# Cocycles for Cantor minimal $\mathbb{Z}^d$-systems

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**Abstract**

We consider a minimal, free action, $\varphi$, of the group $\mathbb{Z}^d$ on the Cantor set $X$, for $d \geq 1$. We introduce the notion of small positive cocycles for such an action. We show that the existence of such cocycles allows the construction of finite Kakutani-Rohlin approximations to the action. In the case, $d = 1$, small positive cocycles always exist and the approximations provide the basis for the Bratteli-Vershik model for a minimal homeomorphism of $X$. Finally, we consider two classes of examples when $d = 2$ and show that such cocycles exist in both.

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# 1 Introduction

We continue our investigations of the structure of minimal dynamical systems on Cantor sets. We consider a Cantor set $X$; that is, $X$ is an infinite, compact metrizable, totally disconnected space having no isolated points with an action of the group $\mathbb{Z}^d, d \geq 1$. This means that for every $n$ in $\mathbb{Z}^d$, we have a homeomorphism $\varphi^n : X \to X$. These satisfy the conditions $\varphi^0(x) = x$, for all $x$ in $X$ and $\varphi^m \circ \varphi^n = \varphi^{m+n}$, for all $m, n$ in $\mathbb{Z}^d$. The action is *free* if, for any $x$ in $X$ and $n$ in $\mathbb{Z}^d$, we have $\varphi^n(x) = x$ only if $n = 0$. The *orbit* of a point $x$ in $X$ is the set $\{\varphi^n(x) \mid n \in \mathbb{Z}^d\}$. The action is minimal if the only closed $\varphi$-invariant subsets $Z$ in $X$ (i.e. $\varphi^n(Z) = Z$, for all $n$ in $\mathbb{Z}^d$) are $X$ and the empty set. Equivalently, the action is minimal if the orbit of every point $x$ is dense in $X$.

We are interested in the properties of the cohomology groups of the action, particularly, the first cohomology group. Even more specifically, we are concerned with the existence of 'small, positive' 1-cocycles. We will review the definition of cohomology and introduce the property of 'having small, positive cocycle' in the next section. We will describe some consequences of this property and give some non-trivial examples where the property holds. We conjecture that the property holds for all free, minimal actions of $\mathbb{Z}^d, d \geq 1$.

Our original interest in the problem arose from the study of the orbit structure of minimal $\mathbb{Z}^2$ Cantor systems. We were able to show that, under the hypothesis of having small positive cocycles, the system was orbit equivalent to a minimal AF-system and hence also to a minimal $\mathbb{Z}$ action [GPS2]. In joint work with Hiroki Matui, we have since given a proof of the result without this hypothesis [GMPS2]. However, we believe that a more complete understanding of the first cohomology, particularly regarding the kind of order structure which we consider here, is important to the development of the theory.

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# 2 Preliminaries

The aim of this section is to provide some general definitions (old and new) and establish some basic properties.

## 2.1 Cocycles and cohomology

We begin with some basic definitions for cohomology. The basic references are [FM, R].

Our cohomology is done in the continuous category; the group $\mathbb{Z}^d$ is given the discrete topology, while $X \times \mathbb{Z}^d$ is given the product topology.

**Definition 2.1.** *Let $\varphi$ be a free action of $\mathbb{Z}^d$ on the compact space $X$. A* one-cocycle taking values in $\mathbb{Z}$ *or just* cocycle *for $\varphi$ is a continuous function*

$$\theta : X \times \mathbb{Z}^d \to \mathbb{Z}$$

*such that, for all $x$ in $X$ and $m, n$ in $\mathbb{Z}^d$, we have*

$$\theta(x, m+n) = \theta(x, m) + \theta(\varphi^m(x), n).$$

*We let $Z^1(X, \varphi)$ denote the set of all cocycles, which is a group under addition. If $f$ is in $C(X, \mathbb{Z})$, then the function*

$$bf(x, n) = f(\varphi^n(x)) - f(x)$$

*is called a* coboundary*. We let $B^1(X, \varphi)$ denote the set of coboundaries. It is easily seen to be a subgroup of $Z^1(X, \varphi)$. We let*

$$H^1(X, \varphi) = Z^1(X, \varphi)/B^1(X, \varphi)$$

*denote the quotient group.*

We let

$$R_\varphi = \{(x, \varphi^n(x)) \mid x \in X, n \in \mathbb{Z}^d\}$$

denote the orbit relation of $\varphi$. That is, $R_\varphi$ is the equivalence relation whose equivalence classes are the orbits of $\varphi$. Since the action is free, there is a natural bijection between $X \times \mathbb{Z}^d$ and $R_\varphi$ sending $(x, n)$ to $(x, \varphi^n(x))$, $x \in X, n \in \mathbb{Z}^d$. Moreover, we use this bijection to carry the topology on $X \times \mathbb{Z}^d$ to $R_\varphi$. There will be times when it will be easier to consider our cocycles as defined on $R_\varphi$. It this case, the cocycle condition of the definition becomes

$$\theta(x, z) = \theta(x, y) + \theta(y, z),$$

for all $(x, y), (y, z)$ in $R_\varphi$. Moreover, if $f$ is in $C(X, \mathbb{Z})$, then

$$bf(x, y) = f(y) - f(x)$$

for all $(x, y)$ in $R_\varphi$.

## 2.2 Positive cocycles and small cocycles

We want to introduce the notion of positivity and strict positivity for cocycles.

**Definition 2.2.** *Let $\varphi$ be a free action of $\mathbb{Z}^d$ on $X$. Let $F$ be an independent set of generators of $\mathbb{Z}^d$; that is, each element of $n$ in $\mathbb{Z}^d$ has a unique presentation*

$$n = \sum_{m \in F} i_m m,$$

*where $i_m$ is in $\mathbb{Z}$, for each $m$ in $F$. Let*

$$\mathbb{Z}^+F = \{\sum_{m \in F} i_m m \mid i_m \geq 0, m \in F\}$$

*be the subsemigroup (or cone) of $\mathbb{Z}^d$ generated by $F$.*

1. *A cocycle $\theta$ is* positive with respect to $F$ *if*

$$\theta(X \times \mathbb{Z}^+F) \geq 0.$$

2. *A cocycle $\theta$ is* proper with respect to $F$ *if the map*

$$\theta : X \times \mathbb{Z}^+F \to \mathbb{Z}$$

*is proper (i.e.,the pre-image of any finite set is compact).*

3. *A cocycle is* strictly positive with respect to $F$ *if it is proper and positive with respect to $F$.*

We want to establish some elementary results related to these notions. The first is simple enough that we leave the proof to the reader.

**Lemma 2.3.** *Let $\varphi$ be a free action of $\mathbb{Z}^d$ on $X$. Let $F$ be an independent set of generators of $\mathbb{Z}^d$. A cocycle $\theta$ is positive with respect to $F$ if and only if $\theta(X \times F) \geq 0$.*

**Lemma 2.4.** *Let $F$ be an independent set of generators of $\mathbb{Z}^d$ and let $\theta$ be a cocycle which is positive with respect to $F$. Then $\theta$ is strictly positive with respect to $F$ if and only if the set*

$$S = \{n \in \mathbb{Z}^+F \mid \theta(x, n) = 0, \text{ for some } x \in X\}$$

*is finite.*

*Proof.* If $\theta$ is proper with respect to $F$, then $\theta^{-1}\{0\} \cap X \times \mathbb{Z}^+F$ is compact and it follows at once that $S$ is finite. We consider the reverse implication. For each $n$ in $S$, write $n = \sum_{m \in F} k_{n,m} m$, where $k_{n,m} \geq 0$. Let $K$ be the maximum of all $k_{n,m}, n \in S, m \in F$. It follows that from the choice of $K$ and the positivity of $\theta$ that $\theta(x, (K+1)m) \geq 1$, for all $m$ in $F$. Using the cocycle condition and a simple induction argument, it follows that $\theta(x, l(K+1)m) \geq l$, for all $m$ in $F$, $l \geq 1$. It follows that for any $x$ in $X$ and $k_m \geq 0, m \in F$, if $\theta(x, \sum_m k_m m) = l$, then $k_m \leq (l+1)(K+1)$, for every $m$ in $F$. Thus, $\theta^{-1}\{l\}$ is compact and $\theta : X \times \mathbb{Z}^+F \to \mathbb{Z}$ is proper. □

**Lemma 2.5.** *Let $\theta$ be a cocycle for $(X, \varphi)$, let $F$ be an independent set of generators of $\mathbb{Z}^d$ and let $h$ be in $C(X, \mathbb{Z})$.*

1. *If $\theta$ is positive with respect to $F$, then $\theta + bh : X \times \mathbb{Z}^+F \to \mathbb{Z}$ is bounded below.*

2. *If $\theta$ is proper with respect to $F$, then so is $\theta + bh$.*

*Proof.* It is easy to see that, since $h$ is bounded, the cocycle $bh$ is a bounded function on $X \times \mathbb{Z}^d$. The result follows immediately. □

If $\theta$ is a cocycle, then it follows from the cocycle condition that $\theta(x, -n) = -\theta(\varphi^{-n}(x), n)$, for all $x$ in $X$ and $n$ in $\mathbb{Z}^d$. From this fact, the next result follows easily. We omit the proof.

**Lemma 2.6.** 1. *If the cocycle $\theta$ is positive with respect to the set of independent generators $F$, then $-\theta$ is positive with respect to $-F$.*

2. *If $\theta$ is proper with respect to $F$ then it is also proper with respect to $-F$.*

We will endow $\mathbb{Z}^d$ with the $l^\infty$-norm, denoted $|n|$; that is, for $n$ in $\mathbb{Z}^d$, $|n| = max\{|n_i| \mid 1 \leq i \leq d\}$.

**Definition 2.7.** *Let $\theta$ be a cocycle for $(X, \varphi)$ and let $M \geq 1$. We say that $\theta \leq M^{-1}$ if $|\theta(x, m)| \leq 1$ for all $x$ in $X$ and $m$ in $\mathbb{Z}^d$ with $|m| \leq M$.*

We are now ready to state the property of having small positive cocycles.

**Definition 2.8.** *Let $\varphi$ be a free action of $\mathbb{Z}^d$ on $X$. We say that $\varphi$ has small positive cocycles if, for every independent set of generators, $F$, and every $M \geq 1$, there is a cocycle $\theta$ such that*

*1. $\theta$ is strictly positive with respect to $F$ and*

*2. $\theta \leq M^{-1}$.*

The following characterization of this condition is quite straightforward and will be useful.

**Proposition 2.9.** *Let $\varphi$ be a free action of $\mathbb{Z}^d$ on $X$. Then $\varphi$ has small positive cocycles if and only if, for every independent set of generators $F$ and every $K \geq 1$, there exists a cocycle $\theta$ such that $\theta$ is strictly positive with respect to $F$ and $\theta(x, K\sum_{m\in F} m) \leq 1$, for all $x$ in $X$.*

*Proof.* First suppose that $\varphi$ has small positive cocycles. Let $F$ and $K$ be as above. Let $M = |K\sum_{m\in F} m|$. By hypothesis, there exists a cocycle $\theta$ which is strictly positive with respect to $F$ and such that $\theta \leq M^{-1}$. But this implies that $\theta(K\sum_{m\in F} m) \leq 1$.

