Non-commutative methods for the K-theory of C^*-algebras of aperiodic patterns from cut-and-project systems

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Abstract

We investigate the C^*-algebras associated to aperiodic structures called model sets obtained by the cut-and-project method. These C^*-algebras are Morita equivalent to crossed product C^*-algebras obtained from dynamics on a disconnected version of the internal space. This construction may be made from more general data, which we call a hyperplane system. From a hyperplane system, others may be constructed by a process of reduction and we show how the C^*-algebras involved are related to each other. In particular, there are natural elements in the Kasparov KK-groups for the C^*-algebra of a hyperplane system and that of its reduction. The induced map on K-theory fits in a six-term exact sequence. This provides a new method of the computation of the K-theory of such C^*-algebras which is done completely in the setting of non-commutative geometry.

1 Introduction

This paper is concerned with the study of aperiodic structures obtained by the so-called cut-and-project method, their associated C^*-algebras and the

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K-theory of these C^*-algebras. Such structures are used as models for physical materials called quasi-crystals (see [HG] or [J]). Around the time of the discovery of the first quasi-crystal [SBGC], original versions of the construction appeared in [DK1, DK2, E1, KKL1, KKL1, KrN, LS]. This was later made more axiomatic by using the concept of models sets introduced by Y. Meyer in 1972 [M1]. The situation is as follows: the physical space in which the atoms are actually lying is seen as a subspace of a larger Euclidean space, in which it is called the parallel direction or simply the physical space. Its orthogonal complement is usually called the internal space. In the large space, there is a lattice (called the reference lattice) which is irrationally related to the parallel direction, meaning that their intersection is just the origin. In the internal space, a window is chosen, a compact set which is the closure of its interior. Most models describing real quasi-crystalline materials use the window obtained by the projection of the unit cube in the reference lattice to the internal space.

In the 1980's, it was realized that the only way to get atoms moving in a perfect quasi-crystal at very low temperatures was through the so-called *flip-flops* (see, in particular, Chapter 3 of Gratias and Katz in [HG]) or *phasons* (see [J]). The cut-and-project construction gives a very convenient representation of the phenomenon: by moving the window in the perpendicular space, every time a new point of the reference lattice enters the part of the large space which projects into the window, another point is expelled. By projecting into the physical space, such a move can be seen as a local jump of an atom from its prior position to a position nearby. The family of positions involved in the jumps is usually located on an affine hyperplane (at least whenever the window obtained by the projection of the unit cube). It was realized quite early (see [Be1, BIT, BCL], for instance) that the window was homeomorphic to the *canonical transversal* (also called the *atomic surface* in quasi-crystalography [HG, J]) when endowed with a topology obtained by creating a gap on each affine hyperplane obtained by translating the hyperplanes which form the maximal faces of the window by the vectors in the lattice after projection to in the internal space. The resulting space is totally disconnected in cases of interest.

We present a specific example: the octagonal or Ammann-Beenker tiling. We follow the notation of [Be2]. Let e_1, e_2, e_3, e_4 be the usual basis for $\mathbb{R}^4$

and our lattice is $\mathcal{L} = \mathbb{Z}^4$. Consider the orthonormal basis

$$v_1 = (-\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{2}, \tfrac{1}{2}) \quad v_2 = (0, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}, -\tfrac{1}{2})$$
$$v_3 = (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{2}, \tfrac{1}{2}) \quad v_4 = (0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}, -\tfrac{1}{2})$$

The physical space or parallel direction, G, is the span of $\{v_1, v_2\}$ and the internal space, H, is the span of $\{v_3, v_4\}$. The subgroup $L \subset H$ is the projection of $\mathcal{L}$ onto the space H and is generated by

$$\pi^\perp(e_1) = \frac{1}{\sqrt{2}}v_3, \pi^\perp(e_2) = \frac{1}{\sqrt{2}}v_4, \pi^\perp(e_3) = -\frac{1}{2}v_3 - \frac{1}{2}v_4, \pi^\perp(e_4) = \frac{1}{2}v_3 - \frac{1}{2}v_4.$$

The following shows the space H, the generators of L and also the window, W, which is the projection of the unit cube of the lattice $\mathcal{L}$ onto H:

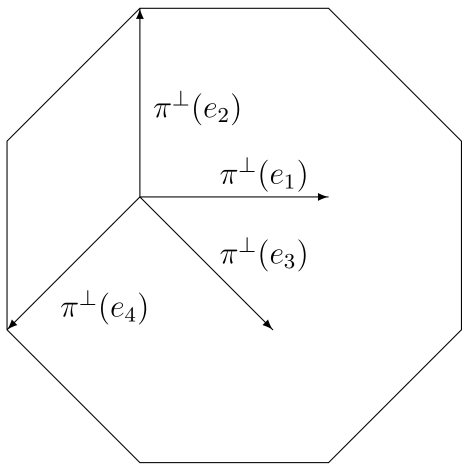

Figure 1: Internal space for the octagonal tilings

Observe that the edges of the window lie in the collection of hyperplanes, denoted $\mathcal{P}$, consisting of $P_k = span_{\mathbb{R}}\pi^\perp(e_k)$, for $k = 1, 2, 3, 4$ and their translates under L.

Beginning with a physical space G, an internal space H, a lattice $\mathcal{L}$ in $G \times H$ and a window W in H, we construct discrete point sets in G as follows. For any x in $G \times H$, we consider the coset $x + \mathcal{L}$, intersect it with $G \times W$ and project the result to G. It is convenient to require that the coset $x + \mathcal{L}$ does not intersect the boundary of $G \times W$. Such a set is called a model set. This collection may then be completed in a natural way to obtain a compact set of uniformly discrete subsets of G with a natural action of G by translation. This is called the hull and is denoted $\Omega(W)$. An element of the space $\Omega(W)$ (or rather a finite part of it) for the octagonal tiling is shown below:

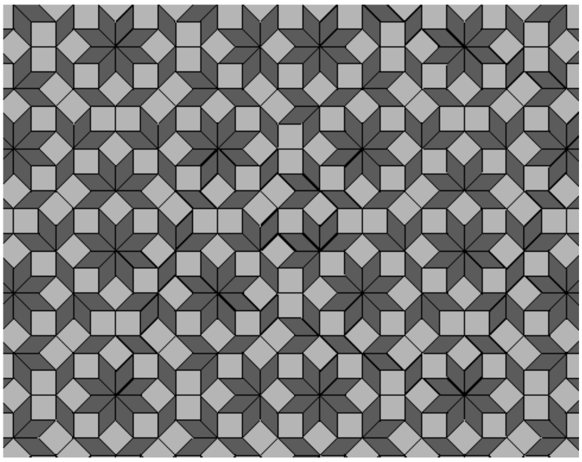

Figure 2: An octagonal tiling

The points of the model set are just the vertices. The edges are drawn as an aid to see the pattern; they are simply the edges given by the generators of $\mathcal{L}$ joining adjacent points in the coset $x+\mathcal{L}$.

We review the basic facts of this construction in section 2. In particular, the space $\Omega(W)$ can be realized as a quotient of $G \times \tilde{H}(W)$ by a natural action of $\mathcal{L}$, where $\tilde{H}(W)$ is a totally disconnected version of the space H. We find it most convenient to follow the idea of [BIT] expressing $\tilde{H}(W)$ as the spectrum of a commutative C^*-algebra generated by the characteristic function of the window and its translates by L. The C^*-algebra of interest is the crossed product $C(\Omega(W)) \times G$. Results of Rieffel immediately imply that this is Morita equivalent to $C_0(\tilde{H}(W)) \times L$ and we concentrate our attention on this C^*-algebra.

These C^*-algebras and variants of them contain operators which are approximants of position and momentum operators for electrons moving in a quasi-crystal represented by these models sets. Moreover, if a self-adjoint operator in this C^*-algebra representing some observable of the system has totally disconnected spectrum, then the gaps in this spectrum may be labelled by the K_0 group of the C^*-algebra. For further discussion of these ideas, we refer the reader to [Be1, Be2, BHZ, KeP]. Thus, a main focus of research has been in computing the K-theory of these C^*-algebras. We mention the references [Be1, BCL, BS], but the most general scheme for computing K-theory, specifically designed for cut-and-project systems, is given in [FHK].

Returning to the construction of the space $\tilde{H}(W)$, in the case where the window is the projection to H of the unit cube in the lattice $\mathcal{L}$, there is an equivalent description in terms of the hyperplanes which form the boundary of W. In section 3, we introduce the notion of a hyperplane system: a Euclidean space, H, a finitely generated subgroup, L and a collection of co-dimension one oriented, affine hyperplanes, $\mathcal{P}$. With this data only, we construct the space $\tilde{H}(L,\mathcal{P})$ with a natural action of L and consider the crossed product. This provides a much more general construction for several reasons: we do not need all the data G and $\mathcal{L}$, we do not need extra hypotheses such as L is dense and finally the topology is given without reference to model sets. As one consequence, it is not necessary that the space $\tilde{H}(L,\mathcal{P})$ be totally disconnected. For this reason, we define it as the spectrum of a commutative C^*-algebra only. In fact, we allow the possibility that our collection of hyperplanes $\mathcal{P}$ is empty, in which case $\tilde{H}(L,\mathcal{P}) = H$ and the C^*-algebra is simply $C_0(H) \times L$. This is a reasonably familiar object; up to Morita equivalence, it is a non-commutative torus. Our generalization to hyperplane systems is actually a fairly obvious one; the justification for introducing it will come in the following sections.

There is a reasonably simple, but somewhat imprecise, description of the space $\tilde{H}(L,\mathcal{P})$ as follows. It is obtained from H by removing each affine hyperplane in $\mathcal{P}$ and replacing it by two copies which are separated by a gap. Each copy is attached to one of the two half-spaces. Of course, there are some subtleties when two or more hyperplanes meet and since the collection of hyperplanes is dense. The example we provided of an octagonal tiling in Figure 2 above actually corresponds to a new 'doubled' point from one of these affine hyperplanes. If one compares this model set to the one arising from its twin doubled point in $\tilde{H}(L,\mathcal{P})$, the two are identical except along a horizontal line passing through the middle of the picture. Observe that across the middle of the pattern there is a sequence of projected three dimensional cubes, each pair separated by either one or two squares. Moving the parameter point to its twin on the other doubled hyperplane, the change in the pattern is that all these cubes flip their orientation as shown below.

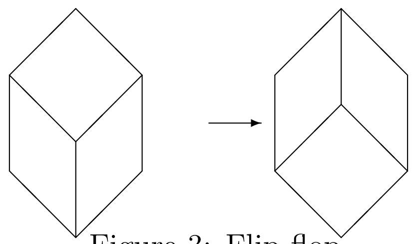

Figure 3: Flip-flop

This is usually refered to as a 'flip-flop'. The points of the model set affected by this move arise from a lower dimensional cut-and-project tiling system which may be regarded as a reduction of the original one to the hyperplane where the pair of doubled points arise. We make this notion of the reduction of a hyperplane system concrete in section 4. Given $(H, L, \mathcal{P})$ and a hyperplane P in $\mathcal{P}$, the reduction of $(H, L, \mathcal{P})$ by P, which we denote by $(H_P, L_P, \mathcal{P}_P)$, is given as follows. As P is a hyperplane, it is the translate of codimension one subspace of H, which we denote H_P. The group L_P is just $L \cap H_P$ and the collection of hyperplanes $\mathcal{P}_P$ is simply the intersections of P with the other elements of $\mathcal{P}$ (translated so that they lie in H_P rather than P). In general, even if we begin with a cut-and-project system, its reductions may not arise from a cut-and-project system. We give a simple geometric condition in Theorem 4.5 for when this holds. It is satisfied, in particular, by the octagonal example we have given. Under these hypotheses, the flip-flops are be described explicitly in terms of the lower dimensional cut-and-project system in Theorem 4.6.

