore, we have $\varphi^{ia+(j-1)b}(x)$ is in A. Repeating this j times, we see that $\varphi^{ia}(x)$ is in A. The function which assigns to any x in X the least positive integer i such that $\varphi^{ia}(x)$ is in A is well-defined and clearly continuous. In a completely analogous way, the function which assigns to any x in X the least positive integer j such that $\varphi^{jb}(x)$ is in B is also well-defined and continuous. Any K which is an upper bound for these functions satisfies the desired properties. $\square$

We conclude this section with a small tool for studying the cohomology of an action. Suppose that μ is an invariant probability measure for our system (X, φ). We will define a map

$$\tau_\mu : H^1(X, \varphi) \to Hom(\mathbb{Z}^2, \mathbb{R}).$$

Of course, the group $Hom(\mathbb{Z}^2, \mathbb{R})$ is isomorphic to $\mathbb{R}^2$. We are implicitly using the L^∞-norm on $\mathbb{Z}^2$ (in for example Definition 3.7) and so $Hom(\mathbb{Z}^2, \mathbb{R})$ is endowed with the dual norm which is the L^1-norm. Specifically, we set

$$|\alpha| = \sup\{\frac{|\alpha(n)|}{|n|} \mid 0 \neq n \in \mathbb{Z}^2\},$$

for all α in $Hom(\mathbb{Z}^2, \mathbb{R})$. Under the identification with $\mathbb{R}^2$, the topology which results is the usual one. Let θ be a cocycle. We define, for any n in $\mathbb{Z}^2$,

$$\tau_\mu(\theta)(n) = \int_X \theta(x, n) d\mu(x).$$

It is easy to see that $\tau_\mu(\theta) = 0$, if θ is a coboundary and so τ_μ descends to a map on $H^1(X, \varphi)$, which we also denote by τ_μ. It is easy to see that if $\theta \leq M^{-1}$, then $|\tau_\mu(\theta)| \leq M^{-1}$. It is also easy to check that $\tau_\mu(\xi_0)(i,j) = i$ and $\tau_\mu(\eta_0)(i,j) = j$, for all (i,j) in $\mathbb{Z}^2$. From this we conclude that the subgroup of $H^1(X, \varphi)$ generated by ξ_0 and η_0 is isomorphic to $\mathbb{Z}^2$ and its range under τ_μ is a discrete cocompact subgroup of $Hom(\mathbb{Z}^2, \mathbb{R})$.

4 The main results

Having established some notation and basic facts and the appropriate notions of positivity and smallness for cocycles, we are ready to state our main result.

Simply put, it say that if a free, minimal $\mathbb{Z}^2$ Cantor system possesses arbitrarily small cocycles for sufficiently many cones, then its orbit relation is affable.

Theorem 4.1. *Let (X, φ) be a free, minimal action of $\mathbb{Z}^2$ on a Cantor set. Suppose that there are positive numbers r_∞, s_∞ with $s_\infty^{-1} - r_\infty^{-1} \geq 1$ satisfying the following. For every $\epsilon > 0$, there are positive $r_\infty + \epsilon > r > r' > r_\infty$ so that for every $M \geq 1$, there is a cocycle θ on (X, φ) such that*

1. *θ is strictly positive on $C(r, r')$, and*
2. *$\theta \leq M^{-1}$.*

Similarly, for every $\epsilon > 0$, there are positive $s_\infty - \epsilon < r < r' < s_\infty$ such that for every $M \geq 1$, there is a cocycle θ such that the same conditions hold. Then the equivalence relation R_φ is affable.

In our examples, there are actually quite a good supply of small positive cocycles for many cones. Our examples satisfy the following stronger hypotheses.

Corollary 4.2. *Let (X, φ) be a free, minimal action of $\mathbb{Z}^2$ on a Cantor set. Suppose that for every a, b in $\mathbb{Z}^2$ which generate $\mathbb{Z}^2$ as a group and $M \geq 1$, there is a cocycle θ such that*

1. *θ is strictly positive on $< a, b >= \{ia + jb \mid i, j \geq 0\}$, and*
2. *$\theta \leq M^{-1}$.*

Then the equivalence relation R_φ is affable.

The proof is quite long and will take several sections to complete. Let us assume the result for the moment and derive some consequences.

Corollary 4.3. *For $i = 1, 2$, let $(X_i, R_i, \mathcal{T}_i)$ be local actions, where, for each i, each X_i is totally disconnected and $(R_i, \mathcal{T}_i)$ is minimal and one of the following:*

1. *an AF-relation,*
2. *arises from a free action of $\mathbb{Z}$, or*
3. *arises from a free action of $\mathbb{Z}^2$ satisfying the hypotheses of 4.1.*

Then the two local actions are orbit equivalent if and only if there is a group isomorphism

$$\alpha : D_m(X_1, R_1) \to D_m(X_2, R_2)$$

such that

$$\begin{aligned} \alpha(D_m(X_1, R_1)^+) &= D_m(X_2, R_2)^+, \\ \alpha([\chi_{X_1}]) &= [\chi_{X_2}]. \end{aligned}$$

Of course, it is natural to ask how generally the hypothesis of Theorem 4.1 holds. We know little about this except that we show that it holds in two examples of interest in the last section. We remark that a necessary condition is that the image of $H^1(X, \varphi)$ under the map τ_μ of section 3 is a dense subgroup of $Hom(\mathbb{Z}^2, \mathbb{R})$. In particular, if there exists (X, φ) with $H^1(X, \varphi) \cong \mathbb{Z}^2$, then the hypothesis does not hold. At this point, we know of no such (free, minimal) action.

5 Preliminaries

The aim of this section is to provide some general constructions which will be used in the proof.

5.1 Cones

For any subset A in $\mathbb{R}^2$, we let $cl(A)$ denote the closure of A and $int(A)$ denote the interior of A.

We will also make use of another type of cone.

Definition 5.1. *For $\infty \geq r \geq r' \geq 0$, we define*

$$D(r,r') = D(r',r) = \{(i,j) \in \mathbb{Z}^2 \mid ri \geq j \geq r'i\},$$

using the convention that $0 \cdot \infty = 0$.

Lemma 5.2. *Let A, B be subsets of $\mathbb{R}^2$ such that $A_1 = cl\{rx \mid x \in A, r \geq 0\}$ and $B_1 = cl\{rx \mid x \in B, r \geq 0\}$ intersect only at 0. Given $K > 0$, there is $L \geq 0$ such that if a is in A and b is in B with either $|a| \geq L$ or $|b| \geq L$, then $|a - b| \geq K$.*

Proof. Let S^1 denote the unit circle in $\mathbb{R}^2$. Then $S^1 \cap A_1$ and B_1 are disjoint. As the former is compact and the latter is closed, there is a positive constant d such that $|x - y| \geq d$, for all x in $S^1 \cap A_1$ and y in B_1. Using the same argument and choosing the minimum of the two values of d, we may also assume the same conclusion of x in A_1 and y in $S^1 \cap B_1$. Given K, let $L = K/d$. Now suppose that a is in A and b is in B and $|a| \geq L$. Then $a/|a|$ is in $S^1 \cap A_1$ and $b/|a|$ is in B_1. Hence we have,

$$|a - b| = |a|\left|\frac{a}{|a|} - \frac{b}{|a|}\right| \geq |a|d \geq Ld = K,$$

as desired. The other case is analogous. $\square$

Definition 5.3. *Let A, B be subsets of $\mathbb{R}^2$ and let $K > 0$. We say that a positive real number L is* K-separating *for A, B if L satisfies the conclusion of 5.2 for A, B and K.*

We note that if L is K-separating for A, B, then so is any $L' \geq L$.

5.2 Cocycles

The first result is to obtain a very crude bound on the values of small cocycles.

Lemma 5.4. *Let θ be a small cocycle on R_φ. For any x in X and m in $\mathbb{Z}^2$, we have*

$$|\theta(x,m)| \leq \frac{|m|+1}{2}.$$

Proof. Let $k = \frac{|m|}{2}$ if $|m|$ is even and $k = \frac{|m|+1}{2}$ if $|m|$ is odd. There is a path in $\mathbb{Z}^2$, $0 = m_0, m_1, \ldots, m_k = m$, with $|m_i - m_{i-1}| \leq 2$, for all $1 \leq i \leq k$. Since θ is small, we have

$$|\theta(x,m)| = |\sum_i \theta(\varphi^{m_{i-1}}(x), m_i - m_{i-1})| \leq \sum_i 1 = k.$$

The result follows. $\square$

The following is our basic structure result for strictly positive, small cocycles.

Lemma 5.5. *Let θ be a cocycle for (X, φ) which is strictly positive on a set*

$$C \supset Q_4 = \{(i,j) \mid i \geq 0, j \leq 0\}$$

and such that $\theta \leq (2M+1)^{-1}$ for some $M \geq 1$. For any x in X and integer N, there is a sequence $\{a_n \in \mathbb{Z}^2 \mid n \in \mathbb{Z}\}$ such that for any integer n, we have

$$\theta(x, B(a_n, M)) = N,$$

and $a_{n+1} - a_n$ is either $(1,0)$ or $(0,1)$. Moreover, for any such sequence, only finitely many a_n are in $C \cup (-C)$.

Proof. First, we construct a_0. Consider the sequence $\theta(x,(i,0))$, for i in $\mathbb{Z}$. As θ is positive on the fourth quadrant, this sequence is non-decreasing in i. Since θ is proper on the fourth quadrant, it is unbounded as i tends to positive or negative infinity. Finally, because θ is small, two successive terms can differ by at most one. So we may find i such that $\theta(x,(i+M,0)) = N$. Reasoning in a similar way with the sequence $\theta(x,(i+M,j))$, we may find an integer j such that $\theta(x,(i+M,j-M)) = N$ and $\theta(x,(i+M,j-M-1)) = N+1$. Now we claim that $\theta(x, B((i,j),M) = N$. It follows from the positivity of θ that $\theta(x, B((i,j),M)) \leq \theta(x,(i+M,j-M)) = N$. On the other hand,

$B((i,j),M) \subset B((i+M,j-M-1),2M+1)$ which implies $\theta(x,B((i,j),M)) \geq \theta(x,(i+M,j-m-1))-1 = N$. This establishes the claim. We set $a_0 = (i,j)$.

We next define a_n inductively, for all $n \geq 1$ as follows. We suppose that a_n has been defined with the desired properties. By hypothesis, $\theta(x, a_n + (M,-M)) = N$ and $(1,0)$ is in C. It follows from the positivity of θ and the fact that θ is small that $\theta(x, a_n + (M+1,-M))$ equals N or $N+1$. Similarly, $\theta(x, a_n + (-M, M+1))$ equals N or $N-1$. The two points $a_n + (M+1,-M)$ and $a_n + (-M, M+1)$ are distance $2M+1$ and so from the condition that $\theta \leq (2M+1)^{-1}$, it is not possible that the former equals $N-1$ and the latter equals $N+1$. Suppose that the latter equals N. In this case, we set $a_{n+1} = a_n + (1,0)$. It is clear the second condition is satisfied. It remains to check the first. It is necessary only to check the values of $\theta(x, a_{n+1} + (i,j))$ for $i = M$, $-M \leq j \leq M$, since for other (i,j) in $B(0,M)$, $a_{n+1} + (i,j)$ is in $B(a_n, M)$. For any $-M \leq j \leq M$, we have

$$
\begin{aligned}
0 &= \theta(x, a_{n+1} + (M-1, M)) \\
&\leq \theta(x, a_{n+1} + (M, j)) \\
&\leq \theta(x, a_{n+1} + (M, -M)) = 0,
\end{aligned}
$$

since the vectors $(1, j-M)$ and $(0, -M-j)$ are in C. In the case that $\theta(x, a_n + (-M+1, M)) = N+1$, then let $a_{n+1} = a_n + (0,1)$. The proof that it satisfies the desired properties is the same as the case above.

The values of a_n for $n \leq -1$ are defined in an analogous fashion. We omit the details. The final statement follows immediately from the fact that θ is proper on C and also on $-C$. □

5.3 Compact subrelations

In our analysis, compact subequivalence relations of R_φ will play a crucial role. We begin by giving some basic terminology and stating some simple properties.

Definition 5.6. *Let R be a compact, open subrelation of R_φ.*

1. *We define the diameter of R by*

$$diam(R) = \sup\{|n| \mid n \in \mathbb{Z}^2, (x, \varphi^n(x)) \in R, \text{ for some } x \in X\}.$$

2. *For $K \geq 0$, we say that R has capacity K if, for every x in X, there is k in $\mathbb{Z}^2$ such that*

$$\{(x, \varphi^m(x)) \mid m \in B(k, K)\} \subset R.$$

We will ultimately construct an increasing sequence of compact open subrelations of R_φ. We will need to know that their union is minimal.

Proposition 5.7. *Let (X, φ) be a free, minimal Cantor $\mathbb{Z}^2$-system and let*

$$R_1 \subset R_2 \subset \dots$$

be a sequence of compact open subequivalence relations of R_φ. Suppose there is a sequence $K_n, n \geq 1$ which tends to infinity such that R_n has capacity K_n, for all $n \geq 1$. Then $R = \cup_{n \geq 1} R_n$ is minimal.

Proof. Let x be an arbitrary point of X. We wish to show that its class in R is dense in X. For each $n \geq 1$, select k_n in $\mathbb{Z}^2$ such that $\varphi^m(x)$ is contained in the R_n equivalence class of x, for all m in $B(k_n, K_n)$. By passing to a subsequence, we may assume that the points $\varphi^{k_n}(x)$ converge to a limit x' in X. So x' is in the closure of the R equivalence class of X. Let l be any element of $\mathbb{Z}^2$. For n sufficiently large, $K_n \geq |l|$, $k_n + l$ is in $B(k_n, K_n)$ and so the point $\varphi^{k_n+l}(x)$ is in the R_n equivalence class of x and hence in the R equivalence class of x. This sequence converges to $\varphi^l(x')$. We conclude that $\varphi^l(x')$ is in the closure of the R equivalence class of x. As l was arbitrary, we see that the entire φ-orbit of x' is contained in the closure of the R equivalence class of x. But as this orbit is dense, the closure of the R equivalence class of x is all of X. ∎

Next, we need the notion of the quotient by a compact open subequivalence relation. The proof of the following result is straightforward and a version is given in [GPS2], so we omit the details.

Proposition 5.8. *Let R be a compact open subequivalence relation of R_φ. Then the quotient space X/R is a Cantor set. Let $\pi : X \to X/R$ denote the quotient map. Then π is proper. Also, $\pi \times \pi(R_\varphi)$, endowed with the quotient topology is an étale relation on X/R and the map $\pi \times \pi$ is proper.*

If R is a compact open subequivalence relation, and θ is a cocycle which is identically zero on R, then it descends in a natural way to a cocycle on the quotient relation. We state the result precisely below; its proof is easy and we omit it.

Proposition 5.9. *Let R be a compact open subequivalence relation of R_φ and $\pi : X \to X/R$ denote the quotient map. Suppose that θ is a cocycle on (X, φ) such that $\theta(R) = 0$. We define θ_R on $\pi \times \pi(R_\varphi)$ by*

$$\theta_R(\pi(x), \pi(y)) = \theta(x, y),$$

for all (x, y) in R_φ. Then θ_R is a well-defined cocycle on $\pi \times \pi(R_\varphi)$.

The next notion related to quotients by compact subrelations is that of a lifting.

Definition 5.10. *Let R be a compact open subequivalence relation of R_φ. A map $\sigma : X/R \to X$ is a* lifting *for R if it is continuous and $\pi \circ \sigma(y) = y$, for all y in X/R, where $\pi : X \to X/R$ is the quotient map.*

Using properties of the Cantor set, one can prove that liftings for compact open subequivalence relations R always exist. We will not need this fact; we will later construct them explicitly in special cases. We note the following result.

Lemma 5.11. *Let R be a compact open subequivalence relation of R_φ and let π denote the quotient map as before. If σ is a lifting for R, then, for every x in X, $\sigma \circ \pi(x) = \varphi^l(x)$, for some l in $\mathbb{Z}^2$ with $|l| \leq diam(R)$.*

Proof. We note that $\pi(\sigma \circ \pi(x)) = \pi \circ \sigma(\pi(x)) = \pi(x)$, which means that $\sigma \circ \pi(x)$ and x are in the same R-equivalence class. The result then follows from the definition of $diam(R)$. $\square$

5.4 From cocycles to compact subrelations

There are a couple of ways in which compact open subequivalence relations and cocycles relate to each other. We have already seen the notion of a cocycle descending to a quotient. Probably the most fundamental here is that cocycles, or more precisely a pair of cocycles with certain properties, naturally give rise to compact open subequivalence relations.

Proposition 5.12. *Let ξ, η be cocycles for (X, φ) and let $C, C' \subset \mathbb{Z}^2$. If ξ is proper on C and η is proper on C' and*

$$C \cup (-C) \cup C' \cup (-C') = \mathbb{Z}^2,$$

then $\xi \times \eta$ is proper and

$$\ker(\xi) \cap \ker(\eta) = \ker(\xi \times \eta) = \{(x, y) \in R_\varphi \mid \xi(x, y) = \eta(x, y) = 0\}$$

is a compact open subequivalence relation of R_φ.

Proof. It is obvious from the cocycle condition that the kernel of a cocycle is a subequivalence relation. Moreover, since cocycles are continuous and $\mathbb{Z}$ is discrete, these kernels are open and closed. It also follows from Lemma 3.5 that if a cocycle is proper on C (or C'), then it is also proper on $-C$ (or $-C'$, respectively). If K is a compact subset of $\mathbb{Z}^2$, then it is contained in $K_1 \times K_2$, for some K_1, K_2 compact subsets of $\mathbb{Z}$. We may write

$$\begin{aligned}(\xi \times \eta)^{-1}(K) &\subset \xi^{-1}(K_1) \cap \eta^{-1}(K_2) \\ &= [C \cap \xi^{-1}(K_1) \cap \eta^{-1}(K_2)] \cup [-C \cap \xi^{-1}(K_1) \cap \eta^{-1}(K_2)] \\ &\quad \cup [C' \cap \xi^{-1}(K_1) \cap \eta^{-1}(K_2)] \cup [-C' \cap \xi^{-1}(K_1) \cap \eta^{-1}(K_2)]\end{aligned}$$

The first set $C \cap \xi^{-1}(K_1)$ is compact because K_1 is compact and ξ is proper on C. In a similar way, each of the other three sets above is compact. This shows that $\xi \times \eta$ is proper.

The last statement follows form this and the fact that $\ker(\xi \times \eta) = (\xi \times \eta)^{-1}\{(0, 0)\}$. ∎

This last result is fairly general. In fact, we will need to say more about specific properties of the relation. We now give a more detailed version of Proposition 5.12, adding some hypothesis of positivity for the cocycles.

Proposition 5.13. *Let $r > r' > s' > s > 0$ and M_1, M_2 be positive. Suppose that ξ is a cocycle which is strictly positive on $C(r, r')$ and $\xi \leq (2M_1 + 1)^{-1}$ and that η is a cocycle which is strictly positive on $-C(s, s')$ and $\xi \leq (2M_2 + 1)^{-1}$. For any x in X and $n = (N_1, N_2)$ in $\mathbb{Z}^2$, there is i in $\mathbb{Z}^2$ such that*

$$\xi(x, B(i, M_1)) = N_1, \qquad \eta(x, B(i, M_2)) = N_2.$$

In particular, $\xi \times \eta$ is surjective.

Proof. We make two applications of Lemma 5.5 to obtain sequences a_n and b_n in $\mathbb{Z}^2$, such that

$$\begin{aligned}\xi(x, B(a_n, M_1)) = N_1, &\qquad \eta(x, B(b_n, M_2)) = N_2, \\ a_{n+1} - a_n = (1, 0) \text{ or } (0, 1), &\qquad b_{n+1} - b_n = (1, 0) \text{ or } (0, 1)\end{aligned}$$

for n in $\mathbb{Z}$. Moreover, only finitely many a_n are in $C(r,r') \cup -C(r,r')$ and only finitely many b_n are in $C(s,s') \cup -C(s,s')$. This means that for all but finitely many n, a_n are in either $D(r,r')$ or $-D(r,r')$ and similarly, all but finitely many b_n are in either $D(s,s')$ or $-D(s,s')$. From the second property of the sequences as above, it follows that there is $N \geq 1$ such that a_n is in $D(r,r')$ and b_n is in $D(s,s')$, for $n \geq N$, while a_n is in $-D(r,r')$ and b_n is in $-D(s,s')$, for $n \leq -N$. The two sequences a_n and b_n move from left to right in $\mathbb{Z}^2$, each step being length one. For large negative n, the a_n's are 'below' the b_n's and for large positive n, they are 'above'. By a kind of discrete analogue of the intermediate value theorem, there exists n, n' such that $a_n = b_{n'}$. Letting $i = a_n$ completes the proof. □

5.5 Adjusting cocycles by compact open subequivalence relations

In the last section, it was shown that certain pairs of cocycles give rise to compact open subequivalence relations of R_φ. Here, we show how cocycles may be perturbed or adjusted by a compact open subequivalence relation, R. Basically, the idea is to adjust a cocycle θ so that it takes value 0 on R. It is worth noting that, although we will denote the adjusted cocycle by θ^R, it depends on the choice of a lifting for R.

Lemma 5.14. *Let R be a compact open subequivalence relation of R_φ and $\pi : X \to X/R$ denote the quotient map. Suppose that $\sigma : X/R \to X$ is a lifting for R. Let θ be a cocycle on R_φ. Define θ^R on R_φ by*

$$\theta^R(x,y) = \theta(\sigma \circ \pi(x), \sigma \circ \pi(y)),$$

for all (x,y) in R_φ. Then θ^R is also a cocycle on R_φ and the difference $\theta^R - \theta$ is a coboundary. We have $\theta^R(R) = 0$. If $\theta(R) = 0$, then $\theta^R = \theta$ and in particular, does not depend on the choice of σ. Moreover, if $\theta \leq M^{-1}$ for some $M > 2diam(R)$, then $\theta^R \leq (M - 2diam(R))^{-1}$.

Proof. We define $f(x) = \theta(x, \sigma \circ \pi(x))$, which is a continuous function on X. It is straightforward to check that $bf + \theta = \theta^R$. We omit the details. It follows that θ^R is a cocycle. It is immediate that $\theta^R(R) = 0$. If $\theta(R) = 0$, then the function f above is zero and hence $\theta^R = \theta$.

Finally, suppose that n is in $B(0, M - 2diam(R))$ in $\mathbb{Z}^2$ and x is in X. We have

$$\theta^R(x, \varphi^n(x)) = \theta(\varphi^i(x), \varphi^{n+j}(x)) = \theta(\varphi^i(x), \varphi^{n+j-i}(\varphi^i(x)))$$

for some i, j with $|i|, |j| \leq diam(R)$, by Lemma 5.11. As $|n + j - i| \leq |n| + |i| + |j| \leq M$, the value of the above expression is at most 1 in absolute value. The conclusion follows. $\square$

Lemma 5.15. *Let $R \subset S$ be compact open subequivalence relations of R_φ. Let $\pi_R : X \to X/R$ be the quotient map and σ_R be a lifting for R. Let $S_R = \pi_R \times \pi_R(S)$, which is a compact equivalence relation on X/R. Let $\pi : X/R \to X/S$ be the quotient map and let $\sigma : X/S \to X/R$ is any lifting for S_R. Then $\sigma_R \circ \sigma$ is a lifting for S. Moreover, if θ is any cocycle on R_φ with $\theta(R) = 0$, then we have $\theta^S(R) = 0$ and*

$$(\theta^S)_R = (\theta_R)^{S_R}.$$

In particular, when $R = S$, we have $S_R = \Delta_{X_R}$ and $(\theta^R)_R = \theta_R$.