Next suppose that $\varphi$ satisfies the condition of the proposition. Let $F$ be an independent set of generators and let $M \geq 1$. Write each element $n$ of $\mathbb{Z}^d$ with $|n| \leq M$ as $n = \sum_{m\in F}(i_{n,m} - j_{n,m})m$, where $i_{n,m}, j_{n,m} \geq 0$. Let $K$ be the maximum of all $i_{n,m}, j_{n,m}$ taken as $m$ varies over $F$ and $|n| \leq M$. Choose a cocycle $\theta$ which is strictly positive with respect to $F$ and such that $\theta(x, K\sum_{m\in F} m) \leq 1$, for all $x$ in $X$.

Now for $|n| \leq M$, we may write

$$n = \sum_{m\in F} i_{n,m}m - \sum_{m\in F} j_{n,m}m = n' - n'',$$

where $n', n'', K\sum_{m\in F} m - n', K\sum_{m\in F} m - n''$ all in $\mathbb{Z}^+F$. It follows that for any $x$ in $X$,

$$\begin{aligned}
0 &\leq \theta(x, n') \\
&\leq \theta(x, n') + \theta(\varphi^{n'}(x), K\sum_{m\in F} m - n') \\
&= \theta(x, K\sum_{m\in F} m) \\
&\leq 1.
\end{aligned}$$

Similarly, for any $x$ in $X$, we have $0 \leq \theta(x, n'') \leq 1$. Finally, we have

$$\theta(x, n) + \theta(\varphi^n(x), n'') = \theta(x, n').$$

From this, we conclude that $|\theta(x, n)| \leq 1$. This completes the proof. $\square$

The value of any cocycle is determined on a generating set for $\mathbb{Z}^d$ and on our generating sets, we are usually looking for the cocycle to take values 0 and 1 only. Hence, the function $\theta(\cdot, m)$ is a characteristic function of some set, and since it must be continuous, that set must be clopen. We can rephrase our cocycle condition in the following form.

**Theorem 2.10.** *Let $\varphi$ be a free minimal action of $\mathbb{Z}^d$ on the Cantor set $X$. Suppose that, for any independent set of generators $F = \{m_1, m_2, \ldots, m_d\}$ and any $K \geq 1$, we may find non-empty clopen sets $A_i, 1 \leq i \leq d$ satisfying the following.*

1. *For any $1 \leq i, j \leq d$, $A_i$ and $\varphi^{-m_i}(A_j)$ are disjoint and we have*

$$A_i \cup \varphi^{-m_i}(A_j) = A_j \cup \varphi^{-m_j}(A_i).$$

2. *The sets*

$$A_1, \varphi^{-m_1}(A_2), \varphi^{-m_1-m_2}(A_3), \ldots, \varphi^{-m_1-\ldots-m_{d-1}}(A_d)$$

*are pairwise disjoint.*

3. *For $0 \leq k \leq K$, the sets*

$$\varphi^{-k\sum_i m_i}(A_1 \cup \varphi^{-m_1}(A_2) \cup \varphi^{-m_1-m_2}(A_3) \cup \ldots \cup \varphi^{-m_1-\ldots-m_{d-1}}(A_d))$$

*are pairwise disjoint.*

4. *For each $1 \leq i \leq d$, we have*

$$\cup_{k\geq 0}\varphi^{-km_i}(A_i) = X.$$

*Then $\varphi$ has small positive cocycles.*

*Proof.* We will show that $\varphi$ satisfies the condition of Proposition 2.9. So we begin with an independent set of generators $F = \{m_1, m_2, \ldots, m_d\}$ and $K \geq 1$. We must produce a cocycle $\theta$ as in 2.9. From the hypothesis we may find sets $A_i, 1 \leq i \leq d$ satisfying the conditions of the Theorem. We define our cocycle as follows. For each $x$ in $X$ and $1 \leq i \leq d$, we set

$$\theta(x, m_i) = \chi_{A_i}(x),$$

where $\chi_E$ denotes the characteristic function of any set $E \subset X$. Next, we extend the definition to $X \times \mathbb{Z}^+ F$. This is done by defining $\theta(x, \sum_i n_i m_i)$, for $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d, n_1, \ldots, n_d \geq 1$, inductively on the value of $|n|$, using the cocycle condition

$$\theta(x, \sum_i n_i m_i + m_j) = \theta(x, m_j) + \theta(\varphi^{m_j}(x), \sum_i n_i m_i),$$

for any $1 \leq j \leq d$. (We have already done the case $|n| = 1$.) To see this is well-defined, one must check that, for any $n$ as above and $1 \leq j, j' \leq d$, we have

$$\theta(x, m_j) + \theta(\varphi^{m_j}(x), \sum_i n_i m_i + m_{j'}) = \theta(x, m_{j'}) + \theta(\varphi^{m_{j'}}(x), \sum_i n_i m_i + m_j).$$

But, by using the induction hypothesis, the term on the left is equal to

$$\theta(x, m_j) + \theta(\varphi^{m_j}(x), m_{j'}) + \theta(\varphi^{m_j + m_{j'}}(x), \sum_i n_i m_i),$$

while the term on the right is equal to

$$\theta(x, m_{j'}) + \theta(\varphi^{m_{j'}}(x), m_j) + \theta(\varphi^{m_j + m_{j'}}(x), \sum_i n_i m_i).$$

Taking the difference, we get

$$\begin{aligned} \theta(x, m_j) \quad & +\theta(\varphi^{m_j}(x), m_{j'}) \\ & - \qquad \theta(x, m_{j'}) - \theta(\varphi^{m_{j'}}(x), m_j) \\ & = \qquad \chi_{A_j}(x) + \chi_{A_{j'}}(\varphi^{m_j}(x)) \\ & \qquad\quad -\chi_{A_{j'}}(x) - \chi_{A_j}(\varphi^{m_{j'}}(x)) \\ & = \qquad \chi_{A_j}(x) + \chi_{\varphi^{-m_j}(A_{j'})}(x) \\ & \qquad\quad -\chi_{A_{j'}}(x) - \chi_{\varphi^{-m_{j'}}(A_j)}(x) \end{aligned}$$

which is zero because of the first condition on the set $A_i$. Finally, one extends the definition to all of $X \times \mathbb{Z}^2$ by setting

$$\theta(x, \sum_i n_i m_i - \sum_i n'_i m_i) = \theta(x, \sum_i n_i m_i) - \theta(\varphi^{\sum_i n_i m_i + \sum_i n'_i m_i}(x), \sum_i n_i m_i).$$

Of course, one must check that this is well-defined and that $\theta$ satisfies the cocycle condition. The proofs are similar to the argument above and we omit the details.

It is clear from our definition of $\theta$ on $X \times \mathbb{Z}^+F$ that it is positive with respect to $F$. Using the definition, it is also easy to see that, for any $x$ in $X$,

$$\theta(x, \sum_i m_i) = \chi_{A_1 \cup \varphi^{-m_1}(A_2) \cup \varphi^{-m_1-m_2}(A_3) \cup \ldots \cup \varphi^{-m_1-\cdots-m_{d-1}}(A_d)}(x),$$

and from this that

$$\theta(x, K\sum_i m_i) = \chi_{\cup_{k=0}^{K} \varphi^{k\sum_i m_i}\left[A_1 \cup \varphi^{-m_1}(A_2) \cup \varphi^{-m_1-m_2}(A_3) \cup \ldots \cup \varphi^{-m_1-\cdots-m_{d-1}}(A_d)\right]}(x),$$

which is clearly bounded above by 1.

It remains for us to prove that $\theta$ is strictly positive. From the last hypothesis and the compactness of $X$, there is $K_i \geq 0$ such that

$$\cup_{k=0}^{K_i} \varphi^{-km_i}(A_i) = X,$$

$1 \leq i \leq d$. Let $K$ be the maximum of the $K_i$. It follows from the cocycle condition and positivity that $\theta(x, Km_i) \geq 1$, for any $x$ in $X$ and $1 \leq i \leq d$. Then if $\theta(x, \sum_i k_i m_i) = 0$, with $k_i \geq 0$ for all $i$, then $k_i < K$. By Lemma 2.4, $\theta$ is strictly positive. □

We complete this section by noting the following positive result in the case $d = 1$.

**Theorem 2.11.** *Every minimal free $\mathbb{Z}$ action on a Cantor set has small positive cocycles.*

*Proof.* Let $a = 1$ or $-1$ (which are the only possibilities for an independent set of generators) and let $K \geq 1$. Choose any clopen set $A$ such that $\varphi^{-ak}(A)$ are pairwise disjoint for $0 \leq k \leq K$. The single set $A$ satisfies the conditions of 2.10. □

# 3 Finite subrelations and induced systems

In this section, we use the existence of small positive cocycles to construct finite approximations to minimal free $\mathbb{Z}^d$ systems. These will be described in detail in Theorem 3.9 below but let us begin informally by describing the result in the special case $d = 1$. We review a construction for Kakutani-Rohlin towers given in [HPS].

One begins with a non-empty clopen set $Y$ and considers the first return map, $\psi$, of $\varphi$ on the set $Y$. Suppose the first return times for $\varphi$ on $Y$ are $J_1, \ldots, J_K$, for some $K$. Based on this, one constructs sets $Y(k, j), 1 \leq k \leq K, 1 \leq j \leq J_k$. These form a partition of $X$ and, for fixed $k$, they are a tower in the sense that $\varphi(Y(k, j)) = Y(k, j+1)$, for all $1 \leq j < J_k$. The set $Y$ is just the union of the $Y(k, J_k)$ while $\varphi(Y)$ is the union of the $Y(k, 1)$. Moreover, by a careful choice of $Y$ the first return times may be made arbitrarily large.

Let $R$ be the smallest equivalence relation containing $(x, \varphi(x))$, for all $x \notin Y$. It may be described concretely as follows. For each $x$ in $\varphi(Y)$, $x$ is in some $Y(k, 1)$. The set $\varphi^j(x), 0 \leq j < J_k$ is an equivalence class in $R$. Notice that $Y$ meets each such class in exactly one point. Moreover, the equivalence relation generated by $R$ and the first return map $\psi$ is exactly $R_\varphi$.

We would like to extend this construction to the case of actions of $\mathbb{Z}^d$. A number of things go wrong if we try to repeat what is done above, the first being the question of what is the first return map. Of course, there are useful versions which employ the notion of Voronoi cells to construct finite equivalence relations [F, GMPS2, Ph]. One drawback of this approach is that it is unclear how to form induced systems analogous to the map $\psi$ in the one-dimensional case, as above. Here, we pursue a different route using small positive cocycles. We make some special choices of generating sets for $\mathbb{Z}^d$ and, with the hypothesis that our system possesses small, positive cocycles, we construct a set $Y$, a compact, open subequivalence relation $R \subset R_\varphi$ and an action of $\mathbb{Z}^d$ on $Y$ which satisfy most of the conditions from the $d = 1$ case above. Indeed, for the case $d = 1$, if one starts with $Y$ as above and regards it as giving a cocycle as in 2.11, then this construction is the same.

