Families of Type III KMS States on a Class of C^*-Algebras containing O_n and $\mathcal{Q}_{\mathbb{N}}$

A.L. Carey[a], J. Phillips[b,c], I.F. Putnam[b], A. Rennie[a]

[a] Mathematical Sciences Institute, Australian National University, Canberra, ACT, AUSTRALIA
[b] Department of Mathematics and Statistics, University of Victoria, Victoria, BC, CANADA
[c] Corresponding Author

Abstract

We construct a family of purely infinite C^*-algebras, $\mathcal{Q}^\lambda$ for $\lambda \in (0,1)$ that are classified by their K-groups. There is an action of the circle $\mathbb{T}$ with a unique KMS state ψ on each $\mathcal{Q}^\lambda$. For $\lambda = 1/n$, $\mathcal{Q}^{1/n} \cong O_n$, with its usual $\mathbb{T}$ action and KMS state. For $\lambda = p/q$, rational in lowest terms, $\mathcal{Q}^\lambda \cong O_n$ $(n = q - p + 1)$ with UHF fixed point algebra of type $(pq)^\infty$. For any $n > 1$, $\mathcal{Q}^\lambda \cong O_n$ for infinitely many λ with distinct KMS states and UHF fixed-point algebras. For any $\lambda \in (0,1)$, $\mathcal{Q}^\lambda \neq O_\infty$. For λ irrational the fixed point algebras, are NOT AF and the $\mathcal{Q}^\lambda$ are usually NOT Cuntz algebras. For λ transcendental, $K_1(\mathcal{Q}^\lambda) \cong K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}^\infty$, so that $\mathcal{Q}^\lambda$ is Cuntz' $\mathcal{Q}_{\mathbb{N}}$, [Cu1]. If λ and λ^{-1} are both algebraic integers, the **only** O_n which appear are those for which $n \equiv 3(mod\ 4)$. For each λ, the representation of $\mathcal{Q}^\lambda$ defined by the KMS state ψ generates a type III_λ factor. These algebras fit into the framework of modular index theory / twisted cyclic theory of [CPR2, CRT] and [CNNR].

Keywords: KMS state, III_λ factor, modular index, twisted cyclic theory, K-theory.

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1. Introduction

In this paper we introduce some new examples of KMS states on a large class of purely infinite C^*-algebras that were motivated by the 'modular index theory' of [CPR2, CNNR]. We were aiming to find algebras that were not Cuntz-Krieger algebras (or the CAR algebra) and which were not previously known to be examples of this phenomenon, in order to explore the possibilities opened by [CNNR]. These algebras, denoted by $\mathcal{Q}^\lambda$ for $0 < \lambda < 1$, are not constructed as graph algebras, but as "corner algebras" of certain crossed product C^*-algebras. The $\mathcal{Q}^\lambda$ have similar structural properties to the Cuntz algebras, however there are important new features, such as

1) when $\lambda = p/q$ is rational in lowest terms, then $\mathcal{Q}^\lambda \cong O_{q-p+1}$ as mentioned in the Abstract,

2) when λ is algebraic, the K-groups depend on the minimal polynomial (and its coefficients) of λ,

3) when λ is transcendental, $\mathcal{Q}^\lambda \cong \mathcal{Q}_{\mathbb{N}}$, Cuntz' algebra, [Cu1].

We prove in Section 3 that the $\mathcal{Q}^\lambda$ are purely infinite, simple, separable, nuclear C^*-algebras, so there is no nontrivial trace on them. Also in Section 3 we determine in many cases the K-groups of these algebras and use classification theory to identify them when these algebras have the same K-groups as others found previously (these facts are summarised in the Abstract). As each $\mathcal{Q}^\lambda$ comes equipped with a gauge action of the circle, our results thus give an uncountable family of distinct circle actions on $\mathcal{Q}_{\mathbb{N}}$, each with its own unique KMS state. Indeed, for all $0 < \lambda < 1$, we find a unique KMS state, [BR2], for this gauge action, and we prove in Section 4 that the GNS representation of $\mathcal{Q}^\lambda$ associated to our KMS state generates a type III_λ von Neumann algebra.

The result of [CPR2] that motivated this paper was the construction of a 'modular spectral triple' with which one may compute an index pairing using the KMS state. In [CNNR] it was shown how modular spectral triples arise naturally for KMS states of circle actions and lead to 'twisted residue cocycles' using a variation on the semifinite residue cocycle of [CPRS2]. It is well known that such twisted cocycles can not pair with ordinary K_1. In [CPR2, CRT] a substitute was introduced which is called 'modular K_1'. The correct definition of modular K_1 was found in [CNNR], and there is a general spectral flow formula which defines the pairing of modular K_1 with our 'twisted residue cocycle'.

There is a strong analogy with the local index formula of noncommutative geometry in the $\mathcal{L}^{1,\infty}$-summable case, however, there are important differences: the usual residue cocycle is replaced by a twisted residue cocycle and the Dixmier trace arising in the standard situation is replaced by a KMS-Dixmier functional. The common ground with [CPRS2] stems from the use of the spectral flow formula of [CP2] to derive the twisted residue cocycle and this has the corollary that we have a homotopy invariant. To illustrate the theory for these examples we compute, for particular modular unitaries in matrix algebras over the algebras $\mathcal{Q}^\lambda$, the precise numerical values arising from the general formalism.

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2. The algebras $\mathcal{Q}^\lambda$ for $0 < \lambda < 1$.

2.1. **The C^*-algebras $C^*(\Gamma_\lambda) = C(\hat{\Gamma}_\lambda)$ and their K-theory.** We will construct our algebras $\mathcal{Q}^\lambda$ as "corner" algebras in certain crossed product C^*-algebras but first we need some preliminaries. For $0 < \lambda < 1$, let Γ_λ be the countable additive abelian subgroup of $\mathbb{R}$ defined by:

$$\Gamma_\lambda = \left\{ \sum_{k=-N}^{k=N} n_k \lambda^k \;\middle|\; N \geq 0 \text{ and } n_k \in \mathbb{Z} \right\}.$$

Loosely speaking, Γ_λ consists of Laurent polynomials in λ and λ^{-1} with integer coefficients. It is not only a dense subgroup of $\mathbb{R}$, but is clearly a unital **subring** of $\mathbb{R}$.

Proposition 2.1. *Let $0 < \lambda < 1$.*
(1) *If $\lambda = p/q$ where $0 < p < q$ are integers in lowest terms, then $\Gamma_\lambda = \mathbb{Z}[1/n]$, where $n = pq$.*
(2) *If λ and λ^{-1} are both algebraic integers, then $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$ is an internal direct sum where $d \geq 2$ is the degree of the minimal (monic) polynomial in $\mathbb{Z}[x]$ satisfied by λ.*
(3) *If λ is transcendental then, $\Gamma_\lambda = \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}\lambda^k$ is an internal direct sum.*
(4) *If $\lambda = 1/\sqrt{n}$ with $n \geq 2$ a square-free positive integer, then $\Gamma_\lambda = \mathbb{Z}[1/n] + \mathbb{Z}[1/n] \cdot \sqrt{n}$ is an internal direct sum.*
(5) *In general, if λ is algebraic with minimal polynomial, $n\lambda^d + \cdots + m = 0$ over $\mathbb{Z}$, then*

$$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \Gamma_\lambda \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}.$$

Hence, $\mathrm{rank}(\Gamma_\lambda) := \dim_{\mathbb{Q}}(\Gamma_\lambda \otimes_{\mathbb{Z}} \mathbb{Q}) = d$.

Proof. In case (1), since $gcd(p,q) = 1$, there exist $a, b \in \mathbb{Z}$ so that $1 = ap + bq$. Therefore, $\frac{1}{q} = \frac{ap+bq}{q} = a\lambda + b \in \Gamma_\lambda$; and similarly, $\frac{1}{p} \in \Gamma_\lambda$. Since, Γ_λ is a commutative ring, for any $k, m \in \mathbb{Z}$ with $k \geq 1$ we have: $\frac{m}{n^k} = \frac{m}{(pq)^k}$ is in Γ_λ. That is, $\mathbb{Z}[1/n] \subseteq \Gamma_\lambda$. On the other hand, for $k \geq 1$ we have

$$\lambda^k = \frac{p^k}{q^k} = p^{2k}\frac{1}{(pq)^k} = p^{2k}\frac{1}{n^k} \in \mathbb{Z}[1/n] \text{ and } \lambda^{-k} = \frac{q^k}{p^k} = q^{2k}\frac{1}{(pq)^k} = q^{2k}\frac{1}{n^k} \in \mathbb{Z}[1/n].$$

That is, $\mathbb{Z}[1/n] = \Gamma_\lambda$.

In case (2), it is not hard to see the minimal polynomial of λ in $\mathbb{Z}[x]$ is not only monic, but also has constant term $= \pm 1$; say, $p(\lambda) = \lambda^d + a\lambda^{d-1} + \cdots \pm 1 = 0$. Clearly, $\lambda \in \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$. Since $\lambda^{-1}p(\lambda) = 0$, we also have $\lambda^{-1} \in \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$. By an easy induction, we have $\lambda^k \in \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$, for all $k \in \mathbb{Z}$. Hence, $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$. The sum is direct by the minimality of the degree of the minimal polynomial.

In case (3) the sum is direct because if λ satisfied a Laurent polynomial over $\mathbb{Z}$, then by multipling by a high power of λ it would also satisfy a genuine polynomial over $\mathbb{Z}$.

Case (4) is an easy calculation which we leave to the reader. Case (5) is proved by similar methods used in case (2). Again, the sum $\mathbb{Z}[\frac{1}{mn}] + \mathbb{Z}[\frac{1}{mn}]\lambda + \cdots + \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}$ is direct by the minimality of the degree of the minimal polynomial. □

Proposition 2.2. *Let* $0 < \lambda < 1$.

(1) *If* $\lambda = p/q$ *is rational in lowest terms so that* $\Gamma_\lambda = \mathbb{Z}[1/n]$, *where* $n = pq$, *then*

$$K_0(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1_{\hat{\Gamma}_\lambda}] \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1/n].$$

(2) *If* λ *and* λ^{-1} *are both algebraic integers, so that* $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$ *is an internal direct sum as above, then*

$$K_0(C(\hat{\Gamma}_\lambda)) = \bigwedge^{even}(\Gamma_\lambda) = \bigoplus_{k=0, k\ even}^{d} \bigwedge^k(\Gamma_\lambda) \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^{odd}(\Gamma_\lambda) = \bigoplus_{k=1, k\ odd}^{d} \bigwedge^k(\Gamma_\lambda).$$

(3) *If* λ *is transcendental then,*

$$K_0(C(\hat{\Gamma}_\lambda)) = \bigwedge^{even}(\Gamma_\lambda) = \bigoplus_{k=0, k\ even}^{\infty} \bigwedge^k(\Gamma_\lambda) \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^{odd}(\Gamma_\lambda) = \bigoplus_{k=1, k\ odd}^{\infty} \bigwedge^k(\Gamma_\lambda).$$

(4) *If* $\lambda = 1/\sqrt{n}$ *with* $n \geq 2$ *a square-free positive integer, then*

$$K_0(C(\hat{\Gamma}_\lambda)) \cong \mathbb{Z} \oplus \mathbb{Z}[1/n] \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) \cong \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]$$

(5) *In general, if* λ *is algebraic with* $n\lambda^d + \cdots + m = 0$ *over* $\mathbb{Z}$ *then the composition of the inclusions*

$$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \Gamma_\lambda \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}$$

induces an inclusion on K*-theory, so that both of the following maps are one-to-one*

$$\bigwedge^{even}(\mathbb{Z}^d) \cong K_0(C^*(\mathbb{Z} \oplus \cdots \mathbb{Z}\lambda^{d-1})) \hookrightarrow K_0(C(\hat{\Gamma}_\lambda)) \ \text{and} \ \bigwedge^{odd}(\mathbb{Z}^d) \cong K_1(C^*(\mathbb{Z} \oplus \cdots \mathbb{Z}\lambda^{d-1})) \hookrightarrow K_1(C(\hat{\Gamma}_\lambda)).$$

Proof. In case (1), $\Gamma_\lambda = \varinjlim \mathbb{Z}$ where each map is multiplication by n, so that $\hat{\Gamma}_\lambda = \varprojlim \mathbb{T}$. Since $K_0(C(\mathbb{T})) = \mathbb{Z}[1]$ is generated by multiples of the trivial rank one bundle, the maps in the direct limit $K_0(C(\hat{\Gamma}_\lambda)) = \varinjlim K_0(C(\mathbb{T}))$ are the identity map in each case, so that $K_0(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1]$. On the other hand, $K_1(C(\mathbb{T}))$ is generated by the maps on $C(\mathbb{T})$, $z \mapsto z^k$, and each map in the direct limit is the same map induced by $z \mapsto z^n$. Thus, $K_1(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1/n]$.

Cases (2) and (3) are well-known facts about the K-theory of tori.

Case (4): first one uses item (4) of the previous Proposition, then the proof of case (1) above in order to apply Proposition 2.11 of [Sc]. The proof is finished off with the easily proved observation that $\mathbb{Z}[1/n] \otimes \mathbb{Z}[1/n] = \mathbb{Z}[1/n]$.

Case (5) the composed embedding is just containment:
$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}$. Since we know that $K_*(C^*(\mathbb{Z})) \to K_*(C^*(\mathbb{Z}[1/mn]))$ is one-to-one (even an isomorphism after tensoring with $\mathbb{Q}$), an application of C. Schochet's Künneth Theorem, [Sc], shows that the induced map on K-theory:

$$K_*(C^*(\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1})) \longrightarrow K_*(C^*(\mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}))$$

is one-to-one (even an isomorphism after tensoring with $\mathbb{Q}$). $\square$

Corollary 2.3. *If λ is algebraic with minimal polynomial of degree d so that* $\mathrm{rank}(\Gamma_\lambda) = d$ *then*

$$\mathrm{rank}(K_0(C(\hat{\Gamma}_\lambda))) = \mathrm{rank}(\bigwedge^{even}(\mathbb{Z}^d)) = 2^{d-1} = \mathrm{rank}(\bigwedge^{odd}(\mathbb{Z}^d)) = \mathrm{rank}(K_1(C(\hat{\Gamma}_\lambda))).$$

Proof. For each $N \geq d-1$, let $\Gamma_N = \mathbb{Z}\lambda^{-N} + \cdots + \mathbb{Z}\lambda^N \subseteq \Gamma_\lambda$. Then each Γ_N is a finitely generated torsion free (and hence free abelian) subgroup of Γ_λ. Moreover,

$$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \Gamma_N \subseteq \Gamma_\lambda \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1},$$

so that by tensoring with $\mathbb{Q}$ the induced inclusions are all equalities, and hence all are $\mathbb{Q}$-vector spaces of dimension d. Since Γ_N is free abelian, $\Gamma_N \cong \mathbb{Z}^d$. Now,

$$K_0(C^*(\Gamma_N)) \cong K_0(C(\mathbb{T}^d)) \cong \bigwedge^{even}(\mathbb{Z}^d) \cong \mathbb{Z}^{2^{d-1}} \quad \text{and} \quad K_1(C^*(\Gamma_N)) \cong K_1(C(\mathbb{T}^d)) \cong \bigwedge^{odd}(\mathbb{Z}^d) \cong \mathbb{Z}^{2^{d-1}}.$$

So, each $K_i(C^*(\Gamma_N)) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$-vector space of dimension 2^{d-1} and the map:

$$K_*(C^*(\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1})) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_*(C^*(\Gamma_N)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is one-to-one and hence an isomorphism of $\mathbb{Q}$-vector spaces. Since the corresponding isomorphism onto $K_*(C^*(\Gamma_{N+1})) \otimes_{\mathbb{Z}} \mathbb{Q}$ factors through $K_*(C^*(\Gamma_N)) \otimes_{\mathbb{Z}} \mathbb{Q}$ the maps

$$K_*(C^*(\Gamma_N)) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_*(C^*(\Gamma_{N+1})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are all isomorphisms. Now, $C^*(\Gamma_\lambda) = \lim_N C^*(\Gamma_N)$ and so $K_i(C^*(\Gamma_\lambda)) = \lim_N K_i(C^*(\Gamma_N))$, and therefore,

$$K_i(C^*(\Gamma_\lambda)) \otimes_{\mathbb{Z}} \mathbb{Q} = \lim_N K_i(C^*(\Gamma_N)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{2^{d-1}}$$

for each $i = 1, 2$. $\square$

Now, let $G_\lambda \supset G_\lambda^0$ be the following countable discrete groups of matrices:

$$G_\lambda = \left\{ \begin{pmatrix} \lambda^n & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \Gamma_\lambda,\ n \in \mathbb{Z} \right\} \supset G_\lambda^0 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \Gamma_\lambda \right\}.$$

Of course, G_λ^0 is isomorphic to the additive group Γ_λ, and G_λ is semidirect product of $\mathbb{Z}$ acting on $G_\lambda^0 \cong \Gamma_\lambda$. We let G_λ act on $\mathbb{R}$ as an "ax+b" group, noting that the action leaves Γ_λ invariant. That is,

$$\text{for } t \in \mathbb{R} \text{ and } g = \begin{pmatrix} \lambda^n & a \\ 0 & 1 \end{pmatrix} \in G_\lambda \text{ define } g \cdot t := \lambda^n t + a.$$

Notation. For such an element $g \in G_\lambda$ we will use the notation $g := [\lambda^n : a]$ in place of the matrix for g and $|g| := \det(g) = \lambda^n$ for the determinant of g. Note: $G_\lambda^0 = \{g \in G_\lambda \mid |g| = 1\} \lhd G_\lambda$.