Proof. By Lemma 5.14, we have $\theta^S(S) = 0$ and since $R \subset S$, it follows that $\theta^S(R) = 0$. Let x, y be in X. By definition, we have

$$\begin{aligned}
(\theta_R)^{S_R}(\pi_R(x), \pi_R(y)) &= \theta_R(\sigma \circ \pi(\pi_R(x)), \sigma \circ \pi(\pi_R(y))) \\
&= \theta_R(\pi_R \circ \sigma_R \circ \sigma \circ \pi(\pi_R(x)), \pi_R \circ \sigma_R \circ \sigma \circ \pi(\pi_R(y))) \\
&= \theta(\sigma_R \circ \sigma \circ \pi(\pi_R(x)), \sigma_R \circ \sigma \circ \pi(\pi_R(y))) \\
&= \theta^S(x, y) \\
&= (\theta^S)_R(\pi_R(x), \pi_R(y)).
\end{aligned}$$

The last part is an easy consequence. $\square$

Our final result in this section is a version of Proposition 5.13 which allows for adjustment of the cocycles.

Proposition 5.16. *Suppose that $r > r' > s' > s$ are positive numbers. Let R and S are compact open subequivalence relations of R_φ. Let π_R and π_S denote the respective quotient maps. Suppose that we have liftings σ_R and σ_S for R and S, respectively. Suppose that ξ and η are cocycles for (X, φ) which are strictly positive on $C(r, r')$ and $C(s, s')$, respectively. Also suppose that $\xi \leq (2(K + diam(R)) + 1)^{-1}$ and $\eta \leq (2(K + diam(S)) + 1)^{-1}$, for some $K \geq 1$. Then $\xi^R \times \eta^S$ is surjective and $\ker(\xi^R \times \eta^S)$ is a compact open subequivalence relation of R_φ with capacity K.*

Proof. Let $n = (n_1, n_2)$ be in $\mathbb{Z}^2$ and let x be in X. Let $N_1 = n_1 - \xi(\sigma_R \circ \pi_R(x), x)$, $N_2 = n_2 - \eta(\sigma_S \circ \pi_S(x), x)$, $M_1 = K + diam(R)$ and $M_2 = K + diam(S)$. We apply 5.13 to obtain i in $\mathbb{Z}^2$ satisfying the conclusion for ξ and η. Then for any k with $|k| \leq K$, we compute

$$\begin{aligned} \xi^R(x, \varphi^{i+k}(x)) &= \xi(\sigma_R \circ \pi_R(x), \sigma_R \circ \pi_R(\varphi^{i+k}(x))) \\ &= \xi(\sigma_R \circ \pi_R(x), x) + \xi(x, \sigma_R \circ \pi_R(\varphi^{i+k}(x))) \\ &= \xi(\sigma_R \circ \pi_R(x), x) + \xi(x, \varphi^{l+i+k}(x)) \end{aligned}$$

for some l with $|l| \leq diam(R)$, by 5.11. Then, $|l + k| \leq K + diam(R) = M_1$ and so by the choice of i, we have

$$\xi^R(x, \varphi^{i+k}(x)) = \xi(\sigma_R \circ \pi_R(x), x) + N_1 = n_1.$$

A similar computation shows that

$$\eta^S(x, \varphi^{i+k}(x)) = n_2.$$

The first part of the conclusion follows at once. The second part follows by setting $n_1 = 0 = n_2$. □

We will need one further piece of notation. In Proposition 5.9, for a cocycle θ and compact open subrelation R such that $\theta(R) = 0$, we defined θ_R. We now drop the hypothesis that $\theta(R) = 0$, but suppose that we have a specific choice for a lift $X/R \to X$. Then we define $\theta_R = (\theta^R)_R$, using our earlier definition and the fact that $\theta^R(R) = 0$. Notice that when $\theta(R) = 0$, this definition agrees with our previous one and is independent of the choice of lift.

5.6 Induced systems

We begin this section by introducing some new notation. If ξ and η are cocycles for X, φ such that $\ker(\xi \times \eta)$ is a compact open subequivalence relation of R_φ, then we let $X_{\xi\times\eta}$ denote the quotient space $X/\ker(\xi \times \eta)$ and $\pi_{\xi\times\eta}$ denote the quotient map. Also, if θ is a cocycle for X, φ such that $\theta(\ker(\xi \times \eta)) = 0$, then we denote $\theta_{\ker(\xi\times\eta)}$ of Proposition 5.9 by $\theta_{\xi\times\eta}$ for simplicity.

Proposition 5.17. *Let ξ and η be cocycles for X, φ such that $\xi \times \eta$ is proper and surjective. Then there is a free minimal action, denoted $\varphi_{\xi\times\eta}$, of $\mathbb{Z}^2$ on $X_{\xi\times\eta}$ defined by setting, for any x in X and n in $\mathbb{Z}^2$,*

$$\varphi^n_{\xi\times\eta}(\pi_{\xi\times\eta}(x)) = \pi_{\xi\times\eta}(\varphi^m(x)),$$

where m is chosen such that $\xi \times \eta(x, m) = n$. Moreover, m may be chosen continuously as a function of the pair x, n. Finally, the map $\pi_{\xi\times\eta} \times \pi_{\xi\times\eta} : R_\varphi \to R_{\varphi_{\xi\times\eta}}$ is continuous, onto and proper.

The proof is straightforward and we omit it. The following result is worth noting; it is a trivial consequence of the definitions and we also leave its proof to the reader.

Lemma 5.18. *Let ξ and η be cocycles for X, φ be such that $\xi \times \eta$ is surjective and proper. Then $\xi_{\xi\times\eta} = \xi_0$ and $\eta_{\xi\times\eta} = \eta_0$ for the system $X_{\xi\times\eta}, \varphi_{\xi\times\eta}$.*

The following result follows easily from the properness of $\xi \times \eta$. We omit the details.

Lemma 5.19. *Let ξ and η be cocycles for X, φ such that $\xi \times \eta$ is surjective and proper. Let $\sigma : X_{\xi\times\eta} \to X$ be a lift for the quotient map. For any $L \geq 1$, there is an $M \geq 1$ such that, if θ is any cocycle on R_φ such that $\theta \leq M^{-1}$, then $\theta_R \leq L^{-1}$.*

5.7 Standard pairs of cocycles

Lemma 5.20. *Let X, φ be a free, minimal Cantor $\mathbb{Z}^2$ system. If ξ is a cocycle which is small and strictly positive on the fourth quadrant, $Q_4 = \{(i,j) \in \mathbb{Z}^2 \mid i \geq 0, j \leq 0\}$, then $\ker(\xi \times \eta_0)$ is a compact, open subequivalence relation of R_φ. For any x in X, its equivalence class in $\ker(\xi \times \eta_0)$ is $\{\varphi^{(i,0)}(x) \mid i^- \leq i \leq i^+\}$, for some integers $i^- \leq 0 \leq i^+$. The values i^-, i^+ are characterized by*

$$\xi(x, (i^-, 0)) = 0, \quad \xi(x, (i^- - 1, 0)) = -1$$
$$\xi(x, (i^+, 0)) = 0, \quad \xi(x, (i^+ + 1, 0)) = 1.$$

Moreover, $\xi \times \eta_0$ is surjective.

Proof. Since ξ is small and strictly positive on the fourth quadrant, the sequence $\xi(x,(i,0))$ is non-decreasing, unbounded, takes value 0 for $i=0$ and since

$$\xi(x,(i+1,0)) - \xi(x,(i,0)) = \xi(\varphi^{(i,0)}(x),(1,0))$$

successive terms differ by at most one. The statements follow easily from this observation. We omit the details. □

The first nice property of cocycles as above is that the quotient map, $\pi_{\xi\times\eta_0}$, has two canonical liftings.

Definition 5.21. *Let ξ be a cocycle for the system X,φ which is small and strictly positive on the fourth quadrant. We define*
$\sigma_{\xi\times\eta_0}, \tau_{\xi\times\eta_0} : X_{\xi\times\eta_0} \to X$ *by*

$$\sigma_{\xi\times\eta_0}(\pi_{\xi\times\eta_0}(x)) = \varphi^{(i^-,0)}(x), \tau_{\xi\times\eta_0}(\pi_{\xi\times\eta_0}(x)) = \varphi^{(i^+,0)}(x),$$

where i^- and i^+ are as in 5.20.

It is easy to check that σ and τ are continuous liftings for the quotient map $\pi_{\xi\times\eta_0}$.

Lemma 5.22. *Let ξ be a cocycle for the system X,φ which is small and strictly positive on the fourth quadrant. Any x in X is the range of $\sigma_{\xi\times\eta_0}$ if and only if $\xi(x,(-1,0)) = -1$ and is in the range of $\tau_{\xi\times\eta_0}$ if and only if $\xi(x,(1,0)) = 1$.*

Proof. First, suppose that $\xi(x,(-1,0)) = -1$. It follows from the definition (as in 5.20) that $i^- = 0$. Then from 5.21, we have $\sigma_{\xi\times\eta_0}(\pi_{\xi\times\eta_0}(x)) = x$. For the converse, let x' is in X and $\pi_{\xi\times\eta_0}(x') = y'$. We apply the results of 5.20 to find i^- so that

$$\begin{aligned}
\xi(\sigma_{\xi\times\eta_0}(y'),(-1,0)) &= \xi(\varphi^{(i^-,0)}(x'),(-1,0)) \\
&= -\xi(x',(i^-,0)) + \xi(x',(i^- - 1,0)) \\
&= 0 + (-1) = -1.
\end{aligned}$$

The only if part of the first statement follows. The second statement is proved in an analogous fashion. □

In the case of standard cocycles, there is a fairly simple relation with the original system and the canonical lifting, σ.

Lemma 5.23. *Let ξ be a small cocycle which is strictly positive on the fourth quadrant. For all x in $X_{\xi\times\eta_0}$, there are integers $i \geq 1$ and $j \geq 0$ such that*

$$\begin{aligned}\sigma_{\xi\times\eta_0}(\varphi^{(1,0)}_{\xi\times\eta_0}(x)) &= \varphi^{(i,0)}(\sigma_{\xi\times\eta_0}(x))\\ \sigma_{\xi\times\eta_0}(\varphi^{(0,1)}_{\xi\times\eta_0}(x)) &= \varphi^{(j,1)}(\sigma_{\xi\times\eta_0}(x))\end{aligned}$$

Proof. We will drop the subscripts from the map $\sigma_{\xi\times\eta_0}$. We consider the first statement, choosing (i,k) in $\mathbb{Z}^2$ such that

$$\sigma(\varphi^{(1,0)}_{\xi\times\eta_0}(x)) = \varphi^{(i,k)}(\sigma(x)).$$

From the definition of the induced action, we have

$$0 = \eta_0(\sigma(x), (i,k)) = k.$$

Then we have also

$$1 = \xi(\sigma(x), (i,0)).$$

The fact that $i \geq 1$ follows from the positivity of ξ.

For the second statement, we again choose (j,k) such that

$$\sigma(\varphi^{(0,1)}_{\xi\times\eta_0}(x)) = \varphi^{(j,k)}(\sigma(x)).$$

Again from the definition of the induced action, we have

$$1 = \eta_0(\sigma(x), (i,k)) = k.$$

Then we have also

$$0 = \xi(\sigma(x), (j,1)).$$

We use the characterization of $\sigma(x)$ in Lemma 5.22 to see

$$\begin{aligned}0 &= \xi(\sigma(x), (j,1))\\ &= \xi(\sigma(x), (-1,0)) + \xi(\varphi^{(-1,0)}(\sigma(x)), (j+1,1))\\ &= -1 + \xi(\varphi^{(-1,0)}(\sigma(x)), (j+1,1)).\end{aligned}$$

If $j < 0$, then $(j+1,1)$ is in $-Q_4$ and this would contradict the positivity of ξ on Q_4. Hence, we have $j \geq 0$ as desired. $\square$

Definition 5.24. *Let (X,φ) be a free, Cantor minimal $\mathbb{Z}^2$-system. Let ξ and η be a pair of cocycles for (X,φ). We say that ξ, η is a* standard pair *if*

1. *ξ is strictly positive on the fourth quadrant and $\xi \leq 4^{-1}$,*

2. *$\eta(\ker \xi \times \eta_0) = 0$,*

3. *the cocycle $\eta_{\xi\times\eta_0}$ for the system $(X_{\xi\times\eta_0}, \varphi_{\xi\times\eta_0})$ is strictly positive on the second quadrant, $Q_2 = \{(i,j) \in \mathbb{Z}^2 \mid i \leq 0, j \geq 0\}$ and $\eta_{\xi\times\eta_0} \leq 4^{-1}$,*

4. *$(1,1), (-1-1) \notin \xi \times \eta(X, B(0,1))$, and*

5. *for any x in X, at least one of $\xi \times \eta(x, (0,1)), \xi \times \eta(x, (0,-1))$ is 0.*

If ξ, η satisfy the first four conditions above, we refer to them as a weak standard pair.

The following result is an immediate consequence of Lemma 5.20; we omit the proof.

Proposition 5.25. *Let ξ, η be a weak standard pair for the system X, φ. Then $\ker(\xi \times \eta)$ is a compact open subequivalence relation of R_φ and $\xi \times \eta$ is surjective.*

If ξ, η is a weak standard pair for the system X, φ, then we have the quotient map $\pi_{\xi\times\eta_0} : X \to X_{\xi\times\eta_0}$, the induced system $\varphi_{\xi\times\eta_0}$ and a canonical lifting $\sigma_{\xi\times\eta_0}$. The cocycle $\eta_{\xi\times\eta_0}$ then defines a cocycle on this quotient. We can then repeat the same process, considering $\ker(\xi_0 \times \eta)$. We observe that, on the quotient system, $X_{\xi\times\eta_0}, \varphi_{\xi\times\eta_0}$, we have $\xi_{\xi\times\eta_0} = \xi_0$, by Lemma 5.18. This means that we have natural identification of $(X_{\xi\times\eta_0})_{\xi_0\times\eta}$ with $X_{\xi\times\eta}$, $\pi_{\xi_0\times\eta_{\xi\times\eta_0}} \circ \pi_{\xi\times\eta_0}$ with $\pi_{\xi\times\eta}$ and $(\varphi_{\xi\times\eta_0})_{\xi_0\times\eta}$ and $\varphi_{\xi\times\eta}$. Continuing with this, we define $\sigma_{\xi\times\eta} = \sigma_{\xi\times\eta_0} \circ \sigma_{\xi_0\times\eta_{\xi\times\eta_0}}$ and $\tau_{\xi\times\eta}$ in a similar way. These are both lifts for $\pi_{\xi\times\eta}$.

Our final result of this section is a method which allows us to pass from a weak standard pair to a standard pair.

Lemma 5.26. *Let ξ, η be a weak standard pair for the system X, φ. Define $A = \{x \in X \mid \xi \times \eta(x, (0,1)) = (0,1)\}$ and $A' = \{x \in X \mid \xi \times \eta(x, (0,-1)) = (0,-1)\}$. Then,*

$$\begin{aligned} R &= \{(x,x) \mid x \in X\} \\ &\quad \cup \{(x, \varphi^{(0,-1)}(x)), (\varphi^{(0,-1)}(x)), x) \mid x \in A\} \\ &\quad \cup \{(x, \varphi^{(0,1)}(x)), (\varphi^{(0,1)}(x)), x) \mid x \in A'\} \end{aligned}$$

is a compact open subequivalence relation in R_φ. Also, ξ, η is a standard pair if and only if $\xi(R) = 0$. In any case, there is a unique lift for the quotient map whose image contains A and A' and using this to define ξ^R, the pair ξ^R, η is a standard pair. Moreover, the maps

$$\pi_{\xi^R \times \eta} \circ \sigma_{\xi \times \eta} : X_{\xi \times \eta} \to X_{\xi^R \times \eta}$$

and

$$\pi_{\xi \times \eta} \circ \sigma_{\xi^R \times \eta} : X_{\xi^R \times \eta} \to X_{\xi \times \eta}$$

are inverse and are conjugacies between the systems $X_{\xi \times \eta}, \varphi_{\xi \times \eta}$ and $X_{\xi^R \times \eta}, \varphi_{\xi^R \times \eta}$. The same is true replacing the maps $\sigma_{\xi \times \eta}, \sigma_{\xi^R \times \eta}$ with $\tau_{\xi \times \eta}, \tau_{\xi^R \times \eta}$.

Proof. To see that R is a subequivalence relation, it suffices to show that the sets A, A', $\varphi^{(0,1)}(A)$ and $\varphi^{(0,-1)}(A')$ are disjoint. This follows quite easily from the condition that $\eta \leq 4^{-1}$. We leave the details to the reader. ☐

5.8 Boundaries for standard pairs of cocycles

One of the main points regarding standard pairs of cocycles is that the equivalence classes in the relation $\ker(\xi \times \eta)$ have boundaries which are easily defined and described.

For convenience, we will use the following notation:

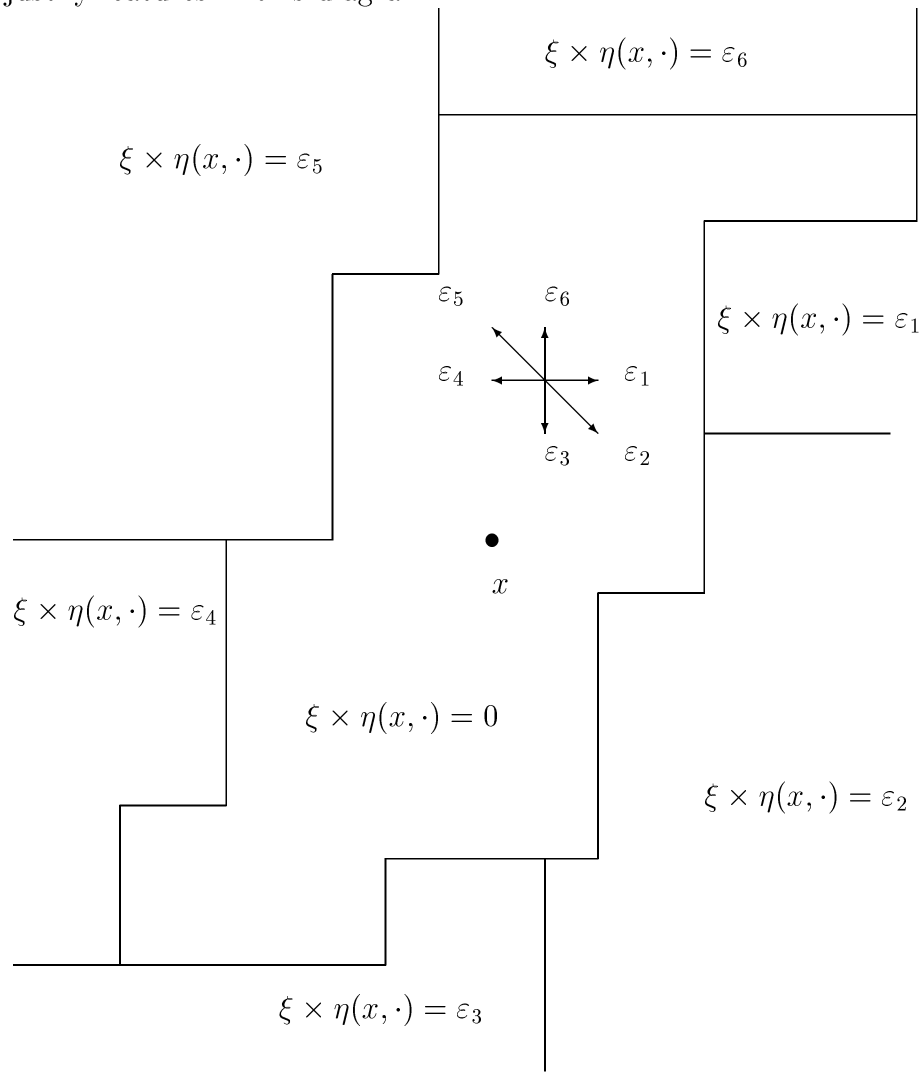
$$\begin{aligned} \varepsilon_1 = (1,0), \quad \varepsilon_2 = (1,-1), \quad \varepsilon_3 = (0,-1), \\ \varepsilon_4 = (-1,0), \quad \varepsilon_5 = (-1,1), \quad \varepsilon_6 = (0,1). \end{aligned}$$

We let $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $E^* = \{\varepsilon_4, \varepsilon_5, \varepsilon_6\}$. We note that the subscript i on the ε_i should always be regarded $mod\, 6$.

We note the following. The proof is an easy direct computation, which we omit.

Lemma 5.27. *For all $1 \leq i \leq 6$, we have $-\varepsilon_i = \varepsilon_{i+3}$, $\varepsilon_{i+1} + \varepsilon_{i-1} = \varepsilon_i$ and $\varepsilon_i - \varepsilon_{i-1} = \varepsilon_{i+1}$, where the indices are understood mod 6.*

We are going to derive a number of properties on the equivalence relation $\ker(\xi \times \eta)$, for a standard pair ξ, η. The following picture should prove helpful. We select any x in X. The picture is of the orbit of x under φ, which is most conveniently drawn as $\mathbb{Z}^2$ (or perhaps $\mathbb{R}^2$). The lines shown partition this orbit according to the values of $\xi \times \eta(x, \cdot)$ or, equivalently, the $R_{\xi \times \eta}$ classes. Most of the properties may be seen there; basically, our proofs are attempting to justify features in this diagram.



Lemma 5.28. *Let ξ, η be a standard pair for the free minimal Cantor $\mathbb{Z}^2$-system (X, φ). Then, for all x in X, we have*

$$\begin{aligned} \xi \times \eta(x, E) &\subset E \cup \{0\} \\ \xi \times \eta(x, E^*) &\subset E^* \cup \{0\} \end{aligned}$$

Proof. We begin by noting that since ξ is positive on Q_4 and η_0 is negative on Q_4, we have

$$\xi \times \eta_0(X, Q_4) \subset Q_4.$$

As η is positive on Q_2 and $Q_2 = -Q_4$, we have

$$\xi \times \eta(X, Q_4) \subset Q_4.$$

A similar argument shows that

$$\xi \times \eta(X, Q_2) \subset Q_2.$$

The statements follows at once from this, the fact that $E \subset Q_4$ and $E^* \subset Q_2$ and the hypotheses that ξ and η are small. □

Later, we will need the following simple technical fact.

Lemma 5.29. *Let ξ, η be a standard pair of cocycles. Then, for any x in X,*

$$\begin{aligned} \xi \times \eta(x, \varepsilon_1) &\neq \varepsilon_3 \\ \xi \times \eta(x, \varepsilon_4) &\neq \varepsilon_6 \end{aligned}$$

Proof. We prove only the first statement. Suppose that $\xi \times \eta(x, \varepsilon_1) = \varepsilon_3$. In particular, we have $\xi(x, \varepsilon_1) = 0$. Also, since $\varepsilon_1 = (1, 0)$, we have $\eta_0(x, \varepsilon_1) = 0$. It follows that $\eta(x, \varepsilon_1) = 0$ and this is a contradiction. □

Recall the canonical lifts $\sigma_{\xi\times\eta}$ and $\tau_{\xi\times\eta}$ for a standard pair ξ, η. In our diagram, these map the equivalence class of x to the lower left and upper right corners, respectively.