It is natural to then continue this construction inductively to produce a kind of Bratteli-Vershik model for $\mathbb{Z}^d$-dynamical systems. We do not pursue this here for several reasons. The first is that, even if we assume that the original system $(X, \varphi)$ has small, positive cocycles, it is not clear that the new system $(Y, \psi)$ will also. This means that the obvious inductive process will not work. Nevertheless, it would seem this result can be used inductively to produce large $AF$-subrelations of $R_\varphi$.

There are three other serious reasons why this model, even if it exists, seems less appealing than the one in the one-dimensional case. The first is that the existence of small, positive cocycles is not known for the case $d > 1$, in full generality. Secondly, in the case $d = 1$, it is possible to construct a model so that the sets $Y$ shrink to a single point and it is not clear that this is possible for $d > 1$. Finally, using the model to *construct* examples of free, minimal $\mathbb{Z}^2$-actions does not seem very tractable.

In view of what we have said above, it will be useful to have a notion of a 'large' subequivalence relation. To this end, we make the following definition.

**Definition 3.1.** *Let $R$ be a subequivalence relation of $R_\varphi$ and let $C \geq 0$. We say that $R$* has capacity $C$ *if, for any $x$ in $X$, there exists $n$ in $\mathbb{Z}^d$ such that*

$$(x, \varphi^m(x)) \in R,$$

*for all $m$ in $\mathbb{Z}^d$ such that $|m - n| \leq C$.*

Recall from [GPS1] that an equivalence relation $R$ on a space $X$, in a given topology, is étale if it is a locally compact groupoid (with the usual operations) and the projection maps from the equivalence relation to the underlying space are open and local homeomorphisms. We begin with the following fairly general result.

**Proposition 3.2.** *Suppose that $F_1, F_2, \ldots, F_I$ are sets of independent generators of $\mathbb{Z}^d$ and*

$$\cup_{i=1}^{I} (\mathbb{Z}^+ F_i \cup (-\mathbb{Z}^+ F_i)) = \mathbb{Z}^d.$$

*Suppose that, for each $1 \leq i \leq I$, $\theta_i$ is a cocycle which is strictly positive with respect to $F_i$. For each $1 \leq i \leq I$, we let*

$$\ker(\theta_i) = \{(x, \varphi^n(x)) \in R_\varphi \mid x \in X, n \in \mathbb{Z}^d, \theta_i(x, n) = 0\}.$$

*Then*

$$R = \cap_{i=1}^{I} \ker(\theta_i)$$

*is a compact, open étale subequivalence relation of $R_\varphi$.*

*Proof.* It follows from the cocycle condition that $\ker(\theta_i)$ is a subequivalence relation of $R_\varphi$, for each $i$, and hence the same is true of their intersection. From the continuity of $\theta_i$, $\ker(\theta_i)$ is also closed and open and hence the same is also true of their intersection. Since $R$ is an open subequivalence relation of an étale equivalence relation, it is also étale [GPS1].

It remains for us to show that $R$ is compact. Recall that the topology on $R_\varphi$ is defined by identifying it with $X \times \mathbb{Z}^d$. Under this identification, $\ker(\theta_i)$ is just $\theta_i^{-1}\{0\}$, for any $i$. Fix $1 \leq j \leq I$ and consider

$$\left[\cap_{i=1}^{I} \theta_i^{-1}\{0\}\right] \cap X \times \mathbb{Z}^+ F_j = \cap_{i=1}^{I} (\theta_i^{-1}\{0\} \cap X \times \mathbb{Z}^+ F_j).$$

The terms in the intersection on the right hand side are all closed; moreover, the term for $i = j$ is compact by the properness of $\theta_j$ on $\mathbb{Z}^+ F_j$. Hence the intersection over $i$ is compact. A similar argument applies, replacing $\mathbb{Z}^+$ with $-\mathbb{Z}^+$. If we union these sets over all $j = 1, \ldots, I$, the result is $\cap_{i=1}^{I} \theta_i^{-1}\{0\}$ from the hypothesis on the sets $F_i$. Thus we see that $R$ is compact. $\square$

We want to apply this result in a very specific situation. Let $F = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d\}$ be the standard set of generators for $\mathbb{Z}^d$. For $1 \leq k \leq d$, consider the set of $d$ elements:

$$\begin{aligned} F_k &= \{\varepsilon_i - 2^{d-k} \sum_{l>k} \varepsilon_l, \\ &\quad \varepsilon_k - 2^{d-k} \sum_{l>k} \varepsilon_l - \sum_{i<k} \varepsilon_i, \\ &\quad \varepsilon_l \mid 1 \leq i < k < l \leq d\}. \end{aligned}$$

It is easy to see that $F_k$ is a generating set of $\mathbb{Z}^d$ and also that $\mathbb{Z}^+ F \subset \mathbb{Z}^+ F_k$.

**Lemma 3.3.** *The generating sets $F_1, F_2, \ldots, F_d$ satisfy the hypothesis of Proposition 3.2.*

*Proof.* Let $n$ be in $\mathbb{Z}^d$. Let $k$ be any integer such that $|n_k| = \sup\{|n_i| \mid 1 \leq i \leq d\}$. Assuming that $n_k \geq 0$, we will show that $n$ is in $\mathbb{Z}^+ F_k$. If $n_k \leq 0$ the same argument shows that $-n$ is in $\mathbb{Z}^+ F_k$ and so $n$ is in $-\mathbb{Z}^+ F_k$.

Using the fact that $n = \sum_{k=1}^{d} n_k \varepsilon_k$, we consider the following expression

$$\begin{aligned} n &= \sum_{i<k} (n_i + n_k) \left[\varepsilon_i - 2^{d-k} \sum_{l>k} \varepsilon_l\right] \\ &\quad + n_k \left[\varepsilon_k - 2^{d-k} \sum_{l>k} \varepsilon_l - \sum_{i<k} \varepsilon_i\right] \\ &\quad \sum_{l>k} \left[n_l + n_k 2^{d-k} + \sum_{i<k} (n_i + n_k) 2^{d-k}\right] \varepsilon_l. \end{aligned}$$

It is then clear from the choice of $k$ and the assumption that $n_k \geq 0$ that $n$ is a non-negative combination of the elements of $F_k$. $\square$

We now assume that our system $(X, \varphi)$ has small positive cocycles. Let $M > 2^d$ be given and find cocycles $\theta_1, \theta_2, \ldots, \theta_d$ such that, for each $1 \leq k \leq d$, $\theta_k \leq M^{-1}$ and $\theta_k$ is strictly positive with respect to $F_k$.

It will be useful for us to use the following (slightly abusive) notation: for $0 \leq k \leq d$, define

$$\mathbb{Z}^k = \{n \in \mathbb{Z}^d \mid n_{k+1} = n_{k+2} = \cdots = n_d = 0\}.$$

Notice that $\mathbb{Z}^0 = \{0\}$. We also define $\pi_k$ to be the usual orthogonal projection of $\mathbb{Z}^d$ onto $\mathbb{Z}^k$.

We want to define a set of continuous maps $\eta_k, 0 \leq k \leq d$,

$$\eta_k : X \times \mathbb{Z}^k \to \mathbb{Z}^d.$$

These will satisfy conditions set out in the Lemma below. We will proceed as follows. We start with $\eta_d(x, n) = n$, for $(x, n)$ in $X \times \mathbb{Z}^d$, and see that this satisfies the first condition of the Lemma. Then we show that, for any $1 \leq k \leq d$, the first condition in the Lemma implies the second. We then show how the Lemma being valid for a particular $k$ allows us to define $\eta_{k-1}$ and it satisfies the first condition.

**Lemma 3.4.** *1. For $1 \leq i \leq k \leq d$ and $n$ in $\mathbb{Z}^k$, we have*

$$\eta_k(x, n + \varepsilon_i) - \eta_k(x, n) = \varepsilon_i + \sum_{l > k} a_{i,l}\varepsilon_l,$$

*where each $a_{i,l}$ is an integer (depending on $x$ and $n$) such that $|a_{i,l}| \leq 2^{d-k-1}$.*

*2. For $1 \leq k \leq d$, $x$ in $X$ and $n$ in $\mathbb{Z}^{k-1}$, the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by*

$$f(j) = \theta_k(x, \eta_k(x, n + j\varepsilon_k))$$

*is proper and, for all $j$, $f(j) \leq f(j + 1) \leq f(j) + 1$.*

The first part of the Lemma is trivial for $k = d$ since $\eta_d$ is the identity on $\mathbb{Z}^d$.

Now, we prove that if the first part holds for some value $1 \leq k \leq d$, the second part follows. With $f$ as defined and $j$ in $\mathbb{Z}$, we have

$$\begin{aligned}
f(j+1) - f(j) &= \theta_k(x, \eta_k(x, n + (j+1)\varepsilon_k)) - \theta_k(x, \eta_k(x, n + j\varepsilon_k)) \\
&= \theta_k(\varphi^{\eta_k(x,n+j\varepsilon_k)}(x), \eta_k(x, n + (j+1)\varepsilon_k) - \eta_k(x, n + j\varepsilon_k)) \\
&= \theta_k(\varphi^{\eta_k(x,n+j\varepsilon_k)}(x), \varepsilon_k + \sum_{l>k} a_{k,l}\varepsilon_l),
\end{aligned}$$

using part 1 of the Lemma. We may write

$$\begin{aligned}
\varepsilon_k + \sum_{l>k} a_{k,l}\varepsilon_l &= \varepsilon_k - 2^{d-k}\sum_{l>k}\varepsilon_l - \sum_{i<k}\varepsilon_i \\
&\quad + \sum_{i<k}(\varepsilon_i - 2^{d-k}\sum_{l>k}\varepsilon_l) \\
&\quad + \sum_{l>k}(k2^{d-k} + a_{k,l})\varepsilon_l,
\end{aligned}$$

which is in $\mathbb{Z}^+F_k$. Hence, we see that $f(j+1) - f(j) \geq 0$ since $\theta_k$ is positive with respect to $F_k$. The fact the difference is at most one follows from the fact that $|\varepsilon_k + \sum_{l>k} a_{k,l}\varepsilon_l| \leq 2^{d-k} < M$ and the condition $\theta_k \leq M^{-1}$. It is also clear from iteration of this formula that the $k$th entry of $\eta_k(x, n + (j + j')\varepsilon_k) - \eta_k(x, n + j\varepsilon_k)$ is $j'$. So as $j$ tends to infinity, so does this vector and the properness of $f$ follows from this fact and $\theta_k$ being proper with respect to $F_k$. This completes the proof that part 2 follows from part 1.

Now, we assume that $\eta_k$ is defined, continuous and satisfies the Lemma above, for some $1 \leq k \leq d$. We define $\eta_{k-1}$ as follows. Fix $x$ in $X$ and let $n$ be in $\mathbb{Z}^{k-1}$. Let $j$ be the largest integer such that

$$\theta_k(x, \eta_k(x, n + j\varepsilon_k)) = 0. \tag{1}$$

Such an integer exists from the first part of Lemma 3.4. We define

$$\eta_{k-1}(x, n) = \eta_k(x, n + j\varepsilon_k).$$

The continuity of $\eta_{k-1}$ follows from that of $\eta_k$ and $\theta_k$. We now give a proof of Lemma 3.4, done by induction on $k = d, d-1, \ldots, 1$. Assume the result is true for $k$.