We use this action on $\mathbb{R}$ to define the transpose action α of G_λ on $\mathcal{L}^\infty(\mathbb{R})$:

$$\alpha_g(f)(t) = f(g^{-1}t) \text{ for } f \in \mathcal{L}^\infty(\mathbb{R}) \text{ and } t \in \mathbb{R}.$$

Now let $C_0^\lambda(\mathbb{R})$ be the separable C^*-subalgebra of $\mathcal{L}^\infty(\mathbb{R})$ generated by the countable family of projections $\mathcal{X}_{[a,b)}$ where $a, b \in \Gamma_\lambda$. That is,

$$C_0^\lambda(\mathbb{R}) = \text{closure}\left(\left\{ \sum_{k=1}^{n} c_k \mathcal{X}_{[a_k,b_k)} \;\middle|\; c_k \in \mathbb{C};\;\; a_k, b_k \in \Gamma_\lambda \right\}\right).$$

We observe that $C_0^\lambda(\mathbb{R})$ is a commutative AF-algebra. Clearly, $C_0(\mathbb{R}) \subset C_0^\lambda(\mathbb{R})$ and since $\alpha_g(\mathcal{X}_{[a,b)}) = \mathcal{X}_{[g(a),g(b))}$ both are invariant under the action α of G_λ. We define the separable C^*-algebras $A^\lambda \supset A_0^\lambda$ as the crossed products:

$$A^\lambda = G_\lambda \rtimes_\alpha C_0^\lambda(\mathbb{R}) = \mathbb{Z} \rtimes (G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R})) \;\supset\; A_0^\lambda = G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R}).$$

Since G_λ and G_λ^0 are amenable these equal the reduced crossed products by [Ped, Theorem 7.7.7]. Let $C_{00}^\lambda(\mathbb{R})$ denote the dense $*$-subalgebra of $C_0^\lambda(\mathbb{R})$ consisting of finite linear combinations of the generating projections, $\mathcal{X}_{[a,b)}$, and let $A_c^\lambda \subset l_\alpha^1(G_\lambda, C_0^\lambda(\mathbb{R})) \subset A^\lambda$ denote the dense $*$-subalgebra of A^λ consisting of finitely supported functions $x : G_\lambda \to C_{00}^\lambda(\mathbb{R})$. Similarly we define $A_{0,c}^\lambda \subset A_0^\lambda$.

Proposition 2.4. *For any $\lambda \in (0,1)$, A_0^λ and A^λ are in the bootstrap class $\mathfrak{N}_{nuc}$.*

Proof. Since $A^\lambda = \mathbb{Z} \rtimes A_0^\lambda$, it suffices to see that A_0^λ is in $\mathfrak{N}_{nuc}$. By the proof of the previous Corollary, we can write Γ_λ as an increasing union of finitely generated torsion-free abelian groups Γ_N which are free abelian group of finite rank so that A_0^λ is the direct limit of crossed products of the separable commutative C^*-algebra $C_0^\lambda(\mathbb{R})$ by $\mathbb{Z}^{m_i}$ and hence is in $\mathfrak{N}_{nuc}$. □

Notation: We remind the reader of the crossed product operations in our setting (Definition 7.6.1 of [Ped]) together with some particular notations we use. To this end, let $x, y \in l_\alpha^1(G_\lambda, C_0^\lambda(\mathbb{R}))$ then we have the product and adjoint formulas:

$$\begin{aligned} (x \cdot y)(g) &= \sum_{h \in G_\lambda} x(h)\alpha_h(y(h^{-1}g)) \;\text{ for }\; g \in G_\lambda; \\ x^*(g) &= \alpha_g((x(g^{-1}))^*) \;\text{ for }\; g \in G_\lambda. \end{aligned}$$

If $x \in l_\alpha^1(G_\lambda, C_0^\lambda(\mathbb{R}))$ is supported on the single element $g \in G_\lambda$ and $x(g) = f \in C_0^\lambda(\mathbb{R})$, then we write $x = f \cdot \delta_g$. Since A_c^λ (respectively, $A_{0,c}^\lambda$) is dense in A^λ (respectively, A_0^λ) we often do our calculations with these elements and we have the following easily verified calculus for them.

Lemma 2.5. *Let $f_1 \cdot \delta_{g_1}, f_2 \cdot \delta_{g_2}, f \cdot \delta_g \in A_c^\lambda$, then:*
(1) $(f_1 \cdot \delta_{g_1}) \cdot (f_2 \cdot \delta_{g_2}) = f_1 \alpha_{g_1}(f_2) \cdot \delta_{g_1 g_2}$
(2) $(f \cdot \delta_g)^* = \alpha_{g^{-1}}(\bar{f}) \cdot \delta_{g^{-1}}$.
(3) *$f \cdot \delta_g$ is self-adjoint if and only if f is self-adjoint and $g = 1$.*
(4) *$f \cdot \delta_g$ is a projection if and only if f is a projection and $g = 1$.*
(5) *$f \cdot \delta_g$ is a partial isometry if and only if $|f|$ is a projection.*
(6) *The product of partial isometries of the form $\mathcal{X}_{[a,b)} \cdot \delta_g$ is a partial isometry of the same form.*
(7) *Consider the partial isometry, $v = \mathcal{X}_{[a,b)} \cdot \delta_g$. Given that v has this form, any two of the following: vv^*, v^*v, g completely determine the interval $[a,b)$ and the element g.*

Definition 2.6. *Let $e \in A_{0,c}^\lambda$ be the projection $e = \mathcal{X}_{[0,1)} \cdot \delta_1$. We define the separable unital C^*-algebras:*

$$\mathcal{Q}^\lambda := eA^\lambda e \supset eA_0^\lambda e =: F^\lambda.$$

We will also have occasion to use the dense subalgebras $\mathcal{Q}_c^\lambda := eA_c^\lambda e$, and $F_c^\lambda := eA_{0,c}^\lambda e$.

Proposition 2.7. *The orthogonal family of projections $e_n = \mathcal{X}_{[n,n+1)} \cdot \delta_1 \in A_0^\lambda$ for $n \in \mathbb{Z}$ are mutually equivalent by partial isometries in A_0^λ of the form $V_{n,k} := \mathcal{X}_{[n,n+1)} \cdot \delta_{g_{n-k}}$ where $g_{n-k} = [1 : (n-k)]$. Moreover, the finite sums $E_N := \sum_{n=-N}^{N-1} e_n = \mathcal{X}_{[-N,N)} \cdot \delta_1$ form an approximate identity for A^λ so that*

$$A^\lambda \cong \mathcal{Q}^\lambda \otimes \mathcal{K}(l^2(\mathbb{Z})) \text{ and } A_0^\lambda \cong F^\lambda \otimes \mathcal{K}(l^2(\mathbb{Z})).$$

Proof. By Lemma 2.5, one easily calculates that:

$$\text{for each pair } n,k \in \mathbb{Z}, \ V_{n,k}V_{n,k}^* = e_n \text{ and } V_{n,k}^*V_{n,k} = e_k.$$

Now for each positive integer N if we have $y \in A_c^\lambda$ that satisfies $\mathrm{supp}(y_h) \subseteq [-N,N)$ for all h, then using Lemma 2.5 again we see that $E_N \cdot y = y$. Since the collection of all such elements $y \in A_c^\lambda$ is dense in A^λ, we see that the increasing sequence of projections $\{E_N\}$ form an approximate identity for A^λ. □

Corollary 2.8. *It follows from Proposition 2.4.7 of* [RS] *and Proposition 2.4 that for any $\lambda \in (0,1)$, $\mathcal{Q}^\lambda$ and F^λ are both in $\mathfrak{N}_{nuc}$.*

Lemma 2.9. *(cf.* [PhR, Proposition 3.1, Lemma 3.6]*) The algebra $C_0^\lambda(\mathbb{R})$ is a commutative separable AF algebra consisting of all functions $f : \mathbb{R} \to \mathbb{C}$ which vanish at ∞ and: are right continuous at each $x \in \Gamma_\lambda$; have a finite left-hand limit at each $x \in \Gamma_\lambda$; and are continuous at each $x \in (\mathbb{R} \setminus \Gamma_\lambda)$. Moreover, if $\phi \in \widehat{C_0^\lambda(\mathbb{R})}$, (the space of all nonzero $*$-homomorphisms: $C_0^\lambda(\mathbb{R}) \to \mathbb{C}$) then there exists a unique $x_0 \in \mathbb{R}$ such that:*
(1) *if $x_0 \in (\mathbb{R} \setminus \Gamma_\lambda)$ then $\phi(f) = f(x_0)$ for all $f \in C_0^\lambda(\mathbb{R})$,*
(2) *if $x_0 \in \Gamma_\lambda$ then either* $\begin{cases} \phi(f) = f(x_0) \text{ for all } f \in C_0^\lambda(\mathbb{R}), \text{ or} \\ \phi(f) = f^-(x_0) = \lim_{x \to x_0^-} f(x) \text{ for all } f \in C_0^\lambda(\mathbb{R}). \end{cases}$

Proof. Since generating functions for $C_0^\lambda(\mathbb{R})$ satisfy each of the properties above which are clearly preserved by passing to uniform limits, we see that any function in $C_0^\lambda(\mathbb{R})$ satisfies these properties. Conversely, it is easy to show that any function satisfying these properties can be uniformly approximated by a finite linear combination of the generators. The remainder of the proof is given in [PhR, Lemma 3.6]. □

Notation. We denote the dual space, $\widehat{C_0^\lambda(\mathbb{R})}$ by $\mathbb{R}_\lambda$ and endow it with the relative weak-$*$ topology, that is the topology of pointwise convergence on $C_0^\lambda(\mathbb{R})$. Of course, $\mathbb{R}_\lambda$ is a locally compact Hausdorff space, and $C_0^\lambda(\mathbb{R}) \cong C_0(\mathbb{R}_\lambda)$.

Proposition 2.10. *The algebras A^λ and A_0^λ (and hence $\mathcal{Q}^\lambda$ and F^λ) are simple C^*-algebras. Moreover, A^λ is purely infinite and hence so is $\mathcal{Q}^\lambda$.*

Proof. Now, both G_λ and G_λ^0 act on $C_0^\lambda(\mathbb{R})$ as countable discrete groups of outer automorphisms. Thus, we can apply Theorem 3.2 of [E] once we check that neither action has any nontrivial invariant ideals in $C_0^\lambda(\mathbb{R})$ and that the actions are **properly outer** in the sense of Definition 2.1 of [E].

To do this we look at the induced action of G_λ and G_λ^0 on $\mathbb{R}_\lambda$. So, for $g \in G_\lambda$ we have g acting on $\mathbb{R}_\lambda$ via $g(\phi) = \phi \circ \alpha_g^{-1}$ so that for $\phi = \phi_x$ given by evaluation at $x \in \mathbb{R}$, we have as expected $g(\phi_x) = \phi_{g(x)}$. Now, if $x \in \Gamma_\lambda$ we use the notation ϕ_{x_-} to denote the $*$-homomorphism $\phi_{x_-}(f) = f^-(x) = f(x_-) = \lim_{y \to x_-} f(y)$. One easily checks that since $g(x) \in \Gamma_\lambda$, we have $g(\phi_{x_-}) = \phi_{g(x)_-}$.
Next we claim that each of the sets $\{\phi_m \mid m \in \Gamma_\lambda\}$ and $\{\phi_{m_-} \mid m \in \Gamma_\lambda\}$ is dense in $\mathbb{R}_\lambda$ in the relative weak-$*$ topology. For example, we show that the second set is dense. To approximate ϕ_x for some $x \in \mathbb{R}$ we let $\{m_n\}$ be a sequence in Γ_λ converging to x from the right in $\mathbb{R}$. Let $f \in C_0^\lambda(\mathbb{R})$ so that f

is right continuous at x. One easily shows that $|\phi_{m_{n\_}}(f) - \phi_x(f)| \to 0$; that is, the sequence $\{\phi_{m_{n\_}}\}$ converges to ϕ_x in the relative weak-$*$ topology.

It is easy to see that the action of G^0_λ on $\mathbb{R}_\lambda$ has dense orbits, and so, of course, the action of G_λ has dense orbits also. This implies that the actions of G^0_λ and G_λ on $C^\lambda_0(\mathbb{R})$ have no nontrivial invariant ideals since the induced action on $\mathbb{R}_\lambda$ has no nontrivial invariant closed sets. We complete the proof by showing that the action is **properly outer** in the sense of Definition 2.1 of [E]. Since there are no nontrivial α-invariant ideals and $C^\lambda_0(\mathbb{R})$ is commutative this is the condition that for each $g \neq 1$ and each nonzero closed two sided ideal $\mathcal{I}$ invariant under α_g we have $\|(\alpha_g - Id)_{|\mathcal{I}}\| = 2$. Since $\mathcal{I}$ is nonzero there is a nonempty open subset, $\mathcal{O}$ of $\mathbb{R}_\lambda$ so that $\hat{\mathcal{I}} = \mathcal{O}$. But since $g \neq 1$ and $\mathcal{O}$ is not finite there exists $y \in \mathcal{O}$ such that $g(y) \neq y$ and $g(y) \in \mathcal{O}$. Let $x = g(y) \in \mathcal{O}$ so that $g^{-1}(x) = y \in \mathcal{O}$ and $x \neq g^{-1}(x)$. So we can choose a continuous compactly supported real-valued function f on $\mathcal{O}$ with $f(x) = 1$, $f(g^{-1}(x)) = -1$ and $\|f\| = 1$. But then $f \in \mathcal{I}$ and

$$2 \geq \|(\alpha_g - Id)_{|\mathcal{I}}\|| \geq \|(\alpha_g - Id)(f)\| = \|\alpha_g(f) - f\| \geq |f(g^{-1}(x)) - f(x)| = 2.$$

Now that we know A^λ is simple, we can easily apply Theorem 9 of [LS] to conclude that A^λ satisfies hypothesis (v) of Proposition 4.1.1 (page 66) of [RS]. For simple C^*-algebras, this is equivalent to being purely infinite by Definition 4.1.2 of [RS]: the authors of [LS] had used one of the earlier definitions of purely infinite in their paper (namely, hypothesis (v)). By Proposition 4.1.8 of [RS] $\mathcal{Q}^\lambda$ is also purely infinite. $\square$

Corollary 2.11. *It follows from Corollaries 8.2.2 and 8.4.1 (Kirchberg-Phillips) of* [RS] *and the fact that A^λ is stable that for any $\lambda \in (0,1)$, A^λ is classified up to isomorphism (among Kirchberg algebras in $\mathfrak{N}_{nuc}$) by its K-theory.*

Since we need to calculate with elements of $\mathcal{Q}^\lambda$ and F^λ, we make the following observations.

Lemma 2.12. *Now, $\mathcal{Q}^\lambda$ (respectively, F^λ) is the norm closure of finite linear combinations of the elements of the form $e(\mathcal{X}_{[a,b)} \cdot \delta_g)e$, where $g \in G_\lambda$ (respectively, $g \in G^0_\lambda$), henceforth called the* generators. *Thus, we calculate*
(1) If $f \cdot \delta_g \in A^\lambda$, (respectively, $f \cdot \delta_g \in A^\lambda_0$) where $f \in C^\lambda_0(\mathbb{R})$, then

$$e(f \cdot \delta_g)e = \mathcal{X}_{[a,b)} f \cdot \delta_g \quad \textit{where} \quad [a,b) = [0,1) \cap [g(0), g(1)).$$

(2) Thus, for $g \in G_\lambda$, (respectively, $g \in G^0_\lambda$) $f \cdot \delta_g$ is in $\mathcal{Q}^\lambda$ (respectively, F^λ) iff $supp(f) \subseteq [0,1) \cap [g(0), g(1))$. In particular, for $g \in G_\lambda$, (respectively, $g \in G^0_\lambda$) $\mathcal{X}_{[a,b)} \cdot \delta_g$ is in $\mathcal{Q}^\lambda$ (respectively, F^λ) iff $[a,b) \subseteq [0,1) \cap [g(0), g(1))$.

Proof. The first item is an easy calculation using part (1) of Lemma 2.5 and the fact that $\alpha_g(\mathcal{X}_{[a,b)}) = \mathcal{X}_{[g(a),g(b))}$. The second item follows easily from the first. $\square$

Proposition 2.13. *If λ is rational, then A^λ_0 and F^λ are AF-algebras. In particular, if $\lambda = p/q$ where $0 < p < q$ are in lowest terms, then F^λ is the UHF algebra n^∞ where $n = pq$. Moreover, the minimal projections in the finite-dimensional subalgebras can all be chosen from the canonical commutative subalgebra $C^\lambda_0(\mathbb{R}) \cdot \delta_I$.*

Proof. We have shown in Proposition 2.1 that if $\lambda = p/q$ where $0 < p < q$ are in lowest terms, then $\Gamma_\lambda = \mathbb{Z}[1/n]$, where $n = pq$. Now, any element in $\mathbb{Z}[1/n]$ has the form $m/n^k = m(1/n^k)$ where $k \geq 1$. Therefore any of the generating partial isometries $\mathcal{X}_{[a,b)} \cdot \delta_{[1:c]} \in A^\lambda_0$ can (by bringing a, b and c to a common denominator) be written (assuming $c > 0$) as a finite linear combination of partial isometries of the form $\mathcal{X}_{[l/n^k,(l+1)/n^k)} \cdot \delta_{[1:1/n^k]}$. For partial isometries in F^λ we would have to restrict $0 \leq l \leq n^k - 1$

and such partial isometries generate an n^k by n^k matrix subalgebra of F^λ. It should now be clear that F^λ is a UHF algebra of type n^∞. $\square$

At this point we define some special elements in $\mathcal{Q}^\lambda$ which behave very much like the isometries $S_\mu \in O_n$, except for the fact that **some** of them are **not** isometries.

Definition 2.14. *Fix $0 < \lambda < 1$ and let k be a positive integer. Define m_k to be the unique positive integer satisfying: $m_k\lambda^k < 1 \leq (m_k+1)\lambda^k$. For $0 \leq m \leq m_k$ define* **partial isometries** *$S_{k,m} \in \mathcal{Q}^\lambda$ via:*

$$S_{k,m} = \mathcal{X}_{[m\lambda^k,(m+1)\lambda^k)} \cdot \delta_{g_{k,m}} \quad \text{where} \quad g_{k,m} = [\lambda^k \ : \ m\lambda^k].$$

Note: for $m < m_k$ the $S_{k,m}$ are actually **isometries**, *and S_{k,m_k} is an isometry iff $1 = (m_k+1)\lambda^k$.*

Remarks. The defining inequalities $m_k\lambda^k < 1 \leq (m_k+1)\lambda^k$ for the positive integer m_k are equivalent to: $0 < \lambda^{-k} - m_k \leq 1$. In particular, these differences are positive and bounded above by 1. In the case of $\mathcal{Q}^{1/n}$ we have $m_k = n^k - 1$. Generally we have $m_1^k \leq m_k < 1 \leq (m_k+1) \leq (m_1+1)^k$.