The main goal of the paper is to show how this reduction to a hyperplane has a natural interpretation for the K-theory of the associated C^*-algebras. This is done in section 5. Given $(H, L, \mathcal{P})$ and its reduction to a hyperplane P in $\mathcal{P}$, we let L'_P be a complimentary subgroup of L_P in L; i.e. $L = L_P \times L'_P$. The set all translates of P under L can be indentified with $P \times L'_P$. This is typically a dense subset of H. In passing to $\tilde{H}(L, \mathcal{P})$, each of these points is replaced by two copies. (As well, the space P is itself disconnected by the other hyperplanes, and we will write $\tilde{P}$ instead. For simplicity, we could imagine the case that there are no other hyperplanes so $\tilde{P} = P$.) These two copies mean that we have two embeddings of $\tilde{P} \times L'_P$ into $\tilde{H}(L, \mathcal{P})$, which we denote by i_0 and i_1. If we endow L'_P with the discrete topology, these maps are continuous, but very far from proper. Let f be a continuous function of compact support on $\tilde{H}(L, \mathcal{P})$. The compositions $f \circ i_0$ and $f \circ i_1$ are continuous and bounded, but they are *not* compactly supported on $\tilde{P} \times L'_P$. This means that they lie in the multiplier algebra of $C_0(\tilde{P} \times L'_P)$. More subtlely and importantly, their difference *does* have compact support on $\tilde{P} \times L'_P$. Finally, this is all equivariant for the action of L on both spaces. This means that we have an element of the the Kasparov group $KK_0(C_0(\tilde{H}(L, \mathcal{P})) \times L), C_0(\tilde{P} \times L'_P) \times L)$, as interpreted by J. Cuntz [Cu]. Moreover, the C^*-algebra $C_0(\tilde{P} \times L'_P) \times L$ is Morita equivalent to that associated with the reduction, $C_0(\tilde{H}_P(L_P, \mathcal{P}_P)) \times L_P$. However, more is true. The results of [Pu] and show there is a six-term exact sequence relating the K-groups of these two C^*-algberas. The third C^*-algebra appearing in this sequence is that obtained from H, L and $\mathcal{P}'$, which is simply $\mathcal{P}$ after removing P and its L-orbit.

In practical terms, this means that the K-theory of our C^*-algebra arising from $(H, L, \mathcal{P})$ may be computed from that of $(H_P, L_P, \mathcal{P}_P)$ and $(H, L, \mathcal{P}')$. The former is simpler because it arises from a lower dimensional hyperplane system and the latter is simpler because it involves fewer hyperplanes. It is important to note in this computation that neither of the two new simpler hyperplane systems need arise from a cut-and-project system. Indeed, continuing this way, we end with the empty collection of hyperplanes which is certainly not arising from a cut-and-project system. We carry out this computation completely for the example of the octagonal tilings in section 6.

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2 Cut-and-project systems

In this section, we present the basic definitions and well-known results concerning projection method tilings. These (or variations of them) can be found in many places, e.g. [Be2, GS, Mo, Se]. They can also be found in [FHK], but the notation there seems less standard.

For $d \geq 1$, $\mathbb{R}^d$ denotes the usual Euclidean space of dimension d. The Euclidean norm of an element x in any Euclidean space is denoted $|x|$. We also use $B(x, R)$ to denote the ball centred at $x \in \mathbb{R}^d$ of radius $R > 0$. A subset W is *regular* if it is non-empty and is the closure of its interior. The boundary of a set W is denoted ∂W.

Let $G = \mathbb{R}^d$ and $H = \mathbb{R}^N$ be two Euclidean spaces. We let π and $\pi^\perp$ denote the two canonical projections of $G \times H$ onto G and H, respectively. There is an obvious action of G on $G \times H$ by translation in the first coordinate.

With a slight abuse of notation, we denote this by $u + x$ (or $x + u$), for x in $G \times H$ and u in G. In other words, we will regard G and H as subsets of $G \times H$.

Definition 2.1. *A* cut-and-project system *is a triple* $(G, H, \mathcal{L})$ *where* $G \cong \mathbb{R}^d$, $H \cong \mathbb{R}^N$ *and* $\mathcal{L} \subset G \times H$ *is a lattice (i.e.. a co-compact discrete subgroup) satisfying the following:*

1. *the restriction of* $\pi : G \times H \to G$ *to* $\mathcal{L}$ *is injective and*

2. *the map* $\pi^{\perp} : G \times H \to H$ *has* $\pi^{\perp}(\mathcal{L})$ *dense in* H*. We usually denote* $\pi^{\perp}(\mathcal{L})$ *by* L*.*

   If, in addition, we have

3. *the restriction of* $\pi^{\perp}$ *to* $\mathcal{L}$ *is injective,*

then we say the system is aperiodic.

Forrest, Hunton and Kellendonk [FHK] work in somewhat greater generality. Also, they take the view that the lattice $\mathcal{L} = \mathbb{Z}^{d+N}$. Of course, this is always the case up to isomorphism of groups, but their point of view is that the space G (which is E in their notation) is skewed, rather than the other way around.

Definition 2.2. *Let* $(G, H, \mathcal{L})$ *be a cut-and-project system and let* W *be a compact, regular subset of* H*. The set of non-singular points, denoted* $\mathcal{N}$*, is*

$$\mathcal{N} = \{x \in G \times H \mid \pi^{\perp}(x + \mathcal{L}) \cap \partial W = \emptyset\}.$$

Furthermore, for x *in* $\mathcal{N}$*, we define* $\Lambda_x(W) \subset G$ *by*

$$\Lambda_x(W) = \pi\{y \in x + \mathcal{L} \mid \pi^{\perp}(y) \in W\}.$$

The elements of $\mathcal{N}$ are also sometimes called generic points and the set $\Lambda_x(W)$ is a model set. We note that usually model sets may be constructed for any x in $G \times H$, without our hypothesis that x is in $\mathcal{N}$. The following is an easy consequence of the definitions and we omit the proof.

Lemma 2.3. *Let* $(G, H, \mathcal{L})$ *be a cut-and-project system and* $W \subset H$ *be compact and regular.*

1. *If* x *is in* $\mathcal{N}$ *and* s *is in* $\mathcal{L}$*, then* $x + s$ *is in* $\mathcal{N}$ *and* $\Lambda_{x+s}(W) = \Lambda_x(W)$*.*

2. *If x is in $\mathcal{N}$ and u is in G, then $x+u$ is in $\mathcal{N}$ and $\Lambda_{x+u}(W) = \Lambda_x(W)+u$.*

The next step is to define a topology on the collection of model sets. We follow [BHZ]. We define $\mathcal{M}(G)$ as being in the dual space of $C_c(G)$, the continuous compactly supported functions on G and give it the weak-* topology. The elements of $\mathcal{M}(G)$ are measures on G. To each set $\Lambda_x(W), x \in \mathcal{N}$, we regard it as an element of $\mathcal{M}(G)$ which is the sum of point measures over its elements. The fact that our sets are uniformly discrete plays an important part; we refer the reader to [BHZ] for details. The weak-* closure of $\Lambda_x(W), x \in \mathcal{N}$ in $\mathcal{M}(G)$ will be compact. It is also important to note that a measure in the closure can is again the sum of point masses over discrete point sets in G. We may suppress the distinction between point sets and measures.

In fact, the space constructed above is metrizable. There are a number of possibilities for the metric and some may be more suitable than others. For our purposes here, it will be sufficient to use the following (see [FHK]), although this is not the most general. Let Λ, Λ' be countable subsets of G. We consider the set of all $\epsilon > 0$ satisfying the following:

$$\text{there exist } v, v' \in B(0,\epsilon), (\Lambda - v) \cap B(0,\epsilon^{-1}) = (\Lambda' - v') \cap B(0,\epsilon^{-1}).$$

We define $d(\Lambda, \Lambda')$ to be the infimum of all such ϵ. The closure of $\Lambda_x(W), x \in \mathcal{N}$ in the topology concides with its completion in this metric.

We also introduce a new topology on the set $\mathcal{N}$ as follows. We embed $\mathcal{N}$ into $G \times H \times \mathcal{M}(G)$ by sending x in $\mathcal{N}$ to $(x, \Lambda_x(W))$. Again the closure of the image is compact. This space is also metrizable and we define for x, y in $\mathcal{N}$, we define

$$d(x, y) = |x - y| + d(\Lambda_x(W), \Lambda_y(W)).$$

The closure of $\mathcal{N}$ in the larger space and the completion in this metric coincide.

The following results summarize the facts we need regarding convergence of models sets in our metric. Proofs can be found in [FH].

Lemma 2.4. *Suppose that $x_n, n \geq 1$, is a sequence in $\mathcal{N} \cap H$.*

1. *The sequence $\Lambda_{x_n}(W), n \geq 1$, is convergent if and only if, for every $R > 0$, there is an $N \geq 1$ such that*

$$\Lambda_{x_m}(W) \cap B(0, R) = \Lambda_{x_n}(W) \cap B(0, R),$$

for all $m, n \geq N$.

2. *If the sequence $\Lambda_{x_n}(W), n \geq 1$, converges to Λ, then a point u in G is in Λ if and only if there exists $N \geq 1$ such that u is in $\Lambda_{x_n}(W)$, for every $n \geq N$.*

3. *The sequence $\Lambda_{x_n}(W), n \geq 1$, is convergent if and only if, for every s in $\mathcal{L}$, there exists $N \geq 1$ such that either $\pi^{\perp}(x_n + s) \in W$, for all $n \geq N$ or else $\pi^{\perp}(x_n + s) \notin W$, for all $n \geq N$.*

Definition 2.5. *For a cut-and-project system, $(G, H, \mathcal{L})$ and compact, regular set W in H, we let $\Omega(W)$ or simply Ω denote the closure of $\{\Lambda_x(W) \mid x \in \mathcal{N}\}$ in $\mathcal{M}(G)$, or equivalently, its completion in the metric d. We also let $\tilde{\Omega}(W)$ or simply $\tilde{\Omega}$ denote the closure of $\mathcal{N}$ in $G \times H \times \mathcal{M}(G)$.*

We summarize the important features of our spaces $\Omega(W)$ and $\tilde{\Omega}(W)$ in the following theorem. We refer the reader to Chapter I of [FHK] for a proof.

Theorem 2.6. *Let $(G, H, \mathcal{L})$ be a cut-and-project system and W be a regular subset of H.*

1. *There is a unique projection $\tilde{\beta} : \tilde{\Omega}(W) \to G \times H$ which extends the identity map on $\mathcal{N}$.*

2. *The actions of $\mathcal{L}$ and G on $\mathcal{N}$ extend to continuous actions on $\tilde{\Omega}(W)$ and $\tilde{\beta}$ is equivariant with respect to these and the obvious translation actions on the image.*

3. *The actions of $\mathcal{L}$ and G are free, wandering and commute with each other.*

4. *There is a unique continuous surjection $\beta : \Omega(W) \to G \times H/\mathcal{L}$ which maps $\Lambda_x(W)$ to $x + \mathcal{L}$, for any x in $\mathcal{N}$.*

5. *The action of G on $\{\Lambda_x(W) \mid x \in \mathcal{N}\}$ by translation extends to a continuous action on $\Omega(W)$ and β is equivariant with respect to it and the obvious translation action on the image.*

6. *$\tilde{\Omega}(W)/\mathcal{L} \cong \Omega(W)$.*

We present an alternate approach to the definitions of the continuous hulls, $\Omega(W)$ and $\tilde{\Omega}(W)$ in terms of the spectrum of a commutative C^*-algebra. This approach was first taken in [BIT] for the Kohmoto model in the one-dimensional case.