Lemma 5.30. *Let ξ, η be a standard pair of cocycles for (X, φ), and let $\sigma_{\xi\times\eta}, \tau_{\xi\times\eta}$ be the standard lifts as above. For any x in $X_{\xi\times\eta}$, we have*

$$\xi \times \eta(\tau_{\xi\times\eta}(x), \varepsilon_1) = \varepsilon_1, \tag{1}$$

$$\xi \times \eta(\tau_{\xi\times\eta}(x), \varepsilon_6) = \varepsilon_6, \tag{2}$$

$$\xi \times \eta(\sigma_{\xi\times\eta}(x), \varepsilon_4) = \varepsilon_4, \tag{3}$$

$$\xi \times \eta(\sigma_{\xi\times\eta}(x), \varepsilon_3) = \varepsilon_3. \tag{4}$$

Proof. Noting that $\tau_{\xi\times\eta}(x) = \tau_{\xi\times\eta_0}(\tau_{\xi_0\times\eta}(x))$, the first equation follows at once from Lemma 5.22. The third follows in an analogous way.

For the second, we claim that, for any x' in $\sigma_{\xi\times\eta_0}(X_{\xi\times\eta_0})$,

$$\pi_{\xi\times\eta_0}(\varphi^{(0,-1)}(x')) = \varphi_{\xi\times\eta_0}(\pi_{\xi\times\eta_0}(x')).$$

From the definition of $\varphi_{\xi\times\eta_0}$, it suffices to check that

$$\xi \times \eta_0(x', (0,-1)) = (0,-1).$$

It is clear that $\eta_0(x', (0,-1)) = -1$. The value of $\xi(x', (0,-1))$ is non-negative since $(0,-1)$ is in Q_4, where ξ is positive. On the other hand, by 5.22 and the fact that x' is in $\sigma_{\xi\times\eta_0}(X_{\xi\times\eta_0})$, we have $\xi(x', (-1,0)) = -1$. Then the value of $\xi(x', (0,-1))$ being positive would contradict the fact that ξ is small. We conclude that $\xi(x', (0,-1)) = 0$, as desired. The proof of the last statement is analogous and we omit the details. $\square$

Lemma 5.31. *Let ξ, η be a standard pair of cocycles for (X, φ) and x be in X. If for some $1 \leq i, j \leq 6$ with $i \neq j$, $\varepsilon_i, \varepsilon_j$ are in $\xi \times \eta(x, E \cup E^*)$, then $j = i \pm 1$. Moreover, we have $\xi \times \eta(x, \varepsilon_i) = \varepsilon_i$ and $\xi \times \eta(x, \varepsilon_j) = \varepsilon_j$.*

Proof. We assume that $\varepsilon_i \in E$; the other case is similar. We may find $\varepsilon_k \in E$ such that $\xi \times \eta(x, \varepsilon_k) = \varepsilon_i$. We may also find $1 \leq l \leq 6$, such that $\xi \times \eta(x, \varepsilon_l) = \varepsilon_j$. As $i \neq j$, clearly $k \neq l$. Using the cocycle condition, we have

$$\begin{aligned}
\varepsilon_i + \varepsilon_{j+3} &= \varepsilon_i - \varepsilon_j \\
&= \xi \times \eta(x, \varepsilon_k) - \xi \times \eta(x, \varepsilon_l) \\
&= \xi \times \eta(\varphi^{\varepsilon_l}(x), \varepsilon_k - \varepsilon_l).
\end{aligned}$$

As $|\varepsilon_k - \varepsilon_l| \leq 2$ and ξ and η are small, we have $|\varepsilon_i + \varepsilon_{j+3}| \leq 1$. If ε_{j+3} is in E, then ε_i and ε_{j+3} must be ε_1 and ε_3. Hence, ε_i and ε_j are either ε_1 and ε_6 or else ε_3 and ε_4. Next, we consider when ε_{j+3} is in E^*. Then ε_j and hence ε_l are in E. Then $\varepsilon_k - \varepsilon_l$ is in $B(0,1)$ and as $(1,1), (-1,-1) \notin \xi \times \eta(x, B(0,1))$, we have $\varepsilon_i - \varepsilon_j \neq (1,1), (-1,-1)$. This eliminates the case that ε_i and ε_j are ε_1 and ε_3. The first statement follows. $\square$

Lemma 5.32. *Let ξ, η be a standard pair of cocycles for (X, φ). Let x be in X.*

1. *For $i = 1, 4$, if $\xi \times \eta(x, \varepsilon_i) = \xi \times \eta(x, \varepsilon_{i+2}) \neq 0$, then $\xi \times \eta(x, \varepsilon_i) = \xi \times \eta(x, \varepsilon_{i+1})$.*

2. *If $\varepsilon_j, \varepsilon_{j-1}$ are in $\xi \times \eta(x, \{\varepsilon_i, \varepsilon_{i-1}\})$ for some $1 \leq i, j \leq 6$, then $i = j$, $\xi \times \eta(x, \varepsilon_i) = \varepsilon_i$ and $\xi \times \eta(x, \varepsilon_{i-1}) = \varepsilon_{i-1}$.*

Proof. For the first part, we consider the case $i = 1$ for simplicity. We claim that $\xi(x, \varepsilon_1) \neq 0$. If $\xi(x, \varepsilon_1) = 0$, then also $\eta_0(x, \varepsilon_1) = 0$ and it follows that $\eta(x, \varepsilon_1) = 0$, which contradicts our hypothesis. As ε_1 is in the fourth quadrant, we must have $\xi(x, \varepsilon_1) = 1 = \xi(x, \varepsilon_3)$. Again by positivity, we have $\xi(x, \varepsilon_1 + \varepsilon_3) \geq 1$. Strict inequality is not possible, as ξ is small. Hence, we have $\xi(x, \varepsilon_2) = 1$. Now we have $\xi(\varphi^{\varepsilon_3}(x), \varepsilon_1) = \xi(x, \varepsilon_2) - \xi(x, \varepsilon_3) = 0$ and also $\eta_0(\varphi^{\varepsilon_3}(x), \varepsilon_1) = 0$. It follows then that $\eta(\varphi^{\varepsilon_3}(x), \varepsilon_1) = 0$. Then we have $\eta(x, \varepsilon_2) = \eta(x, \varepsilon_3) + \eta(\varphi^{\varepsilon_3}(x), \varepsilon_1) = \eta(x, \varepsilon_3)$. This completes the proof.

For the second part, we first consider the case that $\xi \times \eta(x, \varepsilon_{i-1}) = \varepsilon_j$. Then it follows that $\xi \times \eta(x, \varepsilon_i) = \varepsilon_{j-1}$. Then we have

$$\xi \times \eta(\varphi^{\varepsilon_{i-1}}(x), -\varepsilon_{i-1}) = -\varepsilon_j = \varepsilon_{j+3}.$$

On the other hand, we also have

$$\begin{aligned}
\xi \times \eta(\varphi^{\varepsilon_{i-1}}(x), \varepsilon_{i-2}) &= \xi \times \eta(\varphi^{\varepsilon_{i-1}}(x), \varepsilon_{i-1} - \varepsilon_i) \\
&= \xi \times \eta(x, \varepsilon_{i-1}) - \xi \times \eta(x, \varepsilon_i) \\
&= \varepsilon_j - \varepsilon_{j-1} \\
&= \varepsilon_{j+1}
\end{aligned}$$

This contradicts Lemma 5.31.

This leaves us to consider the case that $\xi \times \eta(x, \varepsilon_i) = \varepsilon_j$ and $\xi \times \eta(x, \varepsilon_{i-1}) = \varepsilon_{j-1}$. The same calculation as above shows that $\xi \times \eta(\varphi^{\varepsilon_{i-1}}(x), \varepsilon_{i-2}) = \varepsilon_{j-2}$. Now from Lemma 5.28, we know that ε_i is in E if and only if ε_j is also. The same applies to ε_{i-1} and ε_{j-1} and also to ε_{i-2} and ε_{j-2}. From this it follows that $i = j$. ☐

Definition 5.33. *Let ξ, η be a standard pair of cocycles for (X, φ). For each x in X, let $E(x) = \xi \times \eta(x, E \cup E^*) - \{0\}$.*

Theorem 5.34. *Let ξ, η be a standard pair of cocycles for the system (X, φ).*

1. *For any x in X, $\#E(x) \leq 2$.*

2. *For any x in X, if $\#E(x) = 2$, then $E(x) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some $1 \leq i \leq 6$. Also, we have $\xi \times \eta(x, \varepsilon_i) = \varepsilon_i$ and $\xi \times \eta(x, \varepsilon_{i-1}) = \varepsilon_{i-1}$.*

3. *For any x in X with $\#E(x) = 2$, we have*

$$\begin{array}{lll}
E(x) = \{\varepsilon_1, \varepsilon_6\} & \textit{if and only if} & x \in \tau_{\xi\times\eta}(X_{\xi\times\eta}) \\
E(x) = \{\varepsilon_3, \varepsilon_2\} & \textit{if and only if} & x \in \varphi^{\varepsilon_6}(\tau_{\xi\times\eta}(X_{\xi\times\eta})) \\
E(x) = \{\varepsilon_5, \varepsilon_4\} & \textit{if and only if} & x \in \varphi^{\varepsilon_1}(\tau_{\xi\times\eta}(X_{\xi\times\eta})) \\
E(x) = \{\varepsilon_4, \varepsilon_3\} & \textit{if and only if} & x \in \sigma_{\xi\times\eta}(X_{\xi\times\eta}) \\
E(x) = \{\varepsilon_6, \varepsilon_5\} & \textit{if and only if} & x \in \varphi^{\varepsilon_3}(\sigma_{\xi\times\eta}(X_{\xi\times\eta})) \\
E(x) = \{\varepsilon_2, \varepsilon_1\} & \textit{if and only if} & x \in \varphi^{\varepsilon_4}(\sigma_{\xi\times\eta}(X_{\xi\times\eta}))
\end{array}$$

4. *For each $1 \leq i \leq 6$ and each x in X, there is a unique point x_i in the same* $\ker(\xi \times \eta)$ *equivalence class as x with $E(x_i) = \{\varepsilon_i, \varepsilon_{i-1}\}$.*

Proof. The first part is an immediate consequence of Lemma 5.31.

The second statement is a consequence of Lemma 5.31 and the second part of Lemma 5.32.

We now consider the fourth equivalence of part 3. The 'if' direction follows from Lemma 5.30. For the converse, let us suppose that $\xi\times\eta(x,\varepsilon) = \varepsilon$, for $\varepsilon = \varepsilon_3, \varepsilon_4$. The fact that $\xi(x, \varepsilon_4) = -1$ implies that x is in the range of $\sigma_{\xi\times\eta_0}$. Then we have $-1 = \eta(x, \varepsilon_3) = \eta_{\xi\times\eta_0}(\pi_{\xi\times\eta_0}(x), \varepsilon_3)$. This then implies that $\pi_{\xi\times\eta_0}(x)$ is in the range of $\sigma_{\xi_0\times\eta}$. This implies that x is in the range of $\sigma_{\xi\times\eta}$. This completes the proof of the fourth equivalence. The first is done in a similar fashion and we omit the details.

Now consider the last equivalence. We use the cocycle condition to compute, for x in X,

$$\begin{aligned}
\xi \times \eta(\varphi^{\varepsilon_4}(x), \varepsilon_1) &= \xi \times \eta(\varphi^{\varepsilon_4}(x), -\varepsilon_4) \\
&= -\xi \times \eta(x, \varepsilon_4)
\end{aligned}$$

and also

$$\begin{aligned}
\xi \times \eta(\varphi^{\varepsilon_4}(x), \varepsilon_2) &= \xi \times \eta(x, \varepsilon_4 + \varepsilon_2) - \xi \times \eta(x, \varepsilon_4) \\
&= \xi \times \eta(x, \varepsilon_3) - \xi \times \eta(x, \varepsilon_4).
\end{aligned}$$

Now, if x is in $\sigma_{\xi\times\eta}(X_{\xi\times\eta})$, then it follows from the fourth equivalence, part 2 and the equations above, that $E(\varphi^{\varepsilon_4}(x)) = \{\varepsilon_2, \varepsilon_1\}$. Conversely, if $E(\varphi^{\varepsilon_4}(x)) = \{\varepsilon_2, \varepsilon_1\}$, then the equations above, part two and the fourth equivalence imply that x is in $\sigma_{\xi\times\eta}(X_{xi\times\eta})$.

For part 4, the cases $i = 1, 4$ follow from part 1 and the fact that $\sigma_{\xi\times\eta}$ and $\tau_{\xi\times\eta}$ are lifts for the quotient map by $\ker \xi \times \eta$.

Let us consider $i = 2$, the other cases being similar. Let

$$x_2 = \varphi^{\varepsilon_4}(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_1}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)))).$$

By the last statement of part 1, we have $E(x_2) = \{\varepsilon_2, \varepsilon_1\}$. We also have

$$\begin{aligned}
\xi \times \eta(x, x_2) &= \xi \times \eta(x, \sigma_{\xi\times\eta}(\varphi^{\varepsilon_1}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)\\
&\quad +\xi \times \eta(\sigma_{\xi\times\eta}(\varphi_{\xi\times\eta}(\pi_{\xi\times\eta}(x), x_2)\\
&= \xi_0 \times \eta_0(\pi_{\xi\times\eta}(x), \varphi^{\varepsilon_1}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)\\
&\quad +\xi \times \eta(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_1}_{\xi\times\eta}(\pi_{\xi\times\eta}(x), \varphi^{\varepsilon_4}(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_1}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)))\\
&= \varepsilon_1 + \varepsilon_4\\
&= 0.
\end{aligned}$$

by 5.18 and 5.30. Now if x' is any other point satisfying the desired properties, it is easily checked that $\varphi^{\varepsilon_1}(x_2)$ and $\varphi^{\varepsilon_1}(x')$ are in the same $\ker \xi \times \eta$ equivalence class and both have value $\{\varepsilon_3, \varepsilon_4\}$ under $E(\cdot)$. It follows from part 1 and the fact that $\sigma_{\xi\times\eta}$ is a lift that $x' = x_2$.

For completeness, we note that

$$\begin{aligned}
x_3 &= \varphi^{\varepsilon_6}(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_3}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)))\\
x_5 &= \varphi^{\varepsilon_1}(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_4}_{\xi\times\eta}(\pi_{\xi\times\eta}(x)))\\
x_6 &= \varphi^{\varepsilon_3}(\sigma_{\xi\times\eta}(\varphi^{\varepsilon_6}_{\xi\times\eta}(\pi_{\xi\times\eta}(x))).
\end{aligned}$$

□

Proposition 5.35. *Let x be in $X_{\xi\times\eta}$. For each $i = 1, \ldots, 6$, there is x_i in X such that $\pi_{\xi\times\eta}(x_i) = x$, $\xi \times \eta(x_i, \varepsilon_i) = \varepsilon_i$ and $E(x_i) = \{\varepsilon_i\}$.*

Proof. We begin with the case $i = 1$. Let $x_1 = \varphi^{(-1,0)}(\tau_{\xi\times\eta}(x))$. As $\xi \times \eta(\tau_{\xi\times\eta}(x), \varepsilon_6) = \varepsilon_6$ from Lemma 5.30, it follows from the final condition of a standard pair that $\xi \times \eta(\tau_{\xi\times\eta}(x), -\varepsilon_6) = (0, 0)$. This means that $(\tau_{\xi\times\eta}(x), x_1)$ is in $\ker(\xi \times \eta)$ and hence $\pi_{\xi\times\eta}(x_1) = x$. We also have

$$\begin{aligned}
\varepsilon_1 &= \xi \times \eta(\tau_{\xi\times\eta}(x), \varepsilon_1)\\
&= \xi \times \eta(x_1, (1, 1))\\
&= \xi \times \eta(x_1, \varepsilon_1) + \xi \times \eta(\varphi^{\varepsilon_1}(x_1), (0, 1)).
\end{aligned}$$

Since ξ is positive on Q_4, η is positive on Q_2, and ξ is small, it follows that $\xi(x_1, \varepsilon_1) = \varepsilon_1$. Next, we claim that $E(x_1) = \{\varepsilon_1\}$. In view of 5.34, it suffices to see that $\xi \times \eta(x_1, \varepsilon_6) \neq \varepsilon_6$ and $\xi \times \eta(x_1, \varepsilon_2) \neq \varepsilon_2$. For the former, we have already observed that $\xi \times \eta(x_1, \varepsilon_6) = -\xi \times \eta(\tau_{\xi\times\eta}(x), -\varepsilon_6) = (0,0)$. For the latter, if $\eta(x_1, \varepsilon_2) = -1$, then we would have

$$\begin{aligned}
\eta(\varphi^{\varepsilon_2}(x_1), \varepsilon_2 + \varepsilon_6 + \varepsilon_6) &= -\eta(x_1, \varepsilon_2) + \eta(x_1, \varepsilon_6) + \eta(\tau_{\xi\times\eta}(x), \varepsilon_6) \\
&\geq 1 + 0 + 1 = 2,
\end{aligned}$$

which would contradict the fact that $\eta_{\xi\times\eta_0} \leq 4^{-1}$.

For $i = 2$, we let $x' = \sigma_{\xi\times\eta}(\varphi^{\varepsilon_1}_{\xi\times\eta}(x)))$. First consider the case $\xi \times \eta(x', \varepsilon_3 + \varepsilon_4) = \varepsilon_4$. Then we set $x_2 = \varphi^{\varepsilon_3+\varepsilon_4}(x')$. Then we have $\xi \times \eta(x_2, \varepsilon_1 + \varepsilon_6) = \varepsilon_1$ and so by definition, $\varphi_{\xi\times\eta}(\pi_{\xi\times\eta}(x_2)) = \pi_{\xi\times\eta}(x')$ and it follows that $\pi_{\xi\times\eta}(x_2) = x$. Also we have

$$\begin{aligned}
\xi \times \eta(x_2, \varepsilon_2) &= \xi \times \eta(x_2, \varepsilon_1 + \varepsilon_6 + 3\varepsilon_3) \\
&= \xi \times \eta(x_2, \varepsilon_1 + \varepsilon_6) + \xi \times \eta(x', \varepsilon_3) \\
&\quad + \xi \times \eta(\varphi^{\varepsilon_3}(x'), \varepsilon_3) \\
&= \varepsilon_1 + \varepsilon_3 + 0 \\
&= \varepsilon_2,
\end{aligned}$$

using 5.34 and the final property of standard pairs.

Now we claim that $E(x_2) = \{\varepsilon_2\}$. It suffices to check that $\xi\times\eta(x_2, \varepsilon_j) \neq \varepsilon_j$ for $j = 1, 3$. For $j = 1$, we have

$$\begin{aligned}
\xi \times \eta(x_2, \varepsilon_1) &= \xi \times \eta(x_2, \varepsilon_1 + \varepsilon_6) + \xi \times \eta(x', \varepsilon_3) \\
&= \varepsilon_1 + \varepsilon_3 \\
&\neq \varepsilon_1
\end{aligned}$$

For $j = 3$, we have $\eta(x_2, \varepsilon_3) \leq 0$ because η is positive on Q_2. Strict inequality would violate the smallness of η since $\eta(x', \varepsilon_3 + \varepsilon_4) = -1$. Next, for $i = 2$, we consider the case $\xi \times \eta(x', \varepsilon_3 + \varepsilon_4) \neq \varepsilon_4$. Then we set $x_2 = \varphi^{(-2,0)}(x')$.

Finally, for $i = 3$, we set $x_3 = \varphi^{(1,0)}(\sigma_{\xi\times\eta}(x))$. Checking that x_3 has the desired properties is similar to the other cases and we omit the details. $\square$

Ultimately, we will need to know that the boundaries of equivalence classes in $\ker(\xi \times \eta)$ have small measure. The following results will be used toward that end in the next section.

Lemma 5.36. *Let ξ, η be a standard pair of cocycles for (X, φ). Let $C = \{x \in X \mid E(x) = \{\varepsilon_1, \varepsilon_6\}\}$ and $C' = \{x \in X \mid E(x) = \{\varepsilon_4, \varepsilon_3\}\}$. Then for any ε with $|\varepsilon| = 1$,*

$$\varphi^{\varepsilon}(C) \cap C = \emptyset = \varphi^{\varepsilon}(C') \cap C'.$$

Proof. We prove the first statement only. By applying $-\varepsilon$ to both sides, we may assume that the sum of the entries of ε is non-negative. Let x be in C; then we have

$$\begin{aligned} \xi(x, \varepsilon_1 + \varepsilon) &= \xi(x, \varepsilon_1) + \xi(\varphi^{\varepsilon_1}(x), \varepsilon - \varepsilon_1) + \xi(\varphi^{\varepsilon}(x), \varepsilon_1) \\ &= 2 + \xi(\varphi^{\varepsilon_1}(x), \varepsilon - \varepsilon_1) \end{aligned}$$

If $\varepsilon - \varepsilon_1$ is in the fourth quadrant, then the remaining term is non-negative and this would contradict $\xi \leq 4^{-1}$. A similar calculation, using ε_6 and η instead, would contradict $\eta \leq 4^{-1}$ if $\varepsilon - \varepsilon_6$ is in the second quadrant. The only remaining case is $\varepsilon = (1, 1) = \varepsilon_1 + \varepsilon_6$. Following the same reasoning as in the first case above, we have a contradiction unless $\xi(x, \varepsilon_1 + \varepsilon_6) = 0$. Then we have

$$\xi(\varphi^{\varepsilon_6}(x), \varepsilon_1) = \xi(x, \varepsilon_1 + \varepsilon_6) - \xi(x, \varepsilon_6) = 0 - 0 = 0.$$

As $\eta_0(\varphi^{\varepsilon_6}(x), \varepsilon_1) = 0$, by condition 2 of a standard pair, it follows that $\varphi^{\varepsilon_6}(x)$ and $\varphi^{\varepsilon_1 + \varepsilon_6}(x)$ are in the same class modulo $\ker(\xi \times \eta_0)$. It also follows from the defintions of the quotient system $\varphi_{\xi \times \eta_0}$ that

$$\begin{aligned} \varphi^{\varepsilon_6}_{\xi \times \eta_0}(\pi_{\xi \times \eta_0}(x)) &= \pi_{\xi \times \eta_0}(\varphi^{\varepsilon_6}(x)) \\ \varphi^{\varepsilon_6}_{\xi \times \eta_0}(\pi_{\xi \times \eta_0}(\varphi^{\varepsilon_6}(x))) &= \pi_{\xi \times \eta_0}(\varphi^{\varepsilon_1 + 2\varepsilon_6}(x)) \end{aligned}$$

From this, we arrive at a contradiction to condition $\eta_{\xi \times \eta_0} \leq 4^{-1}$ from property 3 of a standard pair. This completes the proof.