Lemma 2.15. *With the previously defined elements we have:*

$$S_{k,m}^* = \mathcal{X}_{[0,1)} \cdot \delta_{g_{k,m}^{-1}} \quad \text{and} \quad S_{k,m_k}^* = \mathcal{X}_{[0,\lambda^{-k}-m_k)} \cdot \delta_{g_{k,m_k}^{-1}} \quad \text{where for all } m, \ \ g_{k,m}^{-1} = [\lambda^{-k} \ : \ -m].$$

Moreover, for $0 \leq m < m_k$, $S_{k,m}^*S_{k,m} = \mathcal{X}_{[0,1)} \cdot \delta_1 = e$ while $S_{k,m_k}^*S_{k,m_k} = \mathcal{X}_{[0,\lambda^{-k}-m_k)} \cdot \delta_1$.
Finally, for $0 \leq m < m_k$, $S_{k,m}S_{k,m}^* = \mathcal{X}_{[m\lambda^k,(m+1)\lambda^k)} \cdot \delta_1$ while $S_{k,m_k}S_{k,m_k}^* = \mathcal{X}_{[m_k\lambda^k,1)} \cdot \delta_1$, so that

$$\sum_{m=0}^{m_k} S_{k,m}S_{k,m}^* = \mathcal{X}_{[0,1)} \cdot \delta_1 = e.$$

Proof. These are just straightforward calculations based on Lemma 2.5 which we leave to the reader. $\square$

Theorem 2.16. *For each λ with $0 < \lambda < 1$, consider the partial isometries $S_{1,m}$ for $m = 0, 1, ..., m_1$ where $m_1\lambda < 1 \leq (m_1+1)\lambda$. For $m < m_1$, $S_{1,m}$ is an isometry and $\sum_{m=0}^{m_1} S_{1,m}S_{1,m}^* = 1$. For $\lambda = 1/n$, $m_1 = n-1$, S_{1,m_1} is also an isometry, and $\mathcal{Q}^{1/n} \cong O_n$, the usual Cuntz algebra.*

Proof. The first statement is clear. With $\lambda = 1/n$ we have inside $\mathcal{Q}^{1/n}$, n isometries one for each $m = 0, 1, ..., (n-1)$ defined by:

$$S_m = \mathcal{X}_{[\frac{m}{n},\frac{m+1}{n})} \cdot \delta_{g_m} \quad \text{where} \quad g_m = [1/n \ : \ m/n] \quad \text{and so} \quad S_m^* = \mathcal{X}_{[0,1)} \cdot \delta_{g_m^{-1}} \quad \text{where} \quad g_m^{-1} = [n \ : \ -m].$$

Using Lemma 2.12, we easily see that for each m, $S_m \in \mathcal{Q}^{1/n}$.

Then, using item (1) of Lemma 2.5 we calculate:

$$S_m^*S_m = \mathcal{X}_{[0,1)} \cdot \delta_1 = e \quad \text{and} \quad S_mS_m^* = \mathcal{X}_{[\frac{m}{n},\frac{m+1}{n})} \cdot \delta_1 \quad \text{and so} \quad \sum_{m=0}^{n-1} S_mS_m^* = \mathcal{X}_{[0,1)} \cdot \delta_1 = e.$$

Since e is the identity of $\mathcal{Q}^{1/n}$, we have constructed a unital copy of O_n inside $\mathcal{Q}^{1/n}$. Now one shows by induction that for each $k > 0$ the product of exactly k of these n isometries has the form $S_{k,m}$ where $S_{k,m}$ has the same defining equation as S_m above but with n^k in place of n and $m = 0, 1, ..., (n^k - 1)$. These new isometries have range projections $S_{k,m}S_{k,m}^* = \mathcal{X}_{[\frac{m}{n^k},\frac{m+1}{n^k})} \cdot \delta_1$ which therefore lie in this copy of O_n. By adding up some of these projections, we can get any projection of the form $\mathcal{X}_{[a,b)} \cdot \delta_1$ where $0 \leq a < b \leq 1$ and both a, b have the form m/n^k. But any element $a \in \Gamma_{1/n}$ can be written as $a = \frac{m}{n^k}$ for a sufficiently large $k \geq 0$ and some $m \in \mathbb{Z}$ depending on k, and any pair a, b can be brought to a common denominator n^k. Hence any projection of the form $\mathcal{X}_{[a,b)} \cdot \delta_1$ in $\mathcal{Q}^{1/n}$ is in this copy of O_n.

Now, a straightforward calculation gives us:

$$(1) \qquad \sum_{m=1}^{n^k-1} S_{k,m}S_{k,m-1}^* = \sum_{m=1}^{n^k-1} \mathcal{X}_{[\frac{m+1}{n^k},\frac{m+2}{n^k})} \cdot \delta_{[1\,:\,1/n^k]} = \mathcal{X}_{[1/n^k,1)} \cdot \delta_{[1\,:\,1/n^k]} \in O_n.$$

Finally, let $\mathcal{X}_{[a,b)} \cdot \delta_g \in \mathcal{Q}^{1/n}$ be an arbitrary generator. By taking adjoints if necessary we can assume that g has the form $g = [n^k\ :\ *]$ where $k \geq 0$. Since $S_{k,0}$ is an isometry in O_n it suffices to prove that $S_{k,0}(\mathcal{X}_{[a,b)} \cdot \delta_g) \in O_n$. That is, we are reduced to the case $g = [1\ :\ c]$ and again by taking adjoints if necessary we can assume that $c \geq 0$. The case $c = 0$ is done and so we can assume that $c > 0$. So (with possibly new a, b) we have $\mathcal{X}_{[a,b)} \cdot \delta_{[1\,:\,c]}$ where $0 < c \leq 1$ and $[a, b) \subseteq [0, 1) \cap [c, c + 1) = [c, 1)$. But, $\mathcal{X}_{[a,b)} \cdot \delta_{[1\,:\,c]} = \mathcal{X}_{[a,b)}\mathcal{X}_{[c,1)} \cdot \delta_{[1\,:\,c]} = \mathcal{X}_{[a,b)} \cdot \delta_1\mathcal{X}_{[c,1)} \cdot \delta_{[1\,:\,c]}$ and we already know that $\mathcal{X}_{[a,b)} \cdot \delta_1 \in O_n$. Therefore it suffices to see that $\mathcal{X}_{[c,1)} \cdot \delta_{[1\,:\,c]} \in O_n$. However, $c = l/n^k$ for some $0 < l < n^k$ and so:

$$\mathcal{X}_{[c,1)} \cdot \delta_{[1\,:\,c]} = \mathcal{X}_{[l/n^k,1)} \cdot \delta_{[1\,:\,l/n^k]} = \left(\mathcal{X}_{[1/n^k,1)} \cdot \delta_{[1\,:\,1/n^k]}\right)^l$$

which is in O_n by Equation 1. Since all generators for $\mathcal{Q}^{1/n}$ are in O_n we're done. □

2.2. **K-theory of $\mathcal{Q}^\lambda$ for λ rational.** Since A_0^λ is stable and stably isomorphic to the UHF algebra F^λ, each of its projections is equivalent to one in some finite-dimensional subalgebra and hence to some projection in $C_0^\lambda(\mathbb{R})$, and in this case the trace induces an isomorphism from $K_0(A_0^\lambda)$ onto $\Gamma_\lambda = \mathbb{Z}[1/(pq)] \subset \mathbb{R}$. This isomorphism carries the projection $e = \mathcal{X}_{[0,1)} \cdot \delta_1$ which is the identity of $\mathcal{Q}^\lambda$ and F^λ onto $1 \in \mathbb{Z}[1/(pq)]$. Now, since A_0^λ is AF, $K_1(A_0^\lambda) = \{0\}$, and since $A^\lambda = \mathbb{Z} \rtimes_\lambda A_0^\lambda$ we can use the Pimsner-Voiculescu exact sequence to calculate $K_*(A^\lambda) = K_*(\mathcal{Q}^\lambda)$. When we do this we get:

$$K_1(\mathcal{Q}^\lambda) = \{0\}, \text{ and } K_0(\mathcal{Q}^\lambda) = \mathbb{Z}[1/(pq)]/(1-\lambda)\mathbb{Z}[1/(pq)].$$

Proposition 2.17. *For λ rational with $\lambda = p/q$ in lowest terms, we have*

$$K_1(\mathcal{Q}^\lambda) = \{0\}, \text{ and } K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}[1/(pq)]/(1-\lambda)\mathbb{Z}[1/(pq)] \cong \mathbb{Z}_{(q-p)}.$$

Proof. By Proposition 2.1, $\Gamma_\lambda = \mathbb{Z}[1/(pq)]$, so we must show that

$$\mathbb{Z}[1/(pq)]/(1-(1/(pq))\mathbb{Z}[1/(pq)] \cong \mathbb{Z}_{(q-p)}.$$

Since $(q-p) = (1-p/q)q$ and every element of $\mathbb{Z}[1/(pq)]$ is of the form $m/(pq)^N$, it is easy to see that $(q-p)\mathbb{Z}[1/(pq)] = (1-p/q)\mathbb{Z}[1/(pq)]$. Now, $(q-p)$ and $(pq)^N$ are relatively prime for any N and so there exist $a, b \in \mathbb{Z}$ so that $1 = a(q-p) + b(pq)^N$ and hence $m/(pq)^N = (q-p)am/(pq)^N + mb$. That is, $m/(pq)^N$ and mb represent the same element in the quotient. So, every element in the quotient has an integer representative. Two integers c, d represent the same element in the quotient if and only if $c - d = (p-q)n/(pq)^N$, or $(c-d)(pq)^N = n(q-p)$. But then:

$$(c-d) = (c-d)[a(q-p) + b(pq)^N] = (c-d)a(q-p) + b(c-d)(pq)^N = [(c-d)a + bn](q-p).$$

That is, c, d represent the same element in $\mathbb{Z}/(q-p)\mathbb{Z} = \mathbb{Z}_{(q-p)}$. On the other hand if $(c-d)$ is in $(q-p)\mathbb{Z}$ then clearly, $[c] = [d]$ in $\mathbb{Z}[1/(pq)]/(1-(1/(pq))\mathbb{Z}[1/(pq)]$ and we are done. □

Corollary 2.18. *If $\lambda = p/q$ in lowest terms, then*

$$F^\lambda = F^{p/q} \cong UHF((pq)^\infty) \quad \text{and} \quad \mathcal{Q}^\lambda = \mathcal{Q}^{p/q} \cong O_{(q-p+1)}.$$

In particular, if $\lambda = \frac{k}{k+1}$ then

$$F^\lambda \cong UHF((k(k+1))^\infty) \quad \text{and} \quad \mathcal{Q}^\lambda \cong O_2.$$

Proof. Since each $\mathcal{Q}^\lambda$ is separable, nuclear, simple, purely infinite and in the bootstrap category $\mathcal{N}_{nuc}$ once we show that the class of the identity $e \in \mathcal{Q}^\lambda$ is a generator for $K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/(q-p)\mathbb{Z}$, the Kirchberg-Phillips Classification Theorem, Theorem 8.4.1 of [RS], shows that $\mathcal{Q}^\lambda \cong O_{(q-p+1)}$. To this end we observe that since e is mapped to 1 in $\mathbb{Z}[1/pq]$, we must show that [1] is a generator for $K_0(\mathcal{Q}^\lambda) = \mathbb{Z}[1/pq]/(1-(p/q))\mathbb{Z}[1/pq]$. Now, by the proof of the previous proposition, $k[1] = [k \cdot 1] = 0 \in \mathbb{Z}[1/pq]/(1-(p/q))\mathbb{Z}[1/pq]$ if and only if $[k \cdot 1] = 0 \in \mathbb{Z}/(q-p)\mathbb{Z}$ if and only if $k - 0 = m(q-p)$ for some $m \in \mathbb{Z}$ if and only if k is a multiple of $(q-p)$. That is, $[1], [2 \cdot 1], \ldots, [(q-p-1) \cdot 1]$ are all nonzero in $K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/(q-p)\mathbb{Z}$ and hence [1] is a generator. □

2.3. **The K-theory of the Algebras A_0^λ for λ irrational.** The case λ rational is much simpler, and while it does fit into the following scheme, it does not need this deeper machinery. Initially, we (and others) believed that the algebras A_0^λ were AF algebras when λ is irrational. In fact we will show that A_0^λ is **never** AF when λ is irrational. We will set up our examples to fit the situation on page 1487 of [Put2] so that we can apply the six-term exact sequence of Theorem 2.1 on page 1489 of [Put2].

We let $\Gamma = \Gamma_\lambda \cong G_\lambda^0$. Thus, $\Gamma \subset \mathbb{R}$ is a countable dense subgroup of $\mathbb{R}$ which acts on $\mathbb{R}$ by translations. Before looking at the crossed product of Γ acting on $C_0^\lambda(\mathbb{R}) = C_0(\mathbb{R}_\lambda)$ (which gives us A_0^λ) we first consider the crossed product of Γ acting on $C_0(\mathbb{R})$. Since Γ acts on $\mathbb{R}$ by translation we can Fourier transform to get an isomorphism:

$$\Gamma \rtimes C_0(\mathbb{R}) \cong \hat{\mathbb{R}} \rtimes C(\hat{\Gamma}).$$

Then, by Connes' Thom isomorphism we get for $i = 0, 1$:

$$K_i(\Gamma \rtimes C_0(\mathbb{R})) \cong K_i(\hat{\mathbb{R}} \rtimes C(\hat{\Gamma})) \cong K_{i+1}(C(\hat{\Gamma})).$$

Proposition 2.19. *The composition:*

$$K_1(C_0(\mathbb{R})) \xrightarrow{i_*} K_1(\Gamma \rtimes C_0(\mathbb{R})) \xrightarrow{\widehat{\ }} K_1(\hat{\mathbb{R}} \rtimes C(\hat{\Gamma})) \xrightarrow{\cong} K_0(C(\hat{\Gamma}))$$

takes the generator $[u] \in K_1(C_0(\mathbb{R})) = \mathbb{Z} \cdot [u]$; where u is the Bott element in $C_0(\mathbb{R})^1$ defined by $u(t) = \frac{1+it}{1-it}$; to $[1_{\hat{\Gamma}}]$ where $1_{\hat{\Gamma}}$ is the identity function in $C(\hat{\Gamma})$.

Proof. We first work on the right hand side of this sequence of maps. Let $u(t) = 1 + \varepsilon(t)$. Now, by the proof of Connes' Thom isomorphism, the mapping:

$$K_0(C(\hat{\Gamma})) \otimes_{\mathbb{Z}} K_1(C_0(\mathbb{R})) \longrightarrow K_1(\mathbb{R} \rtimes C(\hat{\Gamma}))$$

takes the element, $[1_{\hat{\Gamma}}] \otimes [u]$, to the class $[1 + (\textit{convolution by } \hat{\varepsilon} \cdot 1_{\hat{\Gamma}})]$. Now the left hand side of this displayed mapping is naturally isomorphic to $K_0(C(\hat{\Gamma}))$, via:

$$K_0(C(\hat{\Gamma})) \otimes_{\mathbb{Z}} K_1(C_0(\mathbb{R})) = K_0(C(\hat{\Gamma})) \otimes_{\mathbb{Z}} \mathbb{Z} \cdot [u] = K_0(C(\hat{\Gamma})) \cdot [u] \cong K_0(C(\hat{\Gamma})).$$

Thus, $[1_{\hat{\Gamma}}]$ in $K_0(C(\hat{\Gamma}))$ gets mapped to the class $[1 + (\textit{convolution by } \hat{\varepsilon} \cdot 1_{\hat{\Gamma}})]$ by the Thom isomorphism.

On the other hand, the map $K_1((C_0(\mathbb{R})^1) \longrightarrow K_1((\Gamma \rtimes C_0(\mathbb{R}))^1)$ takes $[u] \longmapsto [\delta_0 \cdot \varepsilon + 1]$ and by the Fourier transform this goes to $[(\textit{convolution by } \hat{\varepsilon} \cdot 1_{\hat{\Gamma}}) + 1]$ in $K_1(\mathbb{R} \rtimes C(\hat{\Gamma}))$. Combining these we get:

$$1 \in \mathbb{Z} \longmapsto [u] \in \mathbb{Z} \cdot [u] = K_1((C_0(\mathbb{R}))^1) = K_1(C_0(\mathbb{R})) \longmapsto [1_{\hat{\Gamma}}] \in K_0((C(\hat{\Gamma})).$$

□

Now, by Proposition 2.1 we know Γ in many cases so that these last groups are quite computable. In the notation of [Put2] we define the transformation groupoids:

$$G := \mathbb{R}_\lambda \rtimes \Gamma, \quad G' := \mathbb{R} \rtimes \Gamma, \quad \text{and} \quad H := \Gamma \rtimes \Gamma.$$

Then, $A_0^\lambda = C_r^*(G)$ is the reduced C^*-algebra of G; $\Gamma \rtimes C_0(\mathbb{R}) = C_r^*(G')$ is the reduced C^*-algebra of G'; and $\mathcal{K}(l^2(\Gamma))$ is the reduced C^*-algebra of H. By the proof of Proposition 2.10 there is a continuous proper surjective map: $\mathbb{R}_\lambda \to \mathbb{R}$, where points in $\mathbb{R}$ which are not in Γ each have a single pre-image, while points $\gamma \in \Gamma$ have exactly two pre-images in $\mathbb{R}_\lambda$, which we denote by γ^- and γ^+. Thus, there are two disjoint embeddings of Γ in $\mathbb{R}_\lambda$:

$$i_0, i_1 : \Gamma \to \mathbb{R}_\lambda \ : \quad i_0(\gamma) = \gamma^-, \quad i_1(\gamma) = \gamma^+.$$

Now in order to mesh with the notation of [Put2], we let $Y := \Gamma$ with the equivalence relation, "="; $X := \mathbb{R}_\lambda$, with the equivalence relation $(i_0(\gamma) \sim i_1(\gamma))$; and quotient $\pi : X \to X' := \mathbb{R}$ where $X' = X/(i_0(\gamma) \sim i_1(\gamma)) = \mathbb{R}$; while the image of the groupoid $G = \mathbb{R}_\lambda \times \Gamma = X \times \Gamma$ under the sujective mapping $\mathbb{R}_\lambda \to \mathbb{R}$, is $G' := \mathbb{R} \times \Gamma = X' \times \Gamma$. Heurisically, a "factor groupoid".