We begin with a cut-and-project system, $(G, H, \mathcal{L})$, and a window, W. Since $\mathcal{N}$, the set of non-singular points, is invariant under the action of G, we have $\mathcal{N} = G \times (\mathcal{N} \cap H)$. First, consider the Hilbert space $l^2(\mathcal{N} \cap H)$ of square summable functions on $\mathcal{N} \cap H$. Each bounded function on $\mathcal{N} \cap H$ defines an operator on this space by pointwise multiplication. Let $\mathcal{A}(W)$ or $\mathcal{A}$ denote the $*$-algebra of operators generated by $C_0(H)$ and all functions of the form $\chi_{W-\pi^{\perp}(s)}$, where s is in $\mathcal{L}$. (In the aperiodic case, it is not necessary to include $C_0(H)$, but we do so for convenience. Having these algebras represented as operators is also for convenence in later arguments.) We let $A(W)$ or A denote the closure of $\mathcal{A}$ in the operator norm. Both $\mathcal{A}$ and A are commutative.

Definition 2.7. *The space $\tilde{H}(W)$ or simply $\tilde{H}$ is the spectrum of the commutative C^*-algebra $A(W)$.*

Put another way, this means that, by the Gelfand-Naimark Theorem, there exists a locally compact Hausdorff space $\tilde{H}$ such that $A \cong C_0(\tilde{H})$. In fact, the space $\tilde{H}$ is the collection of non-zero homomorphisms from A to the complex numbers. The topology is obtained by realizing each such homomorphism as an element of the dual space of A and using the relative weak topology. In our case (where A is separable), this means that a sequence $\phi_n, n \geq 1$, converges to ϕ if $\phi_n(a)$ converges to $\phi(a)$, for every a in A. For the C^*-algebra $C_0(H)$, this space is homeomorphic to H itself and the map associates to a point of H the homomorphism obtained by evaluation at that point. Since $C_0(H) \subset A$, there is a continuous proper map $q : \tilde{H}(W) \to H$ such that $\phi(f) = f(q(\phi))$, for all f in $C_0(H)$ and ϕ in $\tilde{H}(W)$.

We note that there is an injection of $\mathcal{N} \cap H$ in $\tilde{H}$ which sends a point x to the functional $\phi_x(f) = f(x)$, for any f in $\mathcal{A}$. The fact that any such functional extends continuously from $\mathcal{A}$ to A is immediate since it is given as a vector state from the Hilbert space $l^2(\mathcal{N} \cap H)$. (As a remark, it is probably more natural to use the Hilbert space $L^2(H)$, using Lebesgue measure. We avoid this route for two reasons, first we would need to add the hypothesis that the Lebesgue measure of the boundary of W is zero. Secondly, we would need to prove at this point that the homomorphisms $\phi_x, x \in \mathcal{N} \cap H$ extend. This is not difficult, but we easily avoid the issue with our approach.)

There is an action of $\mathcal{L}$ on $\mathcal{A}$, denoted α, by

$$\alpha_s(f)(x) = f(x + \pi^{\perp}(s)),$$

for all x in $\mathcal{N} \cap H$. This extends to A and hence induces an action of $\mathcal{L}$ on $\tilde{H}$. The map q is equivariant for this action.

More general versions of the following appear in sections 3-7 of Chapter I of [FHK], but this will suffice for our purposes here.

Theorem 2.8. *The map sending x in $\mathcal{N}$ to $(\pi(x), \phi_{\pi^{\perp}(x)})$ in $G \times \tilde{H}$ extends to a homeomorphism from $\tilde{\Omega}$ to $G \times \tilde{H}$ which commutes with both G and $\mathcal{L}$ actions.*

The basic C^*-algebra of interest is the crossed product, $C(\Omega(W)) \times G$. However, from Theorem 2.6, $\Omega(W)$ is the quotient of $\tilde{\Omega}(W)$ by the action of $\mathcal{L}$, the groups $\mathcal{L}$ and G are both acting on $\tilde{\Omega}(W)$ and the actions are commuting and wandering. The results of Rieffel (situation 10 of [Ri]) apply directly and we conclude that $C(\Omega(W)) \times G$ is Morita equivalent to $C_0(\tilde{\Omega}(W)/G) \times \mathcal{L}$. Moreover, it is clear from the description above that $\tilde{\Omega}(W)/G$ is just $\tilde{H}(W)$ and the action of $\mathcal{L}$ is just α as above. We summarize with the following statement, but the reader should also consult II.2.9 of [FHK].

Theorem 2.9. *The C^*-algebras $C(\Omega(W)) \times G$ and $C_0(\tilde{H}(W)) \times_{\alpha} \mathcal{L}$ are Morita equivalent.*

This implies, in particular, that $C(\Omega(W)) \times \mathbb{R}^d$ and $C_0(\tilde{H}(W)) \times_{\alpha} \mathcal{L}$ have isomorphic K-theory groups. Henceforth, we will concentrate on the latter.

We want to give some notation for crossed products. If A is a C^*-algebra with an action α of the discrete group L, we write elements of the crossed product as $\sum_{s \in L} a_s u_s$, where each a_s is an element is an element of A and only finitely many are non-zero. The collection of such elements is a dense $*$-subalgebra of the crossed product. The elements $u_s, s \in L$, are unitary operators. In the case that A is non-unital, these lie in the multiplier algebra of $A \times L$ rather than algebra itself. These satisfy the relation

$$u_s a = \alpha_s(a) u_s,$$

for all a in A and s in L.

3 Hyperplane systems

In the last section, we saw how C^*-algebras could be produced from a cut-and-project system and a window. Now, we give a different construction of C^*-algebras. The data in this case is called a hyperplane system. We will then establish some concrete relations between the two constructions; broadly speaking, the new construction will be more general.

In the following defintion, we will consider an oriented, affine hyperplane of codimension one in a Euclidean space H. By this, we mean a set P which is the translate of a subspace H_P by some vector x_P. Of course, H_P is just the set of all differences $u - v$, where u, v are in P. The point x_P may be chosen arbitrarily from P, but we fix a choice. As P has codimension one, it divides the space H into two closed half-spaces whose intersection is P. By an orientation of P, we mean that we have a fixed choice of labelling these as P^0 and P^1.

Definition 3.1. *A* hyperplane system *is a triple* $(H, L, \mathcal{P})$*, where* $H \cong \mathbb{R}^N$*,* $L \subset H$ *is a finitely generated subgroup and* $\mathcal{P}$ *is a countable collection of co-dimension* 1 *oriented affine hyperplanes in* H *which is invariant under the action of* L*. That is, for each* P *in* $\mathcal{P}$ *and* s *in* L*,* $P + s$ *is also in* $\mathcal{P}$*.*

A hyperplane cut-and-project system *is a quadruple,* $(G, H, \mathcal{L}, \mathcal{P})$*, where* $(G, H, \mathcal{L})$ *is a cut-and-project system and* $(H, \pi^{\perp}(\mathcal{L}), \mathcal{P})$ *is a hyperplane system.*

Of course, a hyperplane system does not need G as part of its data. Moreover, the subgroup L does not need to be dense in H. Also, notice that we allow the possibility that $\mathcal{P}$ is empty. We note that for any two co-dimension one hyperplanes P, Q, their intersection is either $P = Q$, the empty set or a co-dimension 2 hyperplane.

Frequently, the elements of $\mathcal{P}$ will be the L-orbits of subspaces of H, but this is not necessary. For any x in H, we let $\mathcal{P}(x)$ denote all elements of $\mathcal{P}$ which contain x. Let us also make clear that when we say that L acts on $\mathcal{P}$, we mean that it preserves the orientation; that is, we have $(P + s)^i = P^i + s$, for all s in L and P in $\mathcal{P}$.

We will proceed to define C^*-algebras from a hyperplane system. In the case of a hyperplane cut-and-project system, we establish a link between these and the C^*-algebras of the last section, at least for the special class of windows which are polytopes whose boundaries are contained in the hyperplanes.

Although at this point, we do not yet have any model sets, we may proceed to define a disconnected version of H, $\tilde{H}$. This can be done in purely topological terms (see [Le, FHK]), but we find it most convenience to follow the ideas of section 2 exploiting the Gelfand-Naimark theorem.

Let $\mathcal{N}_H$ denote the complement of the union of $\mathcal{P}$ in H. (This is a replacement for $\mathcal{N} \cap H$ of the last section.) We consider the Hilbert space $l^2(\mathcal{N}_H)$ and regard bounded functions on $\mathcal{N}_H$ as operators. We let $\mathcal{A}(L, \mathcal{P})$ or simply $\mathcal{A}$ be the $*$-algebra generated by $C_0(H)$ and all functions of the form $f\chi_{P^i}$, where f is in $C_0(H)$, P is in $\mathcal{P}$ and $i = 0, 1$, considered as operators on $l^2(\mathcal{N}_H)$. Since $\mathcal{N}_H$ is a dense G_δ in H, this representation of $C_0(H)$ is faithful. Note that we have $f\chi_{P^0} + f\chi_{P^1} = f$, since the functions agree on $\mathcal{N}_H$. (We remark that we do not want our algebra to contain the function χ_{P^i} since it is not compactly supported.) We let $A(L, \mathcal{P})$ or simply A be the closure of $\mathcal{A}(L, \mathcal{P})$ in the operator norm. Notice that if $(H, L, \mathcal{P})$ and $(H, L, \mathcal{P}')$ are hyperplane systems with $\mathcal{P}' \subset \mathcal{P}$, then $\mathcal{A}(L, \mathcal{P}') \subset \mathcal{A}(L, \mathcal{P})$ and $A(L, \mathcal{P}') \subset A(L, \mathcal{P})$.

Definition 3.2. *Let $(H, L, \mathcal{P})$ be a hyperplane system. We define $\tilde{H}(L, \mathcal{P})$ or simply $\tilde{H}$ to be the spectrum of the commutative C^*-algebra $A(L, \mathcal{P})$.*

As A contains $C_0(H)$, there is a natural continuous surjection $q : \tilde{H} \to H$. Notice that $\tilde{H}$ need not be totally disconnected. In fact, when $\mathcal{P}$ is empty, it is just H.

Our first result is to note that changing the data by translating $\mathcal{P}$ does not seriously affect the construction.

Proposition 3.3. *Let $(H, L, \mathcal{P})$ be a hyperplane system and let x be in H. There exists a homeomorphism*

$$\tau_x : \tilde{H}(L, \mathcal{P}) \to \tilde{H}(L, \mathcal{P} - x)$$

such that $q \circ \tau_x(z) = q(z) + x$, for all z in $\tilde{H}(L, \mathcal{P})$.

Proof. The map sending f to $\tau_x(f)(y) = f(y + x)$ defines an isomorphism from $\mathcal{A}(L, \mathcal{P})$ to in $\mathcal{A}(L, \mathcal{P} - x)$. This extends to the A algebras and therefore is induced by a homeomorphism of their spectra. That it satisfies the last statement is trivial. □

Our next objective is to give a better description of the elements of $\tilde{H}$.

Lemma 3.4. *Let ϕ be a non-zero functional on A and let $x = q(\phi)$.*

1. *For any f in $C_0(H)$, P in $\mathcal{P} \setminus P(x)$ and $i = 0, 1$, we have*

$$\phi(f\chi_{P^i}) = f(x)\chi_{P^i}(x).$$

2. *For any P in $P(x)$ and $i = 0, 1$, we have either $\phi(f\chi_{P^i}) = f(x)$, for all f in $C_0(H)$ or $\phi(f\chi_{P^i}) = 0$, for all f in $C_0(H)$.*

Proof. For the first statement, if P is not in $P(x)$, then x is not in P and so either x is in the interior of P^i or P^{1-i}. Let us suppose the former. We may find a function, g in $C_0(H)$, whose support is compact and contained in P^i and satisfying $g(x) = 1$. Then we have

$$\phi(f\chi_{P^i}) = \phi(f\chi_{P^i})g(x) = \phi(f\chi_{P^i}g) = \phi(fg) = f(x)g(x) = f(x),$$

and we are done. The case x is in P^{1-i} is similar and we omit the details.