To establish the first part of the Lemma holds for $\eta_{k-1}$, we begin with the following claim: for any $x$ in $X$, $n$ in $\mathbb{Z}^k$ and $i \leq k$, we have

$$\begin{aligned}
\eta_k(x, n+\varepsilon_i) - \eta_k(x,n) &\in \mathbb{Z}^+F_k, \\
\eta_k(x, n+\varepsilon_k-\varepsilon_i) - \eta_k(x,n) &\in \mathbb{Z}^+F_k,
\end{aligned}$$

and the norm of both of these is at most $2^{d-k}$. For the first, using the induction hypothesis, we have

$$\begin{aligned}
\eta_k(x, n+\varepsilon_i) - \eta_k(x,n) &= \varepsilon_i + \sum_{l>k} a_{i,l}\varepsilon_l \\
&= \varepsilon_i - \sum_{l>k} 2^{d-k}\varepsilon_l + \sum_{l>k}(2^{d-k} + a_{i,l})\varepsilon_l,
\end{aligned}$$

which is in $\mathbb{Z}^+F_k$, by the definition of $F_k$. The conclusion regarding the norm of this element is immediate from the right hand side of the first line and the hypothesis on $a_{i,l}$. For the second, we again use the induction hypothesis:

$$\begin{aligned}
\eta_k(x, n+\varepsilon_k-\varepsilon_i) - \eta_k(x,n) &= \eta_k(x, n+\varepsilon_k-\varepsilon_i) - \eta_k(x, n-\varepsilon_i) \\
&\quad -(\eta_k(x,(n-\varepsilon_i)+\varepsilon_i) - \eta_k(x,n-\varepsilon_i)) \\
&= \left[\varepsilon_k + \sum_{l>k} a_{k,l}\varepsilon_l\right] - \left[\varepsilon_i + \sum_{l>k} a_{i,l}\varepsilon_l\right] \\
&= \left[\varepsilon_k - \varepsilon_i - \sum_{l>k} 2^{d-k}\varepsilon_l\right] \\
&\quad + \sum_{l>k}(2^{d-k} + a_{i,l} + a_{k,l})\varepsilon_l
\end{aligned}$$

which is in $\mathbb{Z}^+F_k$ as desired, since $|a_{i,l}+a_{k,l}| \leq 2^{d-k-1}+2^{d-k-1} = 2^{d-k}$. Again, the result concerning the norm of this element follows from the second line on the right hand side and the same estimate on $|a_{i,l} + a_{k,l}|$.

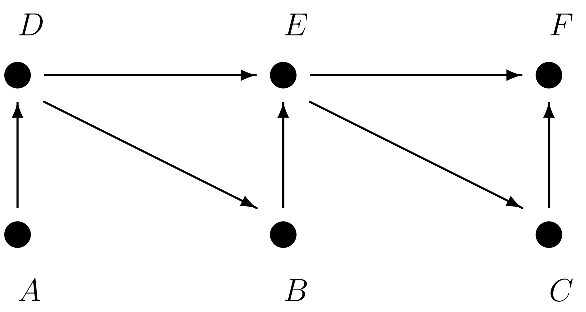
Suppose that $\eta_{k-1}(x,n) = \eta_k(x, n+j\varepsilon_k)$, which we denote by $B$. Let $C = \eta_k(x, n+(j+1)\varepsilon_k)$, so that

$$\theta_k(x,B) = 0, \theta_k(x,C) = 1,$$

from the choice of $j$ in the definition of $\eta_{k_1}$ in 1. For convenience, let

$$\begin{aligned} A &= \eta_k(x, n + (j-1)\varepsilon_k), \\ D &= \eta_k(x, n + \varepsilon_i + (j-1)\varepsilon_k), \\ E &= \eta_k(x, n + \varepsilon_i + j\varepsilon_k), \\ F &= \eta_k(x, n + \varepsilon_i + (j+1)\varepsilon_k). \end{aligned}$$

The following diagram, although it may not be very accurate, may be of help in keeping track of these points.




Each horizontal move to the right is the result of adding $\varepsilon_k$ (and then applying $\eta_k$), while each vertical move up is the result of adding $\varepsilon_i$ (and then applying $\eta_k$). We will use the claims established above to show that each vector in our picture lies in $\mathbb{Z}^+ F_k$.

If we use the first part of the claim above, using $n + (j-1)\varepsilon_k + \varepsilon_i$ and $n + j\varepsilon_k + \varepsilon_i$ in place of $n$, we obtain that $E - D$ and $F - E$, respectively, are in $\mathbb{Z}^+ F_k$ and each has norm at most $2^{d-k}$. Similarly, using $n + (j-1)\varepsilon_k, n + j\varepsilon_k$ and $n + (j+1)\varepsilon_k$, we obtain the same conclusion for $D - A, E - B$ and $F - C$. Finally, using $n + (j-1)\varepsilon_k + \varepsilon_i$ and $n + j\varepsilon_k + \varepsilon_i$ in the second part, we obtain the same conclusion for $B - D$ and $C - E$.

We claim that $\theta_k(x, D) = 0$. We know that $\theta_k(\varphi^D(x), B - D) \geq 0$, since $B - D$ is in $\mathbb{Z}^+ F_k$. On the other hand, since $|C - D| \leq |C - B| + |B - D| \leq$

$2^{d-k} + 2^{d-k} = 2^{d-k+1} < M$, we have

$$\begin{aligned} 1 &\geq \theta_k(\varphi^D(x), C - D) \\ &= \theta_k(\varphi^D(x), B - D) + \theta_k(\varphi^B(x), C - B) \\ &= \theta_k(\varphi^D(x), B - D) + \theta_k(x, C) - \theta_k(x, B) \\ &= \theta_k(\varphi^D(x), B - D) + 1 - 0 \\ &\geq 1. \end{aligned}$$

The conclusion follows as

$$\theta_k(x, D) = \theta_k(x, B) - \theta_k(\varphi^D(x), B - D) = 0 - 0.$$

This means that the $j$ needed in the definition of $\eta_{k-1}(x, n + \varepsilon_i)$ (as given by the appropriate version of 1) must be at least our $j - 1$.

Next, we claim that $\theta_k(x, F) = 1$. Similar to the last case, we have $|F - D| \leq 2^{d-k+1} < M$, and it follows that

$$\begin{aligned} 1 &\geq \theta_k(\varphi^D(x), F - D) \\ &= \theta_k(x, F) - \theta_k(x, D) \\ &= \theta_k(x, F) \\ &= \theta_k(x, C) + \theta_k(\varphi^C(x), F - C) \\ &\geq 1 + 0, \end{aligned}$$

giving the desired conclusion.

We now claim $\theta_k(x, E)$ is either 0 or 1. To see this, we use the facts $F - E$ and $E - D$ are in $\mathbb{Z}^+ F_k$ and

$$\begin{aligned} 1 &= \theta_k(x, F) \\ &= \theta_k(x, E) + \theta_k(\varphi^E(x), F - E) \\ &\geq \theta_k(x, E) \\ &= \theta_k(x, D) + \theta_k(\varphi^D(x), E - D) \\ &\geq 0 + 0. \end{aligned}$$

If $\theta_k(x, E) = 1$, then it follows from the definition that $\eta_{k-1}(x, n+\varepsilon_i) = D$, while if $\theta_k(x, E) = 0$, $\eta_{k-1}(x, n + \varepsilon_i) = E$.

First, we consider the former case for proving the first part of the Lemma and compute

$$\begin{aligned}\eta_{k-1}(x, n+\varepsilon_i) - \eta_{k-1}(x,n) &= D - B \\ &= \eta_k(x, n+\varepsilon_i + (j-1)\varepsilon_k) \\ &\quad -\eta_k(x, n+j\varepsilon_k) \\ &= \eta_k(x, n+\varepsilon_i + (j-1)\varepsilon_k) \\ &\quad -\eta_k(x, n+(j-1)\varepsilon_k) \\ &\quad -(\eta_k(x, n+j\varepsilon_k) \\ &\quad -\eta_k(x, n+(j-1)\varepsilon_k)) \\ &= \varepsilon_i + \sum_{l>k} a_{i,l}\varepsilon_l - \varepsilon_k - \sum_{l>k} a_{k,l}\varepsilon_l \\ &= \varepsilon_i + (-1)\varepsilon_k + \sum_{l>k}(a_{i,l} - a_{k,l})\varepsilon_l.\end{aligned}$$

Further, using $a_{i,k} = -1$, we have $|a_{i,k}| = |-1| \leq 2^{d-k+1}$ and

$$|a_{i,l} - a_{k,l}| \leq |a_{i,l}| + |a_{k,l}| \leq 2^{d-k} + 2^{d-k} = 2^{d-k+1}.$$

So the first condition holds in this case.

For the case $\theta_k(x, E) = 0$ and $\eta_{k-1}(x, n+\varepsilon_i) = E$, we have

$$\begin{aligned}\eta_{k-1}(x, n+\varepsilon_i) - \eta_{k-1}(x,n) &= E - B \\ &= \eta_k(x, n+\varepsilon_i + j\varepsilon_k) - \eta_k(x, n+j\varepsilon_k) \\ &= \varepsilon_i + \sum_{l>k} a_{i,l}\varepsilon_l,\end{aligned}$$

which is clearly of the desired form. This completes the proof that $\eta_{k-1}$ satisfies part 1 of the Lemma.

**Lemma 3.5.** *1. For each $0 \leq k \leq d$, $x$ in $X$ and $n$ in $\mathbb{Z}^k$, we have*

$$\pi_k \circ \eta_k(x, n) = n.$$

*2.*

$$\eta_d(x, \mathbb{Z}^d) \supset \eta_{d-1}(x, \mathbb{Z}^{d-1}) \supset \cdots \supset \eta_1(x, \mathbb{Z}^1) \supset \eta_0(x, \mathbb{Z}^0)$$

*3. If $n$ is in $\eta_i(x, \mathbb{Z}^i)$ and $i < k$, then $\theta_k(x, n) = 0$.*

4. *If $n$ is in $\eta_i(x, \mathbb{Z}^i)$ and $i < k$, then there exists $l$ in $\mathbb{Z}^d$ such that $|l| \leq 2^d$ and $\theta_k(\varphi^n(x), l) > 0$.*

*Proof.* Not surprisingly, the proof of the first part is by induction, beginning with $k = d$, in which case it is clear. Next, assume the result is true for some $1 \leq k \leq d$. Let $n$ be in $\mathbb{Z}^{k-1}$. We know that $\eta_{k-1}(x, n) = \eta_k(x, n + j\varepsilon_k)$. It follows from part 2 of Lemma 3.4 that

$$\eta_{k-1}(x, n) = \eta_k(x, n + j\varepsilon_k) = \eta_k(x, n) + j\varepsilon_k + m,$$

where $m$ is a combination of $\varepsilon_l, l > k$. Applying $\pi_{k-1} = \pi_{k-1} \circ \pi_k$ to both sides yields

$$\begin{aligned}
\pi_{k-1}(\eta_{k-1}(x, n)) &= \pi_{k-1}(\eta_k(x, n)) + \pi_{k-1}(j\varepsilon_k) + \pi_{k-1}(m) \\
&= \pi_{k-1} \circ \pi_k(\eta_k(x, n)) + 0 + 0 \\
&= \pi_{k-1}(n) \\
&= n,
\end{aligned}$$

since $n$ is in $\mathbb{Z}^{k-1}$.