We represent each of these three C^*-algebras on $\mathcal{H} := l^2(\Gamma^+) \oplus l^2(\Gamma^-)$ where $\Gamma^\pm = \{\gamma^\pm \mid \gamma \in \Gamma\}$ in the following way. First we denote the natural orthonormal basis elements of $\mathcal{H}$ by δ_{a^+} and δ_{a^-} for each $a \in \Gamma$. Now the unitary representation U of Γ on $\mathcal{H}$ is $U_\gamma(\delta_{a^\pm}) = \delta_{(a-\gamma)^\pm}$. The actions of $C_0(\mathbb{R}_\lambda)$, $C_0(\mathbb{R})$, and $C_0(\Gamma)$ on $\mathcal{H}$ are as follows for $f_1 \in C_0(\mathbb{R}_\lambda)$, $f_2 \in C_0(\mathbb{R})$, $f_3 \in C_0(\Gamma)$, and $\delta_{a^\pm} \in \mathcal{H}$

$$\pi_1(f_1)(\delta_{a^\pm}) = f_1(a^\pm)\delta_{a^\pm} \quad \pi_2(f_2)(\delta_{a^\pm}) = f_2(a)\delta_{a^\pm} \quad \pi_3(f_3)(\delta_{a^\pm}) = f_3(a)\delta_{a^\pm}.$$

These three covariant pairs of representations, (π_1, U), (π_2, U), and (π_3, U) define representations of $C_r^*(G) = A_0^\lambda$, $C_r^*(G') = \Gamma \rtimes C_0(\mathbb{R})$, and $C_r^*(H) = \mathcal{K}(l^2(\Gamma))$ respectively on $\mathcal{H}$. Since each of these C^*-algebras is simple these representations are faithful.

Now, one checks that the hypotheses of Theorem 2.1 of [Put2] are satisfied. As in [Put1, Put2] one shows that the two **mapping cone algebras** of the inclusions:

$$C_r^*(G') = \Gamma \rtimes C_0(\mathbb{R}) \longrightarrow A_0^\lambda = C_r^*(G) \quad \text{and} \quad C_r^*(H) \longrightarrow C_r^*(H) \oplus C_r^*(H) : \quad (\ x \mapsto (x, x)\ )$$

have isomorphic K-theory. One then pastes these isomorphisms into the mapping cone long exact sequence for $C_r^*(G') = \Gamma \rtimes C_0(\mathbb{R}) \longrightarrow A_0^\lambda = C_r^*(G)$. Next one observes that for any C^*-algebra, B the diagonal embedding $B \longrightarrow B \oplus B$ induces the diagonal embedding $K_*(B) \longrightarrow K_*(B) \oplus K_*(B)$ with quotient isomorphic to $K_*(B)$ (this is true for any abelian group). This implies that $K_*(B) \cong K_{*+1}(M(B, B \oplus B))$ so that we get the six-term exact sequence from [Put2]:

$$\begin{tikzcd}
K_1(C_r^*(H)) \arrow[r] & K_0(C_r^*(G')) \arrow[r] & K_0(C_r^*(G)) \arrow[d] \\
K_1(C_r^*(G)) \arrow[u] & K_1(C_r^*(G')) \arrow[l] & K_0(C_r^*(H)) \arrow[l]
\end{tikzcd}$$

In our set-up this becomes:

$$\begin{tikzcd}
\{0\} \arrow[r] & K_0(\Gamma \rtimes C_0(\mathbb{R})) \arrow[r] & K_0(\Gamma \rtimes C_0(\mathbb{R}_\lambda)) \arrow[d] \\
K_1(\Gamma \rtimes C_0(\mathbb{R}_\lambda)) \arrow[u] & K_1(\Gamma \rtimes C_0(\mathbb{R})) \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

Which by Connes' Thom isomorphism becomes:

$$\begin{tikzcd}
\{0\} \arrow[r] & K_1(C(\hat{\Gamma})) \arrow[r] & K_0(A_0^\lambda) \arrow[d] \\
K_1(A_0^\lambda) \arrow[u] & K_0(C(\hat{\Gamma})) \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

By Proposition 2.19, the nonzero element $[1_{\hat{\Gamma}}]$ in $K_0(C_0(\hat{\Gamma})) \cong K_1(\Gamma \rtimes C_0(\mathbb{R}))$ is mapped to the image of the class $[u]$ in $K_1(\Gamma \rtimes C_0(\mathbb{R}))$ by Connes' Thom isomorphism, and then the image of $[1_{\hat{\Gamma}}]$ in

$K_1(\Gamma \rtimes C_0(\mathbb{R}_\lambda))$ is the same as the image of $[u]$ under the inclusion $K_1(\Gamma \rtimes C_0(\mathbb{R})) \longrightarrow K_1(\Gamma \rtimes C_0(\mathbb{R}_\lambda))$. However, this is clearly the same as the image of $[u]$ under the inclusion $K_1(C_0(\mathbb{R})) \to K_1(C_0(\mathbb{R}_\lambda)) \to K_1(\Gamma \rtimes C_0(\mathbb{R}_\lambda))$. This composition is 0 since $C_0(\mathbb{R}_\lambda)$ is an AF-algebra. That is, the element $[1_{\hat{\Gamma}}]$ in $K_0(C_0(\hat{\Gamma}))$ is mapped to 0 in $K_1(A_0^\lambda)$ and hence is in the image of the map $\mathbb{Z} \longrightarrow K_0(C_0(\hat{\Gamma}))$. Since $[1_{\hat{\Gamma}}]$ generates a copy of $\mathbb{Z}$ in $K_0(C_0(\hat{\Gamma}))$, we have a nonzero homomorphism from $\mathbb{Z}$ to $\mathbb{Z}[1_{\hat{\Gamma}}]$ which is onto and hence one-to-one. By the exactness, the map $K_0(A_0^\lambda) \longrightarrow \mathbb{Z}$ is the zero map.

$$\text{CONCLUSION}: \quad K_0(A_0^\lambda) \cong K_1(C(\hat{\Gamma}_\lambda)) \quad \text{and} \quad K_1(A_0^\lambda) \cong K_0(C(\hat{\Gamma}_\lambda))/[1_{\hat{\Gamma}_\lambda}]\mathbb{Z}.$$

Proposition 2.20. *If λ is irrational, then $K_1(A_0^\lambda) \neq \{0\}$ so that A_0^λ is not an AF-algebra.*

Proof. By items (3) and (5) of Proposition 2.2 we see that when λ is irrational, $K_0(C(\hat{\Gamma}_\lambda))$ is not singly generated so that $K_1(A_0^\lambda) \cong K_0(C(\hat{\Gamma}_\lambda))/[1_{\hat{\Gamma}_\lambda}]\mathbb{Z} \neq \{0\}$. □

2.4. **K-theory computations of particular $\mathcal{Q}^\lambda$ for λ irrational. Example(s) $\lambda = 1/\sqrt{n}$:** for $n > 1$ a square-free integer. Using Proposition 2.1, we get:

$$K_0(F^\lambda) = K_0(A_0^\lambda) = K_1(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]$$

$$K_1(F^\lambda) = K_1(A_0^\lambda) = (K_0(C(\hat{\Gamma}_\lambda)))/\mathbb{Z}[1] = (\mathbb{Z}[1] \oplus \mathbb{Z}[1/n])/\mathbb{Z}[1] = \mathbb{Z}[1/n].$$

To compute the K-theory of $\mathcal{Q}^\lambda$ in this case using the Pimsner-Voiculescu exact sequence, one must first compute the induced automorphism λ_* on $K_1(C(\hat{\Gamma}_\lambda))$ and on $K_0(C(\hat{\Gamma}_\lambda))$ by a more detailed analysis of the proof of [Sc, Proposition 2.11]. In the case of $K_1(C(\hat{\Gamma}_\lambda))$ we get a copy of the group $\Gamma_\lambda = \mathbb{Z}[1/n] + \mathbb{Z}[1/n]\sqrt{n}$ and the action on Γ_λ is just multiplication by $\lambda = 1/\sqrt{n}$. As an action translated to the abstract group $\mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]$, the automorphism becomes $\lambda_*(a,b) = (b, a/n)$. Therefore, $id_* - \lambda_*$ on $K_0(A_0^\lambda) = \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]$ to itself is clearly $1:1$. Now it is an instructive exercise to show that the kernel of the homomorphism

$$(a,b) \in \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n] \mapsto [a+b] \in \mathbb{Z}[1/n]/(1-1/n)\mathbb{Z}[1/n]$$

is exactly the range of the homomorphism

$$id_* - \lambda_* : \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n] \longrightarrow \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n].$$

Hence, we have the isomorphisms:

$$(\mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n])/(id_* - \lambda_*)(\mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]) \cong \mathbb{Z}[1/n]/(1-1/n)\mathbb{Z}[1/n] \cong \mathbb{Z}/(n-1)\mathbb{Z}.$$

where the last isomorphism follows from the proof of Proposition 2.17 with $p = 1$ and $q = n$.

Once we have computed the action of λ_* on $K_1(A_0^\lambda) = \mathbb{Z}[1/n]$ we will be ready to compute $K_*(\mathcal{Q}^\lambda)$. Now, by Proposition 2.11 of [Sc] we have the isomorphism:

$$K_0(C(\hat{\Gamma}_\lambda)) \cong (\mathbb{Z}[1] \otimes_{\mathbb{Z}} \mathbb{Z}[1]) \oplus (\mathbb{Z}[1/n] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]) = \mathbb{Z}[1] \oplus (\mathbb{Z}[1/n] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]).$$

The action of λ_* on $\mathbb{Z}[1]$ is of course the identity. However, the action of λ_* on $(\mathbb{Z}[1/n] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n])$ is just $x \otimes y \mapsto y \otimes x/n$. If one combines this with the multiplication isomorphism $x \otimes y \mapsto xy : \mathbb{Z}[1/n] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n] \longrightarrow \mathbb{Z}[1/n]$ we see that λ_* acts as multiplication by $1/n$ on $\mathbb{Z}[1/n] = \mathbb{Z}[1/n] \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$. Thus, λ_* on the quotient $K_1(A_0^\lambda) = \mathbb{Z}[1/n]$ is just multiplication by $1/n$. Therefore, $id_* - \lambda_*$ becomes multiplication by $(1 - 1/n)$ on $\mathbb{Z}[1/n]$ which is clearly $1:1$. Applying the Pimsner-Voiculescu exact sequence and recalling that $K_i(\mathcal{Q}^\lambda) = K_i(A^\lambda)$ we get the isomorphisms:

$$K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}/(n-1)\mathbb{Z}, \quad \text{and} \quad K_1(\mathcal{Q}^\lambda) \cong \mathbb{Z}/(n-1)\mathbb{Z}, \quad \text{for } \lambda = 1/\sqrt{n}.$$

For $n > 2$ we get $K_1 \neq 0$ and so these are not Cuntz algebras, in fact not even Cuntz-Krieger algebras since K_1 has nonzero torsion. For $\lambda = 1/\sqrt{2}$ however we get $K_0 = 0 = K_1$ and by classification theory, we must have $\mathcal{Q}^{1/\sqrt{2}} \cong O_2$! However, even in this case the fixed point algebra, is NOT AF since it has

$K_1 = \mathbb{Z}[1/2]$, the tape-measure group. So for the simplest irrational number $1/\sqrt{2}$ we get the Cuntz algebra, O_2 with a strange gauge action of $\mathbb{T}$.

Remarks. In the examples below it is important to note that any polynomial of the form $f(x) = x^n + ax^{n-1} + \cdots + bx \pm 1$ has at most $n-1$ roots in the open interval $(0,1)$ because the product of all the roots of f must equal ± 1.

Example(s) quadratic integers and an algorithm: If both λ and λ^{-1} are quadratic integers with $\lambda \in (0,1)$, then $\lambda^2 + a\lambda \pm 1 = 0$ where the integer polynomial $f(x) = x^2 + ax \pm 1$ is irreducible over $\mathbb{Q}$. With these restrictions there are two cases, either $f(x) = x^2 + ax - 1$ where $a > 0$ and $\lambda = 1/2\cdot(\sqrt{a^2+4} - a) \in (0,1)$; or $f(x) = x^2 + ax + 1$ where $a \leq -3$ and $\lambda = 1/2\cdot(-\sqrt{a^2-4} - a) \in (0,1)$.

In the first case, $\lambda^2 + a\lambda - 1 = 0$, with $a > 0$, so that $\lambda + a - \lambda^{-1} = 0$ and $\lambda^{-1} = a + \lambda$. For this case we outline an algorithm using the ideas of the Smith Normal Form and the Pimsner-Voiculescu exact sequence to calculate the K-theory. By Proposition 2.2, and the CONCLUSION before Proposition 2.20, $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda$ and $K_0(A_0^\lambda) \cong K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^1(\Gamma_\lambda) = \Gamma_\lambda \cong \mathbb{Z}^2$. Giving Γ_λ its $\mathbb{Z}$-basis $\{1, \lambda\}$ we see that the action of the automorphism λ_* on $K_0(A_0^\lambda) \cong \Gamma_\lambda$ has matrix: $\begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix}$. So, $(id - \lambda_*) = \begin{bmatrix} 1 & -1 \\ -1 & (a+1) \end{bmatrix} := M$. To compute the kernel and cokernel of this matrix mapping $\mathbb{Z}^2 \to \mathbb{Z}^2$ we row and column-reduce M over $\mathbb{Z}$ to obtain matrices $P, Q \in GL(2, \mathbb{Z})$ so that $PMQ = D$ where D is diagonal over $\mathbb{Z}$. Then $\ker(M) \cong \ker(D)$ and $\mathbb{Z}^2/M(\mathbb{Z}^2) \cong \mathbb{Z}^2/D(\mathbb{Z}^2)$. In this case, we get $D = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$. Hence, on $K_0(A_0^\lambda)$ we have

$$\ker(id - \lambda_*) = \ker(M) \cong \ker(D) = \{0\} \text{ and } \operatorname{coker}(id - \lambda_*) \cong \operatorname{coker}(D) = \mathbb{Z}/a\mathbb{Z}.$$

Now we compute $(id - \lambda_*)$ on

$$K_1(A_0^\lambda) \cong K_0(C(\hat{\Gamma}_\lambda))/\mathbb{Z}\cdot 1_o = (\mathbb{Z}\cdot 1_o \oplus \mathbb{Z}(1 \wedge \lambda))/\mathbb{Z}\cdot 1_o = \mathbb{Z}(1 \wedge \lambda).$$

Now, $\lambda_*(1 \wedge \lambda) = \lambda \wedge \lambda^2 = \lambda \wedge (1 - a\lambda) = \lambda \wedge 1 = (-1)1 \wedge \lambda$. That is, $\lambda_* = -id$ on $K_1(A_0^\lambda) \cong \mathbb{Z}$. Therefore, $(id - \lambda_*) =$ multiplication by 2 on $\mathbb{Z}(1 \wedge \lambda)$ which has $\ker(id - \lambda_*) = \{0\}$ and cokernel$(id - \lambda_*) \cong \mathbb{Z}/2\mathbb{Z}$.

Applying these results to the Pimsner-Voiculescu exact sequence we obtain:

$$K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/a\mathbb{Z} \text{ and } K_1(\mathcal{Q}^\lambda) = \mathbb{Z}/2\mathbb{Z}, \text{ for } \lambda^2 + a\lambda - 1 = 0,\ n \geq 1.$$

None of these examples are Cuntz-Krieger algebras since K_1 is not torsion-free. In particular, when $\lambda = (1/2)(\sqrt{5} - 1)$ is the inverse of the golden mean, we get $K_0 = \{0\}$ and $K_1 = \mathbb{Z}/2\mathbb{Z}$.

In the second case, $\lambda^2 + a\lambda + 1 = 0$, we have as above, $K_0(A_0^\lambda) \cong \Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda$ with $\mathbb{Z}$-basis $\{1, \lambda\}$; the diagonal version of $(id - \lambda_*)$ is $D = \operatorname{diag}[1, (a+2)]$ so that $\ker(id - \lambda_*) = \{0\}$ and $\operatorname{coker}(id - \lambda_*) \cong \mathbb{Z}/(a+2)\mathbb{Z}$. On the other hand, $K_1(A_0^\lambda) \cong \mathbb{Z}(1 \wedge \lambda)$ only now, $\lambda_* = id$ here so that $(id - \lambda_*) = 0$ and hence $\ker(id - \lambda_*) \cong \mathbb{Z}$ while $\operatorname{coker}(id - \lambda_*) \cong \mathbb{Z}$. By Pimsner-Voiculescu we get

$$K_0(\mathcal{Q}^\lambda) = \mathbb{Z} \oplus (\mathbb{Z}/(a+2)\mathbb{Z}) \text{ and } K_1(\mathcal{Q}^\lambda) = \mathbb{Z}, \text{ for } \lambda^2 + a\lambda + 1 = 0,\ a \leq -3.$$

We note that in this case, $\mathcal{Q}^\lambda$ has the correct K-theory to be a Cuntz-Krieger algebra (and is therefore stably isomorphic to one), and that in the case $a = -3$ (i.e., $\lambda = (1/2)(3 - \sqrt{5})$) we have $K_0 = \mathbb{Z} = K_1$.

Example cubic integers: If λ **and** λ^{-1} are cubic integers with $\lambda \in (0,1)$, then $\lambda^3 + a\lambda^2 + b\lambda \pm 1 = 0$ where the integer polynomial $f(x) = x^3 + ax^2 + bx \pm 1$ is irreducible over $\mathbb{Q}$. Such an f is irreducible if and only if $f(1) \neq 0 \neq f(-1)$. There are two cases depending on the constant, ± 1.

First, consider $f(x) = x^3 + ax^2 + bx - 1 = 0$ with $f(1) = a + b \neq 0$ and $f(-1) = a - b - 2 \neq 0$ so that f is irreducible. Now assume $a + b$ is **positive** (but $a \neq b + 2$). Then $f(0) = -1$ and $f(1) = a + b > 0$ so that f has a **unique** root in $(0, 1)$ since it is a cubic.

Next consider the same polynomial, $f(x) = x^3 + ax^2 + bx - 1 = 0$, with $a + b$ **negative** (but $a \neq b + 2$). Since both $f(0)$ and $f(1)$ are negative, in order to have a solution the function f must have a local maximum on $(0, 1)$. There are examples with no solutions in $(0, 1)$; for example, $f(x) = x^3 - 3x - 1$. In order to have a **unique solution**, then considering $f'(x)$, one would need $4a^2 - 12b = 0$: while this has many solutions, they all satisfy $|a| \leq b$ and so we can not have $a + b < 0$. So solutions are **not** unique in this case. But, there are infinitely many cubics with **two distinct** solutions in $(0, 1)$; eg., $f(x) = x^3 - (a + k)x^2 + ax - 1$ for $a \geq k + 4$ and $k \geq 1$ has two solutions in $(0, 1)$, since $f(.5) > 0$.