For the second statement, we fix a function g in $C_0(H)$ such that $g(x) = 1$. First, we have

$$\phi(g\chi_{P^i}) + \phi(g\chi_{P^{1-i}}) = \phi(g\chi_{P^i} + g\chi_{P^{1-i}}) = \phi(g) = g(x) = 1.$$

In addition, we also have

$$\phi(g\chi_{P^i})\phi(g\chi_{P^{1-i}}) = \phi(g\chi_{P^i} \cdot g\chi_{P^{1-i}}) = \phi(0) = 0.$$

We conclude that there are two possibilities, either
$\phi(g\chi_{P^0}) = 1$ and $\phi(g\chi_{P^1}) = 0$, or vice verse. Now for any other f in $C_0(H)$, we have

$$\phi(f\chi_{P^i}) = g(x)\phi(f\chi_{P^i}) = \phi(gf\chi_{P^i}) = \phi(f)\phi(g\chi_{P^i}) = f(x)\phi(g\chi_{P^i}).$$

This completes the proof. □

We may give a presentation of the points of $\tilde{H}$ as follows. Let ϕ be a non-zero functional on A. Let $x = q(\phi)$; this means that, for each f in $C_0(H)$, we have $\phi(f) = f(x)$. Using the second part of the last result, we define $\delta : \mathcal{P}(x) \to \{0, 1\}$ by

$$\phi(f\chi_{P^1}) = \delta(P)f(x),$$

for any f in $C_0(H)$. In view of the last result, ϕ is uniquely determined by the pair (x, δ). Henceforth, we write $\phi = (x, \delta)$. Notice that for a given x, not every function δ arises from some ϕ. In the case $N = 2$, if $\mathcal{P}(x)$ contains k lines, then $q^{-1}\{x\}$ has $2k$ points, not 2^k.

The following provides a description of the points of $\tilde{H}$ and also the topology.

Proposition 3.5. *1. Let x be in H and let $\delta : \mathcal{P}(x) \to \{0,1\}$. The point (x,δ) is in $\tilde{H}$ if and only if, for every $\epsilon > 0$, there exists y in $B(x,\epsilon) \cap \mathcal{N}_H$ such that y is in $P^{\delta(P)}$ for every P in $\mathcal{P}(x)$.*

2. A sequence $x_n, n \geq 1$, in $\mathcal{N}_H$ converges to (x,δ) in $\tilde{H}$ if and only if x_n converges to x in H and, for all n sufficiently large, x_n is in $P^{\delta(P)}$, for every P in $\mathcal{P}(x)$.

Proof. First, we suppose that (x,δ) is in $\tilde{H}$. Let $\epsilon > 0$. Define

$$\mathcal{W}^i = \{P \in \mathcal{P}(x) \mid \delta(P) = i\},$$

for $i = 0,1$. Choose a function f in $C_0(H)$ whose support is contained in $B(x,\epsilon)$ and so that $f(x) = 1$. Define the function g in $\mathcal{A}$ by

$$g = \prod_{P \in \mathcal{W}^1} f\chi_{P^1} \prod_{P \in \mathcal{W}^0} f\chi_{P^0} = \prod_{P \in \mathcal{W}^1} f\chi_{P^1} \prod_{P \in \mathcal{W}^0} (f - f\chi_{P^1}).$$

We compute the value of the functional (x,δ) on g:

$$\begin{aligned}(x,\delta)\Big(\prod_{P \in \mathcal{W}^1} f\chi_{P^1} \prod_{P \in \mathcal{W}^0} (f - f\chi_{P^1})\Big) &= \prod_{P \in \mathcal{W}^1} (x,\delta)(f\chi_{P^1}) \\ &\quad \cdot \prod_{P \in \mathcal{W}^0} (x,\delta)(f - f\chi_{P^1}) \\ &= \prod_{P \in \mathcal{W}^1} \delta(P) \prod_{P \in \mathcal{W}^0} (1 - \delta(P)) \\ &= 1.\end{aligned}$$

From this, we conclude that the function g is non-zero. Hence, there is a point y in $\mathcal{N}_H$ with $g(y) \neq 0$. It is follows from Lemma 3.4 that y satisfies the desired conclusion.

For the converse, it follows from the hypothesis that we may construct a sequence, $y_n, n \geq 1$, in $\mathcal{N}_H$ which converges to x and such that y_n is in $P^{\delta(P)}$, for every P in $\mathcal{P}(x)$. Regarding these points as vector states on A and hence in $\tilde{H}$, we may find a subsequence which is convergent in the weak topology. It follows at once that (x,δ) arises from this limit.

The second statement follows from Lemma 3.4. □

The next objective is give a concrete link between the C^*-algebra of the hyperplane system arising from a cut-and-project system and that of the last section. For this, we need to consider a window W, but we must restrict to a specific class of polytopes, as follows.

Definition 3.6. *A subset W of H is a $\mathcal{P}$-*polytope *if it is non-empty, compact, regular and can be written as*

$$W = \cap_{Q \in \mathcal{W}^0} Q^0 \cap \cap_{Q \in \mathcal{W}^1} Q^1,$$

where $\mathcal{W}^0$ and $\mathcal{W}^1$ are finite subsets of $\mathcal{P}$. Moreover, we say that the collections $\mathcal{W}^0$ and $\mathcal{W}^1$ are minimal if no proper subcollection of their union will have the same intersection.

For Q in $\cup_i \mathcal{W}^i$, we define

$$\partial_Q W = Q \cap W.$$

The boundary of W can be written as the union of the sets $\partial_Q W$ over all such Q.

In some sense, this definition is going the wrong way. It is most usual to begin with a window W which is a polytope in the standard sense and then define $\mathcal{P}$ to be all hyperplanes which are translates of the set of hyperplanes which form the faces of W.

The next result gives a specific link between our space $\tilde{H}$ and the topology on model sets from section 2.

Proposition 3.7. *Let $(G, H, \mathcal{L}, \mathcal{P})$ be a hyperplane cut-and-project system. Suppose the sequence $x_n, n \geq 1$, in $\mathcal{N}_H$ converges to (x, δ) in $\tilde{H}$. Then for any $\mathcal{P}$-polytope W, $\Lambda_{x_n}(W)$ is Cauchy. We define $\Lambda_{(x,\delta)}(W)$ to be its limit in the sense described in section 2.*

Proof. First, observe that $\mathcal{N}_H$ is contained in $\mathcal{N} \cap H$. Since the space $\Omega(W)$ is compact, it suffices for us to show that any two limit points of subsequences of $\Lambda_{x_n}(W)$ are equal. Suppose that Λ and Λ' are limit points of the subsequences $\Lambda_{x_{m_k}}(W), k \geq 1$ and $\Lambda_{x_{n_k}}(W), k \geq 1$, respectively.

Let u be in Λ. By Lemma 2.4, for all k sufficiently large, we have u is in $\Lambda_{m_k}(W)$ and $u = \pi(s)$ for some s in $\mathcal{L}$ and $x_{m_k} + \pi^{\perp}(s)$ is in W. Write W as an intersection as in 3.6 for collections $\mathcal{W}^0, \mathcal{W}^1$. If $x_{m_k} + \pi^{\perp}(s)$ is in W, then it is in P^0, for each P in $\mathcal{W}^0$ and in P^1, for each P in $\mathcal{W}^1$. Fix P in $\mathcal{W}^i$ for the moment. Since x_n converges to (x, δ) in $\tilde{H}$, we know from Proposition 3.5 that for all n sufficiently large, x_n is in $(P - \pi^{\perp}(s))^{\delta(P-\pi^{\perp}(s))} = P^{\delta(P-\pi^{\perp}(s))} - \pi^{\perp}(s)$. This means that $\delta(P - \pi^{\perp}(s)) = i$ and the subsequence x_{n_k} is also in $P^0 - \pi^{\perp}(s)$, hence $x_{n_k} + \pi^{\perp}(s)$ is in P^0, for k large. As this holds for each P, we have $x_{n_k} + \pi^{\perp}(s)$ is in W, for k large. This implies that u is also in Λ'. We have shown $\Lambda \subset \Lambda'$ but the same argument shows the reverse inclusion and the conclusion follows. □

The precise relation between the construction of this section and the last is summarized in the following two results.

Theorem 3.8. *Suppose* $(G, H, \mathcal{L}, \mathcal{P})$ *is an aperiodic hyperplane cut-and-project system and that* W *is a* $\mathcal{P}$*-polytope. Then we have*

$$C_0(\tilde{H}(W)) \subset C_0(\tilde{H}(L, \mathcal{P}))$$

and the map sending $f\, u_s$ *to* $f\, u_{\pi^\perp(s)}$ *extends to an inclusion of* C^**-algebras*

$$C_0(\tilde{H}(W)) \times_\alpha \mathcal{L} \subset C_0(\tilde{H}(L, \mathcal{P})) \times_\alpha L.$$

Proof. Let $\mathcal{W}^0, \mathcal{W}^1$ be the minimal collection of elements of $\mathcal{P}$ as in the definition of W. Let s be in $\mathcal{L}$. Choose f, a function in $C_0(H)$ such that f is identically one on $W - \pi^\perp(s)$. For $i = 0, 1$ and P in $\mathcal{W}^i$, the function $f\chi_{P^i - \pi^\perp(s)}$ is in $A(L, \mathcal{P})$. Then so is their product (over all i and P), which is just $\chi_{W-\pi^\perp(s)}$ and it follows that $A(W) \subset A(L, \mathcal{P})$, which is the first statement.

The second statement is a trivial consequence of the first and the fact aperiodicity means that the map $\pi^\perp : \mathcal{L} \to L$ is an isomorphism. □

In general, the inclusion may be proper (for example, if W is $\mathcal{P}'$-polytope where $\mathcal{P}'$ is some proper $\mathcal{L}$-invariant subset of $\mathcal{P}$), but equality is obtained in the particular case when the window is the so-called canonical acceptance domain as follows.

Let $\mathcal{S}$ be a set of generators for $\mathcal{L}$. We define

$$C_\mathcal{S} = \{\sum_{s \in \mathcal{S}} t_s s \mid 0 \leq t_s \leq 1, s \in \mathcal{S}\},$$

where C is chosen to suggest 'cube'. We will consider the window $\pi^\perp(C_\mathcal{S})$. The set of hyperplanes, $\mathcal{P}$, which form the boundaries of W (and all their translates under $\pi^\perp(\mathcal{L})$) are described explicitly in section IV.2 of [FHK]. There is also a proof of the following.

Corollary 3.9. *Suppose* $(G, H, \mathcal{L})$ *is an aperiodic cut-and-project system,* $\mathcal{S}$ *is a set of generators of* $\mathcal{L}$*,* $W = \pi^\perp(C_\mathcal{S})$ *is as above and* $\mathcal{P}$ *is the associated set of hyperplanes. Then we have*

$$C_0(\tilde{H}(W)) = C_0(\tilde{H}(L, \mathcal{P}))$$

and the map sending $f u_s$ to $f u_{\pi^\perp(s)}$ extends to an isomorphism of C^*-algebras

$$C_0(\tilde{H}(W)) \times_\alpha \mathcal{L} \cong C_0(\tilde{H}(L, \mathcal{P})) \times_\alpha L.$$

Remark 3.10. *Suppose that L is dense in H or at least that its span is all of H. We may then choose subgroups L_0, L_1 of L so that $L = L_0 \oplus L_1$ and L_0 is a lattice in H. In this case, the action of L_0 on $\tilde{H}(L, \mathcal{P})$ is free and wandering and $C_0(\tilde{H}(L, \mathcal{P})) \times L$ is strongly Morita equivalent to $C_0(\tilde{H}(L, \mathcal{P})/L_0) \times L_1$ (this is just an L_1-equivariant version of situation 2 of [Ri]). Moreover, the space $\tilde{H}(L, \mathcal{P})/L_0$ has a natural finite-to-one mapping onto H/L_0, which is a torus.*

Remark 3.11. *In the special case that $\mathcal{P}$ is empty, we have $\tilde{H}(L, \mathcal{P}) = H$. If we also assume L is as in the last remark, $\tilde{H}(L, \mathcal{P})/L_0$ is just a torus and the action of L_1 is by rotation. The C^*-algebra $C_0(\tilde{H}(L, \mathcal{P})/L_0) \times L_1$ is a non-commutative torus whose dimension is the rank of L.*

Remark 3.12. *In the special case that H is one-dimensional and the group L has rank two, then we may choose $L_0 \cong \mathbb{Z} \cong L_1$. The action of L_1 on $\tilde{H}(L, \mathcal{P})/L_0$ is by the restriction of a Denjoy homeomorphisms to its unique minimal set, which is totally disconnected (see [PSS]). Provided that $\mathcal{P}$ is non-empty, the results of [PSS] show that*

$$K_0(C_0(\tilde{H}(L, \mathcal{P}) \times L) \cong \mathbb{Z}^{p+1}, K_1(C_0(\tilde{H}(L, \mathcal{P}) \times L) \cong \mathbb{Z},$$

where p is the number of L-orbits in $\mathcal{P}$.