The second statement follows from the fact that, for any $k, x, n$, we have $\eta_{k-1}(x, n) = \eta_k(x, n + j\varepsilon_k)$, for some $j$.

For the third part, in view of the second, it suffices to consider the case $i = k - 1$. We assume that $n = \eta_{k-1}(x, m) = \eta_k(x, m + j\varepsilon_k)$, for some $m$ in $\mathbb{Z}^i$. Moreover, the $j$ is chosen so that

$$\theta_k(x, \eta_k(x, n + j\varepsilon_k)) = 0.$$

It follows at once that

$$\theta_k(x, n) = \theta_k(x, \eta_{k-1}(x, m)) = \theta_k(x, \eta_k(x, n + j\varepsilon_k)) = 0,$$

as desired.

For the fourth part, in view of the second, we know that $n$ is also in $\eta_{k-1}(x, \mathbb{Z}^{k-1})$. Assume that $n = \eta_{k-1}(x, m) = \eta_k(x, m + j\varepsilon_k)$, for some $m$ in $\mathbb{Z}^{k-1}$ and $j$. We have $\eta_{k-1}(x, n) = \eta_k(x, n + j\varepsilon_k)$ for some $j$. In addition, from the choice of $j$ in the definition of $\eta_{k-1}$, we know

$$0 \neq \theta_k(x, \eta_k(x, m + (j + 1)\varepsilon_k)).$$

It follows from the first part of Lemma 3.4 that

$$\theta_k(x, \eta_k(x, m + (j + 1)\varepsilon_k)) = 1.$$

Hence we let $l = \eta_k(x, m + (j+1)\varepsilon_k) - \eta_k(x, m + j\varepsilon_k)$. It follows from the cocycle condition that

$$\theta_k(\varphi^n(x), l) = 1.$$

Moreover, the estimate $|l| \leq 2^d$ follows from part 2 of Lemma 3.4. $\square$

**Lemma 3.6.** *Suppose that $x$ in $X$ and $m$ in $\mathbb{Z}^d$ satisfy*

$$\theta_k(x, m) = 0,$$

*for all $1 \leq k \leq d$. Then we have*

$$\eta_k(\varphi^m(x), n - \pi_k(m)) = \eta_k(x, n) - m,$$

*for all $0 \leq k \leq d$ and all $n$ in $\mathbb{Z}^k$.*

*Proof.* We proceed by induction, starting with $k = d$, for which the conclusion is obviously true, without using the hypothesis. Now assume the conclusion holds for some $1 \leq k \leq d$ and we will show it is true for $k-1$. To compute $\eta_{k-1}(\varphi^m(x), n - \pi_{k-1}(m))$, we use the induction hypothesis:

$$\begin{aligned} \eta_k(\varphi^m(x), n - \pi_{k-1}(m) + j\varepsilon_k) &= \eta_k(\varphi^m(x), n - \pi_k(m) + (j + m_k)\varepsilon_k) \\ &= \eta_k(x, n + (j + m_k)\varepsilon_k) - m. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \theta_k(\varphi^m(x), \eta_k(\varphi^m(x), n - \pi_{k-1}(m) + j\varepsilon_k)) \\ = & \theta_k(\varphi^m(x), \eta_k(x, n + (j + m_k)\varepsilon_k) - m) \\ = & \theta_k(x, \eta_k(x, n + (j + m_k)\varepsilon_k) - \theta_k(x, m) \\ = & \theta_k(x, \eta_k(x, n + (j + m_k)\varepsilon_k). \end{aligned}$$

Now the largest $j$ for which the left hand side is zero is also the largest $j$ for which the right hand side is zero and for this $j$ we have

$$\begin{aligned} \eta_{k-1}(\varphi^m(x), n - \pi_{k-1}(m)) &= \eta_k(\varphi^m(x), n - \pi_{k-1}(m) + j\varepsilon_k) \\ &= \eta_k(x, n + (j + m_k)\varepsilon_k) - m \\ &= \eta_{k-1}(x, n) - m. \end{aligned}$$

This completes the proof. $\square$

Before beginning the next Lemma, we recall that $\mathbb{Z}^0 = \{0\}$. This result shows that, in certain sense, the value of $\eta_0(x,0)$ determines the $R$-equivalence class of $x$.

**Lemma 3.7.** *Let $x$ be in $X$ and $m$ be in $\mathbb{Z}^d$. The pair $(x, \varphi^m(x))$ is in $R$ if and only if*

$$\eta_0(x,0) - m = \eta_0(\varphi^m(x), 0).$$

*Proof.* If $(x, \varphi^m(x))$ is in $R$, then by definition, $\theta_k(x,m) = 0$, for all $1 \leq k \leq d$. By the last Lemma, just using the conclusion for $k = 0$ and $n = 0$ (which is the only element of $\mathbb{Z}^0$), we have the desired conclusion.

For the converse, we use part 3 of Lemma 3.5 to see that

$$\theta_k(x, \eta_0(x,0)) = 0 = \theta_k(\varphi^m(x), \eta_0(\varphi^m(x), 0)), 1 \leq k \leq d.$$

On the other hand, using the hypothesis and the cocycle condition, we have, for all $k$,

$$\begin{aligned} 0 &= \theta_k(\varphi^m(x), \eta_0(\varphi^m(x), 0)) \\ &= \theta_k(\varphi^m(x), \eta_0(x,0) - m) \\ &= \theta_k(x, \eta_0(x,0)) - \theta_k(x,m) \\ &= -\theta_k(x,m). \end{aligned}$$

By definition, this implies that $(x, \varphi^m(x))$ is in $R$. □

It will be convenient for us to assemble all the $\theta_k$ together and regard them as a single cocycle with values in $\mathbb{Z}^d$. That is, we define

$$\theta : X \times \mathbb{Z}^d \to \mathbb{Z}^d$$

by

$$\theta(x,n)_k = \theta_k(x,n), x \in X, n \in \mathbb{Z}^d, 1 \leq k \leq d.$$

We may extend our definition of the kernel of a cocycle as the pre-image of the identity element, then notice at once that

$$\ker(\theta) = \cap_{k=1}^d \ker(\theta_k).$$

**Lemma 3.8.** *For any $x$ in $X$, we have*

$$\theta(x, \mathbb{Z}^d) = \mathbb{Z}^d.$$

*Proof.* We must show that the map $\theta$ is surjective, for fixed $x$. Let $m$ be in $\mathbb{Z}^d$. We will inductively define $l_d, l_{d-1}, \ldots, l_0$ in $\mathbb{Z}^d$ with the property that

$$\theta_k(x, l_i) = m_k,$$

provided $i < k$. Having done so, we will have $\theta(x, l_0) = m$ as desired.

We begin with $l_d = 0$, which vacuously satisfies the conclusion. Suppose that $l_i$ has been defined and satisfies the condition. Consider

$$f(j) = \theta_i(\varphi^{l_i}(x), \eta_i(\varphi^{l_i}(x), j\varepsilon_i)), j \in \mathbb{Z}.$$

From the first statement of Lemma 3.4, we know that there exists $j_i$ such that

$$\theta_i(\varphi^{l_i}(x), \eta_i(\varphi^{l_i}(x), j_i\varepsilon_i)) = m_i - \theta_i(x, l_i).$$

We define $l_{i-1} = l_i + \eta_i(\varphi^{l_i}(x), j_i\varepsilon_i)$. We verify the condition on $l_{i-1}$ for $k > i - 1$:

$$\begin{aligned} \theta_k(x, l_{i-1}) &= \theta_k(x, l_i) + \theta_k(\varphi^{l_i}(x), l_{i-1} - l_i) \\ &= \theta_k(x, l_i) + \theta_k(\varphi^{l_i}(x), \eta_i(\varphi^{l_i}(x), j_i\varepsilon_i)). \end{aligned}$$

If $i = k$, then the last expression is $m_i$, from our choice of $j_i$. If $k > i$, then the first term in the last expression is $m_k$, since $l_i$ satisfies the condition, while the second is zero because of the third part of Lemma 3.5. □

At this point, we have enough to prove our main approximation result.

**Theorem 3.9.** *Let $\varphi$ be a free, minimal action of $\mathbb{Z}^d$ on the Cantor set $X$. Suppose that $\varphi$ has small positive cocycles. For any $C \geq 0$, there exists a non-empty, clopen subset $Y \subset X$, a compact, open subequivalence relation $R \subset R_\varphi$ and a free minimal action of $\mathbb{Z}^d$, $\psi$, on $Y$ such that*

1. *$R$ has capacity $C$,*
2. *each $R$ equivalence class meets $Y$ in exactly one point,*
3. *the equivalence relation generated by $R_\psi$ and $R$ is exactly $R_\varphi$.*

*Proof.* We use the generating sets $F_k, 1 \leq k \leq d$ as above in Lemma 3.3. We choose $M \geq 2^d + 2C + 1$ and let $\theta_k, 1 \leq k \leq d$ also be as above, that is $\theta_k \leq M^{-1}$ and $\theta_k$ is strictly positive with respect to $F_k$, for all $1 \leq k \leq d$.

We let

$$R = \cap_{k=1}^{d} \ker(\theta_k) = \ker(\theta).$$

That $R$ is a compact, open subequivalence relation follows from Lemma 3.3 and Proposition 3.2.

Define a map $\pi : X \to X$ by

$$\pi(x) = \varphi^{\eta_0(x,0)}(x), x \in X,$$

where $\eta_0$ is defined in Lemma 3.4. It is clear that $\pi$ is continuous. Let $Y = \pi(X)$, which is clopen. We know from part 3 of Lemma 3.5 that $\theta_k(x, \eta_0(x, 0)) = 0$, for all $k$, and so $(x, \varphi^{\eta_0(x,0)}(x)) = (x, \pi(x))$ is in $R$, for any $x$ in $X$.

Next, we claim that $\eta_0(\varphi^{\eta_0(x,0)}(x), 0) = 0$, for all $x$ in $X$. We have already seen that, for $m = \eta_0(x, 0)$, we have $(x, \varphi^m(x))$ is in $R$. The claim follows from the only if part of Lemma 3.7, for this value of $m$. As a consequence, we see that $\eta_0(y, 0) = 0$ and $\pi(y) = y$ if $y$ is in $Y$.