□

Proposition 5.37. *Let ξ, η be a standard pair of cocycles on (X, φ). Let Z be a clopen subset of X which is $\ker(\xi \times \eta)$-invariant in the sense that if x is in Z, then its $\ker(\xi \times \eta)$ equivalence class is in Z. Let*

$$B = \{x \in Z \mid \emptyset \neq E(x) \cap E\}, B^* = \{x \in Z \mid \emptyset \neq E(x) \cap E^*\}.$$

Then, for any $\ker(\xi \times \eta)$-invariant probability measure, μ, on X, we have $\mu(B) \leq \mu(Z)/2$ and $\mu(B^*) \leq \mu(Z)/2$.

Proof. We define a map $\gamma : B \to X$ such that γ is injective, $\gamma(B) \cap B = \emptyset$ and $Graph(\gamma) \subset \ker(\xi \times \eta)$. As Z is $\ker(\xi \times \eta)$-invariant, we have $\gamma(B) \subset Z$. From this it follows that

$$\mu(Z) \geq \mu(B) + \mu(\gamma(B)) = 2\mu(B),$$

and we are done. We construct γ as follows. If $E(x) \subset E$, then we set $\gamma(x) = \varphi^{\varepsilon_5}(x)$. As $\xi \times \eta(x, E^*)$ is contained in $E^* \cup \{0\}$, we must have $\xi \times \eta(x, \varepsilon_5) = 0$ if $E(x) \subset E$. This means that $(x, \gamma(x))$ is in $\ker(\xi \times \eta)$ for such x.

If $E(x)$ is not contained in E, then by Theorem 5.34 we have $E(x)$ is either $\{\varepsilon_1, \varepsilon_6\}$ or $\{\varepsilon_4, \varepsilon_3\}$. Moreover, there is exactly one point in each $R_{\xi\times\eta}$-equivalence class taking each of these values. If $E(x) = \{\varepsilon_1, \varepsilon_6\}$, we define $\gamma(x)$ to be the unique point such that $\xi \times \eta(x, \gamma(x)) = 0$ and $E(\gamma(x)) = \{\varepsilon_6, \varepsilon_5\}$. If $E(x) = \{\varepsilon_4, \varepsilon_3\}$, we define $\gamma(x)$ to be the unique point such that $\xi \times \eta(x, \gamma(x)) = 0$ and $E(\gamma(x)) = \{\varepsilon_5, \varepsilon_4\}$.

We must check that γ is injective. Clearly the restriction of γ to the two cases, $E(x) \subset E$ and its complement, are injective. Let us consider the case $E(x) \subset E$ and $E(x') = \{\varepsilon_1, \varepsilon_6\}$ and suppose that $\gamma(x) = \gamma(x')$. This means that $\gamma(x') = \varphi^{\varepsilon_5}(x)$. For some ε in E, we have $\xi \times \eta(x, \varepsilon) = \varepsilon_i$ with $1 \leq i \leq 3$. Consider

$$\begin{aligned}
\xi \times \eta(\varphi^{\varepsilon}(x), -\varepsilon + \varepsilon_5 + \varepsilon_5) &= \xi \times \eta(\varphi^{\varepsilon}(x), -\varepsilon) \\
&\quad + \xi \times \eta(x, \varepsilon_5) + \xi \times \eta(\varphi^{\varepsilon_5}(x), \varepsilon_5) \\
&= -\varepsilon_i + 0 + \xi \times \eta(\gamma(x'), \varepsilon_5) \\
&= \varepsilon_{i+3} + \varepsilon_5
\end{aligned}$$

using Theorem 5.34. But this contradicts either the smallness of ξ or η (or both). The other case is done in a similar way. We omit the details. $\square$

6 Construction of the AF-relation R

In this section, we will construct an AF-subrelation $R \subset R_\varphi$. This will be done by constructing an increasing sequence of compact open subequivalence relations. These, in turn, arise from pairs of cocycles for (X, φ).

With r_∞ and s_∞ as in the hypotheses of 4.1, we find sequences of positive numbers:

$$r_1 > r'_1 > r_2 > r'_2 > r_3 > r'_3 > \cdots > r_\infty,$$

and

$$s_1 < s'_1 < s_2 < s'_2 < s_3 < s'_3 < \cdots < s_\infty,$$

such that for any $M \geq 2$ and any $n \geq 1$, there is a cocycle ξ which is strictly positive on $C(r_n, r'_n)$ and $\xi \leq M^{-1}$ and a cocycle η which is strictly positive on $C(s_n, s'_n)$ and $\eta \leq M^{-1}$. We also set $r_0 = r'_0 = +\infty$ and $s_0 = s'_0 = 0$.

Let us outline our plan for construction of an AF-relation $R \subset R_\varphi$. For each $n \geq 0$, we will select a pair of cocycles ξ^n and η^n. These will satisfy a long list of properties (numbered 1 to 11). We will have

1. For all $n \geq 0$,

$$R_n = \ker(\xi^n \times \eta^n)$$

is a compact open subequivalence relation.

2. For all $n \geq 0$, $\xi^n \times \eta^n$ is surjective.

We let $X_n = X/R_n$ and π^n be the quotient map from X to X_n. By 5.17, there is a natural induced system on X_n, which we denote by φ_n.

3. For all $n \geq 1$, $\xi^n(R_{n-1}) = \eta^n(R_{n-1}) = 0$ and hence, $R_n \supset R_{n-1}$.

This implies that there is a natural quotient map, which we denote π_n from $X_{n-1} = X/R_{n-1}$ to $X_n = X/R_n$ which satisfies $\pi_n \circ \pi^{n-1} = \pi^n$.

4. For all $n \geq 0$, R_n has capacity n (Definition 5.6).

5. For all $n \geq 1$, $(\xi^n)_{R_{n-1}}, (\eta^n)_{R_{n-1}}$ are a standard pair of cocycles for the system X_{n-1}, φ_{n-1}.

By the results earlier on standard pairs of cocycles, there are canonical lifts $\sigma_n, \tau_n : X_n \to X_{n-1}$, for all $n \geq 1$. We then inductively define maps $\sigma^n : X_n \to X$ and $\tau^n : X_n \to X$ by $\sigma^1 = \sigma_1$, $\tau^1 = \tau_1$, $\sigma^n = \sigma^{n-1} \circ \sigma_n$ and $\tau^n = \tau^{n-1} \circ \tau_n$, for $n \geq 2$. It is easy to check that $\pi^n \circ \sigma^n$ and $\pi^n \circ \tau^n$ are the identity map on X_n.

6. For all $n \geq 1$, $\sigma^n(X_n) \cap \varphi^m(\tau^n(X_n))$ is empty if $|m| < n$.

We will need 'liftings' of the quotient system to X. This is achieved by defining maps u_n and v_n as follows. We define $u_n, v_n : X_n \to \mathbb{Z}^2$ by

$$\begin{aligned} \sigma^n \circ \varphi_n^{(1,0)}(x) &= \varphi^{u_n(x)} \circ \sigma^n(x), \\ \sigma^n \circ \varphi_n^{(0,1)}(x) &= \varphi^{v_n(x)} \circ \sigma^n(x) \end{aligned}$$

for all x in X_n. It follows from Proposition 5.17 that u_n, v_n are continuous functions. We define

$$U_n = \sup\{|u_n(x)| \mid x \in X\} + \sup\{|v_n(x)| \mid x \in X\}.$$

We will have the following conditions.

7. For all $n \geq 0$, $u_n(x) \in D(s_n, s'_n)$, for all x in X_n.

8. For all $n \geq 0$, $v_n(x) \in D(r_n, r'_n)$, for all x in X_n.

9. For all $n \geq 1$, we have $v_n(x) \pm u_n(x')$ is in $D(r_n, r'_n)$, for all x, x' in X_n.

10. For all $n \geq 1$, we have

$$|u_n(x)|, |v_n(x')|, |u_n(x) \pm v_n(x')| \geq U_{n-1},$$

for all x, x' in X_n.

11. For all $n \geq 0$, we have

$$\begin{aligned}[v_n(x) + D(s'_n, s_n)] \quad &\cap \quad \left[(\mathbb{Z}^2 - C(r_{n+1}, r'_{n+1})) + u_n(x')\right] \\ &\subset \quad -C(s'_{n+1}, s_{n+1})\end{aligned}$$

for all x, x' in X_n.

We begin with $\xi^0 = \xi_0$ and $\eta^0 = \eta_0$ as before. Then R^0 is just equality, $X_0 = X$ and π^0, σ^0, τ^0, for convenience, are all the identity. Finally, we have $u_0(x) = (1, 0)$ and $v_0(x) = (0, 1)$, for all x in X and (i, j) in $\mathbb{Z}^2$. It is then easy to see that conditions 1, 2, 4, 6, 7, 8 hold for $n = 0$. We claim that 11 also holds. As $v_0(x) = (1, 0)$, $u_0(x') = (1, 0)$ and $s_0 = s'_0 = 0$, any vector of the form $v_0(x) + D(s'_0, s_0)$ is $(i, 1)$ for some $i \geq 0$. For this to also be in $[\mathbb{Z}^2 - C(r_1, r'_1)] + (1, 0)$, we must have $r_1(i-1) < 1$ or $r'_1(i-1) < 1$ which implies $i < r_1^{-1} + 1 < r_\infty^{-1} + 1 \leq s_\infty^{-1}$ or else $i < r_1'^{-1} + 1 < r_\infty^{-1} + 1 \leq s_\infty^{-1}$. This then implies that $s_1 i \leq 1$ and $s'_1 i \leq 1$. This implies that $(-i, -1)$ is in $C(s_1, s'_1)$ and the conclusion follows.

Next, we assume that ξ^{n-1} and η^{n-1} have been chosen, for some $n \geq 1$ and the conditions above are satisfied. We describe the process of selecting ξ^n and η^n.

The following result follows at once from the definition of u_{n-1}, v_{n-1} and U_{n-1}. We omit the proof.

Lemma 6.1. *For all x in X_{n-1} and k in $\mathbb{Z}^2$ with $|k| \leq 1$, $\sigma^{n-1}(\varphi^k_{n-1}(x)) = \varphi^l(\sigma^{n-1}(x))$, for some l in $\mathbb{Z}^2$ with $|l| \leq U_{n-1}$.*

First, we select M such that the following holds

$$M \geq 2(n + 2diam(R_{n-1}) + U_{n-1}) + 1.$$

We apply 5.19 to ξ^{n-1}, η^{n-1}, to find M sufficiently large so that, if θ is any cocycle,

$$\theta \leq M^{-1} \quad \text{implies} \quad \theta_{R_{n-1}} \leq 4^{-1}. \tag{5}$$

Observe that the sets $D(s_{n-1}, s'_{n-1})$ and $\mathbb{Z}^2 - C(s_n, s'_n)$ satisfy the hypotheses of 5.2. We require that $M - 2U_{n-1}$ is a $(n + diam(R_{n-1}))$-separating constant for these. The sets $C(s_n, s'_n)$ and $\mathbb{Z}^2 - C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$ satisfy the hypothesis of 5.2 and we require that $M - U_{n-1}$ is a U_{n-1}-separating constant for these. The sets $-C(s_n, s'_n)$ and $\mathbb{Z}^2 - [-C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)]$ satisfy the hypothesis of 5.2 and we require that $M - U_{n-1}$ is a U_{n-1}-separating constant for these.

Then we choose ξ to be any cocycle which is strictly positive on $C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)$ and so that $\xi \leq M^{-1}$. Note that $C(r_n, r'_n)$ is contained in $C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)$ and so ξ is also strictly positive there.

Lemma 6.2. *The cocycle $\xi_{R_{n-1}}$ for the system (X_{n-1}, φ_{n-1}) is strictly positive on Q_4 and $\xi_{R_{n-1}} \leq 4^{-1}$.*

Proof. It suffices to check positivity on the generators $(1,0), (0,-1)$ of Q_4. Using the definition of $\xi_{R_{n-1}}$, we have, for any x in X_{n-1},

$$\begin{aligned} \xi_{R_{n-1}}(x, \varphi^{(1,0)}_{n-1}(x)) &= \xi(\sigma^{n-1}(x), \sigma^{n-1}(\varphi^{(1,0)}_{n-1}(x))) \\ &= \xi(\sigma^{n-1}(x), \varphi^{u_{n-1}(x)}(\sigma^{n-1}(x))) \end{aligned}$$

By induction hypothesis 9, $u_{n-1}(x)$ is in $D(s_{n-1}, s'_{n-1})$ which is contained in $C(r_n, r'_n)$ and hence the value of ξ above is non-negative. For x in X_{n-1}, we let $x' = \varphi^{(0,-1)}_{n-1}(x)$ so that

$$\begin{aligned} \xi_{R_{n-1}}(x, \varphi^{(0,-1)}_{n-1}(x)) &= -\xi_{R_{n-1}}(\varphi^{(0,-1)}_{n-1}(x), x) \\ &= -\xi_{R_{n-1}}(x', \varphi^{(0,1)}_{n-1}(x')) \\ &= -\xi(\sigma^{n-1}(x'), \sigma^{n-1}(\varphi^{(0,1)}_{n-1}(x'))) \\ &= -\xi(\sigma^{n-1}(x'), \varphi^{v_{n-1}(x')}(\sigma^{n-1}(x'))) \end{aligned}$$

By induction hypothesis 9, we have $v_{n-1}(x')$ is in $D(r_{n-1}, r'_{n-1})$ which is contained in $-C(r_n, r'_n)$ and hence the value of ξ is non-positive. To see that $\xi_{R_{n-1}}$ is proper on Q_4, we have, for any x in X_{n-1} and $i, j \geq 0$,

$$\xi_{R_{n-1}}(x, \varphi^{(i,-j)}(x)) = \xi(\sigma^{n-1}(x), \varphi^u(\sigma^{n-1}(x)))$$

where u is a sum of i vectors of the form $u_{n-1}(\cdot)$ and j vectors of the form $v_{n-1}(\cdot)$. It follows that as $i+j$ tends to infinity, so does the vector u. Moreover, as $u_{n-1}(\cdot)$ and $v_{n-1}(\cdot)$ are in $C(r_n, r'_n)$, so is u. Properness follows.

The fact that $\xi_{R_{n-1}} \leq 4^{-1}$ follows from the choice of M and $\xi \leq M^{-1}$. $\square$

Having established this result, we now have the quotient $(X_{n-1})_{\xi_{R_{n-1}} \times \eta_0}$, the quotient map $\pi_{\xi_{R_{n-1}} \times \eta_0}$ from X_{n-1} to this space. Moreover, we find standard liftings $\sigma_{\xi_{R_{n-1}} \times \eta_0}$ and $\tau_{\xi_{R_{n-1}} \times \eta_0}$ as in Definition 5.21. For notational purposes, it will be useful to let

$$\begin{aligned}
Y &= (X_{n-1})_{\xi_{R_{n-1}} \times \eta_0}, \\
\psi &= (\varphi_{n-1})_{\xi_{R_{n-1}} \times \eta_0}, \\
\pi &= \pi_{\xi_{R_{n-1}} \times \eta_0}, \\
\sigma &= \sigma_{\xi_{R_{n-1}} \times \eta_0} \\
\tau &= \tau_{\xi_{R_{n-1}} \times \eta_0}
\end{aligned}$$

We let S denote $\ker(\xi^{R_{n-1}} \times \eta^{n-1})$. First, notice that $S \supset R_{n-1}$ since $R_{n-1} = \ker(\xi^{n-1} \times \eta^{n-1})$ and $\xi^{R_{n-1}}(R_{n-1}) = 0$. Next, we have $S = (\pi^{n-1} \times \pi^{n-1})^{-1}(\ker(\xi_{R_{n-1}} \times \eta^{n-1}_{R_{n-1}}))$. By Lemma 5.18, we have $\eta^{n-1}_{R_{n-1}} = \eta_0$ and $\ker(\xi_{R_{n-1}} \times \eta_0)$ is a compact open subequivalence relation of $R_{\varphi_{n-1}}$ and $\xi_{R_{n-1}} \times \eta_0$ is surjective by Lemma 5.20 . It follows that S is a compact open equivalence relation and $\xi^{R_{n-1}} \times \eta^{n-1}$ is surjective. We note that we have a canonical lift $\sigma^{n-1} \circ \sigma_{\xi_{R_{n-1}} \times \eta_0}$ for the quotient map from X to X/S.

Now we observe that as ξ is proper on $C(r_n, r'_n)$, it is also proper on $D(s_n, s'_n)$. Therefore, we may find a constant K such that, if $|\xi(x,k)| \leq 1$, then $|k| \leq K$.

We select N such that the following hold. First of all, we have

$$N \geq 2(n + diam(S) + U_{n-1}) + 1 \tag{6}$$

$$N \geq diam(S) + U_{n-1} + K \tag{7}$$

and, by application of 5.19 to $\xi^{R_{n-1}}, \eta^{n-1}$, if θ is any cocycle

$$\theta \leq N^{-1} \quad \text{implies} \quad \theta_S \leq 4^{-1}. \tag{8}$$

The sets $Q_1 \cap [\mathbb{Z}^2 - C(r_{n+1}, r'_{n+1})]$ and $\mathbb{Z}^2 - \left[-C(s_{n+1}, s'_{n+1})\right]$ satisfy the hypothesis of Lemma 5.2 and we choose N such that $N - U_{n-1} - diam(S)$ is K-separating for these sets. The sets $\mathbb{Z}^2 - [C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)]$ and $C(r_n, r'_n)$ also satisfy the hypothesis and we require that $N - U_{n-1}$ be a K-separating constant of these sets. The same also hold for the sets $\mathbb{Z}^2 - [-C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)]$ and $-C(r_n, r'_n)$

We choose η to be any cocycle which is strictly positive on $-C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$ and such that $\eta \leq N^{-1}$. Note that as $-C(s_n, s'_n)$ is contained in $-C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$, η is also strictly positive there.

We now define new cocycles ξ' and η' for X, φ by

$$\xi' = \xi^{R_{n-1}}, \eta' = \eta^S.$$

Specifically, we have

$$\begin{aligned} \xi'(x,y) &= \xi(\sigma^{n-1} \circ \pi^{n-1}(x), \sigma^{n-1} \circ \pi^{n-1}(y)), \\ \eta'(x,y) &= \eta(\sigma^{n-1} \circ \sigma \circ \pi \circ \pi^{n-1}(x), \sigma^{n-1} \circ \sigma \circ \pi \circ \pi^{n-1}(y)), \end{aligned}$$

for all (x, y) in R_φ. Note that in the case x, y are in the range of σ^{n-1}, we have $\xi(x, y) = \xi'(x, y)$. Also, if x, y are in the range of $\sigma^{n-1} \circ \sigma$, we have $\eta'(x, y) = \eta(x, y)$. We also note that $(\xi')_{R_{n-1}} = \xi_{R_{n-1}}$ and $(\eta')_S = \eta_S$.

Lemma 6.3. *The pair $(\xi')_{R_{n-1}}, (\eta')_{R_{n-1}}$ is a weak standard pair for the system (X_{n-1}, φ_{n-1}).*

Proof. We have already established that $(\xi')_{R_{n-1}} = \xi_{R_{n-1}}$ is strictly positive on Q_4. We apply Lemma 5.15 to note that $(\eta^S)_{R_{n-1}} = (\eta_{R_{n-1}})^{S_{R_{n-1}}}$. We also note that $S = \ker(\xi^{R_{n-1}} \times \eta^{n-1})$ so $S_{R_{n-1}} = \ker((\xi^{R_{n-1}})_{R_{n-1}} \times (\eta^{n-1})_{R_{n-1}})$. By Lemma 5.15, we have $(\xi^{R_{n-1}})_{R_{n-1}} = \xi_{R_{n-1}}$ and by Lemma 5.18, $(\eta^{n-1})_{R_{n-1}} = \eta_0$.

We now consider the desired positivity of $(\eta^S)_{R_{n-1}}$. It follows from the definitions that $((\eta^S)_{R_{n-1}})_{\xi_{R_{n-1}} \times \eta_0} = \eta_S$. Let x be in Y and $x' = \psi^{(-1,0)}(x)$ so that

$$\eta_S(x, \psi^{(-1,0)}(x)) = -\eta_S(x', \psi^{(1,0)}(x')).$$

It follows from Lemma 5.23 that for some $i \geq 0$, we have

$$\begin{aligned} \sigma(\psi^{(1,0)}(x')) &= \varphi_{n-1}^{(i,0)}(\sigma(x')) \\ \sigma^{n-1}(\sigma(\psi^{(1,0)}(x'))) &= \varphi^u(\sigma^{n-1} \circ \sigma(x')) \end{aligned}$$

where u is the sum of i vectors of the form $u_{n-1}(\cdot)$. By hypothesis 7, u is in $D(s_{n-1}, s'_{n-1})$ and hence also in $C(s_n, s'_n)$. The desired positivity of η_S follows since η is positive on $-C(s_n, s'_n)$.

Now, we consider the other generator, $(0,1)$. Again using Lemma 5.23, we have that, for any x in Y, there is $j \geq 0$ such that

$$\sigma(\psi^{(0,1)}(x)) = \varphi_{n-1}^{(j,1)}(\sigma(x))$$

and hence

$$\sigma^{n-1}(\sigma(\psi^{(0,1)}(x))) = \varphi^{v}(\sigma^{n-1} \circ \sigma(x))$$

where v is the sum of a vector of the form $v_{n-1}(\cdot)$ and j vectors of the form $u_{n-1}(\cdot)$. From Lemma 5.22, we also know that

$$\begin{aligned} -1 &= \xi_{R_{n-1}}(\sigma(x), \varphi_{n-1}^{(j-1,1)}(\sigma(x))) \\ &= \xi(\sigma^{n-1}(\sigma(x)), \sigma^{n-1}(\varphi_{n-1}^{(j-1,1)}(\sigma(x))) \\ &= \xi(\sigma^{n-1}(\sigma(x)), \varphi^{v'}(\sigma^{n-1}(\sigma(x))) \end{aligned}$$

where v' is of the form $v - u_{n-1}(\cdot)$, with v as above. It follows from the positivity of ξ that v' is not in $C(r_n, r'_n)$. The vector v is in both $[\mathbb{Z}^2 - C(r_n, r'_n)] + u_{n-1}(\cdot)$ and in $v_{n-1}(\cdot) + D(s_{n-1}, s'_{n-1})$. It follows from hypothesis 11 that v is in $-C(s'_n, s_n)$. The desired positivity follows from that of η on $-C(s'_n, s_n)$.