4 Reduction of hyperplane systems

In this section, we introduce the notion of the reduction of a hyperplane system to one of the hyperplanes. The result is another hyperplane system with lower dimensional data. It is important to note that if the original system is part of a hyperplane cut-and-project system, this extra feature need not pass to the reduction.

Begin with a hyperplane system, $(H, L, \mathcal{P})$. Choose P in $\mathcal{P}$. Recall that $H_P = P - P$ is a $N-1$-dimensional subspace $H_P = P - x_P$ of H. Further, we define

$$L_P = L \cap H_P.$$

The final ingredient in our reduced system is the collection of hyperplanes. We define $\mathcal{P}_P$ to be the collection of all $Q \cap P - x_P$, where Q is in $\mathcal{P}$, with $Q \cap P \neq \emptyset, P$. (Neither H_P nor L_P depends on x_P, and while $\mathcal{P}_P$ does, it is only up to translation. We can then refer to Proposition 3.3 to make the relation precise.) A word of warning is appropriate: the choice of Q in $\mathcal{P}$ in obtaining $Q \cap P - x_P$ is not unique. For example, in the octagonal tiling of section 6, if we use $P = P_1$, we have $P_2 \cap P_1 = P_3 \cap P_1$.

Definition 4.1. *Let $(H, L, \mathcal{P})$ be a hyperplane system. For P in $\mathcal{P}$, the* reduction of $(H, L, \mathcal{P})$ to P *is*

$$(H_P, L_P, \mathcal{P}_P).$$

We note the following easy results without proof.

Proposition 4.2. *Let $(H, L, \mathcal{P})$ be a hyperplane system and let s be in L. We have*

$$H_{P+s} = H_P, L_{P+s} = L_P, \mathcal{P}_{P+s} = \mathcal{P}_P + x_P - x_{P+s}.$$

Lemma 4.3. *If P is in $\mathcal{P}$, then*

$$\mathcal{N}_{H_P} = H_P \setminus \bigcup_{Q \in \mathcal{P}, Q \neq P} (Q - x_P).$$

For any x in $\mathcal{N}_{H_P}$ and s in L, we have $\mathcal{P}(x + x_P + s) = \{P + s\}$.

Lemma 4.4. *Let W be a $\mathcal{P}$-polytope, expressed minimally as*

$$W = (\cap_{Q \in \mathcal{W}^1} Q^1) \cap (\cap_{Q \in \mathcal{W}^0} Q^0).$$

Suppose that P is in $\mathcal{W}^1$ or $\mathcal{W}^0$. Then $\partial_P W - x_P$ is a $\mathcal{P}_P$-polytope.

We now consider a cut-and-project hyperplane system, $(G, H, \mathcal{L}, \mathcal{P})$, and its associated hyperplane system $(H, L, \mathcal{P})$. Our aim is to show that under some (fairly strong) hypotheses, for a given P in $\mathcal{P}$, the reduction of $(H, L, \mathcal{P})$ to P will again arise from a cut-and-project hyperplane system. In this case, there is a precise relation between the associated model sets.

Define

$$\mathcal{L}_P = \{s \in \mathcal{L} \mid \pi^\perp(s) \in H_P\}$$

and subsequently

$$G_P = \{\pi(x) \in span_{\mathbb{R}} \mathcal{L}_P \mid \pi^\perp(x) = 0\} \subset G.$$

Theorem 4.5. *Let $(G, H, \mathcal{L}, \mathcal{P})$ be a hyperplane cut-and-project system and let P be in $\mathcal{P}$. If*

1. $H_P \subset span_{\mathbb{R}}\mathcal{L}_P$ *and*

2. L_P *is dense in* H_P,

then $(G_P, H_P, \mathcal{L}_P)$ is a cut-and-project system. If $(G, H, \mathcal{L})$ is aperiodic, then so is $(G_P, H_P, \mathcal{L}_P)$.

Proof. The first thing to prove is that if $H_P \subset span_{\mathbb{R}}\mathcal{L}_P$, then $\mathcal{L}_P$ is a subset of $G_P \times H_P$. Let s be in $\mathcal{L}_P$, so that $\pi^{\perp}(s)$ is in $H_P \subset span_{\mathbb{R}}\mathcal{L}_P$. It follows that $\pi(s) = s - \pi^{\perp}(s)$ is in G_P. As $\mathcal{L}$ is discrete, so is $\mathcal{L}_P$. Since L_P is dense in H_P, we have $\pi^{\perp}(span_R\mathcal{L}_P) = H_P$ and then $span_R\mathcal{L}_P = G_P \times H_P$.

As $\mathcal{L}_P$ is a subset of $\mathcal{L}$, it follows that the restrictions of π and $\pi^{\perp}$ to the former are injective if their restrictions to the latter are. □

The last result of this section is that, under the hypothesis of the last theorem, there is a precise relation between certain model sets for the original system and those of the reduction. More precisely, suppose that (x, δ_0) and (x, δ_1) are two points in $\tilde{H}$, where x is in P and δ_0 and δ_1 differ only by their values on P. Then the set theoretic difference $\Lambda_{(x,\delta_1)}(W) \setminus \Lambda_{(x,\delta_0)}(W)$ can be expressed in terms of a model set for the reduced system $(G_P, H_P, \mathcal{L}_P, \mathcal{P}_P)$. This result is certainly known in the quasicrycstal community, but we state it here because it has an interesting contrast with a result in the next section, and also give a proof for completeness.

As L is finitely generated, the subgroup L_P is a direct summand and we choose another subgroup L'_P such that $L = L_P \times L'_P$. Of course, there may be many choices; this will not effect the result.

We define maps

$$i^j_P : \tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P \to \tilde{H}(L, \mathcal{P}),$$

for $j = 0, 1$. For (x, δ) in $\tilde{H}_P(L_P, \mathcal{P}_P)$ and s' in L'_P, we define $i^j_P((x, \delta), s') = (x + x_p + s', \bar{\delta}_j)$, where

$$\bar{\delta}_j(Q) = \begin{cases} \delta(Q - x_P - s'), & \text{for } Q \neq P + s', \\ j, & \text{for } Q = P + s' \end{cases}$$

It follows from Proposition 3.5 that both $i^j_P, j = 0, 1$ are well-defined and continuous; we omit the details. In particular, if we consider x which is in

$\mathcal{N}_{H_{\mathcal{P}}}$, we have $\mathcal{P}(x + x_P + s') = \{P + s'\}$ by Proposition 4.3 and we define $i^j_P(x) = (x + x_P + s', j)$, where the second entry is interpreted as the constant function j on the singleton $\{P + s'\}$.

Theorem 4.6. *Let $(G, H, \mathcal{L}, \mathcal{P})$ be a hyperplane cut-and-project system. Let P be an element of $\mathcal{P}$ such that*

1. *$H_P \subset span_{\mathbb{R}}\mathcal{L}_P$ and*
2. *L_P is dense in H_P.*

Let W be a $\mathcal{P}$-polytope associated to the minimal collections $\mathcal{W}^0$ and $\mathcal{W}^1$. Suppose that (x, δ) is in $\tilde{H}_P(L_P, \mathcal{P}_P)$ and s' is in $\mathcal{L}$ with $\pi^\perp(s')$ in L'_P. If there exists s in $\mathcal{L}$ such that $P + \pi^\perp(s)$ is in $\mathcal{W}^1$, then

$$\begin{aligned}\Lambda_{i^1_P((x,\delta),\pi^\perp(s'))}(W) \quad &\setminus \quad \Lambda_{i^0_P((x,\delta),\pi^\perp(s'))}(W) \\ &= \quad \Lambda_{(x,\delta)}(\partial_{P+\pi^\perp(s)}W - x_P - \pi^\perp(s)) + \pi(s - s').\end{aligned}$$

If there is no such s, then

$$\Lambda_{i^1_P((x,\delta),s')}(W) \subset \Lambda_{i^0_P((x,\delta),s')}(W).$$

Proof. Case 1. We begin with the added assumptions that $s' = 0$ and that x is in $\mathcal{N}_{H_P}$. In the right hand side of our formula, the point (x, δ) can be replaced by simply x. Of course, this simplifies things because the set on the right hand side is a genuine model set and not a limit of them. It also implies that $\mathcal{P}(x+x_P) = \{P\}$ and we have $i^j_P(x, 0) = (x+x_P, j)$, for $j = 0, 1$. Choose a sequence $x_n, n \geq 1$, in $P^1 - x_P$ a sequence $y_n, n \geq 1$, in $P^0 - x_P$ both converging to x.

Suppose that $u = \pi(t)$ is in $\Lambda_{(x+x_P,1)}$ and not in $\Lambda_{(x+x_P,0)}$. From the definitions and Lemmas 2.4, this implies that $x_n + x_P + \pi^\perp(t)$ is in W and $y_n+x_P+\pi^\perp(t)$ is not in W, for n large. It follows that $x+x_P+\pi^\perp(t)$ is in ∂W and hence in $\partial_Q W$, for some Q in $\mathcal{W}^i$, where i is either 0 or 1. This implies that $x + x_P$ is in $Q - \pi^\perp(t)$ and since we know $\mathcal{P}(x + x_P) = \{P\}$, we must have $Q - \pi^\perp(t) = P$. Next, since $x_n + x_P + \pi^\perp(t)$ is in W which is contained in $Q^i = P^i + \pi^\perp(t)$. It follows from our choice of x_n that $i = 1$. We have now shown that if the set difference we are considering is non-empty, then there exists an element t satisfying the condition $P + \pi^\perp(t)$ is in $\mathcal{W}^1$. Equivalently, if there is no such t, we have the containment $\Lambda_{(x,\delta_1)}(W) \subset \Lambda_{(x,\delta_0)}(W)$.

Now, we continue to prove that $u = \pi(t)$ is in the right hand side of the first expression. We have $P + \pi^{\perp}(s)$ and $P + \pi^{\perp}(t)$ are both in $\mathcal{W}^1$. If they are distinct, then by simple geometric considerations, either $P + \pi^{\perp}(s) \subset P + \pi^{\perp}(t)$ or the reverse. But proper containment would contradict the minimality of $\mathcal{W}^1$. We conclude that these are equal and so $\pi^{\perp}(t) - \pi^{\perp}(s)$ is in H_P and hence $t - s$ is in $\mathcal{L}_P$. Moreover, we have

$$\pi^{\perp}(x + t - s) = x + \pi^{\perp}(t) - \pi^{\perp}(s) \in \partial_{P+\pi^{\perp}(s)}W - x_P - \pi^{\perp}(s).$$

and it follows that $\pi(x+t-s)$ is in $\Lambda_x(\partial_{P+\pi^{\perp}(s)}W - x_P - \pi^{\perp}(s))$. Moreover, we have

$$u = \pi(t) = \pi(x + t - s) + \pi(s) \in \Lambda_x(\partial_{P+\pi^{\perp}(s)}W - x_P - \pi^{\perp}(s)) + \pi(s).$$

We have proved $\subset$ in the first expression.