We now show that $Y$ meets each $R$ equivalence class exactly once. For any $x$ in $X$, $(x, \pi(x))$ is in $R$ and $\pi(x)$ is in $Y$, so $Y$ meets each $R$ equivalence class. Now suppose that it meets some $R$ equivalence class in two points. Since $R$ is a subequivalence relation of $R_\varphi$, we may assume these points are $x$ and $\varphi^m(x)$, for some $x$ in $X$ and $m$ in $\mathbb{Z}^d$. As $x$ and $\varphi^m(x)$ are both in $Y$, we have

$$\eta_0(x, 0) = 0 = \eta_0(\varphi^m(x), 0).$$

As there are assumed to be in the same $R$ equivalence class, we also have $(x, \varphi^m(x))$ is in $R$, so by Lemma 3.7, we have

$$m = \eta_0(x, 0) - \eta_0(\varphi^m(x), 0) = 0.$$

We note for convenience that we have shown that for any $x, x'$ in $X$, $(x, x')$ is in $R$ if and only if $\pi(x) = \pi(x')$.

We now show that $R$ has capacity $C$. Let $x$ be in $X$. Consider

$$m = \eta_0(x, 0) - (C + 1) \sum_{k=1}^{d} \varepsilon_k.$$

This means that if $n$ is in $\mathbb{Z}^d$ and $|n - m| \leq C$, then $\eta_0(x, 0) - n$ is in $\mathbb{Z}^+ F$ and

$$|\eta_0(x, 0) - n| \leq |\eta_0(x, 0) - m| + |m - n| \leq C + 1 + C = 2C + 1.$$

We claim that $\theta(x,n) = 0$, for all such $n$. This will imply that $(x, \varphi^n(x))$ is in $R$ for all such $n$ and hence, $R$ has capacity $C$. We fix $1 \leq k \leq d$ and compute $\theta_k(x,n)$.

First, note that

$$\begin{aligned} \theta_k(x,n) &= \theta_k(x, \eta_0(x,0)) - \theta_k(\varphi^n(x), \eta_0(x,0) - n) \\ &= 0 - \theta_k(\varphi^n(x), \eta_0(x,0) - n) \end{aligned}$$

so it suffices for us to show that $\theta_k(\varphi^n(x), \eta_0(x,0) - n) = 0$.

Since $\eta_0(x,0) - n$ is in $\mathbb{Z}^+ F$ which is, in turn, contained in $\mathbb{Z}^+ F_k$, we know that

$$\theta_k(\varphi^n(x), \eta_0(x,0) - n) \geq 0.$$

Now suppose that the quantity above is strictly positive. We know from part 4 of Lemma 3.5 that there exists $l$ in $\mathbb{Z}^d$ with $|l| \leq 2^d$ and

$$\theta_k(\varphi^{\eta_0(x,0)}(x), l) > 0.$$

This then implies

$$\begin{aligned} \theta_k(\varphi^n(x), \eta_0(x,0) - n + l) &= \theta_k(\varphi^n(x), \eta_0(x,0) - n) \\ &\quad + \theta_k(\varphi^{\eta_0(x,0)}(x), l) \\ &\geq 1 + 1 = 2. \end{aligned}$$

However, we also know that

$$|\eta_0(x,0) - n + l| \leq |\eta_0(x,0) - n| + |l| \leq 2C + 1 + 2^d \leq M$$

and this contradicts our assumption that $\theta_k < M^{-1}$. This completes the proof that $R$ has capacity $C$.

We now turn to the issue of defining a $\mathbb{Z}^d$-action on $Y$. Let $y$ be in $Y$ and let $m$ be in $\mathbb{Z}^d$. We claim that there exists a unique $n$ in $\mathbb{Z}^d$ such that

$$\begin{aligned} \theta(y,n) &= m, \\ \varphi^n(y) &\in Y. \end{aligned}$$

Assuming this is true, we set $\psi^m(y) = \varphi^n(y)$. To verify this assertion, we first note that, by Lemma 3.8, there exists $l$ in $\mathbb{Z}^d$ such that $\theta(y,l) = m$. Now let $n = l + \eta_0(\varphi^l(y), 0)$ so that

$$\varphi^n(y) = \varphi^{\eta_0(\varphi^l(y),0)}(\varphi^l(y)) = \pi(\varphi^l(y))$$

is in $Y$. We also have

$$\theta(y,n) = \theta(y,l) + \theta(\varphi^l(y), \eta_0(\varphi^l(y),0)) = m + 0,$$

from the choice of $n$ and part 3 of Lemma 3.5. As for the uniqueness of $n$, suppose that $n, n'$ both satisfy the condition. Then, by the cocycle condition, we have

$$0 = \theta(y,n') - \theta(y,n) = \theta(\varphi^n(y), n' - n).$$

It follows that $(\varphi^n(y), \varphi^{n'}(y))$ is in $R$ and both elements are in $Y$. Since $Y$ meets each equivalence class once, we see that $\varphi^n(y) = \varphi^{n'}(y)$ and from the freeness of the action, this means that $n = n'$.

Next, we check that, for any $m, m'$ in $\mathbb{Z}^d$, we have

$$\psi^{m'} \circ \psi^m = \psi^{m+m'}.$$

Let $y$ be in $Y$. We choose $n$ in $\mathbb{Z}^d$ such that $\theta(y,n) = m$ and $\varphi^n(y)$ is in $Y$. This means that $\psi^m(y) = \varphi^n(y)$. Next, choose $n'$ in $\mathbb{Z}^d$ such that $\theta(\psi^m(y), n') = m'$ and $\varphi^{n'}(\psi^m(y))$ is in $Y$. This means that

$$\psi^{m'}(\psi^m(y)) = \varphi^{n'}(\psi^m(y)) = \varphi^{n'}(\varphi^n(y)) = \varphi^{n+n'}(y).$$

On the other hand, it is clear that $\varphi^{n+n'}(y)$ is in $Y$ while

$$\theta(y, n+n') = \theta(y,n) + \theta(\varphi^n(y), n') = m + \theta(\psi^m(y), n') = m + m'.$$

This means that

$$\psi^{m+m'}(y) = \varphi^{n+n'}(y).$$

The conclusion follows.

We now check the third statement of Theorem 3.9. It clearly suffices to show that, for any $x$ in $X$ and $n$ in $\mathbb{Z}^d$, there exists $y$ in $Y$ and $m$ in $\mathbb{Z}^d$ with $(x,y), (\psi^m(y), \varphi^n(x))$ are in $R$. Let $y = \pi(x)$. It follows that $(x,y)$ is in $R$. Let $m = \theta(x,n)$. We know $(\varphi^n(x), \pi(\varphi^n(x)))$ is in $R$. A simple argument like the ones above shows that $\pi(\varphi^n(x)) = \psi^m(\pi(x))$. The conclusion follows at once.

Our final step is showing that the action $\psi$ is free and minimal. First suppose that $\psi^m(y) = y$, for some $y$ in $Y$ and $m$ in $\mathbb{Z}^d$. We know that $y = \psi^m(y) = \varphi^n(y)$, where $n$ is the unique element of $\mathbb{Z}^d$ such that $\theta(y,n) = m$. But since $\varphi$ is free, the only $n$ such that $\varphi^n(y) = y$ is $n = 0$. It follows that $m = \theta(x,0) = 0$. As for the minimality. It suffices to prove that, for any non-empty open set $U$ in $Y$, and any $y$ in $Y$, there is $m$ in $\mathbb{Z}^d$ such that $\psi^m(y)$ is in $U$. From the minimality of $\varphi$, we know there is $n$ in $\mathbb{Z}^d$ such that $\varphi^n(y)$ is in $U$. Let $m = \theta(y, n)$. Using the fact that $U$ is contained in $Y$, it follows at once that $\psi^m(y) = \varphi^n(y)$ is in $U$. $\square$

It should be noted that it is not clear that the system $(Y, \psi)$ given in the last result also has small positive cocycles.

**Remark 3.10.** *Since the equivalence relation $R$ is compact, the quotient $X/R$ is also a Cantor set (see [GPS1]). In fact, $Y$ is homeomorphic to this quotient and $\psi$ can be realized as an action on the quotient. Moreover, since $R = \ker(\theta)$, $\theta$ descends to a cocycle, denoted $\bar{\theta}$ on the quotient relation, which is just $R_\psi$. Under these identifications, we have*

$$\bar{\theta}([x]_R, n) = n,$$

*for all $[x]_R$ in $X/R$ and $n$ in $\mathbb{Z}^d$.*

**Remark 3.11.** *The paper [GMPS2] extends the classification of orbit equivalence to include minimal $\mathbb{Z}^2$-actions on the Cantor set. The main tool is to construct finite approximations to the orbit relation in very much the same spirit as 3.9. This is done by using Voronoi tessellations and variations of them, avoiding the issue of whether or not the system has small positive cocycles. What is lost in this method (as opposed to 3.9) is the existence of a $\mathbb{Z}^2$-action on the quotient $X/R$.*

**Remark 3.12.** *One of the conclusions of [GMPS2] is that every minimal $\mathbb{Z}^2$-action on the Cantor set, say $\varphi$, is orbit equivalent to a minimal $\mathbb{Z}$-action, say $\psi$. Let $h$ denote the map between the underlying spaces. Although we have seen above that $\mathbb{Z}$-actions possess an abundance of cocycles, this does not seem to help in the question of the existence of small positive cocycles for $\varphi$ for the following reason. Let $\theta$ be a cocycle for the action $\psi$. The map $h \times h : R_\varphi \to R_\psi$ is* not, *in general, continuous and so $\theta \circ (h \times h)$ is not a cocycle as it may not be continuous.*

# 4 Examples

In this section we present two classes of minimal free $\mathbb{Z}^2$-actions which satisfy our hypotheses.

**Example 4.1.** *Rotations of the group of $p$-adic integers.*

Here, we consider a prime number $p$ and the group $X$ of $p$-adic integers. We choose a dense copy of $\mathbb{Z}^2$ in $X$ and our action is by rotation by this subgroup. We remark that we believe that the same result is true for the $n$-adic integers, where $n$ is any natural number and more generally for all odometers. But the choice of a prime $p$ will simplify some of our arguments. Let us make this more precise.

We let $\mathbb{Z}_p$ denote the quotient of $\mathbb{Z}$ by $p\mathbb{Z}$. Then, we have $X = \Pi_{k=0}^{\infty} \mathbb{Z}_p$. It is an abelian group ; the operation is addition modulo $p$, with carry over to the right. An element $x = (x_k)_{k=0}^{\infty}$ may be regarded as a formal power series $\sum_k x_k p^k$ with the obvious form for addition. For $x$ in $X$ and non-negative integers $i \leq j$, we let $x_{[i,j]}$ denote the finite sequence $x_i, x_{i+1}, \ldots x_j$. We call such a sequence a *word in $x$ of length $j - i + 1$*.

We choose two elements $\alpha$ and $\beta$ from $X$ such that either $\alpha_0$ or $\beta_0$ is non-zero and so that the only integers $m, n$ which satisfy $m\alpha + n\beta = 0$ are $m = n = 0$. These conditions imply the subgroup generated by $\alpha$ and $\beta$ is dense in $X$. Then our action $\varphi$ is defined by

$$\varphi^{(i,j)}(x) = x - i\alpha - j\beta,$$

for all $x$ in $X$ and $(i, j)$ in $\mathbb{Z}^2$. In our notation, we will identify $\mathbb{Z}^2$ and the subgroup of $X$.