We now calculate the K-theory of $\mathcal{Q}^\lambda$ assuming that λ satisfies $f(x) = x^3 + ax^2 + bx - 1 = 0$, where $a+b \neq 0$, and $a-b \neq 2$. Now, $\lambda^3+a\lambda^2+b\lambda-1 = 0$, so that $\lambda^3 = 1-a\lambda^2-b\lambda$ and $\lambda^{-1} = \lambda^2+a\lambda+b$. Then, $\Gamma_\lambda = \mathbb{Z}+\mathbb{Z}\lambda+\mathbb{Z}\lambda^2$ and $K_0(A_0^\lambda) \cong K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^{odd}(\Gamma_\lambda) = \Gamma_\lambda \oplus (\Gamma_\lambda \wedge \Gamma_\lambda \wedge \Gamma_\lambda) = \Gamma_\lambda \oplus (\mathbb{Z}(1 \wedge \lambda \wedge \lambda^2)) \cong \mathbb{Z}^4$. Giving Γ_λ its natural $\mathbb{Z}$-basis $\{1, \lambda, \lambda^2\}$ the induced homomorphism $(id - \lambda_*)$ on $K_0(A_0^\lambda) \cong \mathbb{Z}^4$ yields the diagonal matrix, $D = \mathrm{diag}[1, 1, (a + b), 0]$ so that on $K_0(A_0^\lambda)$ we have

$$\ker(id - \lambda_*) \cong \ker(D) \cong \mathbb{Z} \text{ and } \mathrm{coker}(id - \lambda_*) \cong \mathrm{coker}(D) = (\mathbb{Z}/(a + b)\mathbb{Z}) \oplus \mathbb{Z}.$$

Now, $K_1(A_0^\lambda) \cong K_0(C(\hat{\Gamma}_\lambda))/\mathbb{Z} \cdot 1_o = \bigwedge^2(\Gamma_\lambda) = \mathbb{Z}(1 \wedge \lambda) + \mathbb{Z}(1 \wedge \lambda^2) + \mathbb{Z}(\lambda \wedge \lambda^2) \cong \mathbb{Z}^3$. By similar computations we get for $K_1(A_0^\lambda) \cong \mathbb{Z}^3$; the matrix $D = \mathrm{diag}[1, 1, (a + b)]$. Hence, on $K_1(A_0^\lambda)$ we have

$$\ker(id - \lambda_*) \cong \ker(D) = \{0\} \text{ and } \mathrm{coker}(id - \lambda_*) \cong \mathrm{coker}(D) = \mathbb{Z}/(a + b)\mathbb{Z}.$$

Applying these results to the Pimsner-Voiculescu exact sequence we obtain:

$$K_0(\mathcal{Q}^\lambda) = \mathbb{Z} \oplus (\mathbb{Z}/(a + b)\mathbb{Z}) \text{ and } K_1(\mathcal{Q}^\lambda) = \mathbb{Z} \oplus (\mathbb{Z}/(a + b)\mathbb{Z}) \text{ for } \lambda^3 + a\lambda^2 + b\lambda - 1 = 0.$$

Remarks. In case $a + b = 1$ (which has infinitely many solutions corresponding to infinitely many distinct invertible cubic integers $\lambda \in (0, 1)$) we get $K_0(\mathcal{Q}^\lambda) = \mathbb{Z} = K_1(\mathcal{Q}^\lambda)$, which as noted above is also true for the invertible quadratic integer, $\lambda = (1/2)(3 - \sqrt{5})$. In the general cubic case with constant term -1 we always have **non-torsion** elements in both K_0 and K_1 : this is the opposite of the case where the constant term is $+1$, where we see below that K_0 and K_1 are both **torsion** groups. A similar phenomenon occurs in the quadratic case above, except that there we get **torsion** in the -1 case and **non-torsion** in the $+1$ case! That this may be a periodic phenomenon is supported by a calculation of two quartic examples: first, the unique solution $\lambda \in (0, 1)$ to the irreducible quartic $f(x) = x^4 - 3x^3 + 1$ gives us $K_0 = \mathbb{Z}$ and $K_1 = \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$; while, second, the unique solution $\lambda \in (0, 1)$ to the irreducible quartic $f(x) = x^4 + 3x^3 - 1$ gives us $K_0 = (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$ and $K_1 = (\mathbb{Z}/9\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, similar to the quadratic case. Proposition 2.21 is further evidence.

When an irreducible polynomial $f(x) = x^n + ax^{n-1} + \cdots \pm 1$ has two roots, $\lambda_1, \lambda_2 \in (0, 1)$, then $\Gamma_{\lambda_1} \cong \Gamma_{\lambda_2}$ as **rings** (but **not** as ordered rings, for that would imply equality). Still, $\mathcal{Q}^{\lambda_1} \cong \mathcal{Q}^{\lambda_2}$ (at least stably) since the calculation of their K-groups are identical. Their KMS states are not equivalent since the type III factors that they generate are not isomorphic, as we will see below.

Proposition 2.21. *Suppose λ satisfies the irreducible (over $\mathbb{Z}$) polynomial, $f(x) = x^n + \cdots \pm 1 = 0$.*
(1) *For n **odd**, if $f(x) = x^n + \cdots + 1$ then $K_0(\mathcal{Q}^\lambda)$ has $\mathbb{Z}/2\mathbb{Z}$ as a summand.*
While, if $f(x) = x^n + \cdots - 1$ then $K_0(\mathcal{Q}^\lambda)$ has $\mathbb{Z}$ as a summand (so, by the next Proposition, $\mathrm{rank}(K_0) = \mathrm{rank}(K_1) \geq 1$ *in this case).*
(2) *For n **even**, if $f(x) = x^n + \cdots + 1$ then $K_1(\mathcal{Q}^\lambda)$ has $\mathbb{Z}$ as a summand (so, by the next Proposition,* $\mathrm{rank}(K_0) = \mathrm{rank}(K_1) \geq 1$ *in this case).*
While, if $f(x) = x^n + \cdots - 1$ then $K_1(\mathcal{Q}^\lambda)$ has $\mathbb{Z}/2\mathbb{Z}$ as a summand.

Proof. In $K_*(A_0^\lambda)$ there is a λ_*-invariant summand, $(1 \wedge \lambda \wedge \lambda^2 \wedge \cdots \wedge \lambda^{n-1})\mathbb{Z}$. Depending on $n(\mathrm{mod}\ 2)$ and the constant term ± 1, the action of λ_* on this summand is $\pm id$. Hence, $(id - \lambda_*)$ here is either 0 or $2(id)$. Applying Pimsner-Voiculescu gives a summand in $K_*(\mathcal{Q}^\lambda)$ of either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. $\square$

Proposition 2.22. *Suppose λ is algebraic.*
(1) *Then, $rank(K_0(\mathcal{Q}^\lambda)) = rank(K_1(\mathcal{Q}^\lambda))$ so that $\mathcal{Q}^\lambda$ is not stably isomorphic to O_∞.*
(2) *If λ and λ^{-1} are both algebraic integers and $\mathcal{Q}_\lambda$ is stably isomorphic to a Cuntz algebra O_n, then the minimal polynomial of λ has odd degree and constant term $+1$. Moreover, n is congruent to $3(mod\ 4)$ and all such Cuntz algebras appear this way.*

Proof. To see part (1) we tensor the Pimsner-Voiculescu exact sequence by $\mathbb{Q}$ (which preserves exactness) to obtain an exact hexagon of $\mathbb{Q}$-vector spaces:

$$\begin{tikzcd} V_1 \arrow[r, "\theta_1"] & V_1 \arrow[r, "\tau_1"] & K_1^{\mathbb{Q}} \arrow[d, "\mu_1"] \\ K_0^{\mathbb{Q}} \arrow[u, "\mu_0"] & V_0 \arrow[l, "\tau_0"] & V_0 \arrow[l, "\theta_0"] \end{tikzcd}$$

where $V_i = K_i(A_0^\lambda) \otimes \mathbb{Q}$, and $K_i^{\mathbb{Q}} = K_i(A^\lambda) \otimes \mathbb{Q}$. Then

$$\dim(K_0^{\mathbb{Q}}) = \mathrm{rank}(\mu_0) + \mathrm{nullity}(\mu_0) = \mathrm{rank}(\mu_0) + \mathrm{rank}(\tau_0) \ \text{ and } \ \dim(K_1^{\mathbb{Q}}) = \mathrm{rank}(\mu_1) + \mathrm{rank}(\tau_1)$$

and

$$\mathrm{rank}(\tau_0) + \mathrm{rank}(\theta_0) = \dim(V_0) = \mathrm{rank}(\theta_0) + \mathrm{rank}(\mu_1) \ \text{ so that } \ \mathrm{rank}(\tau_0) = \mathrm{rank}(\mu_1).$$

Similarly, $\mathrm{rank}(\tau_1) = \mathrm{rank}(\mu_0)$, so that:

$$\dim(K_0^{\mathbb{Q}}) = \mathrm{rank}(\mu_0) + \mathrm{rank}(\tau_0) = \mathrm{rank}(\mu_1) + \mathrm{rank}(\tau_1) = \dim(K_1^{\mathbb{Q}}).$$

That is, $\mathrm{rank}(K_0(\mathcal{Q}^\lambda)) = \dim K_0(\mathcal{Q}^\lambda) \otimes_{\mathbb{Z}} \mathbb{Q} = \dim K_0(A^\lambda) \otimes_{\mathbb{Z}} \mathbb{Q} = \cdots \mathrm{rank}(K_1(\mathcal{Q}^\lambda))$. By Proposition 2.21, if the minimal polynomial of λ has even degree, then $K_1(\mathcal{Q}_\lambda) \neq \{0\}$, and so $\mathcal{Q}_\lambda$ cannot be stably isomorphic to a Cuntz algebra. If $\mathcal{Q}_\lambda$ is stably isomorphic to O_n then n is finite by part (1)and by Proposition 2.21, the order of $K_0(\mathcal{Q}_\lambda)$ must be even and therefore n must be odd. Furthermore, the minimal polynomial must have constant term $+1$. In order for $K_0(\mathcal{Q}_\lambda)$ to be a finite cyclic group of even order, it must be of the form $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ where m is **odd** since $\mathbb{Z}/2\mathbb{Z}$ is a summand. Let $m = 2k+1$ then

$$n = \sharp[\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}] + 1 = 2m + 1 = 4k + 3$$

as claimed.

In the examples below where $\lambda^3 + a\lambda^2 + b\lambda + 1 = 0$, and either $a - b = 1$ and $b \leq -2$ OR $a = b = -1$ and $b \leq -1$, we obtain (stably, at least) all the Cuntz algebras O_n where $n \equiv 3(mod\ 4)$. $\square$

Now consider the case of irreducible cubics of the form $f(x) = x^3 + mx^2 + nx + 1$; so $f(1) = m + n + 2 \neq 0$ and $f(-1) = m - n \neq 0$. Since $f(0) = 1$, if we have $f(1) = m + n + 2 < 0$, then we have as above a **unique** root in $(0, 1)$.

If $f(1) = m + n + 2 > 0$, we can have distinct roots. For example, if $n = -4$ and $m = 3$, then, $f(x) = x^3 + 3x^2 - 4x + 1$ has two roots in $(0, 1)$. If $n << 0$, we get several solutions m for each n: eg., $n = -7$ implies that any m with $6 \leq m \leq 9$ will yield a polynomial with two roots in $(0, 1)$.

We now calculate the K-theory of $\mathcal{Q}^\lambda$ assuming λ satisfies $f(\lambda) = \lambda^3 + m\lambda^2 + n\lambda + 1 = 0$. Again, $K_0(A_0^\lambda) \cong \mathbb{Z}^4$, but now the diagonal matrix $D = \mathrm{diag}[1, 1, (m+n+2), 2]$. On $K_1(A_0^\lambda) \cong \mathbb{Z}^3$, the matrix $D = \mathrm{diag}[1, 1, (m - n)]$. Both matrices are $1:1$ since $m + n + 2 \neq 0 \neq m - n$. We get:

$$K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/(n + m + 2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ \text{ and } \ K_1(\mathcal{Q}^\lambda) = \mathbb{Z}/(m - n)\mathbb{Z} \ \text{ for } \ \lambda^3 + m\lambda^2 + n\lambda + 1 = 0.$$

To obtain Cuntz algebras, we need $m-n=\pm 1$. It turns out $f(1)>0$ can not occur, so we must have $f(1)=m+n+2<0$ hence there is a unique root λ in $(0,1)$. Combining this inequality with $m-n=\pm 1$ we get exactly two infinite families of solutions; $m=n+1$ for $n\leq -2$, OR $m=n-1$ for $n\leq -1$. In either case, the sequence of numbers $\{|m+n+2|\}$ is the same: $\{2k+1|k\geq 0\}$. For this sequence we get the K_0 groups: $\mathbb{Z}/(2k+1)\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\cong\mathbb{Z}/(4k+2)\mathbb{Z}$. Since the K_1 groups are all $\{0\}$, by construction, the algebras $\mathcal{Q}^\lambda$ are (at least stably) the Cuntz algebras, O_{4k+3} for $k\geq 0$. That is, O_3, O_7, O_{11}, etc.

Example, λ transcendental:

Lemma 2.23. *Let $\varphi:\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}\longrightarrow\mathbb{Z}$ be the surjective homomorphism, $\phi(\{a_n\}):=\sum_{n\in\mathbb{Z}}a_n$; and let $S\in Aut(\bigoplus_{n\in\mathbb{Z}}\mathbb{Z})$ be the shift $S(\{a_n\}_{n\in\mathbb{Z}}):=\{a_{n-1}\}_{n\in\mathbb{Z}}$. Then, $(id-S)$ is $1:1$ and $\ker(\varphi)=\mathrm{Im}(id-S)$, so that $(\bigoplus_{n\in\mathbb{Z}}\mathbb{Z})/\mathrm{Im}(id-S)\cong\mathbb{Z}$.*

Proof. As a model for $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}$ we use $\mathbb{Z}[x,x^{-1}]=\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}x^n$, the ring of Laurent polynomials over $\mathbb{Z}$ (i.e., the group ring over $\mathbb{Z}$ of the group $\{x^n|n\in\mathbb{Z}\}$). Here, φ is the augmentation map, S is multiplication by x, and $(id-S)$ is multiplication by $(1-x)$ which is $1:1$. Now,

$$\sum_{n=-N}^{N}a_nx^n\in\ker(\varphi)\Leftrightarrow\sum_{n=-N}^{N}a_n=0\Leftrightarrow\sum_{n=-N}^{N}a_nx^{n+N}\in\ker(\varphi).$$

Let $p(x)=\sum_{n=-N}^{N}a_nx^{n+N}\in\mathbb{Z}[x]$ so $p(1)=\sum_{n=-N}^{N}a_n=0$. Hence, $p(x)$ factors: $p(x)=(1-x)q(x)$ where initially $q(x)\in\mathbb{Q}[x]$. Since $p(x)\in\mathbb{Z}[x]$ it is easy to see that in fact, $q(x)\in\mathbb{Z}[x]$ also. Then,

$$\sum_{n=-N}^{N}a_nx^n=x^{-N}p(x)=(1-x)x^{-N}q(x)\in(1-x)\mathbb{Z}[x,x^{-1}]=\mathrm{Im}(id-S).$$

That is, $\ker(\varphi)\subseteq\mathrm{Im}(id-S)$, and the other containment is immediate. □

Proposition 2.24. *If λ is transcendental then*

$$K_0(\mathcal{Q}^\lambda)\cong\bigoplus_{n=1}^{\infty}\mathbb{Z}\cong K_1(\mathcal{Q}^\lambda).$$

Proof. In this case, by Proposition 2.2 and the CONCLUSION before Proposition 2.20 we have:

$$K_0(A_0^\lambda)=\bigoplus_{k=1}^{\infty}\bigwedge^{2k-1}(\Gamma_\lambda)\ \text{ and }\ K_1(A_0^\lambda)=\bigoplus_{k=1}^{\infty}\bigwedge^{2k}(\Gamma_\lambda)\ \text{ where }\ \Gamma_\lambda=\bigoplus_{n=-\infty}^{\infty}\mathbb{Z}\lambda^n.$$

Now, each individual summand $\bigwedge^m(\Gamma_\lambda)$ is invariant under λ_* and yields (for $m>1$) an infinite direct sum of (λ_*-invariant) examples of the previous lemma where the action of λ_* is just the shift. The general case is notation-heavy, so we do the examples, $\bigwedge^2$ and $\bigwedge^3$. Letting $\mathbb{Z}_+$ denote the positive integers, we have:

$$\bigwedge^2(\Gamma_\lambda)=\bigoplus_{k\in\mathbb{Z}_+}\left(\bigoplus_{n\in\mathbb{Z}}(\lambda^n\wedge\lambda^{n+k})\mathbb{Z}\right)\ \text{ and }\ \bigwedge^3(\Gamma_\lambda)=\bigoplus_{(k_1,k_2)\in\mathbb{Z}_+^2}\left(\bigoplus_{n\in\mathbb{Z}}(\lambda^n\wedge\lambda^{n+k_1}\wedge\lambda^{n+k_1+k_2})\mathbb{Z}\right).$$

The case $m=1$ is just the group $\Gamma_\lambda=\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}\lambda^n$ which yields a single instance of the lemma.

Applying the lemma we see that $(id-\lambda_*)$ is $1:1$ on both $K_0(A_0^\lambda)$ and $K_1(A_0^\lambda)$ and that both $K_0(A_0^\lambda)/(id-\lambda_*)(K_0(A_0^\lambda))$ and $K_1(A_0^\lambda)/(id-\lambda_*)(K_1(A_0^\lambda))$ are isomorphic to a countable direct sum of copies of $\mathbb{Z}$. An application of the Pimsner-Voiculescu exact sequence completes the proof. □

Remark. The classification theory of Kirchberg algebras implies that for λ transcendental we have a new realisation of the algebras found in [Cu1] and denoted $\mathcal{Q}_{\mathbb{N}}$ there.