Finally, we must check that $\xi_{R_{n-1}} \times (\eta^S)_{R_{n-1}}$ does not take the values $\pm(1,1)$ on $X_{n-1} \times B(0,1)$. Let x be in X_{n-1}. It suffices to consider the vectors $(1,0), (1,-1), (0,-1), (1,1)$ in $B(0,1)$. First, we consider ε to be one of $(1,0), (1,-1)$ or $(0,-1)$. Each of these is in the fourth quadrant and so the value of $\xi_{R_{n-1}}(x,\varepsilon)$ is non-negative. We conclude that $\xi_{R_{n-1}} \times (\eta^S)_{R_{n-1}}(x,\varepsilon) \neq (-1,-1)$. Next, we observe that $\eta_0(x,\varepsilon)$ is non-positive. Thus, $\xi_{R_{n-1}} \times \eta_0(x,\varepsilon)$ is again in the fourth quadrant. Since the cocycle $(\eta^S)_{R_{n-1}}$ is constant on $\ker(\xi_{R_{n-1}} \times \eta_0)$ and is non-positive on the fourth quadrant for the system Y, ψ, we see that $(\eta^S)_{R_{n-1}}(x,\varepsilon)$ is non-positive. We conclude that $\xi_{R_{n-1}} \times \eta_{R_{n-1}}(x,\varepsilon) \neq (1,1)$. It remains to check the case $\epsilon = (1,1)$. We use that fact that $\xi_{R_{n-1}} \times \eta_{R_{n-1}}(x,(1,1))$ is of the form $\xi \times \eta(\sigma^{n-1}(x), u_{n-1}(\cdot) + v_{n-1}(\cdot))$ and by hypothesis 9, $u_{n-1}(\cdot) + v_{n-1}(\cdot)$ is in $D(r_{n-1}, r'_{n-1})$ which is contained in $-C(r_n, r'_n)$ and in $-C(s_n, s'_n)$. It follows that the value of $\xi_{R_{n-1}}(x,(1,1))$ is non-positive and that of $(\eta^S)_{R_{n-1}}(x,(1,1))$ is non-negative. This completes the proof. $\square$

We are now ready to define our cocycles ξ^n and η^n. Let R' denote the equivalence relation on X_{n-1} determined by the pair $\xi_{R_{n-1}}, (\eta^S)_{R_{n-1}}$ as in Lemma 5.26. We use the standard lifting for the quotient of 5.26 to define

$$\xi^n = \xi^{(\pi^{n-1}\times\pi^{n-1})^{-1}(R')}, \eta^n = \eta'.$$

Proposition 6.4. *The pair (ξ^n, η^n) satisfy properties 1, 2, 3, 4 and 5.*

Proof. Property 5 follows at once from our construction of ξ^n, η^n and Lemma 5.26. Properties 1 and 2 follow from this and Proposition 5.25. Property 3 follows at once from the definitions. It remains to show 4.

We first claim that $diam((\pi^{n-1} \times \pi^{n-1})^{-1}(R')) \leq 2diam(R_{n-1}) + U_{n-1}$.

Let x be in X and k be in $\mathbb{Z}^2$. Suppose that $(\pi^{n-1}(x), \pi^{n-1}(\varphi^k(x)))$ is in R'. It follows from the definition of R' given in Lemma 5.26 that either we have $\pi^{n-1}(x) = \pi^{n-1}(\varphi^k(x))$, in which case $|k| \leq diam(R_{n-1})$, or else $\pi^{n-1}(x) = \varphi^{\varepsilon}_{n-1}(\pi^{n-1}(\varphi^k(x)))$, for $\varepsilon = \pm(0,1)$. In the latter case, we apply σ^{n-1} to both sides and obtain $\varphi^l(x) = \varphi^v(\varphi^m(\varphi^k(x)))$, where v is of the form $v_{n-1}(\cdot)$ and $|l|, |m| \leq diam(R_{n-1})$. Then we have $l = v + m + k$ and hence $|k| \leq |v| + |l| + |m| \leq U_{n-1} + 2diam(R_{n-1})$, which completes the proof of the claim.

Property 4 then follows from an application of Proposition 5.16 (using $K = n$). ∎

We define $X' = X/\ker(\xi' \times \eta') = X_{n-1}/\ker((\xi_{R_{n-1}})^{R'} \times \eta'_{R_{n-1}})$. We let $\pi' : X_{n-1} \to X'$ be the quotient map and σ' and τ' be the canonical liftings. Let φ' be the induced system on X'. Recall from the definitions given in subsection 5.7, that the quotient map π' is the composition of two maps, from X_{n-1} to $X_{n-1}/\ker\xi_{R_{n-1}} \times \eta_0$ and the second to X'. The first of these we are now denoting by π. Similarly the lifting σ' is the composition of two liftings for these same maps. The first lifting is now denoted σ. This means that

$$\sigma \circ \pi \circ \sigma' = \sigma'.$$

From Lemma 5.26, the maps $\pi_n \circ \sigma'$ and $\pi' \circ \sigma_n$ are conjugacies between (X_n, φ_n) and (X', φ'). We may define maps $u', v' : X' \to \mathbb{Z}^2$ by

$$\begin{aligned} \varphi^{u'(x)}(\sigma^{n-1} \circ \sigma'(x)) &= \pi' \circ \pi^{n-1}(\varphi_n^{(1,0)}(x)), \\ \varphi^{v'(x)}(\sigma^{n-1} \circ \sigma'(x)) &= \pi' \circ \pi^{n-1}(\varphi_n^{(0,1)}(x)), \end{aligned}$$

for all x in X.

Lemma 6.5. *We have*

$$\begin{aligned}\sigma^n(X_n) &= \sigma^{n-1}\circ\sigma'(X'),\\ \tau^n(X_n) &= \tau^{n-1}\circ\tau'(X'),\\ u_n(X_n) &= u'(X'),\\ v_n(X_n) &= v'(X'),\end{aligned}$$

Proof. We prove the first statement only; the others are similar. We have

$$\begin{aligned}\sigma^n(X_n) &= \sigma^{n-1}\circ\sigma_n(X_n)\\ &= \sigma^{n-1}\circ\sigma_n(\pi_n\circ\sigma'(X'))\\ &= \sigma\circ\sigma'(X').\end{aligned}$$

□

Lemma 6.6. *We have*

$$\xi' \le (M - 2diam(R_{n-1}))^{-1}, \eta' \le (N - 2diam(S))^{-1},$$

and

$$\xi' \le (n + 2U_{n-1})^{-1}, \eta' \le (n + 2U_{n-1})^{-1}.$$

Proof. The first two results follow from Lemma 5.14 and the definitions. The last two follow from the choice of $M \ge 2(n + diam(R_{n-1}) + U_{n-1}) + 1$ and $N \ge 2(n + diam(S) + U_{n-1}) + 1$. □

Lemma 6.7. *The pair* (ξ^n, η^n) *satisfies property 6.*

Proof. Property 6 involves only the ranges of the functions σ^n and τ^n. In view of 6.5, it suffices to show that ξ', η' satisfy 6. Consider x in $\sigma^{n-1}\circ\sigma'(X')$ and suppose that $\varphi^m(x)$ is in $\tau^{n-1}\circ\tau'(X')$, for some m in $\mathbb{Z}^2$. We consider the value of $\xi'(x, \varphi^m(x))$. If it is greater than 1 in absolute value, then since $\xi' \le (n + 2U_{n-1}))^{-1}$, $|m| \ge n$ as desired. Next, suppose the value is either 0 or 1. Let $x' = \sigma^{n-1}(\varphi_{n-1}^{(-1,0)}(\pi^{n-1}(x)))$. It follows from the choice of U_{n-1} that $x' = \varphi^k(x)$ for some k with $|k| \le U_{n-1}$. Consider

$$\xi'(x, x') = \xi_{R_{n-1}}(\pi^{n-1}(x), \pi^{n-1}(x')) = \xi_{R_{n-1}}(\pi^{n-1}(x), \varphi_{n-1}^{(-1,0)}(\pi^{n-1}(x))).$$

Note that $\pi^{n-1}(x)$ is in $\sigma'(X')$. We apply part 3 of Lemma 5.30 to conclude that $\xi'(x, x') = -1$. Next, let $x'' = \sigma^{n-1}(\varphi_{n-1}^{(1,0)}(\pi^{n-1}(\varphi^m(x))))$. Arguing in a similar way, we have $x'' = \varphi^l(\varphi^m(x))$ for some l with $|l| \leq U_{n-1}$. We note that $\pi^{n-1}(\varphi^m(x))$ is in $\tau'(X')$ and again we apply 5.30 to show that $\xi^n(\varphi^m(x), x'') = 1$. Putting this together, we have $x'' = \varphi^{-k+l+m}(x')$ and

$$\begin{aligned} \xi'(x', x'') &= \xi'(x', x) + \xi'(x, \varphi^m(x)) + \xi'(\varphi^m(x), x'') \\ &= \xi'(x, \varphi^m(x)) + 2 \\ &\geq 2. \end{aligned}$$

As $\xi' \leq (n + 2U_{n-1})^{-1}$, we conclude that $|-k+l+m| \geq n + 2U_{n-1}$. Since $|k|, |l| \leq U_{n-1}$, this then implies that $|m| \geq n$ as desired. The only remaining case is $\xi'(x, \varphi^m(x)) = -1$. We consider the value of $\eta'(x, \varphi^m(x))$. Arguing in a completely analogous way, we see that $|m| \geq n$, except for the case that $\eta(x, \varphi^m(x)) = -1$.

We are left to consider the case that $\xi'(x, \varphi^m(x)) = \eta'(x, \varphi^m(x)) = -1$. We let $x_1 = \pi^{n-1}(\varphi^m(x)), x_2 = \sigma(\pi(x_1))$ which are in X_{n-1}. We note that $\pi^{n-1}(\sigma^{n-1}(x_1)) = x_1 = \pi^{n-1}(\varphi^m(x))$ and so $\sigma^{n-1}(x_1)$ and $\varphi^m(x)$ are in the same R_{n-1} equivalence class and so

$$\sigma^{n-1}(x_1) = \varphi^k(\varphi^m(x))$$

for some k with $|k| \leq diam(R_{n-1})$. As $\varphi^m(x)$ is in the range of $\tau^{n-1} \circ \tau'$, we have x_1 is in the range of τ' and hence in the range of τ and $x_1 = \tau \circ \pi(x_1)$. We also have $x_2 = \sigma \circ \pi(x_1)$. In particular, x_1 and x_2 are in the same $\ker(\xi_{R_{n-1}} \times \eta_0)$ class. Moreover, by Lemma 5.20 and the definitions of τ and σ, we have $x_2 = \varphi_{n-1}^{(-i,0)}(x_1)$, for some $i \geq 1$. Applying σ^{n-1} to both sides of this equation, we have

$$\sigma^{n-1}(x_2) = \varphi^{-u}(\sigma^{n-1}(x_1)), \tag{9}$$

where u is the sum of i vectors in $D(s_{n-1}, s'_{n-1})$, and hence is also in $D(s_{n-1}, s'_{n-1})$.

We next consider

$$\begin{aligned} \xi_{R_{n-1}}(\varphi_{n-1}^{(-1,0)}(x_2), \varphi_{n-1}^{(1,0)}(x_1)) &= -\xi_{R_{n-1}}(x_2, \varphi_{n-1}^{(-1,0)}(x_2)) \\ &\quad + \xi_{R_{n-1}}(x_2, x_1) + \xi_{R_{n-1}}(x_1, \varphi_{n-1}^{(1,0)}(x_1)). \end{aligned}$$

The middle term is 0, because x_1 and x_2 are in the same $R_{\xi_{R_{n-1}} \times \eta_0}$ class. The first and third terms are both 1, by Lemma 5.22. On the other hand we may use the definition of ξ_n to compute

$$\begin{aligned}
\xi_{R_{n-1}}(\varphi_{n-1}^{(-1,0)}(x_2), \varphi_{n-1}^{(1,0)}(x_1)) &= \xi(\sigma^{n-1}(\varphi_{n-1}^{(-1,0)}(x_2)), \sigma^{n-1}(\varphi_{n-1}^{(1,0)}(x_1))) \\
&= \xi(\varphi^{j_2}(\sigma^{n-1}(x_2)), \varphi^{j_1}(\sigma^{n-1}(x_1))) \\
&= \xi(\varphi^{j_2-u}(\sigma^{n-1}(x_1)), \varphi^{j_1}(\sigma^{n-1}(x_1))),
\end{aligned}$$

for some j_1, j_2 with $|j_1|, |j_2| \leq U_{n-1}$. Now, $\xi \leq M^{-1}$ implies that $|j_1 - j_2 + u| \geq M$, which in turn implies that $|u| \geq M - 2U_{n-1}$.

Now, we use that fact that $\eta'(x, \varphi^m(x)) = -1$. We note that x is in the range of $\sigma^{n-1} \circ \sigma'$ so that, from the definition of η', we have

$$\begin{aligned}
1 &= \eta'(\varphi^m(x), x) \\
&= \eta(\sigma^{n-1} \circ \sigma \circ \pi \circ \pi^{n-1}(\varphi^m(x)), x) \\
&= \eta(\sigma^{n-1}(x_2), x) \\
&= \eta(\varphi^{-u}(\sigma^{n-1}(x_1)), x) \\
&= \eta(\varphi^{-u+k+m}(x), x).
\end{aligned}$$

From the positivity of η, we see that $u - k - m$ is not in $-C(s_n, s'_n)$ and hence $-u + K + m$ is in $\mathbb{Z}^2 - C(s_n, s'_n)$. Now u is in $D(s_{n-1}, s'_{n-1})$ and $|u| \geq M - 2diam(R_{n-1})$, which is a $(n + diam(R_{n-1}))$-separating constant for $D(s_{n-1}, s'_{n-1})$ and $\mathbb{Z}^2 - C(s_n, s'_n)$. We conclude that $|k+m| \geq n + diam(R_{n-1})$. Recall that $|k| \leq diam(R_{n-1})$ which then implies that $|m| \geq n$ as desired. This completes the proof. □

Lemma 6.8. *Let x be in $\sigma^n(X_n)$. There exists k, l in $\mathbb{Z}^2$ with $|k|, |l| \leq U_{n-1}$ such that*

$$\xi(x, \varphi^k(x)) = -1, \eta(x, \varphi^l(x)) \leq -1.$$

Proof. By Lemma 6.5, we have $x = \sigma^{n-1} \circ \sigma'(x')$, for some x' in X'. Applying part (3) of Lemma 5.30 to the system X_{n-1}, φ_{n-1} and the standard pair $\xi_{R_{n-1}}, (\eta')_{R_{n-1}}$. we have

$$\begin{aligned}
-1 &= \xi_{R_{n-1}}(\sigma'(x'), \varepsilon_4) \\
&= \xi(\sigma^{n-1}(\sigma'(x')), \sigma^{n-1}(\varphi_{n-1}^{\varepsilon_4}(\sigma'(x')))) \\
&= \xi(x, \varphi^k(x)),
\end{aligned}$$

where k is in $\mathbb{Z}^2$ with $|k| \leq U_{n-1}$, by Lemma 6.1.

For the second part, we begin again with Lemma 5.30 to assert that

$$\begin{aligned}
-1 &= (\eta')_{R_{n-1}}(\sigma'(x'), \varepsilon_3) \\
&= \eta(\sigma^{n-1} \circ \sigma \circ \pi(\sigma'(x')), \sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')))) \\
&= \eta(\sigma^{n-1} \circ \sigma'(x'), \sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')))) \\
&= \eta(x, \sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')))).
\end{aligned}$$

By Lemma 6.1, we have

$$\sigma^{n-1}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')) = \varphi^l(\sigma^{n-1} \circ \sigma'(x')) = \varphi^l(x),$$

for some l with $|l| \leq U_{n-1}$. Next, we note that

$$\sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))) = \varphi_{n-1}^{(-i,0)}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')),$$

by 5.21, where $i \geq 0$. Applying σ^{n-1} to both sides and using 7 again, we obtain

$$\sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))) = \varphi^{-u} \circ \sigma^{n-1}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')),$$

where u is the sum of i vectors in $D(s_{n-1}, s'_{n-1})$ and hence is in $D(s_{n-1}, s'_{n-1})$. Putting this together, we have

$$\begin{aligned}
\eta(x, \varphi^l(x)) &= \eta(x, \sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))) \\
&\quad + \eta(\sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))), \varphi^l(x)) \\
&= \eta(\sigma^{n-1} \circ \sigma \circ \pi(x'), \sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))) \\
&= +\eta(\sigma^{n-1} \circ \sigma \circ \pi(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))), \sigma^{n-1}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')))) \\
&= -1 - \eta(\sigma^{n-1}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x'))), \varphi^{-u} \circ \sigma^{n-1}(\varphi_{n-1}^{\varepsilon_3}(\sigma'(x')))) \\
&\leq -1,
\end{aligned}$$

since $-u$ is in $-D(s_{n-1}, s_{n-1})$ which is contained in $-C(s_n, s'_n)$, where η is positive. $\square$

Lemma 6.9. *For any x in X_n, $|u_n(x)| \geq M - U_{n-1}$ and $|v_n(x)| \geq N - diam(S) - U_{n-1}$. Moreover, $u_n(x)$ is not in $-C(r_n, r'_n)$ and $v_n(x)$ is not in $C(s_n, s'_n)$.*

Proof. As noted before, it suffices to show the result for $u'(x')$ and $v'(x')$, with x' in X', instead of $u_n(x), v_n(x)$. First, we use from the definition of u', we have

$$\sigma^{n-1} \circ \sigma'(\varphi'^{(1,0)}(x')) = \varphi^{u'(x')}(\sigma^{n-1} \circ \sigma'(x')).$$

Let $x = \sigma^{n-1} \circ \sigma'(x')$. From the definitions of ξ' and the action φ', we have

$$\xi'(x, \varphi^{u'(x')}(x)) = \xi'(\sigma^{n-1} \circ \sigma'(x'), \sigma^{n-1} \circ \sigma'(\varphi'^{(1,0)}(x))) = 1.$$

Next, we note that x and $\varphi^{u'(x')}(x)$ are in the range of σ^{n-1}. This implies that

$$1 = \xi'(x, \varphi^{u'(x')}(x)) = \xi(x, \varphi^{u'(x')}(x))$$

It follows at once from the positivity of ξ that $u_n(x)$ is not in $-C(r_n, r'_n)$.

By Lemma 6.8, there is k with $|k| \leq U_{n-1}$ such that $\xi(x, \varphi^k(x)) = -1$. Then by the cocycle condition

$$\xi(\varphi^k(x), \varphi^{u'(x')}(x)) = 2.$$

Since $\xi \leq M^{-1}$, it follows that $|-k + u'(x')| > M$ and the conclusion follows from this. □

Lemma 6.10. *For all x in X_n, $u_n(x)$ is in $D(s_n, s'_n)$ and $|u_n(x)| \leq K$.*

Proof. Again, it suffices show that for x' in X', $u'(x')$ is in $D(s_n, s'_n)$. Let $x = \sigma^{n-1} \circ \sigma'(x')$. Then by the definition of $u'(x')$, we have $\varphi^{u'(x')}(x) = \sigma^{n-1} \circ \sigma'(\varphi'^{(1,0)}(x'))$. In particular, $\varphi^{u'(x')}(x)$ is in the range of $\sigma^{n-1} \circ \sigma'$. By the definition of the action φ', we have $\eta'(x, \varphi^{u'(x')}(x)) = 0$ and $\xi'(x, \varphi^{u'(x')}(x')) = 1$. Since both arguments are in the image of $\sigma^{n-1} \circ \sigma'$, we also have $\eta(x, \varphi^{u'(x')}(x)) = 0$ and $\xi(x, \varphi^{u'(x')}(x)) = 1$. It follows from the definition of K that $|u'(x')| \leq K$. Using Lemma 6.8, we may find l with $|l| \leq U_{n-1}$ and $\eta(x, \varphi^l(x)) < 0$. Then by the cocycle condition, we have

$$\eta(\varphi^l(x), \varphi^{u'(x')}(x)) > 0.$$

From the positivity of η, it then follows that $u_n - l$ is in $\mathbb{Z}^2 - C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$. As $|u'(x')| \geq M - U_{n-1}$ by Lemma 6.9 and since $M - U_{n-1}$ was chosen to be U_{n-1}-separating for $\mathbb{Z}^2 - C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$ and $C(s_n, s'_n)$, it follows that $u'(x')$ is not in $C(s_n, s'_n)$. By reversing the roles of x and $\varphi^{u'(x')}(x)$ and arguing in a completely analogous fashion, using a value of l' with $|l'| \leq U_{n-1}$ we have $\eta(x, \varphi^{u'(x')+l'}(x)) < 0$. It follows that $u'(x') + l'$ is in $\mathbb{Z}^2 - [-C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)$. As $M - U_{n-1}$ was chosen to be U_{n-1}-separating for $\mathbb{Z}^2 - [-C((2s_n + s'_n)/3, (s_n + 2s'_n)/3)]$ and $-C(s_n, s'_n)$ it follows that $u'(x')$ is not in $-C(s_n, s'_n)$. Since $u'(x')$ is in neither $\pm C(s_n, s'_n)$, nor $-C(r_n, r'_n)$ from the last statement of Lemma 6.9, we have $u'(x')$ is in $D(s_n, s'_n)$. □

Lemma 6.11. *Let x be in X_n and let u be any element of $\mathbb{Z}^2$ with $|u| \leq K$. Then $v_n(x) + u$ is in $D(r_n, r'_n)$. In particular, conditions 8 and 9 hold.*

Proof. Again, we show the conclusion for the function $v'(\cdot)$ rather than $v_n(\cdot)$. Let x' be in X' and let $x = \sigma^{n-1} \circ \sigma'$. By definition, we have

$$\begin{aligned} \xi(x, \varphi^{v(x')}(x)) &= \xi(\sigma^{n-1} \circ \sigma'(x'), \varphi^{v(x')} \circ \sigma^{n-1} \circ \sigma'(x')) \\ &= \xi(\sigma^{n-1} \circ \sigma'(x') \circ \varphi'^{(0,1)}(x')). \end{aligned}$$

By Lemma 6.8, we my find k and l with $|k|, |l| \leq U_{n-1}$ and

$$\xi(x, \varphi^k(x)) = \xi^n(\varphi^{v'(x')}(x), \varphi^{v'(x')+l}(x)) = -1.$$

It follows that

$$\xi(\varphi^{v'(x')}(x), \varphi^k(x)) = -1, \xi(x, \varphi^{v'(x')+l}(x)) = -1.$$

It follows from the positivity of ξ that $k - v'(x')$ and $v'(x') - l$ are in $\mathbb{Z}^2 - C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)$. As $N - U_{n-1}$ is a K-separating constant for the pair $\mathbb{Z}^2 - [C((2r_n + r'_n)/3, (r_n + 2r'_n)/3)]$ and $C(r_n, r'_n)$, we have that none of $v'(x') + u$, $-v'(x') - u$ nor $v'(x')$ is in $C(r_n, r'_n)$. From Lemma 6.9, $v'(x')$ is not in $C(s_n, s'_n)$. It follows that $v'(x')$ is in $D(r_n, r'_n)$. By Lemma 6.9, $|v'(x')| \geq N - diam(S) - U_{n-1} \geq K$ and it follows that $v'(x') + u$ is also in $D(r_n, r'_n)$. ☐

Lemma 6.12. *Condition 11 holds for n.*

Proof. Let v be in

$$[v'(x') + D(s_n, s'_n)] \cap [u'(x'') + [\mathbb{Z}^2 - C(r_{n+1}, r'_{n+1})]]$$

for some x', x'' in X'. Since $v'(x')$ is in $D(r_n, r'_n)$, v is in the first quadrant and $|v| \geq |v'(x')| \geq N - diam(s) - U_{n-1}$. By Lemma 6.10, we have $|u'(x'')| \leq K$. The constant $N - diam(S) - U_{n-1}$ is a K separating constant for $Q_1 \cap [\mathbb{Z}^2 - C(r_{n+1}, r'_{n+1})]$ and $\mathbb{Z}^2 - [-C(s_{n+1}, s'_{n+1})]$; $v = v'(x') - u'(x'')$ being in $\mathbb{Z}^2 - [-C(s_{n+1}, s'_{n+1})]$ would then be a contradiction. The conclusion follows. ☐

From 6.4 and Proposition 5.7, we have the following result.