Now suppose that u is in $\Lambda_x(\partial_{P+\pi^{\perp}(s)}W - x_P - \pi^{\perp}(s))$. This means that $u = \pi(x+t)$ for some t in $\mathcal{L}_P$ where $\pi^{\perp}(x+t)$ is in the interior of $\partial_{P+\pi^{\perp}(s)}W - x_P - \pi^{\perp}(s))$, with respect to H_P. This means that $x + x_P + \pi^{\perp}(s+t)$ is in the interior of $\partial_{P+\pi^{\perp}(s)}W$ relative to $P + \pi^{\perp}(s)$. This, in turn, means that $x + x_P + \pi^{\perp}(s+t)$ is in the interior of Q^i, for each $Q \neq P + \pi^{\perp}(s)$ in $\mathcal{W}^i$ and $i = 0, 1$. This also implies that $x_n + x_P + \pi^{\perp}(s+t)$ is in the interior of Q^i, for each Q in $\mathcal{W}^i$ (including $P + \pi^{\perp}(s)$) and $i = 0, 1$, and n large. On the other hand, $y_n + x_P + \pi^{\perp}(s+t)$ is not in $P^1 + \pi^{\perp}(s)$.. It follows that $x_n + x_P + \pi^{\perp}(s+t)$ is in W, while, $y_n + x_P + \pi^{\perp}(s+t)$ is not in W, for n large. Then we have $\pi(x_n + x_P + s + t) = \pi(s) + \pi(t) = u + \pi(s)$ is in $\Lambda_{i^1_P((x,\delta),0)}(W)$ and not in $\Lambda_{i^0_P((x,\delta),0)}(W)$. This completes the proof in case 1.

Case 2. We remove the condition that $s' = 0$, but still require x is in $\mathcal{N}_{H_P}$. This implies that $\mathcal{P}(x + x_P + \pi^{\perp}(s')) = \{P + \pi^{\perp}(s')\}$. The entire argument above may be applied to the point x and the element $P + \pi^{\perp}(s')$ in $\mathcal{P}$. In this case, we have $P + \pi^{\perp}(s') + \pi^{\perp}(s - s')$ is in $\mathcal{W}^1$. We choose $x_{P+\pi^{\perp}(s')} = x_P + \pi^{\perp}(s')$. Moreover, we have $i^j_{P+\pi^{\perp}(s')}(x, 0) = i^j_P(x, \pi^{\perp}(s'))$. The result follows at once.

Case 3. We finally consider the general case. Any point (x, δ) is the limit of a sequence x_n in $\mathcal{N}_{H_P}$. We know the conclusion holds for each x_n and the result for (x, δ) holds by virtue of Lemma 2.4. □

5 K-theory

There are already techniques available for the computation of the K-theory of the C^*-algebras associated with a cut-and-project system [FHK, BS]. Here, we present a variation on these results. In principle, this is more general in that they apply to hyperplane systems, but it is likely that the results of [FHK] could be generalized in this fashion.

We begin with a hyperplane system, $(H, L, \mathcal{P})$, and an element P in $\mathcal{P}$. Let $(H_P, L_P, \mathcal{P}_P)$ denote the reduction on P. In addition, we let

$$\mathcal{P}' = \mathcal{P} \setminus \{P + s \mid s \in L\}.$$

As earlier, $\tilde{H}(L, \mathcal{P})$, $\tilde{H}(L, \mathcal{P}')$ and $\tilde{H}(L_P, \mathcal{P}_P)$ are the spaces associated with each. For brevity, it will sometimes be useful to use the notation

$$\begin{aligned} B(H, L, \mathcal{P}) &= C_0(\tilde{H}(L, \mathcal{P})) \times L, \\ B(H, L, \mathcal{P}') &= C_0(\tilde{H}(L, \mathcal{P}')) \times L, \\ B(H_P, L_P, \mathcal{P}_P) &= C_0(\tilde{H}_P(L_P, \mathcal{P}_P)) \times L_P. \end{aligned}$$

The aim of this section is to give a six-term exact sequence relating the K-groups of these three C^*-algebras. One map, in particular, is given in a very natural way by a class in the Kasparov group $KK(B(H, L, \mathcal{P}), B(H_P, L_P, \mathcal{P}_P))$. The result is an immediate application of the techniques in [Pu] for the transformation groupoids $\tilde{H}(L, \mathcal{P}) \times L, \tilde{H}(L, \mathcal{P}') \times L$ and $\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P$.

Recall from the last section that $L'_P \subset L$ is chosen such that $L = L_P \times L'_P$. We consider the space $\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P$ with action of L given by $(x, t') + (s + s') = (x + s, t' + s')$, for all x in $\tilde{H}_P(L_P, \mathcal{P}_P)$, t', s' in L'_P and s in L_P. We note that L'_P is given the discrete topology and that this action is free. For any s' in L'_P, we let $\delta_{s'}$ denote the function on L'_P which is 1 at s' and 0 elsewhere. If f is in $\tilde{H}_P(L_P, \mathcal{P}_P)$ and s' is in L'_P, we denote by $f \otimes \delta_{s'}$ the obvious function on $\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P$. The linear span of such functions is dense in $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P)$.

We regard $\tilde{H}_P(L_P, \mathcal{P}_P)$ as a subset of $\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P$ by identifying it with $\tilde{H}_P(L, \mathcal{P}_P) \times \{0\}$. It is an abstract transversal in the sense of Muhly, Renault and Williams [MRW] and the groupoids $(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P) \times L$ and $\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P$ are equivalent. This implies that their C^*-algebras are Morita equivalent. In fact, one can show quite explicitly that the map which sends $(f \otimes \delta_{t'}) u_{s+s'}$ to $f u_s \otimes e_{t', t'+s'}$ extends to an isomorphism

$$C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P) \times L \cong (C_0(\tilde{H}_P(L_P, \mathcal{P}_P)) \times L_P) \otimes \mathcal{K}(l^2(L'_P)),$$

where we use the notation $e_{s'_1, s'_2}$ to denote the rank one operator which maps a vector ξ in $l^2(L'_P)$ to $< \xi, \delta_{s'_2} > \delta_{s_1}$.

We have already defined the two maps

$$i_P^j : \tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P \to \tilde{H}(L, \mathcal{P}),$$

for $j = 0, 1$. We extend i_P^0, i_P^1 to maps on the groupoids

$$i_P^j : (\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P) \times L \to \tilde{H}(L, \mathcal{P}) \times L,$$

for $j = 0, 1$, which are the identity on L. There should be no confusion if we use the same notation.

The pair (i_P^0, i_P^1) defines, in a very natural way, an element of the Kasparov group $KK_0(B(H, L, \mathcal{P}), B(H_P, L_P, \mathcal{P}_P))$ which we describe now. First, we use the definition of KK provided by Cuntz [Cu]. Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space. We consider $M(B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(\mathcal{H}))$, the multiplier algebra of $B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(\mathcal{H})$. It contains $B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(\mathcal{H})$ as an ideal. A quasi-homomorphism from $B(H, L, \mathcal{P})$ to $B(H_P, L_P, \mathcal{P}_P)$ is a pair, (ρ_0, ρ_1) of $*$-homomorphisms from $B(H, L, \mathcal{P})$ to $M(B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(\mathcal{H}))$, such that, for every a in $B(H, L, \mathcal{P})$, $\rho_0(a) - \rho_1(a)$ is in $B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(\mathcal{H})$.

In our case, we use the Hilbert space $l^2(L'_P)$ and we have already noted that

$$(C_0(\tilde{H}_P(L_P, \mathcal{P}_P)) \times L_P) \otimes \mathcal{K}(l^2(L'_P)) \cong C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P) \times L.$$

Notice that if f is in $C_0(\tilde{H}(L, \mathcal{P}))$, then, for $j = 0, 1$, the function $i_P^j(f) = f \circ i_P^j$ is a continuous bounded function on $\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P$. Of course, it fails to vanish at infinity, but it does lie in the multiplier algebra of $C_0(\tilde{H}_P(L_P, \mathcal{P}_P \times L'_P))$. The action of L extends to the multiplier algebra and the map i_P^j is equivariant. That is, we define

$$i_P^j : B(H, L, \mathcal{P}) \to M(C_0(\tilde{H}_P(L_P, \mathcal{P}_P \times L'_P)) \times L),$$

by

$$i_P^j(\sum_{s \in L} f_s u_s) = \sum_{s \in L} (f_s \circ i_P^j) u_s$$

for $j = 0, 1$ and $\sum_{s \in L} f_s u_s$ in $B(H, L, \mathcal{P})$ as before.

Lemma 5.1. *Let f be in $C_0(H) \subset C_0(\tilde{H}(L, \mathcal{P}))$, Q be in $\mathcal{P}$ and $i = 0, 1$. If there exists s' in L'_P such that $Q = P + s'$, then we have*

$$i_P^0(f\chi_{Q^i}) - i_P^1(f\chi_{Q^i}) = (-1)^i f' \otimes \delta_{s'},$$

where f' *in* $C_0(H_P) \subset C_0(\tilde{H}_P(L_P, \mathcal{P}_P))$ *is defined by*

$$f'(x) = f(x + x_P + s'), x \in H_P.$$

If there is no such s', *then*

$$i^0_P(f\chi_{Q^i}) - i^1_P(f\chi_{Q^i}) = 0.$$

In particular, $i^0_P(f\chi_{Q^i}) - i^1_P(f\chi_{Q^i})$ *is in* $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P)$.

Proof. For $j = 0, 1$, we have

$$i^j_P(f\chi_{Q^i})((x, \delta), s') = (f\chi_{Q^i})(x + x_P + s', \bar{\delta}_j).$$

This first case is when $x + x_P + s'$ is not in Q. In this case, the result is $f(x + x_P + s')\chi_{Q^i}(x + x_P + s')$ and is independent of j. The second case is when $x + x_P + s'$ is in Q and $Q \neq P + s'$. In this case, the result is $f(x + x_P + s')\delta(Q)$ and again is independent of j. Finally, if $x + x_P + s'$ is in Q and $Q = P + s'$, the result is $f(x + x_P + s')$ if $j = i$ and zero otherwise. The conclusion follows. □

Since functions of the form $f\chi_{Q^i}$ as in the Lemma generate the C^*-algebra $C_0(\tilde{H}(L, \mathcal{P}))$, which is an ideal in its multiplier algebra, it follows that $i^0_P(a) - i^1_P(a)$ is in $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P)$, for every a in $C_0(\tilde{H}(L, \mathcal{P}))$. It follows immediately that $i^0_P(a) - i^1_P(a)$ is in $B(H_P, L_P, \mathcal{P}_P)$, for every a in $B(H, L, \mathcal{P})$.

Following [Pu], we now want to take the quotient of $\tilde{H}(L, \mathcal{P})) \times L$ by the equivalence relation $i^0_P((x, \delta), s') \sim i^1_P((x, \delta), s')$, for all $((x, \delta), s')$. In fact, it is easy to see that for any $((y_0, \delta_0), s_0)$ and $((y_1, \delta_1), s_1)$ in $\tilde{H}(L, \mathcal{P})) \times L$, they are related by $\sim$ if and only if $y_0 = y_1$, $s_0 = s_1$ and $\delta_0|\mathcal{P}'(y_0) = \delta_1|\mathcal{P}'(y_1)$. This is the essential idea of the proof of the following; the remainder is a matter of checking topological details which are fairly simple by using Proposition 3.5 and we omit them.