We claim that this action satisfies the small positive cocycle property. We will actually verify the property in Theorem 2.10.

Choose an independent set of generators of $\mathbb{Z}^2$. We will denote these by $a$ and $b$ which are elements of $X$. As $a$ and $b$ must generate a dense subgroup of $X$, at least one of $a_0$ and $b_0$ is non-zero. Let us suppose the former. Then the subgroup generated by $a$ alone is dense. This means that there is an automorphism of $X$ carrying $a$ to $(1, 0, 0, \ldots)$. Henceforth, we assume that $a = (1, 0, 0, \ldots)$. Observe that, for any $x$ in $X$ and positive integer $k$, the values of $(x + ia)_{[0,k]}$ are all distinct for $0 \leq i < p^{k+1}$.

**Lemma 4.2.** *Let $b$ be in $X$ and let $K$ be a positive integer and let $a = (1, 0, 0, \ldots)$. Suppose that there are no non-trivial integer solutions, $i, j$, of the equation $ia + jb = 0$. Then there exists positive integers $K < M, N$ such that $2M < N$ and*

$$(-b)_{M-K} < (-b)_{N-K}$$

*regarding these as integers between* 0 *and* $p-1$ *and*

$$(-b)_{M-k} = (-b)_{N-k}$$

*for all* $k = 0, 1, 2, \ldots, K-1$.

*Proof.* We consider the collection of all words $w$ in $b$ of length $K$ and we divide these into three classes. The first is all words that only occur finitely many times in $b$; that is $w = b_{k,k+K-1}$ for only finitely many $k \geq 0$. The second class is all words $w$ such that there are distinct symbols $0 \leq i, j < p$ such that the words $iw$ (concatenation) and $jw$ both appear infinitely many times in $b$. The third class consists of words $w$ such that there is a symbol $i_w$ such that $i_w w$ occurs infinitely many times, but $iw$ occurs only finitely many times for $i \neq i_w$. If there exists such a word in the second class, then we are clearly done. We are left to consider the case that the second class is empty. Then we can select $L \geq 1$ such that each word in the first class does not appear in $b_{[L,\infty)}$ and that for each word $iw$ with $w$ in the third class and $i \neq i_w$ does not appear in $b_{[L,\infty)}$. We claim that $b$ is eventually periodic, that is, there is some $J \geq 1$ such that $b_n = b_{n+J}$, for all $n \geq L$. To see this, consider the second appearance of a word $w$ from the third class in $b_{[L,\infty)}$. From the choice of $L$, this word must be preceded by $i_w$. Let $w'$ be the word obtained by dropping the last symbol from $i_w w$. This word is also of length $K$ and provided we are still at entries greater than $L$, it is again in the third class and it must be preceded by $i_{w'}$. Continuing in this way, we see that the predecessors of $w$ are unique. Eventually the word $w$ occurs in this string, since we began at the second occurrence of $w$. But this argument applies to every occurrence of $w$. The conclusion follows. Since $a = (1, 0, 0, \ldots)$, we may find such integer $i$ such that $ia + b$ is periodic, say of period $J \geq 1$. Multiplying a sequence in $X$ by $p^J$ has the effect of shifting the entries over by $J$ and leaving 0 in the first $J$ positions. As it is periodic, this leaves all of $ia + b$ unchanged except for the first $J$ positions. Then we may find $i' \geq 0$ such that $i'a = (b_0, b_1, \ldots, b_{J-1}, 0, 0 \ldots)$. Then we have

$$i'a + p^J(ia + b) = (ia + b).$$

But this contradicts our hypothesis on $a$ and $b$. This completes the proof. $\square$

Having chosen a positive integer $K$, we select $M, N$ as in the Lemma and define

$$A = \{x \in X \mid x_{[0,M]} = x_{[N-M,N]}\} - \{x \in X \mid x_i = p-1, \text{ for all } 0 \leq i \leq N\}.$$

We define two functions, $\lambda, \mu : A \to \mathbb{Z}$ by

$$\begin{aligned}
\lambda(x) &= \inf\{i \geq 1 \mid x + ia \in A\} \\
\mu(x) &= \inf\{i \geq 1 \mid x - b + ia \in A\}
\end{aligned}$$

for all $x$ in $A$. Since rotation by $a$ is minimal, both of these quantities are well defined. Also, because $A$ is clearly clopen, both functions are continuous.

The key Lemma is the following.

**Lemma 4.3.** *With $A$, $\lambda$ and $\mu$ as above, we have*

*1.*

$$\lambda(x) \geq p^{M+1}, \text{ for all } x \in A,$$

*2.*

$$\mu(x) \leq p^{M-K+1} + 2, \text{ for all } x \in A.$$

*Proof.* We begin with the first statement.

**Case 1:** $x_k < p - 1$, for some $M < k < N - M$. For $1 \leq i < p^{M+1}$, the values of $(x + ia)_{[0,M]}$ are all distinct from $x_{[0,M]}$. However, for these values of $i$, when computing $x + ia$, there is no carry over past the $k$th coordinate and this means that $x_{[N-M,N]} = x_{[0,M]}$. Hence, $x + ia$ is not in $A$.

**Case 2:** $x_k = p - 1$, for all $M < k < N - M$. As $x$ is in $A$, we must have $x_k < p - 1$, for some $0 \leq k \leq N$. By hypothesis, we have either $k \leq M$ or $N - M \leq k \leq N$. But, in the former case, as $x$ is in $A$, we have $x_k = x_{N-M+k}$. In either case, we conclude that $x_k < p - 1$, for some $0 \leq k \leq M$ and some $N - M \leq k \leq N$. Let $I$ be the first positive integer for which the computation of $x + Ia$ involves carry over past coordinate $M$. Note that $I \leq p^{M+1}$. Since $x_k < p - 1$ for some $k \leq M$, $I > 1$. For $1 \leq i < I$, we have $(x + ia)_{[0,M]} \neq x_{[0,M]}$ while $(x + ia)_{[N-M,N]} = x_{[N-M,N]}$. From this we conclude that $x + ia$ is not in $A$. Next consider $I \leq i \leq p^{M+1}$. Here, we have $(x + ia)_{[N-M,N]}$ is obtained from $x_{[N-M,N]} = x_{[0,M]}$ by adding $(1, 0, 0, \ldots, 0)$. That is, we have $(x + ia)_{[N-M,N]} = (x + a)_{[0,M]}$. But as $1 < I \leq i \leq p^{M+1}$, we know that $(x + a)_{[0,M]} \neq (x + ia)_{[0,M]}$ and again we conclude that $x + ia$ is not in $A$. This completes the proof of the first statement.

We now consider the second statement. Since $(-b)_{M-K} < (-b)_{N-K}$, we may find a positive integer $I < p^{M-K+1}$ so that

$$(-b + Ia)_{[0,M-K]} = (-b)_{[N-M,N-K]}.$$

This means that there is no carry over past coordinate $M-K$ in $-b+Ia$. We now claim that, for any $x$ in $A$, at least one of $x-b+Ia$, $x-b+(I+1)a$ or $x-b+(I+2)a$ is in $A$. This will complete the proof.

We see at once from our choice of $I$ that

$$(-b + Ia)_{[0,M-K]} = (-b)_{[N-M,N-K]} = (-b + Ia)_{[N-M,N-K]}$$

since the addition has no carry over past coordinate $M-K$. Using again that there is no carry over past coordinate $M-K$ and our original choice of $M, N$, we have $(-b+Ia)_{[M-K+1,M]} = (-b)_{[M-K+1,M]} = (-b)_{[N-K+1,N]} = (-b+Ia)_{[N-K+1,N]}$. Together, we see that $(-b+Ia)_{[0,M]} = (-b+Ia)_{[N-M,N]}$. We add $-b+Ia$ to an element $x$ in $A$. If there is no carry over from coordinate $N-M-1$ to $N-M$, then $x-b+Ia$ has the same property, $(x-b+Ia)_{[0,M]} = (x-b+Ia)_{[N-M,N]}$. Either $x-b+Ia$ is in $A$ or else $(x-b+Ia)_k = p-1$ for all $0 \leq k \leq N$. In the latter case, it is immediate that $x-b+Ia+a$ is in $A$. Finally, if there is carry over from $N-M-1$ to $N-M$ when adding $x$ to $-b+Ia$, then adding $a$ once more will affect the value on the first interval, but not on the last, and will result in $(x-b+Ia+a)_{[0,M]} = (x-b+Ia+a)_{[N-M,N]}$. We are again reduced to one of the two cases above: either $x-b+Ia+a$ or $x-b+Ia+2a$ is in $A$. □

The first consequence we note is that, for $i = 0, 1, \ldots, p^{M+1}-1$, the sets $A+ia = \varphi^{-ia}(A)$ are pairwise disjoint. Next, we claim that the map from $A$ to itself which sends $x$ to $x-b+\mu(x)a$ is injective. If not, we have $x-b+\mu(x)a = x'-b+\mu(x')a$, for some $x, x'$ in $A$. If $\mu(x) > \mu(x')$, then $x+(\mu(x)-\mu(x'))a = x'$ which is in $A$. But this means that $\lambda(x) \leq \mu(x)-\mu(x')$, which contradicts the estimates of the last lemma. In an analogous way, $\mu(x') > \mu(x)$ is impossible and we conclude that $\mu(x) = \mu(x')$. From this it follows that $x = x'$ as desired. As $A$ is clopen and there exists a $\varphi$-invariant probability measure on $X$, the map above is also onto. To say this another way, we have $x$ is in $A$ if and only if $x-b+\mu(x)a$ is in $A$. It will be convenient later to denote $\xi(x) = x-b+\mu(x)a$, for $x$ in $A$; we have shown $\xi$ is a homeomorphism of $A$.

We define

$$B = \{x + ia \mid x \in A, 0 \leq i < \mu(x)\}.$$

It follows at once from the last paragraph that

$$\begin{aligned}
A \cup \varphi^{-a}(B) &= A \cup \{x' + a \mid x' \in B\} \\
&= A \cup \{x + ia \mid x \in A, 0 < i \leq \mu(x)\} \\
&= \{x + ia \mid x \in A, 0 \leq i \leq \mu(x)\} \\
&= \{x + \mu(x)a \mid x \in A\} \cup B \\
&= \{x + \mu(x)a \mid x - b + \mu(x)a \in A\} \cup B \\
&= \{x + \mu(x)a \mid x + \mu(x)a \in A + b\} \cup B \\
&= (A + b) \cup B \\
&= \varphi^{-b}(A) \cup B.
\end{aligned}$$

Notice that the facts that $A$ and $\varphi^{-a}(B)$ are disjoint, as are $\varphi^{-b}(A)$ and $B$, are clear from the above computation. We have verified the first two conditions of Theorem 2.10. We now consider the third condition; specifically, that the sets $\varphi^{-k(a+b)}(A \cup \varphi^{-a}(B))$ are pairwise disjoint for $0 \leq k < p^{K-3}$. This will suffice in Theorem 2.10, since $K$ is arbitrary.