Theorem 6.13. *Let X, φ be a minimal free Cantor $\mathbb{Z}^2$ system. Given a sequence of pairs of cocycles (ξ^n, η^n) satisfying the conditions above, the relation $R = \cup_{n=1}^{\infty} R_n$ is a minimal AF-relation.*

7 The boundary of the relation R

In this section, we define what we will refer to as the boundary of the relation R.

Let X, φ be a free Cantor minimal $\mathbb{Z}^2$-system. Let (ξ^n, η^n) be a sequence of pairs of cocycles satisfying the conditions of Section 6. For each n, we let

$$\xi_n = (\xi^n)_{R_{n-1}}, \eta_n = (\eta^n)_{R_{n-1}}$$

which are a standard pair for the system (X_{n-1}, φ_{n-1}). We let E_n denote the function $E(\cdot)$ of Definition 5.33 on X_{n-1} for the cocycles ξ_n, η_n. We will also define the following. We define, for each x in X and $n \geq 1$,

$$E^n(x) = \xi^n \times \eta^n(x, E \cup E^*) - \{0\}.$$

For convenience, we set $E^0(x) = E \cup E^*$, for all x in X.

Lemma 7.1. *For each x in X and $n \geq 1$, we have*

$$E^n(x) = \xi_n \times \eta_n(\pi^{n-1}(x), E^{n-1}(x)) - \{0\}$$

and

$$E^n(x) \subset E_n(\pi^{n-1}(x)).$$

If $\#E^n(x) = 2$, then $E^n(x) = E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some $1 \leq i \leq 6$.

Proof. For $1 \leq i \leq 6$ and $n \geq 1$, define $\varepsilon_i^n = \xi^n \times \eta^n(x, \varepsilon_i)$. It follows from the definition of the quotient system X_n, φ_n that $\varphi_n^{\varepsilon_i^n}(\pi^n(x)) = \pi^n(\varphi^{\varepsilon_i}(x))$. By definition, we have $E^n(x) = \{\varepsilon_i^n \mid 1 \leq i \leq 6\} - \{0\}$. From the definition of ξ_n and η_n, we also have $\varepsilon_i^n = \xi_n \times \eta_n(\pi^{n-1}(x), \pi^{n-1}(\varphi^{\varepsilon_i}(x)))$. Combining these gives

$$\begin{aligned}
E^n(x) &= \{\varepsilon_i^n \mid 1 \leq i \leq 6\} - \{0\} \\
&= \{\xi_n \times \eta_n(\pi^{n-1}(x), \pi^{n-1}(\varphi^{\varepsilon_i}(x))) \mid 1 \leq i \leq 6\} - \{0\} \\
&= \{\xi_n \times \eta_n(\pi^{n-1}(x), \varphi^{\varepsilon_i^{n-1}}(\pi^{n-1}(x))) \mid 1 \leq i \leq 6\} - \{0\} \\
&= \{\xi_n \times \eta_n(\pi^{n-1}(x), \varepsilon_i^{n-1}) \mid 1 \leq i \leq 6\} - \{0\} \\
&= \{\xi_n \times \eta_n(\pi^{n-1}(x), \varepsilon) \mid \varepsilon \in E^{n-1}(x) - \{0\}\} - \{0\}
\end{aligned}$$

and the conclusion for the first statement follows.

The second part follows at once from the first and the fact that $E^{n-1}(x) \subset E \cup E^*$.

For the final part, if $\#E^n(x) = 2$, then also $\#E_n(\pi^{n-1}(x)) \geq 2$, and by part 2 of 5.34 it follows that $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\} = E^n(x)$, for some $1 \leq i \leq 6$. Similarly, we have $E_{n+1}(\pi^n(x)) = E^{n+1}(x) = \{\varepsilon_j, \varepsilon_{j-1}\}$, for some j. Then we have

$$\begin{aligned} \{\varepsilon_j, \varepsilon_{j-1}\} &= E^{n+1}(x) \\ &= \xi_{n+1} \times \eta_{n+1}(\pi^n(x), E^n(x)) \\ &= \xi_{n+1} \times \eta_{n+1}(\pi^n(x), \{\varepsilon_i, \varepsilon_{i-1}\}). \end{aligned}$$

It follows from part 2 of Lemma 5.31 that $i = j$. Continuing inductively establishes the last part. □

Lemma 7.2. *Let x be in X. The sequence $\#E^n(x), n \geq 2$, is a non-increasing sequence of integers, each at most 2. If it is constantly 2, then for some i, $E^n(x) = \{\varepsilon_i, \varepsilon_{i+1}\}$, for all $n \geq 1$. If there is $N \geq 1$ such that $\#E^n(x) = 1$ for all $n \geq N$, then either $E^n(x) \subset E$, for all $n \geq N$, or $E^n(x) \subset E^*$ for all $n \geq N$.*

Proof. The first part follows from the last two results and 5.32. This second part follows from 7.1 and 5.28. □

The difference between the relations R_φ and R is controlled by a 'boundary'. These sets fall into two groups, the boundary $B_i, 1 \leq i \leq 6$ and the corners $C_i, 1 \leq i \leq 6$.

Definition 7.3. *For each $1 \leq i \leq 6$, we define*

$$B_i = \{x \in X \mid \xi^1 \times \eta^1(x, \varepsilon_i) = \varepsilon_i, \#E^n(x) = 1, \text{ for all } n \geq 1\}$$

and

$$C_i = \{x \in X \mid E^n(x) = \{\varepsilon_i, \varepsilon_{i-1}\}, \text{ for all } n \geq 1\}.$$

Also, we let $B = \cup_{i=1}^3 B_i, B^* = \cup_{i=4}^6 B_i$.

The following alternate characterization of the C_i sets will be useful.

Lemma 7.4. *For each $1 \leq i \leq 6$, we have*

$$C_i = \{x \in X \mid E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}, \text{ for all } n \geq 1\}.$$

Proof. Suppose that x is in C_i. Then, $E^n(x) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for all $n \geq 1$ and it follows from Lemma 7.1 that $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for all $n \geq 1$.

Suppose that, for $n \geq 1$, we have
$E_n(\pi^{n-1}(x)) = E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$. We prove inductively that $E^n(x) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for all $n \geq 1$. It is true for $n = 1$ since $E^1(x) = E_1(\pi^0(x))$. Assume it is true for $n-1$, with $n \geq 2$. We know that

$$\{\varepsilon_i, \varepsilon_{i-1}\} = E_n(\pi^{n-1}(x)) = \xi_n \times \eta_n(\pi^{n-1}(x), E \cup E^*).$$

It follows from the last part of Lemma 5.32 that

$$\begin{aligned} \{\varepsilon_i, \varepsilon_{i-1}\} &= \xi_n \times \eta_n(\pi^{n-1}(x), \{\varepsilon_i, \varepsilon_{i-1}\}) - \{0\} \\ &= \xi_n \times \eta_n(\pi^{n-1}(x), E^{n-1}(x)) - \{0\} \\ &= E^n(x) \end{aligned}$$

using the induction hypothesis and the first part of Lemma 7.1. ☐

Lemma 7.5. *For all $1 \leq i \leq 6$, B_i and C_i are closed, pairwise disjoint subsets of X. Moreover, we have*

$$E^n(\varphi^{\varepsilon_i}(B_i)) = -E^n(B_i),$$

for all $n \geq 1$, and in particular,

$$\varphi^{\varepsilon_i}(B_i) = B_{i+3}.$$

Proof. The fact that the sets are closed follows from the continuity of the cocycles,and hence the continuity of the functions E^n. It is obvious from the definitions that the sets are pairwise disjoint.

It follows from the cocycle condition that, for any n, $\xi^n \times \eta^n(x, \varepsilon_i) = -\xi^n \times \eta^n(\varphi^{\varepsilon_i}(x), -\varepsilon_i)$. The last statement follows at once from this and $-\varepsilon_i = \varepsilon_{i+3}$. ☐

Lemma 7.6. *Let x be in X. Suppose that, for some $n \geq 1$ and $1 \leq i \leq 6$,*

$$E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\},$$

then

$$\begin{aligned} E_n(\varphi_{n-1}^{\varepsilon_i}(\pi^{n-1}(x))) &= \{\varepsilon_{i+4}, \varepsilon_{i+3}\}, \\ E_n(\varphi_{n-1}^{\varepsilon_{i-1}}(\pi^{n-1}(x))) &= \{\varepsilon_{i+2}, \varepsilon_{i+1}\}. \end{aligned}$$

In particular, we have

$$\begin{aligned}\varphi^{\varepsilon_i}(C_i) &= C_{i+4}\\ \varphi^{\varepsilon_{i-1}}(C_i) &= C_{i+2}.\end{aligned}$$

Proof. The first statement follows from Theorem 5.34. The last part follows at once from the first and Lemma 7.4. $\square$

Definition 7.7. *We define $\beta : B \to B^*$ by $\beta|B_i = \varphi^{\varepsilon_i}$, for $1 \leq i \leq 3$. We also define, for $i = 1, 4$, $\gamma_{i+2} : C_i \to C_{i+2}, \gamma_{i+4} : C_i \to C_{i+4}$ by $\gamma_{i+2} = \varphi^{\varepsilon_{i-1}}|C_i$ and $\gamma_{i+4} = \varphi^{\varepsilon_i}|C_i$.*

Lemma 7.8. 1. *For $n \geq 1$, x in X_{n-1} and m in $\mathbb{Z}^2$, we have*

$$|\xi_n \times \eta_n(x, m)| \leq \frac{|m| + 1}{2}.$$

2. *For $n \geq 1$, x in X and m in $\mathbb{Z}^2$, we have*

$$|\xi^n \times \eta^n(x, m)| \leq |\xi^{n-1} \times \eta^{n-1}(x, m)|,$$

with strict inequality if the right hand side is greater than 1.

3. *For x in X and m in $\mathbb{Z}^2$, the sequence $\xi^n \times \eta^n(x, m)$ is either eventually 0, eventually in E or eventually in E^*.*

Proof. For the first part, let $k = \frac{|m|}{2}$ if $|m|$ is even and $k = \frac{|m|+1}{2}$ if $|m|$ is odd. There is a path $0 = m_0, m_1, \ldots, m_k = m$, with $|m_i - m_{i-1}| \leq 2$, for all $1 \leq i \leq k$. Since ξ_n, η_n are small, we have

$$|\xi_n \times \eta_n(x, m)| = |\sum_i \xi_n \times \eta_n(\varphi_{n-1}^{m_{i-1}}(x), m_i - m_{i-1})| \leq \sum_i 1 = k.$$

The result follows.

For the second part, let $m_n = \xi^n \times \eta^n(x, m)$. By the definition of φ_n, this means that $\varphi_n^{m_n}(\pi^n(x)) = \pi^n(\varphi^m(x))$, for all n. Then, using the definition of ξ_n, η_n, we have

$$\begin{aligned}m_n &= \xi^n \times \eta^n(x, m)\\ &= \xi_n \times \eta_n(\pi^{n-1}(x), \pi^{n-1}(\varphi^m(x)))\\ &= \xi_n \times \eta_n(\pi^{n-1}(x), \varphi^{m_{n-1}}(\pi^{n-1}(x)))\\ &= \xi_n \times \eta_n(\pi^{n-1}(x), m_{n-1})\end{aligned}$$

Now we may use part 1 to conclude that $|m_n| \leq \frac{|m_{n-1}|+1}{2}$. The conclusion follows from this.

From the second part, we see that for some n sufficiently large, $|\xi^n \times \eta^n(x, m)| \leq 1$. If it is zero, then $(x, \varphi^m(x))$ is in $\ker(\xi^n \times \eta^n) = R_n$. Then it is also in $R_{n'}$ for all $n' \geq n$ and hence $\xi^{n'} \times \eta^{n'}(x, m) = 0$. The other cases follow immediately from Lemma 7.2. □

Proposition 7.9. *Let x be in X.*

1. *If there exists x' in B such that (x, x') is in R, then there is $N \geq 1$ such that, for all, $n \geq N$, $E_n(\pi^{n-1}(x)) \cap E \neq \emptyset$.*

2. *For any $1 \leq i \leq 6$, there exists x' in C_i such that (x, x') is in R if and only if there is $N \geq 1$ such that, for all, $n \geq N$, $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$.*

Proof. First suppose that there is x' in B such that (x, x') is in R. Then (x, x') is in R_{N-1} for some N. Then, for any $n \geq N$, $\pi^{n-1}(x) = \pi^{n-1}(x')$ and it follows that $E_n(\pi^{n-1}(x)) = E_n(\pi^{n-1}(x'))$. From Lemma 7.1, we have $E_n(\pi^{n-1}(x')) \supset E^n(x')$, which is non-empty by the definition of B. Also from the definition of B, $E^1(x')$ is in E. It then follows inductively from Lemma 7.2 that $E^m(x')$ is in E for all $m \geq 1$. This implies that $E_n(\pi^{n-1}(x)) \cap E$ is non-empty for all $n \geq N$.

Suppose that (x, x') is in R with x' in C_i. Then (x, x') is in R_{N-1}, for some N. Then we have $\pi^{n-1}(x) = \pi^{n-1}(x')$ and $E_n(\pi^{n-1}(x)) = E_n(\pi^{n-1}(x'))$, for all $n \geq N$. By Lemma 7.4, we have $E_n(\pi^{n-1}(x')) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for all $n \geq 1$.

As for the converse, we first consider the case $i = 4$. suppose that $E_n(\pi^{n-1}(x)) = \{\varepsilon_4, \varepsilon_3\}$, for all $n \geq N$. Let $x' = \sigma^N(\pi^N(x))$. Clearly, $\pi^N(x') = \pi^N(x)$ and so (x, x') is in R_N and hence in R. It follows inductively from Theorem 5.34 that $E_n(\pi^{n-1}(x')) = \{\varepsilon_4, \varepsilon_3\}$, for all $N \geq n \geq 1$, and hence x' is in C_4 by Lemma 7.4.

Hence, x' is in C_4. The case $i = 1$ is done in a similar way, replacing σ^N by τ^N. □

Proposition 7.10. *The equivalence relation R_φ is generated by R and graph(β).*

Proof. Let x be in X and k be in $\mathbb{Z}^2$. We will consider the pair $(x, \varphi^k(x))$ in R_φ. Consider the sequence $\xi^N \times \eta^N(x, k)$, for $N \geq 1$. If, for some $N \geq 1$, the value is 0, then (x, φ^k) is in $R_N = \ker(\xi^N \times \eta^N)$ and hence in R.

We now consider the case that $\xi^N \times \eta^N(x,k) \neq 0$, for all $N \geq 1$. By Lemma 5.28, there is some N such that $\xi^N \times \eta^N(x,k) = \varepsilon_i$, for some $1 \leq i \leq 6$. We now assume that $1 \leq i \leq 3$. In the other case, interchanging x and $\varphi^k(x)$ will replace ε_i with $-\varepsilon_i = \varepsilon_{i+3}$. By applying Proposition 5.35 to the point $\pi^{N-1}(x)$, we may find x_{N-2} in X_{N-2} such that $\pi_{N-1}(x_{N-2}) = \pi^{N-1}(x)$ and $\xi_{N-1} \times \eta_{N-1}(x, \varepsilon_i) = \varepsilon_i$ and $E_{N-1}(\pi^{N-2}(x_{N-2})) = \{\varepsilon_i\}$. By repeating $N-1$ applications of 5.35, we find a sequence of points x_m in X_m such that $\pi_{m+1}(x_m) = x_{m+1}$, $\xi_{m+1} \times \eta_{m+1}(x_m, \varepsilon_i) = \varepsilon_i$ and $E_{m+1}(x_m) = \{\varepsilon_i\}$, for all $0 \leq m < N-1$. Finally, x_0 is in X. We have $\pi^{N-1}(x_0) = \pi^{N-1}(x)$ and so (x, x_0) is R_{N-1} and hence in R. Moreover, $\xi^n \times \eta^n(x_0, \varepsilon_i) = \{\varepsilon_i\}$ and $E^n(x_0) = \{\varepsilon_i\}$, for $1 \leq n < N$, by application of Lemma 7.1. Next, we observe that

$$\pi^{N-1}(\varphi^{\varepsilon_i}(x_0)) = \varphi_{N-1}^{\varepsilon_i}(\pi^{N-1}(x_0)) = \varphi_{N-1}^{\varepsilon_i}(\pi^{N-1}(x)) = \pi^{N-1}(\varphi^k(x)).$$

In particular, $(\varphi^{\varepsilon_i}(x_0), \varphi^k(x))$ is in R_{N-1} and hence in R. It follows then that for any $n \geq N$, $\xi^n \times \eta^n(x_0, \varphi^{\varepsilon_i}(x)) = \xi^n \times \eta^n(x, \varphi^k(x))$. This means that $E^n(x_0)$ is non-empty for all n. As $E^1(x_0)$ contained a single element, ε_i and $\#E^n(x_0)$ is non-increasing, it follows that $\#E^n(x_0) = 1$, for all n. Hence x_0 is in $B_i \subset B$. Note that from Lemma 7.5, $\varphi^{\varepsilon_i}(x_0) = \beta(x_0)$ is in B^* and $(x_0, \varphi^{\varepsilon_i}(x_0))$ is in $graph(\beta)$. This completes the proof. $\square$

We must show that the sets C_i and B_i have measure zero, for all R-invariant probability measures on X. We begin with the C_i.

Lemma 7.11. *Let $1 \leq i \leq 6$. The sets $\varphi^j(C_i)$, for j in $\mathbb{Z}^2$ are pairwise disjoint.*

Proof. We consider C_4. It suffices to show that if k is in $\mathbb{Z}^2$ and $x, x' = \varphi^k(x)$ are both in C_4, then $k = 0$. By Lemma 7.8, there is $n \geq 1$ such that $\xi^n \times \eta^n(x,k)$ is either 0 or is E or E^*. If it equals 0, then $(x, \varphi^k(x))$ is in R_n and $\pi^n(x) = \pi^n(x')$. As x, x' are in C_4, we have for every $m \geq 1$,

$$E^m(x) = E_m(\pi^{m-1}(x)) = E_m(\pi^{m-1}(x')) = E^m(x') = \{\varepsilon_4, \varepsilon_3\}.$$

By applying Theorem 5.34 inductively for $m = n, n-1, \ldots, 0$, we have

$$\pi^m(x) = \sigma_m \circ \cdots \sigma_n \circ \pi^n(x) = \sigma_m \circ \cdots \sigma_n \circ \pi^n(x') = \pi^m(x').$$

Finally, for $m = 0$, we have $x = x'$. This implies that $k = 0$.

Next, we consider the case $\xi^n \times \eta^n(x,k) = \varepsilon_i \in E$. Then we have $\varphi^{\varepsilon_i}_{n-1}(\pi^{n-1}(x)) = \pi^{n-1}(x')$ and

$$E^n(x) = E_n(\pi^{n-1}(x)) = E^n(x') = E_n(\pi^{n-1}(x')) = \{\varepsilon_4, \varepsilon_3\}.$$

This contradicts Lemma 5.36.

The case of C_1 is done in a similar way and the other four cases follows from Lemma 7.6. $\square$

Lemma 7.12. *$C_i \times C_j \cap R$ is non-empty if and only if $i - j$ is even.*

Proof. The 'if' direction follows immediately from the last statement of Lemma 7.6. For the 'only if' part, assume that i is even while j is odd. It follows from the definition of C_i, part 2 of Theorem 5.34 and a simple induction that, for any $n \geq 1$, C_i is in the φ-orbit of $\sigma^n(X_n)$ while C_j is in the φ-orbit of $\tau^n(X_n)$. But it follows from property 6 of our sequence of cocycles ξ^n, η^n that there is no orbit which meets both $\sigma^n(X_n)$ and $\tau^n(X_n)$ for all $n \geq 1$. $\square$

Lemma 7.13. *Let μ be a R-invariant probability measure on X. For any $1 \leq i \leq 6$, we have $\mu(C_i) = \mu(B) = \mu(B^*) = 0$.*

Proof. For any $n \geq 1$, let μ_n be the measure on X_n induced by μ. Fix $N \geq 1$ and define $B_N = X_N$. We then define B_n in X_n inductively for $k = N-1, \ldots, 0$, finishing with B_0 in X. Let

$$B_{n-1} = \{x \in \pi_n^{-1}(B_n) \mid E_n(x) \cap E \neq \emptyset\}.$$

It is clear from 7.4 and the definitions that C_1, C_2, C_3 and B are all contained in B_0. Moreover, for each n, we may apply Proposition 5.37 to conclude that

$$\mu_{n-1}(B_{n-1}) \leq \mu_{n-1}(\pi_n^{-1}(B_n))/2.$$

From this it follows that $\mu(B_0) \leq 2^{-N}\mu_N(B_N) = 2^{-N}$ and hence each set C_1, C_2, C_3 and B has μ-measure less than or equal to 2^{-N}. As N was arbitrary, the conclusion follows. $\square$

8 Construction of the AF-relation R'

Unfortunately, the relation R is slightly too large for us to apply the absorption theorem of [GPS2]. We define an open subequivalence relation R' as follows.

We define a set of labels $\Lambda = \{1, 1', 2, 3, 4, 4', 5, 6\}$. This is endowed with a cyclic order: $1 < 1' < 2 < 3 < 4 < 4' < 5 < 6 < 1$. We write $[i, j]$ to denote all k such that $i \leq k \leq j$. We also let $p(i)$ denote the predecessor of i in Λ.

For each $n \geq 1$, we will inductively define a continuous function

$$\lambda^n : X \to \Lambda.$$

We begin with $n = 1$ and first set λ^1 to satisfy: $\lambda^1(x) = i$ if $E^1(x) = \{\varepsilon_i, \varepsilon_{i-1}\}$ or if $E^1(x) = \{\varepsilon_i\}$, for some $1 \leq i \leq 6$. Values of λ^1 on other points in X may be made arbitrarily, so long as λ^1 is continuous.

For $n \geq 2$ and x in X, having defined $\lambda^{n-1}(x)$, we define $\lambda^n(x)$ as follows. First consider x such that $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some $1 \leq i \leq 3$. Then we set

$$\lambda^n(x) = \begin{cases} i & \text{if } i \leq \lambda^{n-1}(x) \leq 4 \\ p(i) & \text{if } 4' \leq \lambda^{n-1}(x) \leq p(i) \end{cases}$$

Similarly, if $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some $4 \leq i \leq 6$. Then we set

$$\lambda^n(x) = \begin{cases} i & \text{if } i \leq \lambda^{n-1}(x) \leq 1 \\ p(i) & \text{if } 1' \leq \lambda^{n-1}(x) \leq p(i) \end{cases}$$

Next, we set $\lambda^n(x) = i$ if $E_n(\pi^{n-1}(x)) = \{\varepsilon_i\}$. Finally, we set $\lambda^n(x) = 1$ for all other values of x.

We first establish some simple facts about the functions λ^n.