Theorem 5.2. *The map* α *sending* (y, δ) *in* $\tilde{H}(L, \mathcal{P})$ *to* $(y, \delta|\mathcal{P}'(y))$ *in* $\tilde{H}(L, \mathcal{P}')$ *is continuous, proper and* L*-invariant. If we also denote by* α *the map between the groupoids* $\tilde{H}(L, \mathcal{P}) \times L$ *and* $\tilde{H}(L, \mathcal{P}') \times L$ *which is* α *in the first coordinate and the identity in the second, then it induces an isomorphism of topological groupoids between the quotient of* $\tilde{H}(L, \mathcal{P}) \times L$ *by* $i^0_P((x, \delta), s') \sim i^1_P((x, \delta), s')$, *for all* $((x, \delta), s')$, *and* $(\tilde{H}(L, \mathcal{P}')) \times L$.

Notice that our map α induces a *-homomorphism, also denoted α,

$$\alpha : C_0(\tilde{H}(L, \mathcal{P}')) \times L \to C_0(\tilde{H}(L, \mathcal{P})) \times L$$

by $\alpha(\sum_s f_s u_s) = \sum_s (f_s \circ \alpha)\, u_s$.

The hypotheses of Theorem 2.1 of [Pu] are satisfied and we conclude the following holds.

Theorem 5.3. *Let $(H, L, \mathcal{P})$ be a hyperplane system, P be an element of $\mathcal{P}$ and $\mathcal{P}' = \mathcal{P} \setminus \{P + s \mid s \in L\}$. There is a six-term exact sequence*

$$\begin{array}{ccccc}
K_0(B(H, L, \mathcal{P}')) & \xrightarrow{\alpha_*} & K_0(B(H, L, \mathcal{P})) & \xrightarrow{[i_P^0, i_P^1]_*} & K_0(B(H_P, L_P, \mathcal{P}_P)) \\
\uparrow & & & & \downarrow \\
K_1(B(H_P, L_P, \mathcal{P}_P)) & \xleftarrow[{[i_P^0, i_P^1]_*}]{} & K_1(B(H, L, \mathcal{P})) & \xleftarrow[\alpha_*]{} & K_1(B(H, L, \mathcal{P}'))
\end{array}$$

where $[i_P^0, i_P^1]$ is the element of $KK_0(B(H, L, \mathcal{P}), B(H_P, L_P, \mathcal{P}_P))$ described above.

Finally, we note that this computation of the map can be carried out quite explicitly, at least for projections which are functions on $\tilde{H}(L, \mathcal{P})$, as follows.

Theorem 5.4. *Suppose that $W = (\cap_{Q \in \mathcal{W}^0} Q^0) \cap (\cap_{Q \in \mathcal{W}^1} Q^1)$ is a $\mathcal{P}$-polytope, expressed minimally. For $j = 0, 1$, if there exists (a necessarily unique) $s_j' \in L_P'$ such that $P + s_j' \in \mathcal{W}^j$, then set $W_j = \partial_{P + s_j'} W - x_P - s_j'$ and set W_j to be the empty set otherwise. Then we have*

$$[i_P^0, i_P^1]_*([\chi_W]) = [\chi_{W^0}] - [\chi_{W_1}].$$

Proof. Let $p_j = i_P^j(\chi_W)$, for $j = 0, 1$. These are projections in the multiplier algebra of $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P') \times L$ and their difference is in $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P') \times L$. Cuntz describes $K_0(C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P') \times L)$ in terms of such pairs. If a pair $[p, p']$ have both p, p' in the algebra itself, then this pair is the same as $[p] - [p']$ in the usual description of K-zero. In our case, we exploit the extra feature that p_0 and p_1 commute (since they are both in the commutative algebra $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L_P')$). In Cuntz' approach, the pair $(p_0 p_1, p_0 p_1)$ is the trivial element. Let $p_j' = \chi_{W_j} \otimes \delta_{s_j'}$ if s_j' exists and is zero otherwise. Each of these is in $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times$

L'_P). Under the explicit isomorphism between $C_0(\tilde{H}_P(L_P, \mathcal{P}_P) \times L'_P) \times L$ and $B(H_P, L_P, \mathcal{P}_P) \otimes \mathcal{K}(l^2(L'_P))$, the projection p'_j is mapped to $\chi_{W_j} \otimes \delta_{s'_j}$ and its class in $K_0(B(H_P, L_P, \mathcal{P}_P))$ is $[\chi_{W_j}]$.

Since s'_0 and s'_1 are clearly distinct if they both exist, we have $p'_0 p'_1 = 0$. We will show that $p_0 - p_1 = p'_0 - p'_1$. Assuming this for the moment, if we subtract p_0 from both sides and multiply by p'_0, we get $-p'_0 p_1 = p'_0(1 - p_0)$. The left hand side is clearly negative, while the right is positive. We conclude that they are both 0. Analogous results hold replacing p'_0 with p'_1. From this it follows that p'_0 and p'_1 are both orthogonal to $p_0 p_1$ and we have

$$p_0 p_1 + p'_j = p_0 p_1 + p_j p'_j = p_j(p_{1-j} + p'_j) = p_j(p'_{1-j} + p_j) = 0 + p_j = p_j.$$

This means that the pair $(p_0 p_1, p_0 p_1)$ may be added to the orthogonal pair (p'_0, p'_1) and the result is (p_0, p_1). We conclude that in K-zero, we have

$$[i^0_P, i^1_P]_*([\chi_W]) = [(p_0, p_1)] = [(p'_0, p'_1)] = [p'_0] - [p'_1] = [\chi_{W0}] - [\chi_{W_1}].$$

It remains to prove that $p_0 - p_1 = p'_0 - p'_1$. Find a function f in $C_0(H)$ which is identically equal to one on the compact set W. We can express

$$\chi_W = (\Pi_{Q \in \mathcal{W}^0} f \chi_{Q^0})(\Pi_{Q \in \mathcal{W}^1} f \chi_{Q^1})$$

Let us assume for the moment that both s'_0 and s'_1 exist. It follows from Lemma 5.1 that

$$i^0_P(f\chi_{Q^i}) = i^1_P(f\chi_{Q^i}),$$

for all $Q \neq P + s'_0, P + s'_1$. Also, let $f'_i(x) = f(x + x_P + s'_i), i = 0, 1$ be as in Lemma 5.1, so that

$$i^0_P(f\chi_{P^i + s'_i}) - i^1_P(f\chi_{P^i + s'_i}) = (-1)^i f'_i \otimes \delta_{s'_i}$$

for $i = 0, 1$.

Next, for $Q \neq P + s'_0, P + s'_1$ and $j, k = 0,,$ we claim that

$$i^j_P(f\chi_{Q^k}) f'_i \otimes \delta s'_i = (f'_i)^2 \chi_{Q^k \cap P - x_P - s'_i} \otimes \delta_{s'_i}.$$

We evaluate both sides at $(x, s') \in \mathcal{H}(L_P, \mathcal{P}_P) \times L'_P$. It is clear that both sides are zero unless $s' = s'_i$. Moreover, since Q is not in the L-orbit of P, we have

$$i^j_P(f\chi_{Q^k})(x, s'_i) = f(x + x_P + s'_i)\chi_{Q^k}(x + x_P + s'_i) = f'_i(x)\chi_{Q^k \cap P - x_P - s'_i}(x).$$

This establishes the claim. Let a denote the product of all factors $f\chi_{Q^k}$ over $k=0,1$ and Q in $\mathcal{W}^k$ with $Q \neq P+s_i'$. It follows from the claim above that

$$i_P^j(a)f_i' \otimes \delta_{s_i'} = (f_i')^n \chi_{\partial_{P+s_i'}W - x_P - s_i'} \otimes \delta_{s_i'},$$

where n is the number of factors involved. Since f is identically 1 on W, f_i' is identically 1 on $\partial_{P+s_i'}W - x_P - s_i'$ and so

$$\chi_{\partial_{P+s_i'}W - x_P - s_i'} f_i' = \chi_{\partial_{P+s_i'}W - x_P - s_i'}.$$

Next, we consider the product

$$f\chi_{P^0+s_0'} f\chi_{P^1+s_1'} = f^2 \chi_{(P^0+s_0')\cap(P^1+s_1')}.$$

As this is a factor in χ_W, it must be non-zero. The hyperplanes $P+s_0'$ and $P+s_1'$ are parallel and for the two opposite half-spaces to have non-trivial intersection, $P+s_1'$ must be contained in P^0+s_0' and $P+s_0'$ must be contained in P^1+s_1'. It follows that

$$\begin{aligned} i_P^j(f\chi_{P^0+s_0'}))f_1' \otimes \delta_{s_1'} &= i_P^j(f)f_1' \otimes \delta_{s_1'} \\ i_P^j(f\chi_{P^0+s_0'}))f_0' \otimes \delta_{s_0'} &= i_P^j(f)f_0' \otimes \delta_{s_1'}. \end{aligned}$$

We compute

$$\begin{aligned} p_0 - p_1 &= i_P^0(f\chi_W) - i_P^1(f\chi_W) \\ &= i_P^0(f\chi_{P^0+s_0'} f\chi_{P^1+s_1'}a) - i_P^1(f\chi_{P^0+s_0'} f\chi_{P^1+s_1'}a) \\ &= (i_P^0(f\chi_{P^0+s_0'})i_P^0(f\chi_{P^1+s_1'}a) - i_P^1(f\chi_{P^0+s_0'})i_P^0(f\chi_{P^1+s_1'}a)) \\ &\quad +(i_P^1(f\chi_{P^0+s_0'}a)i_P^0(f\chi_{P^1+s_1'}) - i_P^1(f\chi_{P^0+s_0'}a)i_P^1(f\chi_{P^1+s_1'})) \\ &= ((i_P^0(f\chi_{P^0+s_0'}) - i_P^1(f\chi_{P^0+s_0'})))i_P^0(f\chi_{P^1+s_1'}a) \\ &\quad +i_P^1(f)i_P^1(f\chi_{P^0+s_0'}a)(i_P^0(f\chi_{P^1+s_1'}) - i_P^1(f\chi_{P^1+s_1'})) \\ &= (f_0' \otimes \delta_{s_0'})i_P^0(f\chi_{P^1+s_1'}a) - i_P^1(f\chi_{P^0+s_0'}a)(f_1' \otimes \delta_{s_1'}) \\ &= (f_0')^{n+2}\chi_{\partial_{P^0+s_0'}W - x_P - s_0'} \otimes \delta_{s_0'} - (f_1')^{n+2}\chi_{\partial_{P^1+s_1'}W - x_P - s_1'} \otimes \delta_{s_1'} \\ &= \chi_{W_0} \otimes \delta_{s_0'} - \chi_{W_1} \otimes \delta_{s_1'} \\ &= p_0' - p_1'. \end{aligned}$$

This completes the proof in the case that both s_0' and s_1' exist. The proofs in the three remaining cases are similar or even somewhat easier since the argument two paragraphs above is no longer needed. We omit the details.

□

Remark 5.5. *It is interesting to compare this result with Theorem 4.6. This is a dynamical/operator theory analogue of that result for model sets.*

Remark 5.6. *The complexity of a hyperplane system* $(H, L, \mathcal{P})$ *can be roughly measured by three numbers: the dimension of* H*, the rank of* L*, and the number of distinct* L*-orbits in* $\mathcal{P}$*. (The last may be infinite.) We compare the three systems* $(H, L, \mathcal{P}), (H, L, \mathcal{P}')$ *and* $(H_P, L_P, \mathcal{P}_P)$*. The second is simpler than the first; the first two numbers remain the same while the third is reduced by one. The third is also simpler than the first; the first number is reduced by one, the second by at least one (except in rather trivial cases) and the third is reduced by at least one.*

We may regard Theorem 5.3 as computing the K-theory of the C^**-algebra associated with* $(H, L, \mathcal{P})$ *in terms of the other two (simpler) systems. In principle, the same techniques could be applied to these simpler systems. In the end, there would presumably be a spectral sequence. We do not pursue this here. It is important to note that even in the case that* $(H, L, \mathcal{P})$ *arises from a cut-and-project system as earlier, the intermediate systems may not, but are merely hyperplane systems. For example, it is quite possible that* $L_P = 0$*.*

6 Example: the octagonal tiling

We now use the methods of the last section to compute the K-theory of the C^*-algebra of the octagonal tiling, as described in the introduction. Along they way, we establish a number of results in the general setting. We do not give complete proofs, which are somewhat lengthy. One aspect which is nice for the octagonal case is that as we proceed along, all the groups are free abelian, so it most convenient for us to simply list their generators. (The reader should note this is not always the case.)