2.5. **The dual action of $\mathbb{T}^1$ on A^λ and its restriction to the gauge action on $\mathcal{Q}^\lambda$.** Recall, $G_\lambda^0 = \{g \in G_\lambda \,|\, |g| = 1\}$ is a normal subgroup of G_λ. The subgroup of G_λ of elements of the form $[\lambda^n : 0]$ is isomorphic to $\mathbb{Z}$ and acts on G_λ^0 by conjugacy:

$$[\lambda^n : 0][1 : b][\lambda^{-n} : 0] = [1 : \lambda^n b].$$

Thus $G_\lambda = \mathbb{Z} \rtimes G_\lambda^0$ is a semidirect product and we can write A^λ as an iterated crossed product:

$$A^\lambda = G_\lambda \rtimes_\alpha C_0^\lambda(\mathbb{R}) = \mathbb{Z} \rtimes (G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R})) = \mathbb{Z} \rtimes A_0^\lambda.$$

The dual action γ of $\mathbb{T}^1$ on A^λ is relative to this latter crossed product so that for each $z \in \mathbb{T}^1$ and x in the Banach $*$-algebra, $l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R}))$ we have:

$$\gamma_z(x)(g) = z^n x(g) \text{ if } x \in l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R}));\ g \in G_\lambda \text{ and } |g| = \lambda^n.$$

Since A^λ is defined to be the completion of this Banach $*$-algebra in its universal representation, the action γ extends uniquely to an action (also denoted by γ) of $\mathbb{T}^1$ as automorphisms of A^λ. The fixed point subalgebra of the dual action is, of course, exactly $A_0^\lambda = G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R})$.

Since the projection e is in A_0^λ, the action γ restricts to an action of $\mathbb{T}^1$ on $\mathcal{Q}^\lambda = eA^\lambda e$, which we will also denote by γ. We call this the **gauge action** of $\mathbb{T}^1$ on $\mathcal{Q}^\lambda$. Now, γ is clearly a strongly continuous action of $\mathbb{T}^1$ on $\mathcal{Q}^\lambda$. Averaging over γ with respect to normalised Haar measure gives a positive, faithful expectation Φ of $\mathcal{Q}^\lambda$ onto the fixed-point algebra which is clearly F^λ:

$$\Phi(a) := \frac{1}{2\pi}\int_{\mathbb{T}^1} \gamma_z(a)\, d\theta \text{ for } a \in \mathcal{Q}^\lambda, \text{ and } z = e^{i\theta}.$$

Proposition 2.25. *The fixed point algebra, $F^\lambda = eA_0^\lambda e$ is the norm closure of finite linear combinations of elements of the form:*

$$\mathcal{X}_{[a,b)} \cdot \delta_g \textit{ where } g = [1 : c] \textit{ and } [a,b) \subseteq [0,1) \cap [c, 1+c),$$

for $a, b, c \in \Gamma_\lambda$. Recall, $A_0^\lambda \cong \mathcal{K}(l^2(\mathbb{Z})) \otimes F^\lambda$.

Proof. Applying the integral formula for Φ to a finite linear combination of the generators for $\mathcal{Q}^\lambda$ we see that the only terms that survive are those where $|g| = 1$: that is, g has the above form. Then we apply item (2) of Lemma 2.12 to obtain the condition on the interval $[a,b)$. ☐

Corollary 2.26. *The stabilised algebra $\mathcal{Q}^\lambda \otimes \mathcal{K}$ is a crossed product of the stabilised fixed-point algebra $F^\lambda \otimes \mathcal{K}$ by an action of $\mathbb{Z}$. For $\lambda = 1/n$ this is a theorem of J. Cuntz.*

Proof. By Proposition 2.7, $A_0^\lambda \cong F^\lambda \otimes \mathcal{K}$, and $A^\lambda \cong \mathcal{Q}^\lambda \otimes \mathcal{K}$. By the discussion at the beginning of subsection 3.1, $A^\lambda \cong \mathbb{Z} \rtimes A_0^\lambda$ and the proof is complete. See [Cu, Section 2]. ☐

Remarks. If we combine the previous observation that F^λ is the fixed point subalgebra of $\mathcal{Q}^\lambda$ under the gauge action with Corollary 2.18 we get, for example, $O_2 \cong \mathcal{Q}^{2/3}$ with a gauge action whose fixed point subalgebra $F^{2/3}$ is a UHF algebra of type 6^∞. Interestingly, $F^{3/4}$ is UHF of type $12^\infty = 6^\infty$ which is therefore isomorphic to $F^{2/3}$. So we have two gauge actions on O_2 with isomorphic UHF fixed point subalgebras, with distinct, inequivalent KMS states: one where $\beta = \log(3/2)$ and the other where $\beta = \log(4/3)$ by Proposition 2.30 below. Moreover, the two von Neumann algebras generated by the GNS representations of O_2 are not isomorphic as they are type III_λ factors for λ equalling $2/3$ and $3/4$, respectively, by Theorem 2.35 below.

2.6. **The γ-invariant semifinite weight on A^λ and its restriction to $\mathcal{Q}^\lambda$.** The aim of this subsection is to exhibit the unique KMS states for the gauge action on $\mathcal{Q}^\lambda$. We first recall the definition of KMS states.

Definition 2.27. *Let A be a C^*-algebra with a continuous action $\gamma : \mathbb{R} \to Aut(A)$. Let ψ be a state on A and $\beta \in \mathbb{R}$ a real number. We define ψ to be a* KMS_β *state for the action γ if*

$$\psi(x\,\gamma_{i\beta}(y)) = \psi(yx)$$

for all $x, y \in \mathcal{A}$ a dense γ-invariant $*$-subalgebra of A_γ, the subalgebra of analytic elements for the action γ. We refer to [BR1, Section 2.5] *for basic information on the subalgebra of analytic elements, A_γ and to* [BR2, Section 5.3] *for all the basic information on* KMS *states.*

Since G_λ is discrete it is well-known that the map

$$x \mapsto x(1) : l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R})) \to C_0^\lambda(\mathbb{R})$$

extends uniquely to a faithful conditional expectation $E : A^\lambda \to C_0^\lambda(\mathbb{R})$. Composing E with the densely defined (norm) lower semicontinuous weight on $C_0^\lambda(\mathbb{R})$ given by integration, gives us a densely defined (norm) lower semicontinuous weight on A^λ which we denote by $\bar\psi$. In particular, for $x \in l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R}))$ we have:

$$\bar\psi(xx^*) = \int_{\mathbb{R}} xx^*(1)(t)dt = \int_{\mathbb{R}} \left(\sum_{h\in G_\lambda} x(h)\overline{x(h)}\right)(t)dt = \sum_{h\in G_\lambda}\left(\int_{\mathbb{R}} |x(h)(t)|^2 dt\right).$$

So that $\bar\psi$ is faithful. We observe that $\bar\psi$ **is not a trace**, since $\bar\psi(x^*x) = \sum_{h\in G_\lambda} |h^{-1}| \int_{\mathbb{R}} |x(h)(t)|^2 dt$.

Proposition 2.28. *The weight $\bar\psi$ on A^λ restricts to a faithful semifinite trace $\bar\tau$ on A_0^λ and also restricts to a state denoted by ψ on $\mathcal{Q}^\lambda$ satisfying:*
(1) The gauge action γ of $\mathbb{T}^1$ on $\mathcal{Q}^\lambda$ leaves the state ψ invariant.
(2) The state ψ restricted to the fixed point algebra, F^λ is a faithful (finite) trace denoted by τ; which is, of course, the restriction of $\bar\tau$ on A_0^λ to F^λ.
(3) With $\Phi : \mathcal{Q}^\lambda \to F^\lambda$ the canonical expectation, we have $\psi = \tau \circ \Phi$.

Proof. Since $\bar\psi(e) = \int_{\mathbb{R}} \mathcal{X}_{[0,1)}(t)dt = 1$, we see that $\bar\psi$ restricted to $\mathcal{Q}^\lambda$ is a faithful state. To see item (1), it suffices to see that the gauge action on $l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R}))$ leaves $\bar\psi$ invariant. To this end, let $x \geq 0$ be in $l^1_\alpha(G_\lambda, C_0^\lambda(\mathbb{R}))$, and let $z \in \mathbb{T}^1$. Then

$$E(\gamma_z(x)) = \gamma_z(x)(1) = z^0 x(1) = x(1) = E(x)$$

and so

$$\bar\psi(\gamma_z(x)) = \int_{\mathbb{R}} \gamma_z(x)(1)(t)dt = \int_{\mathbb{R}} E(x)(t)dt = \bar\psi(x).$$

To see item (2) we use Proposition 2.25 and the above computation that shows that while $\bar\psi$ is not generally a trace, to see that **it is a trace** when the group elements all have determinant 1.

To see item (3), it suffices to see that for any $x \in \mathcal{Q}^\lambda$ we have $E(x) = E(\Phi(x))$, but this is the same as $x(1) = \Phi(x)(1)$ which is clear since $\det(1) = 1$. $\square$

Now, since the state ψ is invariant under the action γ, this action is unitarily implemented on $\mathcal{L}^2(\mathcal{Q}^\lambda, \psi)$. For $z \in \mathbb{T}^1$ and $x \in \mathcal{Q}_c^\lambda$ we define:

$$(u_z(x))_h = z^n x_h \text{ for } h \in G_\lambda \text{ with } |h| = \lambda^n.$$

We define the spectral subspaces of this unitary group on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ in the usual way. For each $k\in\mathbb{Z}$ let Φ_k be the operator on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$:

$$\Phi_k(x)=\frac{1}{2\pi}\int_{\mathbb{T}^1} z^{-k}u_z(x)d\theta,\ z=e^{i\theta},\ x\in\mathcal{L}^2(\mathcal{Q}^\lambda,\psi).$$

We observe that if $x=f\cdot\delta_g$ is a typical generator of $\mathcal{Q}^\lambda$ considered as a vector in $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ then we have:

$$\Phi_k(f\cdot\delta_g)=\begin{cases} f\cdot\delta_g & \text{if } |g|=\lambda^k\\ 0 & \text{otherwise}\end{cases}$$

More generally, on $\mathcal{H}:=\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$, we have $\Phi_k(\mathcal{H})=\{x\in\mathcal{H}\mid u_z(x)=z^kx\ \text{ for all }\ z\in\mathbb{T}^1\}$.

Lemma 2.29. *For each $k\in\mathbb{Z}$ the subspace span$\{f\cdot\delta_g\in\mathcal{Q}^\lambda\mid |g|=\lambda^k\}$ is dense in the range of Φ_k. The operators Φ_k are mutually orthogonal projections on $\mathcal{H}$ which sum to the identity operator $1=\pi(e)$.*

Proof. The proof of the first statement is similar to the proof of Proposition 2.25. The mutual orthogonality of the Φ_k follows from the fact that $\langle f_1\cdot\delta_{g_1}|f_2\cdot\delta_{g_2}\rangle_\psi=0$ unless $g_1=g_2$. ☐

Proposition 2.30. *The dense $*$-subalgebra of $\mathcal{Q}^\lambda$ consisting of finite linear combinations of the partial isometries $\mathcal{X}_{[a,b)}\cdot\delta_g$ is contained in the subset of entire elements, $\mathcal{Q}^\lambda_\gamma$, for the action γ considered as an action of $\mathbb{R}:t\mapsto\gamma_{e^{it}}$. Moreover, ψ is a* KMS$_\beta$ *state for this action where $\beta=\log(\lambda^{-1})$. In fact, ψ is the* **unique** KMS *state for this action (regardless of β).*

Proof. Let $y=\mathcal{X}_{[a,b)}\cdot\delta_g\in\mathcal{Q}^\lambda$ where $\det(g)=\lambda^k$. Then, $t\mapsto\gamma_{e^{it}}(y)=e^{ikt}y$; $t\in\mathbb{R}$ obviously extends to the entire function $w\mapsto\gamma_{e^{iw}}(y)=e^{ikw}y$; $w\in\mathbb{C}$. For $w=\log(\lambda^{-1})i$, this equation becomes $\gamma_{e^{iw}}(y)=\gamma_\lambda(y)=\lambda^ky$. Letting $\beta=\log(\lambda^{-1})$, we have $\gamma_{\beta i}(y)=\lambda^ky$. Now, let $x=\mathcal{X}_{[c,d)}\cdot h$ so we want to see that: $\lambda^k\psi(xy)=\psi(x\gamma_{\beta i}(y))=\psi(yx)$. That is, we **want** $\lambda^k\psi(xy)=\psi(yx)$. Now both sides of this equation are zero unless $h=g^{-1}$. But, when $h=g^{-1}$, we have

$$xy=\mathcal{X}_{[c,d)}\cdot\mathcal{X}_{[g^{-1}(a),g^{-1}(b))}\cdot\delta_I\ \text{ while }\ yx=\mathcal{X}_{[a,b)}\cdot\mathcal{X}_{[g(c),g(d))}\cdot\delta_I.$$

Moreover,

$$s\in[c,d)\cap[g^{-1}(a),g^{-1}(b))\Longleftrightarrow g(s)\in[g(c),g(d))\cap[a,b).$$

Since $\det(g)=\lambda^k$ the transformation g increases the measure by a factor of λ^k and the result follows. That is, ψ is a KMS$_\beta$ state for the action γ of $\mathbb{R}$ for $\beta=\log(\lambda^{-1})$.

Now let ϕ be a KMS state on $\mathcal{Q}^\lambda$ for the action γ. Since $\mathcal{Q}^\lambda$ is purely infinite it has no nontrivial traces and so ϕ must be KMS for some nonzero β. Hence by [BR2, Proposition 5.3.3], ϕ is invariant under the action of γ. Now, if $\mathcal{X}_{[a,b)}\cdot\delta_g\in\mathcal{Q}^\lambda$ with $\det(g)=\lambda^k$, then we have for all $z\in\mathbb{T}$:

$$\phi(\mathcal{X}_{[a,b)}\cdot\delta_g)=\phi(\gamma_z(\mathcal{X}_{[a,b)}\cdot\delta_g))=z^k\phi(\mathcal{X}_{[a,b)}\cdot\delta_g).$$

That is, if $\det(g)\neq1$ we must have $\phi(\mathcal{X}_{[a,b)}\cdot\delta_g)=0$, and so ϕ is supported on F^λ. Since F^λ is γ-invariant and ϕ is KMS for some nonzero β, ϕ is a trace on F^λ by [BR2, 5.3.28].

Now, if $x=\mathcal{X}_{[a,b)}\cdot\delta_g\in F^\lambda$ and $g\neq I$, then we claim that $\phi(x)=0$. For suppose $g=[1:c]$ with $c>0$. Then there is a positive integer n such that $a+nc<b\leq a+(n+1)c$ and so

$$x=\mathcal{X}_{[a,b)}\cdot\delta_g=\mathcal{X}_{[a,c)}\cdot\delta_g+\mathcal{X}_{[a+c,a+2c)}\cdot\delta_g+\cdots+\mathcal{X}_{[a+nc,b)}\cdot\delta_g:=v_0+v_1+\cdots+v_n.$$

Now each of these partial isometries v_k satisfies $v_k^2=0$, and so $\phi(v_k)=\phi(v_kv_k^*v_k)=\phi(v_k^2v_k^*)=0$ since ϕ is a trace on F^λ. Thus, $\phi(x)=0$ as claimed.

Hence ϕ is supported on the commutative subalgebra

$$\mathcal{C} := \overline{\text{span}}\{f \cdot \delta_I \mid f \in C_0^\lambda(\mathbb{R}) \text{ and } supp(f) \subseteq [0,1)\}.$$

Morever, if f_1, f_2 are characteristic functions of subintervals of $[0,1)$ with endpoints in Γ_λ and having the same length they give equivalent elements $f_i \cdot \delta_I$ in F^λ and therefore have the same value under ϕ.

Now, since $A_0^\lambda \cong F^\lambda \otimes \mathcal{K}$ we can define a lower semicontinuous, densely defined trace, $\tilde{Tr}$ on A_0^λ via $\tilde{Tr} = \phi \otimes Tr$, where Tr is the trace on $\mathcal{K}$. So, for $\mathcal{X}_{[a,b)} \cdot \delta_I \in F^\lambda$ we have $\tilde{Tr}(\mathcal{X}_{[a,b)} \cdot \delta_I) = \phi(\mathcal{X}_{[a,b)} \cdot \delta_I)$. Then, for $k_1 < k_2 \in \mathbb{Z}$ the element $\mathcal{X}_{[k_1,k_2)} \cdot \delta_I$ is the sum of $(k_2 - k_1)$ projections in A_0^λ each equivalent to $\mathcal{X}_{[0,1)} \cdot \delta_I$ which has trace equal to 1; that is, $\tilde{Tr}(\mathcal{X}_{[k_1,k_2)} \cdot \delta_I) = (k_2 - k_1)$. Now, for any $a < b$ in Γ_λ, we have $\mathcal{X}_{[a,b)} \cdot \delta_I \sim \mathcal{X}_{[0,b-a)} \cdot \delta_I$ and so $\tilde{Tr}(\mathcal{X}_{[a,b)} \cdot \delta_I) = \tilde{Tr}(\mathcal{X}_{[0,b-a)} \cdot \delta_I)$, and these values are finite since both these projections are dominated by $\mathcal{X}_{[-N,N)} \cdot \delta_I$ for a sufficiently large integer N. It now suffices to prove the following. **Claim:** $\tilde{Tr}(\mathcal{X}_{[a,b)} \cdot \delta_I) = b - a$ for $a < b \in \Gamma_\lambda$.