Lemma 8.1. *Let (x, y) be in R_{n-1}, for some $n \geq 2$.*

1. *If (x, y) is in R_{n-1} and $\lambda^{n-1}(x) = \lambda^{n-1}(y)$, then $\lambda^n(x) = \lambda^n(y)$.*
2. *$\lambda^n(x)$ and $\lambda^n(y)$ are either equal or adjacent symbols, and neither of the pairs $1, 1'$ nor $4, 4'$.*
3. *If $\lambda^n(x) = i$ and $\lambda^n(y) = p(i)$, then $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$.*
4. *If $\lambda^{n+1}(x) = i$ and $\lambda^{n+1}(y) = p(i)$, then $\lambda^n(x) = i$ and $\lambda^n(y) = p(i)$.*
5. *If, for some i, we have*

$$E_n(\pi^{n-1}(x)) = E_{n+1}(\pi^n(x)) = \{\varepsilon_i, \varepsilon_{i-1}\},$$

then

$$\lambda^{n+1}(x) = \lambda^n(x).$$

Proof. As (x, y) is in R_{n-1}, we have, for all $m \geq n$, $\pi^{m-1}(x) = \pi^{m-1}(y)$ and hence, $E_m(\pi^{m-1}(x)) = E_m(\pi^{m-1}(y))$. Part 1 follows from the observation that λ^n depends only on $E_n(\pi^{n-1}(x))$ and $\lambda^{n-1}(x)$. The second and third parts follow easily from the definition of λ^n. For the fourth part, it follows from part 1 that $\lambda^n(x) \neq \lambda^n(y)$. Hence they are of the form $j, p(j)$, for some j. But from the definition, λ^{n+1} will only distinguish two adjacent symbols if $i = j$ and in this case, we also have the desired conclusion. For the final part, $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$ implies that $\lambda^n(x)$ is either i or $p(i)$. The conclusion follows from the definition of λ^{n+1}. □

We are now ready to define our new relation R'.

Definition 8.2. *For each $n \geq 1$, we define*

$$R'_n = \{(x, y) \in R_n \mid \lambda^n(x) = \lambda^n(y)\},$$

and

$$R' = \cup_{n=1}^{\infty} R'_n.$$

Observe that, by part 1 of Lemma 8.1, we have $R'_{n-1} \subset R'_n$, for all $n \geq 2$.

A crucial step in our application of the absorption theorem is establishing certain properties of our AF-relation on the sets B and B^*. Toward that end, we show how the functions λ^n behave on these sets.

Lemma 8.3. *1. If x is in B, then $1 \leq \lambda^n(x) \leq 3$, for all $n \geq 1$.*

2. If x is in B and $\lambda^n(x) = 1$, for some $n \geq 1$, then

$$E_{n+1}(\pi^n(x)) \neq \{\varepsilon_4, \varepsilon_3\}.$$

3. If x is in B^*, then $4 \leq \lambda^n(x) \leq 6$, for all $n \geq 1$.

4. If x is in B^* and $\lambda^n(x) = 4$, for some $n \geq 1$, then

$$E_{n+1}(\pi^n(x)) \neq \{\varepsilon_1, \varepsilon_6\}.$$

Proof. First, for any x in B (or B^*, respectively), $E^n(x)$ is a single element of E (E^*, respectively). We claim that if $\lambda^n(x) = 1$, for some $n \geq 1$, then $E_{n+1}(\pi^n(x)) \neq \{\varepsilon_4, \varepsilon_3\}$. As 1 is not the predecessor of any j, the only possibilities for $E_n(\pi^{n-1}(x))$ are $\{\varepsilon_1\}$ and $\{\varepsilon_1, \varepsilon_6\}$. In either case, since we know that $E^n(x)$ is a singleton subset of this and contained in E, we must have $E^n(x) = \{\varepsilon_1\}$. It follows from Lemma 7.1 and Lemma 5.29 that $E^{n+1}(x) \neq \{\varepsilon_3\}$ and the conclusion follows. We now prove the first statement by induction. The case $n = 1$ follows from the definition of B and λ^1. Assume it is true for $n-1$, for some $n \geq 2$. We know that $E_n(\pi^{n-1}(x))$ contains $E^n(x)$, by Lemma 7.1. Hence, it is either $\{\varepsilon_i\}$, for some $1 \leq i \leq 3$, in which case we are done, or else $\{\varepsilon_i, \varepsilon_{i-1}\}$, for some $1 \leq i \leq 4$. First consider the case $2 \leq i \leq 3$. The value of $\lambda^n(x)$ is either i or $p(i)$ and we are done. Next consider the case, $i = 1$. By the induction hypothesis, $1 \leq \lambda^{n-1}(x) \leq 4$, and hence $\lambda^n(x) = 1$. Finally, consider the case $i = 4$. We know from the claim above that $\lambda^{n-1}(x) \neq 1$. It follows from the definition that $\lambda^n(x) = 3$. The proofs of the last two statements are analogous to the first and we omit them. $\square$

Lemma 8.4. *Let x be in $B \cup B^*$, and let $n \geq 1$. If, for some i, $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, then we have*

$$\lambda^n(x) = \begin{cases} i & \text{if } E^n(x) = \{\varepsilon_i\}, \\ p(i) & \text{if } E^n(x) = \{\varepsilon_{i-1}\} \end{cases}$$

Proof. We consider only the case x is in B, the other is similar. We know that $E^n(x) \subset E_n(\pi^{n-1}(x))$ and also, from the definition of B, that $E^n(x)$ is a singleton. We consider the statement, for each positive n, that if $E_n(\pi^{n-1}(x)) = \{\varepsilon_i\}$, then $\lambda^n(x) = i$ and if $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, then the value of $\lambda^n(x)$ is given as above. We prove this statement by induction on n.

In the case $n = 1$, we have $E^n(x) = E_n(\pi^{n-1}(x))$, which is a singleton and the result follows from the definition of λ^1. Suppose the statement is true for n. The case of $E_{n+1}(\pi^n(x))$ being a singleton, the conclusion follows from the definition. Next, suppose that $E_{n+1}(\pi^n(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some i. Since this contains $E^{n+1}(x)$ which is in E by Lemmas 5.28 and 7.1, we have $1 \leq i \leq 4$. In the case $i = 4$, it follows from part 2 of Lemma 8.3 that $\lambda^n(x) \neq 1$. In the definition of λ^{n+1}, $4 \leq \lambda^n(x) \leq 1$ is impossible and so $\lambda^{n+1}(x) = p(4) = 3$. On the other hand, as $E^{n+1}(x)$ is a subset of $E_{n+1}(\pi^n(x))$ and in E, it follows that $E^{n+1}(x) = \{\varepsilon_3\}$ and we are done. Next, consider the case $i = 1$. Again as $E^{n+1}(x)$ is a singleton subset of $E_{n+1}(\pi^n(x))$ and is contained in E, we have $E^{n+1}(x) = \{\varepsilon_1\}$. On the other hand $\lambda^{n+1}(x)$ is either 1 or $p(1) = 6$, but the latter is not possible by part 1 of Lemma 8.3. Hence we have $\lambda^{n+1}(x) = 1$. Finally, we consider the case $2 \leq i \leq 3$. Suppose that $E_n(\pi^{n-1}(x)) = E_{n+1}(\pi^n(x))$. Then by part 5 of Lemma 8.1, we have $\lambda^{n+1}(x) = \lambda^n(x)$. On the other hand, applying Lemma

5.28 and Lemma 7.1, we have

$$\begin{aligned} E^{n+1}(x) &= \xi_{n+1} \times \eta_{n+1}(\pi^n(x), E^n(x)) - \{0\} \\ &= E^n(x) - \{0\} \\ &= E^n(x). \end{aligned}$$

The conclusion follows from the induction hypothesis. Next suppose that $E_n(\pi^{n-1}(x)) \neq E_{n+1}(\pi^n(x))$. Then either $E_n(\pi^{n-1}(x)) = \{\varepsilon_j\}$ for some $1 \leq j \leq 3$ or $E_n(\pi^{n-1}(x)) = \{\varepsilon_j, \varepsilon_{j-1}\}$ for some $1 \leq j \leq 3$ and $j \neq i$. Let us first suppose the former occurs for some $i \leq j \leq 3$, or the latter for $i < j \leq 3$. In the former case, the value of $\lambda^n(x)$ is j and in the latter, it is either j or $p(j) \geq i$. In any case, we have $i \leq \lambda^n(x) \leq 3$ and it follows that $\lambda^{n+1}(x) = i$. On the other hand, we consider the value of $E^n(x)$. It is $\{\varepsilon_k\}$, where $k = j$ or $k = j - 1$. In any case, we have $i \leq k \leq 3$. Since, $E^{n+1}(x)$ is a subset of $E_{n+1}(\pi^n(x))$, it is either $\{\varepsilon_i\}$ or $\{\varepsilon_{i-1}\}$. If it were the latter, then we use Lemma 7.1 to see

$$\xi_{n+1} \times \eta_{n+1}(\pi^n(x), \varepsilon_k) = \varepsilon_{i-1}.$$

Since $E_{n+1}(\pi^n(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, part 2 of Theorem 5.34 asserts that

$$\xi_{n+1} \times \eta_{n+1}(\pi^n(x), \varepsilon_i) = \varepsilon_i, \xi_{n+1} \times \eta_{n+1}(\pi^n(x), \varepsilon_{i-1}) = \varepsilon_{i-1}.$$

The fact that $i - 1 \leq i \leq k$ means that we have contradiction to part 1 of Lemma 5.31. We conclude that $E^{n+1}(x) = \{\varepsilon_i\}$ as desired. The proof in the remaining case when $E_n(\pi^{n-1}(x)) = \{\varepsilon_j\}$ or $E_n(\pi^{n-1}(x)) = \{\varepsilon_j, \varepsilon_{j-1}\}$ for some $1 \leq j < i$ is similar and we omit the details. □

Lemma 8.5.

1. *Suppose that x, y are in B with (x, y) in R_{n-1}, for some $n \geq 1$. Then $\lambda^n(x) = \lambda^n(y)$ if and only if $(\beta(x), \beta(y))$ is in R_n.*

2. *Suppose that x, y are in B^* with (x, y) in R_{n-1}, for some $n \geq 1$. Then $\lambda^n(x) = \lambda^n(y)$ if and only if $(\beta^{-1}(x), \beta^{-1}(y))$ is in R_n.*

3. *Suppose that x is in B and there is N such that, for all $n \geq N$, $E_n(\pi^{n-1}(x)) = \{e_2, e_1\}$ and $\lambda^n(x) = 2$, then $\lambda^n(\beta(x)) = 5$, for all $n \geq N$.*

4. *Suppose that x is in B^* and there is N such that, for all $n \geq N$, $E_n(\pi^{n-1}(x)) = \{e_5, e_4\}$ and $\lambda^n(x) = 5$, then $\lambda^n(\beta^{-1}(x)) = 2$, for all $n \geq N$.*

Proof. We prove the first statement. Suppose that x is in B_i and y is in B_j. For convenience, we define

$$\varepsilon^n(x) = \xi^n \times \eta^n(x, \varepsilon_i), \varepsilon^n(y) = \xi^n \times \eta^n(y, \varepsilon_j).$$

It follows from the definition of the action φ_n that

$$\pi^n(\varphi^{\varepsilon_i}(x)) = \varphi_n^{\varepsilon^n(x)}(\pi^n(x)), \pi^n(\varphi^{\varepsilon_j}(y)) = \varphi_n^{\varepsilon^n(y)}(\pi^n(y)).$$

It also follows from the definition on B, Lemma 7.1 and simple induction argument that

$$E^n(x) = \{\varepsilon^n(x)\}, E^n(y) = \{\varepsilon^n(y)\}.$$

Since (x, y) is in R_{n-1}, we have $\pi^{n-1}(x) = \pi^{n-1}(y)$. In the case that $E_n(\pi^{n-1}(x)) = E_n(\pi^{n-1}(y))$ is a singleton, then it follows that $E^n(x) = E^n(y)$ and also $\lambda^n(x) = \lambda^n(y)$. It remains to consider the case that $\#E_n(\pi^{n-1}(x)) = 2$. In this case, it follows immediately from Lemma 8.4 that $\lambda^n(x) = \lambda^n(y)$ if and only if $E^n(x) = E^n(y)$, which in turn is true if and only if $\varepsilon^n(x) = \varepsilon^n(y)$. Recall from the definition of β that $\beta(x) = \varphi^{\varepsilon_i}(x)$ and $\beta(y) = \varphi^{\varepsilon_j}(y)$. Hence we have

$$\begin{aligned}
\pi^n(\beta(x)) &= \pi^n(\varphi^{\varepsilon_i}(x)) \\
&= \varphi_n^{\varepsilon^n(x)}(\pi^n(x)) \\
&= \varphi_n^{\varepsilon^n(x)-\varepsilon^n(y)}(\varphi_n^{\varepsilon^n(y)}(\pi^n(x))) \\
&= \varphi_n^{\varepsilon^n(x)-\varepsilon^n(y)}(\varphi_n^{\varepsilon^n(y)}(\pi^n(y))) \\
&= \varphi_n^{\varepsilon^n(x)-\varepsilon^n(y)}(\pi^n(\varphi^{\varepsilon_j}(y))) \\
&= \varphi_n^{\varepsilon^n(x)-\varepsilon^n(y)}(\pi^n(\beta(y)))
\end{aligned}$$

Then $(\beta(x), \beta(y))$ is in R_n if and only if $\pi^n(\beta(x)) = \pi^n(\beta(y))$ if and only if $\varepsilon^n(x) = \varepsilon^n(y)$. The proof of the second part is analogous.

Consider x as in part 3, and suppose that x is in B_i, for some $1 \leq i \leq 3$. It follows that $\beta(x) = \varphi^{\varepsilon_i}(x)$ is in B_{i+3} and

$$\varepsilon^n(\beta(x)) = \xi^n \times \eta^n(\varphi^{\varepsilon_i}(x), \varepsilon_{i+3}) = -\xi^n \times \eta^n(x, \varepsilon_i) = -\varepsilon^n(x),$$

for all n. Then for $n \geq N$, it follows that $\varepsilon^n(\beta(x)) = \varepsilon_5$. As $E_n(\pi^{n-1}(x)) = \{\varepsilon_2, \varepsilon_1\}$, it follows from Theorem 5.34 that $\pi^{n-1}(x)$ is in $\varphi_{n-1}^{\varepsilon_4}(\sigma_n(X_n))$ and so $\pi^{n-1}(\beta(x)) = \varphi^{\varepsilon_2}(\pi^{n-1}(x))$ is in $\varphi_{n-1}^{\varepsilon_3}(\sigma_n(X_n))$, as $\varepsilon_4 + \varepsilon_2 = \varepsilon_3$. Hence $E_n(\pi^{n-1}(\beta(x))) = \{e_6, e_5\}$. The conclusion then follows from Lemma 8.4 (with $i = 6$) that $\lambda(\beta(x)) = p(6) = 5$. This completes the proof of part 3. Part 4 is done in an analogous way. $\square$

The next step is to establish to difference between the relations R and R'.

Lemma 8.6. *Suppose that (x, y) is in R and not in R'. Then there is an $N \geq 1$ and $1 \leq i \leq 6$ such that*

$$E_n(\pi^{n-1}(x)) = E_n(\pi^{n-1}(y)) = \{\varepsilon_i, \varepsilon_{i-1}\},$$

for all $n > N$, and

$$\lambda^n(x) = i, \lambda^n(y) = p(i),$$

for all $n > N$, or

$$\lambda^n(y) = i, \lambda^n(x) = p(i),$$

for all $n > N$. In particular, for any x in X whose φ-orbit is disjoint from $\cup_{i=1}^{6} C_i$, the R-equivalence class of X is equal to the R'-equivalence class of x.

Proof. If (x, y) is in R, then it is in R_N for some $N \geq 1$. Note that if $\lambda^n(x) = \lambda^n(y)$, for some $n \geq N$ then (x, y) is in R'_n, hence we have $\lambda^n(x) \neq \lambda^n(y)$, for all $n \geq N$. We may now appeal inductively to the last part of Lemma 8.1 to see that the conclusion holds. The last statement follows from the first and part 2 of Proposition 7.9. □

Theorem 8.7. *1. The equivalence relation R' is an open subequivalence relation of R and hence is an AF-relation.*

2. The equivalence relation R' is minimal.

3. Every R' invariant probability measure on X is also R-invariant.

Proof. It is easy to see each R'_n is open in R_n, and the first statement follows (see Proposition 3.12 of [GPS2]). Next, we show that R'_n has capacity $n-2$, for all $n \geq 2$. Suppose that x is in X. If $\#E_n(\pi^{n-1}(x)) < 2$, then the value of $\lambda^n(\cdot)$ is constant on the R_n-equivalence class of x and hence the R'_n-equivalence class of x coincides with its R_n-equivalence class and hence, for some $k \in \mathbb{Z}^2$, contains $\varphi^i(x)$, for $i \in B(k, n)$. We are left to consider the case that $E_n(\pi^{n-1}(x)) = \{\varepsilon_i, \varepsilon_{i-1}\}$, for some i, and hence $\lambda^n(x)$ is either i or $i-1$. Without loss of generality, we assume $1 \leq i \leq 3$. First, consider the case that $\lambda^n(x) = i$. By part 3 of Theorem 5.34, we may find x' in the R_{n-1}-equivalence class of x such that $E_{n-1}(\pi^{n-2}(x')) = \{\varepsilon_4, \varepsilon_3\}$. Then we have $\lambda^{n-1}(x')$ is either 4 or 3. But in either case, $\lambda^n(x') = i$, by definition of λ^n, and so x' is in the same R'_n class as x. However, the same argument shows that for any other point x'' with $\pi^{n-2}(x'') = \pi^{n-2}(x')$, x'' is in the same R'_n class as x. That is, the R'_n class of x contains the R_{n-2} class of x'. The conclusion follows. Now consider the case that $\lambda^n(x) = p(i)$. Then we select x' in the same R_{n-1} class with x with $E_{n-1}(\pi^{n-2}(x')) = \{\varepsilon_6, \varepsilon_5\}$. It follows that $\lambda^{n-1}(x')$ is either 6 or 5. In either case, $\lambda^n(x') = p(i)$. The same argument applies to any other point in the same R_{n-2} class as x'. Part 2 of the Theorem follows from the fact R'_n has capacity $n-2$ and Proposition 5.7.

For the third part, let μ be any R'-invariant probability measure on X. Let $1 \leq i \leq 6$; we will show that $\mu(C_i) = 0$. By Lemma 7.11, the sets $\varphi^j(C_i), j \in \mathbb{Z}^2$ are pairwise disjoint. It follows that for all $n \geq 2$, C_i meets each R'_n equivalence class at most once. If x is in C_i, we may find $k(x)$ in $\mathbb{Z}^2$ such that $(x, \varphi^{k(x)+k'}(x))$ is in R'_n, for all k' in $B(0, n-2)$. Moreover, k may be chosen to be a continuous function of x. For each k' in $B(0, n-2)$, the sets $\{(x, \varphi^{k(x)+k'}(x)) \mid x \in C_i\}$ are in R'_n and have pairwise disjoint ranges. As μ is R'_n-invariant and there are $(2n-3)^2$ points k' in $B(0, n-2)$, we must have

$$\mu(C_i) \leq (2n-3)^{-2}.$$

Since this is true for all $n \geq 2$, the claim follows. So the μ measure of the R'-equivalence classes of all the C_i is zero. But on the rest of X, R' and R are equal. Part 3 follows from this. $\square$

Theorem 8.8. *The sets B and B^* are étale subsets for the relation R'. Moreover, the map β induces an isomorphism between $R' \cap (B \times B)$ and $R' \cap (B^* \times B^*)$.*

Proof. We first show that B is étale for R'. To do this, we take (x, y) in $R' \cap B \times B$. We must find a neighbourhood V of (x, y) in R' such that, for any (x', y') in V, x' is in B if and only if y' is in B. Since (x, y) is in R_φ, we have $y = \varphi^k(x)$ for some k in $\mathbb{Z}^2$. As it is in R', it is in R'_N, for some $N \geq 1$. By continuity of the various functions involved, we may choose an open set U in X such that E^n, $E^n \circ \varphi^k$, λ^n and $\lambda^n \circ \varphi^k$, for $1 \leq n \leq N$, are all constant on U. We also choose U sufficiently small so that $V = \{(x', \varphi^k(x')) \mid x' \in U\}$ is in R'_N. We claim that this V satisfies the desired property.

Suppose that x' is in B and in U. We need to show that $\varphi^k(x')$ is also in B. It follows from the choice of U that $E^n(y') = E^n(y)$, for $1 \leq n \leq N$.

We also have $\lambda^N(y') = \lambda^N(y) = \lambda^N(x) = \lambda^N(x')$. From Lemma 8.4 we have $E^N(x') = E^N(x) = E^N(y) = E^N(y')$. We claim that for all $n \geq N$, we have $E^n(x') = E^n(y')$. The statement is true for N. We know that $\pi^n(x') = \pi^n(y')$ for all $n \geq N$. The claim then follows easily by induction from Lemma 7.1.

To prove the last statement, we know already that β maps B homeomorphically to B^*. We will show that

$$\beta \times \beta(R'_n \cap B \times B) \subset R'_{n+1}$$

for all $n \geq 1$. From this it follows that $\beta \times \beta$ is continuous. An analogous argument applies to $(\beta \times \beta)^{-1}$ and this implies the conclusion.

Suppose that (x, y) is in $R'_n \cap B \times B$. Say that x is in B_i and y is in B_j so that $\beta(x) = \varphi^{\varepsilon_i}(x)$ and $\beta(y) = \varphi^{\varepsilon_j}(y)$. It follows from Lemma 8.5 that $(\beta(x), \beta(y))$ is in R_{n+1}. As $\lambda^n(x) = \lambda^n(y)$, $\lambda^{n+1}(x) = \lambda^{n+1}(y)$ by part 1 of Lemma 8.1, Lemma 8.4 implies that $E^{n+1}(x) = E^{n+1}(y)$. Lemma 7.5 implies that

$$E^{n+1}(\beta(x)) = -E^{n+1}(x) = -E^{n+1}(y) = E^{n+1}(\beta(y)),$$

for all $n \geq 1$. We may then apply Lemma 8.4 again to conclude that $\lambda^{n+1}(\beta(x)) = \lambda^{n+1}(\beta(y))$, and hence $(\beta(x), \beta(y))$ is in R'_{n+1}. □

We now define $\tilde{R}$ to be the equivalence relation generated by R' and $graph(\beta)$.

Theorem 8.9. *The relation $\tilde{R}$ is orbit equivalent to R' and hence is affable.*

Proof. This is an application of the absorption Theorem 2.7 with the equivalence relation R', closed sets B and B^* and map β. To see the first hypothesis is satisfied, we note from parts 1 and 3 of Lemma 8.3 that the values of λ on B and B^* are always distinct, so no pair in $B \times B^*$ can be in R'. For the second hypothesis, any R'-invariant probability measure on X is also R-invariant by part 3 of Theorem 8.7 and therefore has value 0 on B and B^* by Lemma 7.13. Hypotheses 3 and 4 are established in the last Theorem 8.8. □

Lemma 8.10. *1. For all $1 \leq i \neq j \leq 6$, $C_i \times C_j \cap \tilde{R}$ is empty.*

2. For all $\tilde{R}$-invariant measures μ and all $1 \leq i \leq 6$, $\mu(C) = 0$.

3. The equivalence relation generated by $\tilde{R}$ and $graph(\gamma_i)$, $i = 2, 3, 5, 6$, is R_φ.

Proof. We note first that, for any x in C_i and $n \geq 1$, $\lambda^n(x) = i$, follows from the definitions of C_i and λ^n. It follows that $(C_i \times C_j) \cap R'$ is empty for all $i \neq j$. Next, we note that from Lemma 7.12, $(C_i \times C_j) \cap R_\varphi$ is empty unless i and j are both even or both odd. The same is therefore true for $\tilde{R}$. Let us consider the case i, j even. The other is similar. As $\lambda^n(C_2) = 2$, for all $n \geq 1$, it is impossible for any point of C_2 to be in the same R' class as a point of B^*. Similarly, no point of either C_4 or C_6 can be in the same R' class as a point of B. We are left to consider the case that there exists x_2 in C_2, x_j in C_j with $j = 4$ or 6, x in B with (x_2, x) and $(\beta(x), x_j)$ both in R'. Suppose that (x_2, x) is in R'_N, for some N, and hence in R_N also. By the defintion of R'_n, we have $\lambda^n(x) = \lambda^n(x_2) = 2$, for all $n \geq N$ and $E_{n+1}(\pi^n(x)) = E_{n+1}(\pi^n(x_2)) = \{\varepsilon_2, \varepsilon_1\}$, for all $n \geq N$. It then follows from part 3 of Lemma 8.5 that $\lambda^n(\beta(x)) = 5$, for all $n \geq N$, and hence it is not possible for $\beta(x)$ to be in the same R'_n class as any point in either C_4 or C_6.