Let us just start with some notation which is particular to the octagonal case. Here H is dimension two, while L is generated by $\pi^\perp(e_1), \pi^\perp(e_2), \pi^\perp(e_1), \pi^\perp(e_4)$, which we now denote by l_1, l_2, l_3, l_4, for convenience. The four hyperplanes $P_k, k = 1, 2, 3, 4$ are in fact subspaces so that $H_{P_k} = P_k$. For simplicity, we denote this by H_k. We also denote its stabilzer under L as $L_{P_k} = L_k$. It is fairly easy to see that L_1 is generated by l_1 and $l_3 - l_4$. There are similar description of the others. If one considers all non-trivial pairwise intersections of hyperplanes in the collection $\mathcal{P}$, this set of points is, of course, invariant under L. It consists of exactly three L-orbits, that of $H_1 \cap H_2$ (which is the origin), $(H_1 + l_4) \cap H_2$ and $(H_3 + l_1) \cap H_4$. We denote these three points by V_0, V_1 and V_3 respectively. We let L_i denote the stabilzer of H_i, for each $1 \leq i \leq 4$.

First of all, we note that in the case that the collection of hyperplanes $\mathcal{P}$ is empty, then $\tilde{H}(L, \emptyset) = H$. It follows that

$$C_0(\tilde{H}(L,\emptyset)) \times L \cong C_0(H) \times L \cong C^*(L) \times \hat{H} \cong C(\hat{L}) \times \hat{H},$$

where the second isomorphism is via Fourier transform. As L is isomorphic to $\mathbb{Z}^4$, we have $\hat{L} \cong \mathbb{T}^4$, the 4-torus, while $\hat{H} \cong H$, since it is a Euclidean space. Applying Connes' analogue of the Thom isomorphism, the K-theory of this C^*-algebra is the same as that of the torus (since $\dim H$ is even). We conclude from this that

$$K_0(B(H,L,\emptyset))) \cong \wedge^{even} L, K_1(B(H,L,\emptyset))) \cong \wedge^{odd} L.$$

Since our group L and various subgroups of it will be acting, not just on H, but also on various subspaces determined by the hyperplanes, we will denote elements of the K-theory above by l^H, for l in L and exterior powers of such symbols. We may list the generators of $K_0(B(H,L,\emptyset)))$ as

$$\begin{array}{cccc} 1^H, & l_1^H \wedge l_2^H, & l_1^H \wedge l_3^H, & l_1^H \wedge l_4^H, \\ l_2^H \wedge l_3^H, & l_2^H \wedge l_4^H, & l_3^H \wedge l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

and similarly for K_1.

The next step to consider is the exact sequence of Theorem 5.3 in the special case that $\mathcal{P}$ is the L-orbit of a single hyperplane P, and hence $\mathcal{P}'$ is empty. It can be computed explicitly from the exact sequence and in notation above, it simply sends l^{H_P} in $K_i(B(H_P, L_P, \emptyset))$ to l^H in $K_{i+1}(B(H,L,\emptyset))$ and similarly for exterior powers. (Notice that the dimension shift from the exact sequence is taken care of by the isomorphism above and the fact that $\dim H = \dim H_P + 1$.

For the octagonal tiling, we apply this for $P = H_1$. Notice that $L_1 = H_1 \cap L$ is generated by l_1 and $l_3 - l_4$. Let $\mathcal{P}_1$ be the L-orbit of H_1. The generators of $K_0(B(H_1, L_1, \emptyset))$ are $l_1^{H_1}, (l_3 - l_4)^{H_1}$. The generators of $K_1(B(H_1, L_1, \emptyset))$ are $1^{H_1}, l_1^{H_1} \wedge (l_3 - l_4)^{H_1}$. These groups are mapped injectively into the groups $K_i(B(H,L,\emptyset))$ and the quotients are $K_i(B(H,L,\mathcal{P}_1))$. The generators for $K_0(B(H,L,\mathcal{P}_1))$ are

$$\begin{array}{ccc} l_1^H \wedge l_2^H, & l_1^H \wedge l_3^H = l_1^H \wedge l_4^H, & l_2^H \wedge l_3^H, \\ l_2^H \wedge l_4^H, & l_3^H \wedge l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

The generators for $K_1(B(H, L, \mathcal{P}_1))$ are

$$\begin{array}{ccc} l_2^H, & l_3^H = l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H, \\ l_1^H \wedge l_2^H \wedge l_4^H, & l_1^H \wedge l_3^H \wedge l_4^H, & l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

We next turn to disconnecting along the hyperplane H_3 and its L-orbit. Notice that L_3 is generated by l_3 and $l_1 + l_2$. Let $\mathcal{P}_2$ denote the L-orbits of H_1 and H_3. We will apply Theorem 5.3 to $\mathcal{P}_2$, with $P = H_3$ and so $\mathcal{P}' = \mathcal{P}_1$. The first task is to compute
$K_*(B(H_3, L_3, (\mathcal{P}_2)_{H_3}))$. In this case, we have $(\mathcal{P}_2)_{H_3}$ is just the L_3-orbit of V_0. As a result, the generators of $K_0(B(H_3, L_3, (\mathcal{P}_2)_{H_3}))$ are $l_3^{H_3}, (l_1 + l_2)^{H_3}$ and the generator of $K_1(B(H_3, L_3, (\mathcal{P}_2)_{H_3}))$ is $(l_1 + l_2)^{H_3} \wedge l_3^{H_3}$. Again, the maps from these groups into the K-theory of $B(H, L, \mathcal{P}_1)$ are injective and the K-theory of $B(H, L, \mathcal{P}_2)$ is just the quotient. We are able to list the generators of $K_0(B(H, L, \mathcal{P}_2))$ as

$$\begin{array}{ccc} l_1^H \wedge l_2^H, & l_1^H \wedge l_3^H = & l_1^H \wedge l_4^H = -l_2^H \wedge l_3^H, \\ l_2^H \wedge l_4^H, & l_3^H \wedge l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

The generators for $K_1(B(H, L, \mathcal{P}_2))$ are

$$\begin{array}{cc} l_1^H \wedge l_2^H \wedge l_3^H, & l_1^H \wedge l_2^H \wedge l_4^H, \\ l_1^H \wedge l_3^H \wedge l_4^H, & l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

We next turn to disconnecting along the hyperplane H_2 and its L-orbit. Let $\mathcal{P}_3$ denote the L-orbits of H_1, H_3 and H_2. We will apply Theorem 5.3 to $\mathcal{P}_3$, with $P = H_2$ and so $\mathcal{P}' = \mathcal{P}_2$. The first task is to compute $K_*(B(H_2, L_2, (\mathcal{P}_3)_{H_2}))$. It is easy to see that $(\mathcal{P}_3)_{H_2}$ is simply the L_2-orbits of the points V_0, V_1. We would like to apply the same Theorem to compute this, for the hyperplane system $(H_2, L_2, \{V_0, V_1\} + L_2)$. Ordinarily, this would require two application of the Theorem, since there are two distinct hyperplane orbits which must be removed. However, since these hyperplanes are parallel, we may make a single application, letting $\mathcal{P}' = \emptyset$ being the result of removing them both and the third term is simply the direct sum of the K-theory groups of $B(\{V_0\}, 0, \emptyset)$ and $B(\{V_1\}, 0, \emptyset)$. The K_0 group is just the free abelian group on $1_0^V, 1^{V_1}$ and K_1 is trivial. From this, we obtain generators for $K_0(B(H_2, L_2, (\mathcal{P}_2)_{H_2}))$:

$$1^{V_0} - 1^{V_1}, l_2^{H_2}, (l_3 + l_4)^{H_2}$$

where the first is a slightly rough notation for an element whose image under the KK-map is $1^{V_0} - 1^{V_1}$, which is in the kernel of the map into $K_1(B(H_2, L_2, \emptyset))$. We also obtain a single generator for $K_1(B(H_2, L_2, (\mathcal{P}_3)_{H_2}))$, which is $l_2^{H_2} \wedge (l_3 + l_4)^{H_2}$. In this case, the map from $K_0(B(H_2, L_2, (\mathcal{P}_3)_{H_2}))$ to $K_1(B(H, L, \mathcal{P}_2))$ is zero while the map from $K_1(B(H_2, L_2, (\mathcal{P}_3)_{H_2}))$ to $K_0(B(H, L, \mathcal{P}_2))$ is injective. We conclude that a set of generators for $K_0(B(H, L, \mathcal{P}_3))$ is

$$\begin{array}{ccc} 1^{V_0} - 1^{V_1}, & l_2^{H_2} & (l_3 + l_4)^{H_2} \\ l_1^H \wedge l_3^H = & l_1^H \wedge l_4^H = & l_2^H \wedge l_3^H = -l_2^H \wedge l_4^H, \\ l_1^H \wedge l_2^H, & l_3^H \wedge l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

while a set of generators for $K_1(B(H, L, \mathcal{P}_3))$ is the same as for $K_1(B(H, L, \mathcal{P}_2))$ above.

Finally, we consider $\mathcal{P}$ which is the translation of all four hyperplanes and reduce of $P = H_4$, so that $\mathcal{P}' = \mathcal{P}_3$. The computation of $K_*(B(H_4, L_4, (\mathcal{P})_{H_4}))$ is analogous to that above for $K_*(B(H_2, L_2, (\mathcal{P}_3)_{H_2}))$ and the generators of its K_0-group are

$$1^{V_0} - 1^{V_2}, l_4^{H_4}, (l_1 - l_2)^{H_4}$$

while the generator of its K_1-group is $(l_1 - l_2)^{H_4} \wedge l_4^{H_4}$. Here, the map from both of these groups is zero and so we may write generators for $K_0(B(H, L, \mathcal{P}))$ as

$$\begin{array}{ccc} 1^{V_0} - 1^{V_1}, & l_2^{H_2} & (l_3 + l_4)^{H_2} \\ 1^{V_0} - 1^{V_2}, & l_4^{H_4} & (l_1 - l_2)^{H_4} \\ l_1^H \wedge l_3^H = & l_1^H \wedge l_4^H = & l_2^H \wedge l_3^H = -l_2^H \wedge l_4^H, \\ l_1^H \wedge l_2^H, & l_3^H \wedge l_4^H, & l_1^H \wedge l_2^H \wedge l_3^H \wedge l_4^H. \end{array}$$

and the generators for $K_1(B(H, L, \mathcal{P}_2))$ as

$$\begin{array}{lll} l_1^H \wedge l_2^H \wedge l_3^H, & l_1^H \wedge l_2^H \wedge l_4^H, & \\ l_1^H \wedge l_3^H \wedge l_4^H, & l_2^H \wedge l_3^H \wedge l_4^H, & (l_1 - l_2)^{H_4} \wedge l_4^{H_4}. \end{array}$$

Remark 6.1. *As will be clear to the experts at this point, the proper way to organize calculations like the one above is by means of a spectral sequence. We do not pursue this for two reasons. First, it would take considerable effort and, secondly, it seems that the result would be very similar to the spectral sequence obtained by Forrest, Hunton and Kellendonk in [FHK]. There are* *some differences; the computation of [FHK] is actually computing the cohomology of the hull, which is a slightly different thing. Our version for the K-theory would have the effect of 'bundling together' all the even and all the odd cohomology groups into the two K-theory groups.*

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