By the previous discussion we can assume that $a = 0$ so that $b > 0$. Given $\varepsilon > 0$ we choose positive integers m, n such that

$$\frac{1}{m} \leq \varepsilon \text{ and } \frac{n-1}{m} \leq b < \frac{n}{m},$$

so that $(n-1) \leq bm < n$ and $(n-1), bm, n \in \Gamma_\lambda$. Hence

$$(n-1) = \tilde{Tr}(\mathcal{X}_{[0,(n-1))} \cdot \delta_I) \leq \tilde{Tr}(\mathcal{X}_{[0,bm)} \cdot \delta_I) \leq \tilde{Tr}(\mathcal{X}_{[0,n)} \cdot \delta_I) = n.$$

But,

$$\mathcal{X}_{[0,bm)} \cdot \delta_I = \mathcal{X}_{[0,b)} \cdot \delta_I + \mathcal{X}_{[b,2b)} \cdot \delta_I + \cdots + \mathcal{X}_{[(m-1)b,bm)} \cdot \delta_I$$

and these projections are mutually equivalent in A_0^λ. That is,

$$(n-1) \leq m\tilde{Tr}(\mathcal{X}_{[0,b)} \cdot \delta_I) \leq n \quad \text{so that} \quad \frac{n-1}{m} \leq \tilde{Tr}(\mathcal{X}_{[0,b)} \cdot \delta_I) \leq \frac{n}{m}.$$

Hence, $|\tilde{Tr}(\mathcal{X}_{[0,b)} \cdot \delta_I) - b| < \frac{1}{m} \leq \varepsilon$. That is, $b = \tilde{Tr}(\mathcal{X}_{[0,b)} \cdot \delta_I) = \phi(\mathcal{X}_{[0,b)} \cdot \delta_I)$ and ϕ agrees with the given trace τ on F^λ and therefore ϕ agrees with ψ on $\mathcal{Q}^\lambda$. $\square$

Remarks. The above proof shows that the algebra F^λ has a unique (faithful) tracial state τ, and that A_0^λ has a unique (faithful) lower semicontinuous, densely defined trace normalized so that it has value 1 at $e = \mathcal{X}_{[0,1)} \cdot \delta_I$.

2.7. **The von Neumann algebra $\pi(A^\lambda)^{-wo}$ acting on $\mathcal{L}^2(A^\lambda, \bar{\psi})$ is a type III_λ factor.** To prove this we will show that it is unitarily equivalent to a version of the Murray-von Neumann "group-measure space" construction of type III factors on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$: see [D, Chapter 1, Section 9]. We conclude that it is a III_λ factor by an appeal to Connes' thesis [C1]. In order to be consistent with our use of right C^*-modules later, we will do our GNS constructions so that our inner products are linear in the **second** variable.

Proposition 2.31. *The $*$-algebra A_c^λ is a Tomita algebra with the inner product:*

$$\langle y|x\rangle_{\bar{\psi}} = \bar{\psi}(y^*x) = \sum_{h\in G_\lambda} |h|^{-1}\langle x_h|y_h\rangle_{\mathcal{L}^2(\mathbb{R})}.$$

Here we denote x_h in place of $x(h)$ to simplify notation. In this setting we have for $x \in A_c^\lambda$:
(1) *Sharp:* $S(x)_h = \alpha_h(\overline{x_{h^{-1}}})$;
(2) *Flat:* $F(x)_h = |h|\alpha_h(\overline{x_{h^{-1}}})$;
(3) *Delta:* $\Delta(x)_h = |h|x_h$.

Proof. We refer to [Ta] for Takesaki's version of the axioms for a Tomita algebra. Since $Sharp$ is defined to be the adjoint operation on the algebra, item (1) is immediate. A straightforward calculation shows that for all $x, y \in A_c^\lambda$ we have that the defining equation for $Flat$ holds, namely:$\langle S(y)|x\rangle_{\bar{\psi}} = \langle F(x)|y\rangle_{\bar{\psi}}$ so that item (2) holds. By definition, $\Delta = FS$ and so a simple calculation shows that $\Delta(x)_h = |h|x_h$ and (3) holds. From this formula for Δ we see that for each $z \in \mathbb{C}$ we have $\Delta^z(x)_h = |h|^z x_h$ and a straightforward calculation shows that $\Delta^z(x \cdot y) = (\Delta^z(x)) \cdot (\Delta^z(y))$ so that each Δ^z is an algebra homomorphism of A_c^λ as required. That each left multiplication $\pi(x)$ is bounded when x is supported on a single group element is straightforward and the generalization to finitely supported elements is then trivial. The fact that it is a $*$-representation holds as it does for the GNS representation for any weight.

In order to see that products are dense we recall that we have local units. That is, for each positive integer N we have defined $E_N = \mathcal{X}_{[-N,N)} \cdot \delta_1$, and have noted that for each $y \in A_c^\lambda$ that satisfies $\mathrm{supp}(y_h) \subseteq [-N, N)$ for all h, we have $E_N \cdot y = y$. Axioms IV, V, VI in [Ta] are simple calculations involving the definitions of S, F, and Δ which we leave to the reader.

Since our inner products are linear in the second variable, we modify Tomita's Axiom VIII to read: $z \mapsto \langle x|\Delta^x(y)\rangle_{\bar{\psi}}$ is analytic on $\mathbb{C}$ for all $x, y \in A_c^\lambda$. We easily calculate that $\langle x|\Delta^z(y)\rangle_{\bar{\psi}} = \sum_h |h|^{z-1}\langle y_h|x_h\rangle_{\mathcal{L}^2(\mathbb{R})}$. This function is analytic since the sum is finite. □

Lemma 2.32. *The representation of A_c^λ on $\mathcal{L}^2(A_c^\lambda, \bar{\psi})$ decomposes as the integrated form of a covariant pair of representations:*

$$(1)\ \ \tilde{\pi} : C_{00}^\lambda(\mathbb{R}) \to \mathcal{B}(L^2(A_c^\lambda, \bar{\psi})),\ \text{ where : }\ (\tilde{\pi}(f)(y))_h = f \cdot y_h\ \text{ for }\ f \in C_{00}^\lambda(\mathbb{R})\ \text{ and }\ y \in A_c^\lambda;$$
$$(2)\ \ U : G_\lambda \to U(L^2(A_c^\lambda, \bar{\psi}))\ \text{ where : }\ (U_g(y))_h = \alpha_g(y_{g^{-1}h})\ \text{ for }\ g \in G_\lambda\ \text{ and }\ y \in A_c^\lambda.$$

Proof. It is straightforward to verify that U is a unitary representation of G_λ and that $\tilde{\pi}$ is a $*$-representation of $C_{00}^\lambda(\mathbb{R})$. To see the covariance condition:

$$(U_g\tilde{\pi}(f)U_{g^{-1}}(y))_h = \cdots = \alpha_g(f \cdot \alpha_{g^{-1}}(y_{gg^{-1}})) = \alpha_g(f) \cdot y_h = (\tilde{\pi}(\alpha_g(f))y)_h.$$

That is, $U_g\tilde{\pi}(f)U_{g^{-1}} = \tilde{\pi}(\alpha_g(f))$.

Now, by Proposition 7.6.4 of [Ped] the integrated form of this covariant pair is the representation:

$$(\tilde{\pi} \times U)(y) = \sum_h \tilde{\pi}(y_h)U_h\ \text{ for }\ y \in A_c^\lambda.$$

Now, we evaluate this operator on the vector $x \in A_c^\lambda$:

$$[((\tilde{\pi} \times U)(y))(x)]_k = \sum_h [\tilde{\pi}(y_h)U_h(x)]_k = \sum_h y_h\alpha_h(x_{h^{-1}k}) = (y \cdot x)(k) = (\pi(y)(x))_k.$$

That is, $(\tilde{\pi} \times U)(y) = \pi(y)$ the operator left multiplication by y. □

2.7.1. *A representation of A^λ on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$.* We define a covariant pair of representations of $C_0^\lambda(\mathbb{R})$ and G_λ on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ as follows:

$$(1)\text{ for }\ f \in C_0^\lambda(\mathbb{R})\ \text{ let }\ \overline{\pi}(f) = 1 \otimes M_f,\ \text{ and }\ (2)\text{ for }\ g \in G_\lambda\ \text{ let }\ \overline{U}_g = \Lambda(g) \otimes V_g$$

where Λ is the left regular representation of G_λ on $l^2(G_\lambda)$:

$$(\Lambda(g)\xi)(h) = \xi(g^{-1}h)\ \text{ for }\ \xi \in l^2(G_\lambda);$$

and V is the unitary action of G_λ on $\mathcal{L}^2(\mathbb{R})$ induced by the action of G_λ on $\mathbb{R}$:

$$(V_g(f))(t) = |g|^{-1/2}f(g^{-1}t);\ \text{ for }\ f \in \mathcal{L}^2(\mathbb{R}).$$

Using these equations one easily checks the covariance condition for $g \in G_\lambda$ and $f \in C_0^\lambda(\mathbb{R})$:

$$\overline{U}_g \overline{\pi}(f) \overline{U}_g^* = \overline{\pi}(\alpha_g(f)).$$

Clearly the representation $\overline{\pi}$ extends uniquely by weak-operator continuity to the usual representation $1 \otimes M$ of $\mathcal{L}^\infty(\mathbb{R})$ on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ and is covariant with the unitary representation $\overline{U}$ of G_λ for the action α of G_λ on $\mathcal{L}^\infty(\mathbb{R})$. Clearly, the von Neumann algebra on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ generated by the unitaries $\overline{U}_g$ and the operators $1 \otimes M_f$ for $g \in G_\lambda$ and $f \in C_{00}^\lambda(\mathbb{R})$, is the same as the von Neumann algebra generated by the unitaries $\overline{U}_g$ and the operators $1 \otimes M_f$ for $g \in G_\lambda$ and $f \in \mathcal{L}^\infty(\mathbb{R})$. The second item of the following Proposition is clear.

Proposition 2.33. *(1) The representation $\pi = (\tilde{\pi} \times U)$ of A^λ on $\mathcal{L}^2(A_c^\lambda, \bar{\psi})$ is unitarily equivalent to the representation $(\overline{\pi} \times \overline{U})$ of A^λ on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$.*
(2) $(\overline{\pi} \times \overline{U})(A^\lambda)''$ is the von Neumann crossed product (in the sense of the group-measure space construction of Chapter 1 Section 9 of [D]*) $G_\lambda \rtimes_\alpha \mathcal{L}^\infty(\mathbb{R})$.*
(3) This von Neumann algebra is a type III *factor.*

Proof. To see item (3) we use the proof of [D, Theorem 2, Section 9, Chapter 1] where instead of the $ax+b$ group G with $a, b \in \mathbb{Q}$ and $a > 0$ and its subgroup G_0 (with $a = 1$), we use G_λ and its subgroup G_λ^0 (with $|g| = 1$), to conclude that our von Neumann algebra is a type III factor.

To see item (1), we first define a unitary $W : \mathcal{L}^2(A_c^\lambda, \bar{\psi}) \to l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ as follows:

$$W\left(\sum_{i=1}^m f_i \cdot \delta_{h_i}\right) = \sum_{i=1}^m |h_i|^{-1/2} \delta_{h_i} \otimes f_i.$$

On the left side of this equation we are using the formalism $f \cdot \delta_h$ for singly supported elements in A_c^λ with $f \in C_{00}^\lambda(\mathbb{R})$ and $h \in G_\lambda$. On the right of this equation we are using δ_h to denote the canonical orthonormal basis elements in $l^2(G_\lambda)$ and regarding $f \in C_{00}^\lambda(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R})$. Clearly, W is well-defined and linear with dense range. One easily checks that: for all $x, y \in A_c^\lambda$ we have

$$\langle y|x\rangle_{\bar{\psi}} = \langle W(x)|W(y)\rangle_{l^2 \otimes \mathcal{L}^2}$$

recalling that the inner product on A_c^λ is linear in the second coordinate. Thus W is a unitary and its inverse (adjoint) defined at first on the elements in $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ of the form $\sum_{i=1}^m \delta_{h_i} \otimes f_i$ with the $f_i \in C_{00}^\lambda(\mathbb{R})$, is given by:

$$W^*\left(\sum_{i=1}^m \delta_{h_i} \otimes f_i\right) = \sum_{i=1}^m |h_i|^{1/2} f_i \cdot \delta_{h_i}.$$

One then verifies the following two equations for $f \in C_{00}^\lambda(\mathbb{R})$ and $g \in G_\lambda$:

$$(1)\ \ W\tilde{\pi}(f)W^* = 1 \otimes M_f = \overline{\pi}(f) \quad \text{and} \quad (2)\ \ WU_gW^* = \Lambda(g) \otimes V_g = \overline{U}_g.$$

The second equation is more subtle and requires the observation: $U_g(f \cdot \delta_h) = |g|^{1/2} V_g(f) \cdot \delta_{gh}$.
This completes the proof of the proposition. $\square$

2.7.2. *The factor $\pi(A^\lambda)''$ acting on $\mathcal{L}^2(A_c^\lambda, \bar{\psi})$ is type* III_λ. We work in the unitarily equivalent setting of $(\overline{\pi} \times \overline{U})(A^\lambda)''$ acting on $l^2(G_\lambda) \otimes \mathcal{L}^2(\mathbb{R})$ afforded by Proposition 2.33. Recall that the subgroup of G_λ of matrices of the form $[\lambda^n : 0]$ is isomorphic to $\mathbb{Z}$ and acts on the normal subgroup G_λ^0 by conjugacy, and so $G_\lambda = \mathbb{Z} \rtimes G_\lambda^0$ is a semidirect product and we can write a canonical right coset decomposition of G_λ :

$$G_\lambda = \bigcup_{n \in \mathbb{Z}} G_\lambda^0 \cdot [\lambda^n : 0].$$

This gives us an internal orthogonal decomposition of $l^2(G_\lambda)$:

$$l^2(G_\lambda) = \sum_{n\in\mathbb{Z}} \oplus\, l^2\left(G_\lambda^0 \cdot [\lambda^n : 0]\right) \cong l^2(\mathbb{Z}) \otimes l^2(G_\lambda^0).$$

Here the latter isomorphism is given explicitly on basis elements by the map which takes the δ-function at $g\cdot[\lambda^n : 0]$ to $\delta_n \otimes \delta_g$ for $n \in \mathbb{Z}$ and $g \in G_\lambda^0$.

One checks that the restriction of the representation $(\overline{\pi}\times\overline{U})$ of $A^\lambda = G_\lambda \rtimes C_0^\lambda(\mathbb{R})$ to $A_0^\lambda := G_\lambda^0 \rtimes C_0^\lambda(\mathbb{R})$ on $l^2(G_\lambda)\otimes\mathcal{L}^2(\mathbb{R})$ is unitarily equivalent to the representation on $l^2(\mathbb{Z})\otimes l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$ via the covariant pair:

$$1_\mathbb{Z}\otimes\Lambda(h)\otimes V_h = 1_\mathbb{Z}\otimes\overline{U}_h \;\text{ for }\; h\in G_\lambda^0 \;\text{ and}$$
$$1_\mathbb{Z}\otimes 1\otimes M_f = 1_\mathbb{Z}\otimes\overline{\pi}(f) \;\text{ for }\; f\in C_0^\lambda(\mathbb{R}).$$

Therefore, the von Neumann subalgebra of $(\overline{\pi}\times\overline{U})(A^\lambda)''$ generated by $(\overline{\pi}\times\overline{U})(A_0^\lambda)$ is isomorphic to the von Neumann algebra on $l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$ generated by the operators $\Lambda(h)\otimes V_h$ for $h\in G_\lambda^0$ and $1\otimes M_f$ for $f\in C_0^\lambda(\mathbb{R})$. This is clearly the same as the von Neumann algebra generated by the operators $\Lambda(h)\otimes V_h$ for $h\in G_\lambda^0$ and $1\otimes M_f$ for $f\in\mathcal{L}^\infty(\mathbb{R})$, and this von Neumann algebra is a factor of type II_∞ by the methods of [D, Chapter 1, Section 9]. Thus, $(\overline{\pi}\times\overline{U})(A_0^\lambda)''$ is a type II_∞ subfactor of the type III factor, $(\overline{\pi}\times\overline{U})(A^\lambda)''$. Moreover, the faithful normal semifinite trace on $(\overline{\pi}\times\overline{U})(A_0^\lambda)''$ is given by the restriction of $\bar{\psi}$.

Finally, conjugation by the unitary, $\overline{U}_g$ for $g=[\lambda : 0]$, which lies in our type III factor, defines an automorphism β of the type II_∞ subfactor which scales the trace by λ. If $\mathcal{N}_0$ is our type II_∞ factor acting on $l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$ then our type III factor, say $\mathcal{A}_\lambda$ acting on $l^2(\mathbb{Z})\otimes l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$ is unitarily equivalent to the von Neumann crossed product $\mathcal{A}_\lambda \cong \mathbb{Z}\rtimes_\beta\mathcal{N}_0$ and hence is a type III$_\lambda$ factor by [C1, Theorem 4.4.1]. We have proved the following Proposition.

Proposition 2.34. *The von Neumann algebra $\pi(A^\lambda)''$ acting on $\mathcal{L}^2(A_c^\lambda,\bar{\psi})$ is a type* III$_\lambda$ *factor. Moreover, it is unitarily equivalent to $(\overline{\pi}\times\overline{U})(A^\lambda)''$ acting on $l^2(G_\lambda)\otimes\mathcal{L}^2(\mathbb{R})$. The von Neumann subalgebra of $(\overline{\pi}\times\overline{U})(A^\lambda)''$ generated by $(\overline{\pi}\times\overline{U})(A_0^\lambda)$ is a type II_∞ factor. The space $l^2(G_\lambda)\otimes\mathcal{L}^2(\mathbb{R})$ factors as $l^2(\mathbb{Z})\otimes l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$ and with this factorization, our II_∞ factor has the form $\mathcal{N}_0 = 1_\mathbb{Z}\otimes\tilde{\mathcal{N}_0}$ where $\tilde{\mathcal{N}_0}$ acts on $l^2(G_\lambda^0)\otimes\mathcal{L}^2(\mathbb{R})$. Thus, our type* III$_\lambda$ *factor is unitarily equivalent to the von Neumann crossed product $\mathbb{Z}\rtimes_\beta\mathcal{N}_0$ where the automorphism β of $\mathcal{N}_0$ is given by $\beta = Ad(\overline{U}_g)$ where $g=[\lambda : 0]$.*

2.8. The von Neumann algebra, $\pi_0(\mathcal{Q}^\lambda)^{-wo}$ acting on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ is of type III$_\lambda$.