For the final part, R_φ is the smallest equivalence relation containing R and $graph(\beta)$, by Proposition 7.10, while R is the smallest equivalence relation containing R' and $graph(\gamma_i)$, $i = 2, 3, 5, 6$. Hence R_φ is the smallest equivalence relation containing R', $graph(\beta)$ and $graph(\gamma_i)$, $i = 2, 3, 5, 6$. Hence it is also the smallest equivalence relation containing $\tilde{R}$ and $graph(\gamma_i)$, $i = 2, 3, 5, 6$. $\square$

We can now give the final step in the proof of the main theorem, that the relation R_φ is affable. It is four applications of the absorption theorem 2.7, beginning with $\tilde{R}$, closed sets C_2 and C_4 and map γ_4. It is important to note that the topology on $\tilde{R}$ which we use is that which arises from Theorem 8.9 making it an AF-relation and *not* the topology from R_φ. Hypothesis 1 is satisfied by part 1 of Lemma 8.10. The second hypothesis is satisfied by part 2 of 8.10. By part 3 of Lemma 8.10, the relation $\tilde{R}$ restricted to C_2 and C_4 is equality and so the last two hypotheses are satisfied trivially. (This is rather fortunate since we don't actually know much about the new topology we are using on $\tilde{R}$ which makes it an AF-relation.) Let the relation generated by $\tilde{R}$ and $graph(\beta)$ be denoted R_1. It is affable. We again apply the absorption theorem to R_1, closed sets C_2 and C_6 and map γ_6. Again, by lemma 8.10 the relation R_1 restricted to C_2 and C_6 is equality, so the last two hypotheses are trivially satisfied. Two more applications (the first using C_1, C_3 and γ_3 and the last using C_1, C_5 and γ_5) then yield that R_φ is affable. This completes the proof.

9 Examples

In this section we present two classes of minimal free $\mathbb{Z}^2$-actions which satisfy our hypotheses and are therefore affable. We also remark that any extension of a system satisfying our hypotheses, will also satisfy them. Hence, our main result also applies to any extension of the examples below.

Example 9.1. *Rotations of the group of p-adic integers.*

Here, we consider a prime number p and the group X of p-adic integers. We choose a dense copy of $\mathbb{Z}^2$ in X and our action is by rotation by this subgroup. We remark that we believe that the same result is true for the n-adic integers, where n is any natural number and more generally for all odometers. But the choice of a prime p will simplify some of our arguments. Let us make this more precise.

We let $\mathbb{Z}_p$ denote the quotient of $\mathbb{Z}$ by $p\mathbb{Z}$. Then, we have $X = \Pi_{k=0}^{\infty}\mathbb{Z}_p$. It is an abelian group ; the operation is addition modulo p, with carry over to the right. An element $x = (x_k)_{k=0}^{\infty}$ may be regarded as a formal power series $\sum_k x_k p^k$ with the obvious form for addition. For x in X and non-negative integers $i \leq j$, we let $x_{[i,j]}$ denote the finite sequence $x_i, x_{i+1}, \ldots x_j$. We call such a sequence a *word in x of length $j - i + 1$*.

We choose two elements α and β from X such that either α_0 or β_0 is non-zero and so that the only integers m, n which satisfy $m\alpha + n\beta = 0$ are $m = n = 0$. These conditions imply (in fact, they are equivalent to) the subgroup generated by α and β is dense in X. Then our action φ is defined by

$$\varphi^{(i,j)}(x) = x - i\alpha - j\beta,$$

for all x in X and (i, j) in $\mathbb{Z}^2$. In our notation, we will identify $\mathbb{Z}^2$ and the subgroup of X. We claim that this action satisfies the hyptotheses of Theorem 3.9 and hence Corollary 4.2. Let a and b be generators of $\mathbb{Z}^2$. The elements a and b may also be regarded as elements of X, and this is how we will treat them in our notation. As a and b must generate a dense subgroup of X, at least one of a_0 and b_0 is non-zero. Let us suppose the former. Then the subgroup generated by a alone is dense. This means that there is an automorphism of X carrying a to $(1, 0, 0, \ldots)$. Henceforth, we assume that $a = (1, 0, 0, \ldots)$. Observe that, for any x in X and positive integer k, the values of $(x + ia)_{[0,k]}$ are all distinct for $0 \leq i < p^{k+1}$. We also choose a positive integer K. We will show that we may find non-empty clopen sets A and B satisfying Theorem 3.9 with $N = p^K$.

Lemma 9.2. *Let b be in X and let K be a positive integer and let $a = (1, 0, 0, \ldots)$. Suppose that there are no non-trivial integer solutions, i, j, of the equation $ia + jb = 0$. Then there exists positive integers $K < M, N$ such that $2M < N$ and*

$$(-b)_{M-K} < (-b)_{N-K}$$

regarding these as integers between 0 and $p - 1$ and

$$(-b)_{M-k} = (-b)_{N-k}$$

for all $k = 0, 1, 2, \ldots, K - 1$.

Proof. We consider the collection of all words w in b of length K and we divide these into three classes. The first is all words that only occur finitely many times in b; that is $w = b_{k,k+K-1}$ for only finitely many $k \geq 0$. The second class is all words w such that there are distinct symbols $0 \leq i, j < p$ such that the words iw (concatenation) and jw both appear infinitely many times in b. The third class consists of words w such that there is a symbol i_w such that $i_w w$ occurs infinitely many times, but iw occurs only finitely many times for $i \neq i_w$. If there exists such a word in the second class, then we are clearly done. We are left to consider the case that the second class is empty. Then we can select $L \geq 1$ such that each word in the first class does not appear in $b_{[L,\infty)}$ and that for each word iw with w in the third class and $i \neq i_w$ does not appear in $b_{[L,\infty)}$. We claim that b is eventually periodic, that is, there is some $J \geq 1$ such that $b_n = b_{n+J}$, for all $n \geq L$. To see this, consider the second appearance of a word w from the third class in $b_{[L,\infty)}$. From the choice of L, this word must be preceded by i_w. Let w' be the word obtained by dropping the last symbol from $i_w w$. This word is also of length K, and provided we are still at entries greater than L, it is again in the third class and it must be preceded by $i_{w'}$. Continuing in this way, we see that the predecessors of w are unique. Eventually the word w occurs in this string, since we began at the second occurrence of w. But this argument applies to every occurrence of w. The conclusion follows. Since $a = (1, 0, 0, \ldots)$, we may find such integer i such that $ia + b$ is periodic, say of period $J \geq 1$. Multiplying a sequence in X by 2^J has the effect of shifting the entries over by J and leaving 0 in the first J positions. As it is periodic, this leaves all of $ia + b$ unchanged except for the first J positions. Then we may find $i' \geq 0$ such that $i'a = (b_0, b_1, \ldots, b_{J-1}, 0, 0 \ldots)$. Then we have

$$i'a + 2^J(ia + b) = (ia + b).$$

But this contradicts our hypothesis on a and b. This completes the proof. $\square$

Having chosen a positive integer K, we select M, N as in the Lemma and define

$$A = \{x \in X \mid x_{[0,M]} = x_{[N-M,N]}\} - \{x \in X \mid x_i = p-1, \text{ for all } 0 \leq i \leq N\}.$$

We define two functions, $\lambda, \mu : A \to \mathbb{Z}$ by

$$\begin{aligned} \lambda(x) &= \inf\{i \geq 1 \mid x + ia \in A\} \\ \mu(x) &= \inf\{i \geq 1 \mid x - b + ia \in A\} \end{aligned}$$

for all x in A. Since rotation by a is minimal, both of these quantities are well defined. Also, because A is clearly clopen, both functions are continuous.

The key lemma is the following.

Lemma 9.3. *With A, λ and μ as above, we have*

1. $\lambda(x) \geq p^{M+1}$, *for all* $x \in A$,
2. $\mu(x) \leq p^{M-K+1} + 2$, *for all* $x \in A$.

Proof. We begin with the first statement.

Case 1: $x_k < p - 1$, for some $M < k < N - M$. For $1 \leq i < p^{M+1}$, the values of $(x + ia)_{[0,M]}$ are all distinct from $x_{[0,M]}$. However, for these values of i, when computing $x + ia$, there is no carry over past the kth coordinate and this means that $x_{[N-M,N]} = x_{[0,M]}$. Hence, $x + ia$ is not in A.

Case 2: $x_k = p - 1$, for all $M < k < N - M$. As x is in A, we must have $x_k < p-1$, for some $0 \leq k \leq N$. By hypothesis, we have either $k \leq M$ or $N - M \leq k \leq N$. But, in the former case, as x is in A, we have $x_k = x_{N-M+k}$. In either case, we conclude that $x_k < p - 1$, for some $0 \leq k \leq M$ and some $N - M \leq k \leq N$. Let I be the first positive integer for which the computation of $x + Ia$ involves carry over past coordinate M. Note that $I \leq p^{M+1}$. Since $x_k < p-1$ for some $k \leq M$, we have $I > 1$. For $1 \leq i < I$, we have $(x + ia)_{[0,M]} \neq x_{[0,M]}$ while $(x + ia)_{[N-M,N]} = x_{[N-M,N]}$. From this we conclude that $x + ia$ is not in A. Next consider $I \leq i \leq p^{M+1}$. Here, we have $(x + ia)_{[N-M,N]}$ is obtained from $x_{[N-M,N]} = x_{[0,M]}$ by adding $(1, 0, 0, \ldots, 0)$. That is, we have $(x+ia)_{[N-M,N]} = (x+a)_{[0,M]}$. But as $1 < I \leq i \leq p^{M+1}$, we know that $(x+a)_{[0,M]} \neq (x+ia)_{[0,M]}$ and again we conclude that $x+ia$ is not in A. This completes the proof of the first statement.

We now consider the second statement. Since we have $(-b)_{M-K} < (-b)_{N-K}$, we may find a positive integer $I < p^{M-K+1}$ so that $(-b+Ia)_{[0,M-K]} = (-b)_{[N-M,N-K]}$. We now claim that, for any x in A, at least one of $x-b+Ia$, $x-b+(I+1)a$ or $x-b+(I+2)a$ is in A. This will complete the proof.

We see at once from our choice of I that $(-b+Ia)_{[0,M-K]} = (-b)_{[N-M,N-K]} = (-b+Ia)_{[N-M,N-K]}$ since the addition has no carry over past coordinate $M-K$. Using again that there is no carry over past coordinate $N-M$ and our original choice of M, N, we have $(-b+Ia)_{[M-K+1,M]} = (-b)_{[M-K+1,M]} = (-b)_{[N-K+1,N]} = (-b+Ia)_{[M-K+1,M]}$. Together, we see that $(-b+Ia)_{[0,M]} = (-b+Ia)_{[N-M,N]}$. We add $-b+Ia$ to an element x in A. If there is no carry over from coordinate $N-M-1$ to $N-M$, then $x-b+Ia$ has the same property, $(x-b+Ia)_{[0,M]} = (x-b+Ia)_{[N-M,N]}$. Either $x-b+Ia$ is in A or else $(x-b+Ia)_k = p-1$ for all $0 \leq k \leq N$. In the latter case, it is immediate that $x-b+Ia+a$ is in A. Finally, if there is carry over from $N-M-1$ to $N-M$ when adding x to $-b+Ia$, then adding a once more will affect the value on the first interval, but not on the last, and will result in $(x+b+Ia+a)_{[0,M]} = (x-b+Ia+a)_{[N-M,N]}$. We are again reduced to one of the two cases above: either $x-b+Ia+a$ or $x-b+Ia+2a$ is in A. $\square$

The first consequence we note is that, for $i = 0, 1, \ldots, p^{M+1}-1$, the sets $A+ia = \varphi^{-ia}(A)$ are pairwise disjoint. Next, we claim that the map from A to itself which sends x to $x-b+\mu(x)a$ is injective. If not, we have $x-b+\mu(x)a = x'-b+\mu(x')a$, for some x, x' in A. If $\mu(x) > \mu(x')$, then $x+(\mu(x)-\mu(x'))a = x'$ which is in A. But this means that $\lambda(x) \leq \mu(x)-\mu(x')$, which contradicts the estimates of the last lemma. In an analogous way, $\mu(x') > \mu(x)$ is impossible and we conclude that $\mu(x) = \mu(x')$. From this it follows that $x = x'$ as desired. As A is clopen and there exists a φ-invariant probability measure on X, the map above is also onto. To say this another way, we have x is in A if and only if $x-b+\mu(x)a$ is in A.

We define
$$B = \{x+ia \mid x \in A, 0 \leq i < \mu(x)\}.$$

It follows at once from the last paragraph that

$$\begin{aligned}
A \cup \varphi^{-a}(B) &= A \cup \{x' + a \mid x' \in B\} \\
&= A \cup \{x + ia \mid x \in A, 0 < i \leq \mu(x)\} \\
&= \{x + ia \mid x \in A, 0 \leq i \leq \mu(x)\} \\
&= \{x + \mu(x)a \mid x \in A\} \cup B \\
&= \{x + \mu(x)a \mid x - b + \mu(x)a \in A\} \cup B \\
&= \{x + \mu(x)a \mid x + \mu(x)a \in A + b\} \cup B \\
&= (A + b) \cup B \\
&= \varphi^{-b}(A) \cup B.
\end{aligned}$$

Notice that the facts that A and $\varphi^{-a}(B)$ are disjoint, as are $\varphi^{-b}(A)$ and B, are clear from the above computation. It is easy to check that A and B satisfy the final condition of Theorem 3.9 with $n = p^K$ since

$$\sup\{\mu(x) \mid x \in A\} \leq p^K \inf\{\lambda(x) \mid x \in A\}.$$

We omit the details.

We remark that in this example, there is a short exact sequence

$$0 \to \mathbb{Z} \to H^1(X, \varphi) \xrightarrow{q} \mathbb{Z}[\frac{1}{p}] \to 0.$$

The map q is defined as follows. For any θ in $Z^1(X, \varphi)$, we have

$$q([\theta]) = \int_X \theta(x, (1, 0)) d\mu(x),$$

where μ is Haar measure on X.

Example 9.4. *Rotations of a disconnected circle.*

Let α, β be two numbers such that $\{1, \alpha, \beta\}$ is linearly independent over the rational numbers. For simplicity, we will assume that α, β are both between 0 and $\frac{1}{2}$.

We consider the natural action of $\mathbb{Z}^2$ on the circle, $\mathbb{R}/\mathbb{Z}$, by rotating by α and by β. We select a single orbit, say that of 0, and cut the circle at these points, replacing each by two points separated by a gap. The old point will be come the right endpoint of the gap and a new point will be the left end of the gap. Let us make this more precise as follows.

We consider the subgroup of $\mathbb{R}$, $Cut = \{i + j\alpha + k\beta \mid i, j, k \in \mathbb{Z}\}$. We let $\tilde{X} = \mathbb{R} \cup \{a' \mid a \in Cut\}$. We give $\tilde{X}$ a linear order by setting $a' < b$, $a < b'$ and $a' < b'$ as appropriate, whenever $a < b$. Finally, we set $a' < a$, for all a in $\mathbb{R}$. The space $\tilde{X}$ is given the order topology. Notice that for $x < y$ in Cut, $[x, y) = [x, y']$ is a clopen set in $\tilde{X}$. The natural action of the group $\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ extends in a natural way to $\tilde{X}$. We let $X = \tilde{X}/\mathbb{Z}$, which has an action of $\alpha\mathbb{Z} + \beta\mathbb{Z}$. This is our Cantor minimal $\mathbb{Z}^2$ system, φ,

$$\varphi^{(i,j)}(x) = x - i\alpha - j\beta,$$

where x is a real number, interpreted modulo $\mathbb{Z}$, and

$$\varphi^{(i,j)}(x') = (x - i\alpha - j\beta)',$$

for x in Cut.

We claim that φ satisfies the hypotheses of Theorem 3.9 and hence Corollary 4.2. Choose a pair of generators, a, b, of $\mathbb{Z}^2$. Now φ^a and φ^b are again rotations of our cut-up circle X and, for simplicity, we let a and b denote real numbers such that $\varphi^a(x) = x - a$ and $\varphi^b(x) = x - b$, both interpreted modulo the integers. Since a and b are generators of $\mathbb{Z}^2$, the subgroup of $\mathbb{R}$ generated by $a, b, 1$ is the same as that generated by $\alpha, \beta, 1$.

Consider for the moment, the homeomorphism, η of $\mathbb{R}^2/\mathbb{Z}^2$ defined by $\eta(x, y) = (x + a, y + b)$. From Theorem 1, page 97 of [CFS], this action is minimal if and only if there is no non-trivial character of $\mathbb{R}^2/\mathbb{Z}^2$ which annihilates (a, b). The non-existence of such a character is an immediate consequence of the fact that $\{1, \alpha, \beta\}$ are linearly independent over the rationals. We conclude that η is minimal.

Let N be a positive integer. From the minimality of η, we may find a positive integer q such that $\eta^q(0, 0) \in (0, \frac{1}{2N}) \times (0, \frac{1}{2N})$. This means that

$$0 < qa - i < \frac{1}{2N}, 0 < qb - j < \frac{1}{2N}$$

for some integers i, j, or equivalently,

$$0 < a - \frac{i}{q} < \frac{1}{2Nq}, 0 < b - \frac{j}{q} < \frac{1}{2Nq}.$$

From this it follows that, for any $0 \leq m, n < N$ and $k \in \mathbb{Z}$, we have

$$\begin{array}{lcl} \frac{k}{q} \leq & \frac{k}{q} + m(a - \frac{i}{q}) + n(b - \frac{j}{q}) \leq & \frac{k}{q} + \frac{N}{2Nq} + \frac{N}{2Nq} \\ \frac{k}{q} \leq & \frac{k - im - jn}{q} + ma + nb \leq & \frac{k+1}{q} \end{array}$$

Consider the finite set of distinct points in $\mathbb{R}/\mathbb{Z}$,

$$\{\frac{k}{q} + ma + nb \mid 0 \leq m, n < N, k \in \mathbb{Z}\}$$

and choose δ to be less than half the distance between any two of these (in $\mathbb{R}/\mathbb{Z}$). For each $0 \leq k < q$, choose a point x_k with

$$0 < x_k - \frac{k}{q} < \delta, x_k \in Cut.$$

We interpret x_k for any k in $\mathbb{Z}$ by considering k modulo q. This means that we have

$$x_k < x_{k-im-jn} + ma + nb < x_{k+1},$$

for any $k = 0, 1, \ldots, q-1$ and $0 \leq m, n < N$.

We define

$$A = \bigcup_{k=0}^{q-1} [x_k, x_{k-i} + a), B = \bigcup_{k=0}^{q-1} [x_k, x_{k-j} + b),$$

where $k-i$ and $k-j$ are interpreted $mod\, q$. From the estimates above, we see that the intervals appearing in the union in the definition of A are pairwise disjoint. The analogous statement is true of B. We calculate

$$\begin{aligned} A \cup \varphi^{-a}(B) &= \{\cup_k [x_k, x_{k-i} + a)\} \cup \{\cup_{k'} [x_{k'} + a, x_{k'-j} + a + b)\} \\ &= \cup_k \{[x_k, x_{k-i} + a) \cup [x_{k-i} + a, x_{k-i-j} + a + b)\} \\ &= \cup_k [x_k, x_{k-i-j} + a + b). \end{aligned}$$

Again using arguments similar to those above, the intervals involved in the above union are pairwise disjoint. A similar computation shows that

$$\varphi^{-b}(A) \cup B = A \cup \varphi^{-a}(B).$$

Moreover, for any $0 \leq n < N$, we have

$$\varphi^{-n(a+b)}(A \cup \varphi^{-a}(B)) = \cup_k [x_k + (n-1)(a+b), x_{k-i-j} + n(a+b)).$$

Again from the estimates above, these sets are pairwise disjoint.

We remark that in this example, we have $H^1(X, \varphi) \cong \mathbb{Z}^3$. See [FH] for a proof.

References

[BBG] J. Bellissard, R. Benedetti and J.-M. Gambaudo, *Spaces of Tilings, Finite Telescopic Approximations and Gap-Labelling*, preprint.

[BG] R. Benedetti and J.-M. Gambaudo, *On the dynamics of G-solenoids: application to Delone sets*, Ergod. Th. & Dynam. Sys.**23** (2003), 673-691.

[CFW] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergod. Th. & Dynam. Sys. **1** (1981), 431-450.

[CFS] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, *Ergodic Theory*, Grundlehren der Math. Wiss. **245**, Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[D] H.A. Dye, *On groups of measure preserving transformations I*, Amer. J. Math. **81**(1959), 119-159.

[FM] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I and II*, Trans. Amer. Math. Soc. **234**(1977), 289-359.

[F] A. Forrest, *A Bratteli diagram for commuting homeomorphisms of the Cantor set*, Internat. J. Math. **11**(2000), 177-200.

[FH] A. Forrest and J. Hunton, *The cohomology and K-theory of commuting homeomorphisms of the Cantor*, Ergod. Th. & Dynam. Sys. **19** (1999), 611-625.

[GPS1] T. Giordano, I.F. Putnam and C.F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. reine andgew. Math. **469** (1995), 51-111.

[GPS2] T. Giordano, I.F. Putnam and C.F. Skau, *Affable equivalence relations and orbit structure of Cantor dynamical systems*, Ergod. Th. & Dynam. Sys. **23** (2004), 441-475.

[JKL] A. Jackson, A.S. Kechris and A. Louveau, *Countable Borel equivalence relations*, J. Math. Logic **2** (2002), 1-80.

[OW1] D.S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions I: The Rohlin lemma*, Bull. A.M.S. **2** (1980), 161-164.

[OW2] D.S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. d'Anal. Math. **48**(1987), 1-141.

[PPZ] J. Peebles, I.F. Putnam and I.F. Zwiers, *A survey of orbit equivalence for Cantor minimal dynamics*, in preparation.

[Ph] N.C. Phillips, *Crossed product of the Cantor set by a free, minimal action of* $\mathbb{Z}^2$, Comm. Math. Phys., to appear.

[R] J. Renault, *A Groupoid Approach to* C^**-algebras*, Lecture Notes in Mathematics **739**, Springer, Berlin, 1980.