Recall the homeomorphism of $A$, $\xi(x) = x - b + \mu(x)a$. Replacing $x$ by $\xi^{-1}(x)$, we have $x = \xi^{-1}(x) - b + \mu(\xi^{-1}(x))a$. Define integer valued functions $\mu_k, k \geq 0$, on $A$ by $\mu_0 = 0$ and

$$\mu_k(x) = k + \mu(x) + \mu(\xi(x)) + \ldots + \mu(\xi^{k-1}(x)),$$

for $k \geq 1$ and $x$ in $A$. We claim that

$$\varphi^{-k(a+b)}(A \cup \varphi^{-a}(B)) = \{x + ia \mid x \in A, \mu_k(x) \leq i < \mu_{k+1}(x)\}.$$

We have shown this already for $k = 0$; let us now assume it is true for $k$ and prove it for $k+1$. We will only prove the containment $\subset$, since this is the only part of the claim we will use later, however the reverse containment is valid and a very similar argument proves it also. From the induction hypothesis, it suffices for us to consider $x$ in $A$, $\mu_k(x) \leq i < \mu_{k+1}(x)$ and show that $\varphi^{-(a+b)}(x + ia)$ is the set on the right hand side, with $k + 1$ replacing $k$. We have

$$\begin{aligned}
\varphi^{-(a+b)}(x + ia) &= x + b + a + ia \\
&= \xi^{-1}(x) - b + \mu(\xi^{-1}(x))a + b + a + ia \\
&= \xi^{-1}(x) + (i + 1 + \mu(\xi^{-1}(x)))a.
\end{aligned}$$

Of course, $\xi^{-1}(x)$ is in $A$ while, for any $l \geq 0$, we have

$$\begin{aligned}\mu_{l+1}(\xi^{-1}(x)) &= l+1+\mu(\xi^{-1}(x))+\mu(x)+\ldots+\mu(\xi^{l-1}(x)) \\ &= 1+\mu(\xi^{-1}(x))+\mu_l(x).\end{aligned}$$

It follows that $\mu_{k+1}(\xi^{-1}(x)) \leq i+1+\mu(\xi^{-1}(x)) < \mu_{k+2}(\xi^{-1}(x))$ and so the desired conclusion.

Next, we observe that since $\mu(x) \leq p^{M-K+1}+2$, by Lemma 4.3, we have $0 \leq \mu_k \leq k(p^{M-K+1}+3) < p^{K-3}(p^{M-K+1}+3) \leq p^{K-3}(p^{M-K+1}+p^{M-K+1}) \leq p^M$. Suppose that the sets $\varphi^{-k(a+b)}(A \cup \varphi^{-a}(B))$ and $\varphi^{-l(a+b)}(A \cup \varphi^{-a}(B))$, with $0 \leq k,l \leq p^{K-3}$, have a common element. From our claim above, we have $x,y$ in $A$, $\mu_k(x) \leq i < \mu_{k+1}(x)$ and $\mu_l(y) \leq j < \mu_{l+1}(y)$ with $x+ia = y+ja$. From the estimate above of the values of $\mu_k$ and the fact that the sets $\varphi^{-ia}(A)$ are pairwise disjoint for $0 \leq i < p^{M+1}$, we see that $i=j$ and $x=y$. It follows that $k=l$ as desired.

Finally, we verify the last condition of Theorem 2.10. The case for the generator $a$ is straightforward since the map $\varphi^{-a}$ is minimal and $A$ is non-empty and open. Now we consider the generator $b$. It follows from the minimality of the action and the fact that $B$ is non-empty and open that we may find a finite set $F \subset \mathbb{Z}^2$ such that $\cup_{(i,j)\in F}\varphi^{-ia-jb}(B) = X$. Let $I$ be the minimum of the first entries of $F$. We observe that $A$ is a subset of $B$ and hence

$$\varphi^{-a}(B) \subset A \cup \varphi^{-a}(B) = \varphi^{-b}(A) \cup B \subset \varphi^{-b}(B) \cup B.$$

Continuing in this fashion, we may replace the finite set $F$ with a finite set $F'$, where each element has first entry $I$, and $\cup_{(i,j)\in F'}\varphi^{-ia-jb}(B) = X$. Applying $\varphi^{Ia}$ to both sides yields the desired result.

We remark that in this example, there is a short exact sequence

$$0 \to \mathbb{Z} \to H^1(X,\varphi) \xrightarrow{q} \mathbb{Z}[\frac{1}{p}] \to 0.$$

The map $q$ is defined as follows. For any $\theta$ in $Z^1(X,\varphi)$, we have

$$q([\theta]) = \int_X \theta(x,(1,0))d\mu(x),$$

where $\mu$ is Haar measure on $X$.

**Example 4.4.** *Rotations of a disconnected circle.*

Let $\alpha, \beta$ be two numbers such that $\{1, \alpha, \beta\}$ is linearly independent over the rational numbers. For simplicity, we will assume that $\alpha, \beta$ are both between 0 and $\frac{1}{2}$.

We consider the natural action of $\mathbb{Z}^2$ on the circle, $\mathbb{R}/\mathbb{Z}$, by rotating by $\alpha$ and by $\beta$. We select a single orbit, say that of 0, and cut the circle at these points, replacing each by two points separated by a gap. The old point will be come the right endpoint of the gap and a new point will be the left end of the gap. Let us make this more precise as follows.

We consider the subgroup of $\mathbb{R}$, $Cut = \{i + j\alpha + k\beta \mid i, j, k \in \mathbb{Z}\}$. We let $\tilde{X} = \mathbb{R} \cup \{a' \mid a \in Cut\}$. We give $\tilde{X}$ a linear order by setting $a' < b$, $a < b'$ and $a' < b'$ as appropriate, whenever $a < b$. Finally, we set $a' < a$, for all $a$ in $\mathbb{R}$. The space $\tilde{X}$ is given the order topology. Notice that for $x < y$ in $Cut$, $[x, y) = [x, y']$ is a clopen set in $\tilde{X}$. The natural action of the group $\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{Z}$ extends in a natural way to $\tilde{X}$. We let $X = \tilde{X}/\mathbb{Z}$, which has an action of $\alpha\mathbb{Z} + \beta\mathbb{Z}$. This is our Cantor minimal $\mathbb{Z}^2$ system, $\varphi$,

$$\varphi^{(i,j)}(x) = x - i\alpha - j\beta,$$

where $x$ is a real number, interpreted modulo $\mathbb{Z}$, and

$$\varphi^{(i,j)}(x') = (x - i\alpha - j\beta)',$$

for $x$ in $Cut$.

We claim that $\varphi$ has small positive cocycles. To see this, we must consider a pair of generators, $a, b$, of $\mathbb{Z}^2$. Now $\varphi^a$ and $\varphi^b$ are again rotations of our cut-up circle $X$ and we let $a$ and $b$ denote real numbers such that $\varphi^a(x) = x - a$ and $\varphi^b(x) = x - b$, both interpreted modulo the integers. Since $a$ and $b$ are generators of $\mathbb{Z}^2$, the subgroup of $\mathbb{R}$ generated by $a, b, 1$ is the same as that generated by $\alpha, \beta, 1$.

Consider for the moment, the homeomorphism, $\eta$ of $\mathbb{R}^2/\mathbb{Z}^2$ defined by $\eta(x, y) = (x + a, y + b)$. From Theorem 1, page 97 of [CFS], this action is minimal if and only if there is no non-trivial character of $\mathbb{R}^2/\mathbb{Z}^2$ which annihilates $(a, b)$. The non-existence of such a character is an immediate consequence of the fact that $\{1, \alpha, \beta\}$ are linearly independent over the rationals. We conclude that $\eta$ is minimal.

Let $N$ be a positive integer. From the minimality of $\eta$, we may find a positive integer $q$ such that $\eta^q(0, 0) \in (0, \frac{1}{2N}) \times (0, \frac{1}{2N})$. This means that

$$0 < qa - i < \frac{1}{2N}, 0 < qb - j < \frac{1}{2N}$$

for some integers $i, j$, or equivalently,

$$0 < a - \frac{i}{q} < \frac{1}{2Nq}, 0 < b - \frac{j}{q} < \frac{1}{2Nq}.$$

From this it follows that, for any $0 \leq m, n < N$ and $k \in \mathbb{Z}$, we have

$$\begin{array}{lcl} \frac{k}{q} \leq & \frac{k}{q} + m(a - \frac{i}{q}) + n(b - \frac{j}{q}) \leq & \frac{k}{q} + \frac{N}{2Nq} + \frac{N}{2Nq} \\ \frac{k}{q} \leq & \frac{k-im-jn}{q} + ma + nb \leq & \frac{k+1}{q} \end{array}$$

Consider the finite set of distinct points in $\mathbb{R}/\mathbb{Z}$,

$$\{\frac{k}{q} + ma + nb \mid 0 \leq m, n < N, k \in \mathbb{Z}\}$$

and choose $\delta$ to be less than half the distance between any two of these (in $\mathbb{R}/\mathbb{Z}$). For each $0 \leq k < q$, choose a point $x_k$ with

$$0 < x_k - \frac{k}{q} < \delta, x_k \in Cut.$$

We interpret $x_k$ for any $k$ in $\mathbb{Z}$ by considering $k$ modulo $q$. This means that we have

$$x_k < x_{k-im-jn} + ma + nb < x_{k+1},$$

for any $k = 0, 1, \ldots, q-1$ and $0 \leq m, n < N$.

We define

$$A = \bigcup_{k=0}^{q-1} [x_k, x_{k-i} + a), B = \bigcup_{k=0}^{q-1} [x_k, x_{k-j} + b),$$

where $k - i$ and $k - j$ are interpreted $\mod q$. From the estimates above, we see that the intervals appearing in the union in the definition of $A$ are pairwise disjoint. The analogous statement is true of $B$. We calculate

$$\begin{aligned} A \cup \varphi^{-a}(B) &= \{\cup_k [x_k, x_{k-i} + a)\} \cup \{\cup_{k'} [x_{k'} + a, x_{k'-j} + a + b)\} \\ &= \cup_k \{[x_k, x_{k-i} + a) \cup [x_{k-i} + a, x_{k-i-j} + a + b)\} \\ &= \cup_k [x_k, x_{k-i-j} + a + b). \end{aligned}$$

Again using arguments similar to those above, the intervals involved in the above union are pairwise disjoint. A similar computation shows that

$$\varphi^{-b}(A) \cup B = A \cup \varphi^{-a}(B).$$

Moreover, for any $0 \leq n < N$, we have

$$\varphi^{-n(a+b)}(A \cup \varphi^{-a}(B)) = \cup_k [x_k + (n-1)(a+b), x_{k-i-j} + n(a+b)).$$

Again from the estimates above, these sets are pairwise disjoint.

This shows the first three conditions of Theorem 2.10 hold; the fourth is clear since each rotation $\varphi^{-a}, \varphi^{-b}$ is minimal and the sets $A$ and $B$ and open and non-empty.

We remark that in this example, we have $H^1(X, \varphi) \cong \mathbb{Z}^3$. See [FH] for a proof.

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