Theorem 2.35. *The von Neumann algebra, $\pi_0(\mathcal{Q}^\lambda)^{-wo}$ acting on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ is of type* III$_\lambda$. *Moreover, the von Neumann subalgebra, $\pi_0(F^\lambda)^{-wo}$ is a type II_1 factor with unique faithful normal state given by the restriction of the vector state, ψ which is the same as τ on F^λ. By the general theory of type* III *factors, $\pi_0(\mathcal{Q}^\lambda)^{-wo}$ is isomorphic to $\pi(A^\lambda)^{-wo}$ acting on $\mathcal{L}^2(A_c^\lambda,\bar{\psi})$.*

Proof. Recall that $\mathcal{Q}^\lambda = eA^\lambda e$ where $e=\mathcal{X}_{[0,1)}\cdot\delta_1\in A^\lambda$. Then

$$\pi(e)(\pi(A^\lambda)^{-wo})\pi(e) = (\pi(e)\pi(A^\lambda)\pi(e))^{-wo} = \pi(\mathcal{Q}^\lambda)^{-wo}$$

and the cut-down of the type III factor $\pi(A^\lambda)^{-wo}$ (on its separable Hilbert space) by the nonzero projection $\pi(e)$ is isomorphic to $\pi(A^\lambda)^{-wo}$ since $\pi(e)$ is Murray-von Neumann equivalent to the identity operator. Of course the **cut-down mapping** by $\pi(e)$ is **not** an isomorphism. Moreover, by left Hilbert algebra theory, the operator **right** multiplication by e which is denoted by $\pi'(e)$ is in the commutant of $\pi(A^\lambda)^{-wo}$ acting on $\mathcal{L}^2(\mathcal{Q}^\lambda,\bar{\psi})$ and since we are in a factor the **mapping** $\pi(A^\lambda)^{-wo}\to\pi'(e)\pi(A^\lambda)^{-wo}$ **is an isomorphism** by [D, Chapter 1, Section 2, Prop. 2]. Restricting this isomorphism to $\pi(\mathcal{Q}^\lambda)^{-wo}$ gives us an isomorphism $\pi(\mathcal{Q}^\lambda)^{-wo}\to\pi'(e)\pi(\mathcal{Q}^\lambda)^{-wo}$ which acts on the

Hilbert space $\pi'(e)\pi(e)(\mathcal{L}^2(A^\lambda,\bar{\psi}))$, which has as a dense subspace $\pi'(e)\pi(e)(A_c^\lambda) = eA_c^\lambda e \subset eA^\lambda e = \mathcal{Q}^\lambda$ with the inner product given by $\bar{\psi}$ which is the same as the inner product on $eA_c^\lambda e$ given by the state ψ. The completion of this space is, of course, $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ with the action of $\mathcal{Q}^\lambda$ being the GNS representation afforded by the state ψ. We denote this representation of $\mathcal{Q}^\lambda$ on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ by π_0 to distinguish it from the representation π of A^λ on the larger space, $\mathcal{L}^2(A_c^\lambda,\bar{\psi})$.

Similar considerations applied to the type II_∞ subfactor, $\pi(A_0^\lambda)^{-wo} \subset \pi(A^\lambda)^{-wo}$ on $\mathcal{L}^2(A_c^\lambda,\bar{\psi})$, show that:

$$\pi(e)(\pi(A_0^\lambda)^{-wo})\pi(e) = (\pi(e)\pi(A_0^\lambda)\pi(e))^{-wo} = \pi(F^\lambda)^{-wo}.$$

Now the projection $\pi(e)$ is actually in the type II_∞ subfactor $\pi(A_0^\lambda)^{-wo}$ of $\pi(A^\lambda)^{-wo}$ and has finite (ψ) trace $=1$ there. Therefore, $\pi(F^\lambda)^{-wo}$ is a type II_1 factor on $\mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$ with trace given by the vector state ψ. We remark that this is clearly a larger space than the subspace, $\mathcal{L}^2(F^\lambda,\tau) \subset \mathcal{L}^2(\mathcal{Q}^\lambda,\psi)$. ☐

Proposition 2.36. *The $*$-algebra $\mathcal{Q}_c^\lambda$ is a Tomita algebra with the inner product: $\langle y|x\rangle_\psi = \psi(y^*x)$. Again we denote x_h in place of $x(h)$ to simplify notation. In this setting we have for $x \in \mathcal{Q}_c^\lambda$:*
(1) *Sharp:* $S(x)_h = \alpha_h(\overline{x_{h^{-1}}})$;
(2) *Flat:* $F(x)_h = |h|\alpha_h(\overline{x_{h^{-1}}})$;
(3) *Delta:* $\Delta(x)_h = |h|x_h$.

Proof. This is really a corollary of Proposition 2.7, as $\mathcal{Q}_c^\lambda$ is just a Tomita-subalgebra of A_c^λ. ☐

3. The modular spectral triple of the algebra $\mathcal{Q}^\lambda$

Having introduced the main features of the algebras $\mathcal{Q}^\lambda$, we now turn briefly to the modular index theory of [CNNR, CPR2, CRT]. We begin with some semifinite preliminaries.

3.1. Semifinite noncommutative geometry.

We need to explain some semifinite versions of standard definitions and results following [CPRS2]. Let ϕ be a fixed faithful, normal, semifinite trace on a von Neumann algebra $\mathcal{N}$. Let $\mathcal{K}_\mathcal{N}$ be the ϕ-compact operators in $\mathcal{N}$ (that is the norm closed ideal generated by the projections $E \in \mathcal{N}$ with $\phi(E) < \infty$).

Definition 3.1. *A* **semifinite spectral triple** $(\mathcal{A},\mathcal{H},\mathcal{D})$ *is given by a Hilbert space $\mathcal{H}$, a $*$-algebra $\mathcal{A} \subset \mathcal{N}$ where $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$, and a densely defined unbounded self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that $[\mathcal{D},a]$ is densely defined and extends to a bounded operator in $\mathcal{N}$ for all $a \in \mathcal{A}$ and $(\lambda-\mathcal{D})^{-1} \in \mathcal{K}_\mathcal{N}$ for all $\lambda \notin \mathbb{R}$. The triple is said to be* **even** *if there is $\Gamma \in \mathcal{N}$ such that $\Gamma^* = \Gamma$, $\Gamma^2 = 1$, $a\Gamma = \Gamma a$ for all $a \in \mathcal{A}$ and $\mathcal{D}\Gamma + \Gamma\mathcal{D} = 0$. Otherwise it is* **odd**.

Note that if $T \in \mathcal{N}$ and $[\mathcal{D},T]$ is bounded, then $[\mathcal{D},T] \in \mathcal{N}$.

We recall from [FK] that if $S \in \mathcal{N}$, the **t-th generalized singular value** of S for each real $t > 0$ is given by

$$\mu_t(S) = \inf\{\|SE\| \ : \ E \text{ is a projection in } \mathcal{N} \text{ with } \phi(1-E) \leq t\}.$$

The ideal $\mathcal{L}^1(\mathcal{N},\phi)$ consists of those operators $T \in \mathcal{N}$ such that $\|T\|_1 := \phi(|T|) < \infty$ where $|T| = \sqrt{T^*T}$. In the Type I setting this is the usual trace class ideal. We will denote the norm on $\mathcal{L}^1(\mathcal{N},\phi)$ by $\|\cdot\|_1$. An alternative definition in terms of singular values is that $T \in \mathcal{L}^1(\mathcal{N},\phi)$ if $\|T\|_1 := \int_0^\infty \mu_t(T)dt < \infty$. When $\mathcal{N} \neq \mathcal{B}(\mathcal{H})$, $\mathcal{L}^1(\mathcal{N},\phi)$ need not be complete in this norm but it is complete in the norm $\|\cdot\|_1 + \|\cdot\|_\infty$. (where $\|\cdot\|_\infty$ is the uniform norm). We use the notation

$$\mathcal{L}^{(1,\infty)}(\mathcal{N},\phi) = \left\{T \in \mathcal{N} \ : \|T\|_{\mathcal{L}^{(1,\infty)}} := \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)ds < \infty\right\}.$$

The reader should note that $\mathcal{L}^{(1,\infty)}(\mathcal{N},\phi)$ is often taken to mean an ideal in the algebra $\widetilde{\mathcal{N}}$ of ϕ-measurable operators affiliated to $\mathcal{N}$. Our notation is however consistent with that of [?] in the special case $\mathcal{N} = \mathcal{B}(\mathcal{H})$. With this convention the ideal of ϕ-compact operators, $\mathcal{K}(\mathcal{N})$, consists of those $T \in \mathcal{N}$ (as opposed to $\widetilde{\mathcal{N}}$) such that $\mu_\infty(T) := \lim_{t\to\infty}\mu_t(T) = 0$.

Definition 3.2. *A semifinite spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ relative to $(\mathcal{N},\phi)$ with $\mathcal{A}$ unital is $(1,\infty)$-summable if $(\mathcal{D}-\lambda)^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\phi)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

It follows that if $(\mathcal{A},\mathcal{H},\mathcal{D})$ is $(1,\infty)$-summable then it is n-summable (with respect to the trace ϕ) for all $n > 1$. We next need to briefly discuss Dixmier traces. For more information on semifinite Dixmier traces, see [CPS2, CRSS]. For $T \in \mathcal{L}^{(1,\infty)}(\mathcal{N},\phi)$, $T \geq 0$, the function

$$F_T : t \to \frac{1}{\log(1+t)}\int_0^t \mu_s(T)ds$$

is bounded. There are certain $\omega \in L^\infty(\mathbb{R}_*^+)^*$, [CPS2, ?], which define (Dixmier) traces on $\mathcal{L}^{(1,\infty)}(\mathcal{N},\phi)$ by setting

$$\phi_\omega(T) = \omega(F_T), \quad T \geq 0$$

and extending to all of $\mathcal{L}^{(1,\infty)}(\mathcal{N},\phi)$ by linearity. For each such ω we write ϕ_ω for the associated Dixmier trace. Each Dixmier trace ϕ_ω vanishes on the ideal of trace class operators. Whenever the function F_T has a limit at infinity, all Dixmier traces return that limit as their value. This leads to the notion of a measurable operator [?, LSS], that is, one on which all Dixmier traces take the same value.

3.2. **The Kasparov module and modular spectral triple.** We have seen that the algebras $\mathcal{Q}^\lambda$ do not possess a faithful gauge invariant trace but that there is a KMS_β where $\beta = -\log(\lambda)$ for the gauge action, γ, namely $\psi := \tau\circ\Phi : \mathcal{Q}^\lambda \to \mathbb{C}$, where $\Phi : \mathcal{Q}^\lambda \to F^\lambda$ is the expectation and $\tau : F^\lambda \to \mathbb{C}$ is a faithful normalised trace. In fact, ψ is the only KMS state for the gauge action (for any β), by Proposition 2.30. We show below that the generator of the gauge action $\mathcal{D}$ acting on a suitable C^*-F^λ-module X gives us a Kasparov module $(X,\mathcal{D})$ whose class lies in $KK^{1,\mathbb{T}}(\mathcal{Q}^\lambda, F^\lambda)$. In some examples, including the case $\lambda \in \mathbb{Q}$, we have $K_1(\mathcal{Q}^\lambda) = \{0\}$ and so pairing with ordinary K_1 would be fruitless. However, following [CPR2, CNNR] we may compute a numerical pairing using a 'modular spectral triple' constructed from the Kasparov module.

We now review this construction adapted to the present situation. Let $\mathcal{H} = \mathcal{L}^2(\mathcal{Q}^\lambda)$ be the GNS Hilbert space given by the faithful state ψ with the inner product on $\mathcal{Q}^\lambda$ defined by $\langle a,b\rangle = \psi(a^*b) = (\tau\circ\Phi)(a^*b)$. Then $\mathcal{D}$ is a self-adjoint unbounded operator on $\mathcal{H}$, [CPR2]. The representation of $\mathcal{Q}^\lambda$ on $\mathcal{H}$ by left multiplication (which we now denote by π in place of π_0) is bounded and nondegenerate: the left action of an element $a \in \mathcal{Q}^\lambda$ by $\pi(a)$ satisfies $\pi(a)b = ab$ for all $b \in \mathcal{Q}^\lambda$. This distinction between elements of $\mathcal{Q}^\lambda$ as vectors in $\mathcal{L}^2(\mathcal{Q}^\lambda)$ and operators on $\mathcal{L}^2(\mathcal{Q}^\lambda)$ is sometimes crucial. The dense subalgebra $\mathcal{Q}_c^\lambda := eA_c^\lambda e$ which is the finite span of elements in $\mathcal{Q}^\lambda$ of the form $\mathcal{X}_{[a,b)}\cdot\delta_g$ is in the smooth domain of the derivation $\delta = \mathrm{ad}(|\mathcal{D}|)$. We remind the reader that the KMS condition on the modular automorphism group of the state ψ, [Ta], (for $t = i$) is: $\psi(xy) = \psi(\sigma_i(\pi(y))x) = \psi(\sigma(y)x)$ for $x,y \in \pi(\mathcal{Q}^\lambda)$, where $\sigma(y) = \Delta^{-1}(y)$.

Lemma 3.3. *The group of modular automorphisms of the von Neumann algebra $\pi(\mathcal{Q}^\lambda)''$ is given on the generators by*

$$\sigma_t(\pi(f\cdot\delta_g)) := \Delta^{it}\pi(f\cdot\delta_g)\Delta^{-it} = \pi(\Delta^{it}(f\cdot\delta_g)) = |g|^{it}\pi(f\cdot\delta_g) = \det(g)^{it}\pi(f\cdot\delta_g). \tag{2}$$

Proof. This is immediate from Lemma 2.8 if we note that $|g| = \det(g)$. $\square$

Corollary 3.4. *With $\mathcal{Q}^\lambda$ acting on $\mathcal{H} := \mathcal{L}^2(\mathcal{Q}^\lambda)$ and with $\mathcal{D}$ the generator of the natural unitary implementation of the gauge action of $\mathbb{T}^1$ on $\mathcal{Q}^\lambda$, we have $\Delta = \lambda^{\mathcal{D}}$* or *$e^{it\mathcal{D}} = \Delta^{it/\log\lambda}$.*

To simplify notation, we let $A = \mathcal{Q}^\lambda$ and $F = F^\lambda = A^\gamma$, the fixed point algebra for the $\mathbb{T}^1$ gauge action, γ. For convenience we will suppress the notations $\mathcal{D} \otimes 1_k$ and so on. The algebras A_c, F_c are defined as the finite linear span of the generators. Right multiplication makes A into a right F-module, and similarly A_c is a right module over F_c. We define an F-valued inner product $(\cdot|\cdot)_R$ on both these modules by $(a|b)_R := \Phi(a^*b)$.

Definition 3.5. *Let X be the right F C^*-module obtained by completing A (or A_c) in the norm*

$$\|x\|_X^2 := \|(x|x)_R\|_F = \|\Phi(x^*x)\|_F.$$

The algebra A acting by left multiplication on X provides a representation of A as adjointable operators on X. Let X_c be the copy of $A_c \subset X$. The $\mathbb{T}^1$ action on X_c is unitary and extends to X, [CNNR, PR]. For all $k \in \mathbb{Z}$, the projection operator onto the k-th spectral subspace of the $\mathbb{T}^1$ action is also denoted (somewhat carelessly) Φ_k on X:

$$\Phi_k(x) = \frac{1}{2\pi}\int_{\mathbb{T}^1} z^{-k} u_z(x)d\theta, \quad z = e^{i\theta}, \quad x \in X.$$

Observe that Φ_0 restricts to Φ on A and on generators of $\mathcal{Q}^\lambda$ we have

$$\Phi_k(f \cdot \delta_g) = \begin{cases} f \cdot \delta_g & \text{if } |g| = \lambda^k \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Of course $\mathcal{L}^2(\mathcal{Q}^\lambda)$ and X have a common dense subspace $\mathcal{Q}_c^\lambda$ on which these projections are identical. Let $A_k = \Phi_k(A)$ and observe from (3) that $\overline{A_k^* A_k} = F = \overline{A_k A_k^*}$ so that the gauge action γ on $\mathcal{Q}^\lambda$ has **full spectral subspaces**.

We quote the following result from [PR], the proof in our case is the same.

Lemma 3.6. *The operators Φ_k are adjointable endomorphisms of the F-module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k\Phi_l = \delta_{k,l}\Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k\in K}\Phi_k$ converges strictly to a projection in the endomorphism algebra. The sum $\sum_{k\in\mathbb{Z}}\Phi_k$ converges to the identity operator on X. For all $x \in X$, the sum $x = \sum_{k\in\mathbb{Z}}\Phi_k x = \sum_{k\in\mathbb{Z}} x_k$ converges in X.*

The unbounded operator of the next proposition is of course the generator of the $\mathbb{T}^1$ action on X. We refer to Lance's book, [L, Chapters 9,10], for information on unbounded operators on C^*-modules.

Proposition 3.7. [PR] *Let X be the right C^*-F-module of Definition 3.5. Define $\mathcal{D} : X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{k\in\mathbb{Z}} x_k \in X : \|\sum_{k\in\mathbb{Z}} k^2 (x_k|x_k)_R\| < \infty\}.$$

For $x \in X_{\mathcal{D}}$ define $\mathcal{D}(x) = \sum_{k\in\mathbb{Z}} kx_k$. Then $\mathcal{D} : X_{\mathcal{D}} \to X$ is a is self-adjoint, regular operator on X.

This should be compared to the following Hilbert space version.

Proposition 3.8. *The generator $\mathcal{D}$ of the one-parameter unitary group $\{u_z \mid z \in \mathbb{T}^1\}$ on $\mathcal{L}^2(\mathcal{Q}^\lambda, \psi)$ has eigenspaces given by the ranges of the Φ_k and $\mathcal{D}(x) = kx$ iff $\Phi_k(x) = x$. In particular*

$$\mathrm{dom}(\mathcal{D}) = \{x = \sum_k x_k \mid \Phi_k(x_k) = x_k \ \textit{and} \ \sum_k k^2\|x_k\|^2 < \infty\},$$

and $\mathcal{D}(\sum_k x_k) = \sum_k kx_k$.

Remark. On generators in $\mathcal{Q}^\lambda$ regarded as elements of either X or $L^2(\mathcal{Q}^\lambda,\psi)$ we have $\mathcal{D}(f\cdot\delta_g) = (\log_\lambda(|g|))f\cdot\delta_g$.

To continue, we recall the underlying right C^*-F^λ-module, X, which is the completion of $\mathcal{Q}^\lambda$ for the norm $\|x\|_X^2 = \|\Phi(x^*x)\|_{F^\lambda}$. Introduce the rank one operators on X : $\Theta^R_{x,y}$ by $\Theta^R_{x,y}z = x(y|z)_R$. Then using the operators $S_{k,m}$ defined above, we obtain formulas for the projections Φ_k similar to those of [PR, Lemma 4.7] with some important differences. First recall [CPR2, Lemma 3.5].

Lemma 3.9. *Any F^λ-linear endomorphism T of the module X which preserves the copy of $\mathcal{Q}^\lambda$ inside X, extends uniquely to a bounded operator on the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathcal{Q}^\lambda)$.*

In particular, the finite rank endomorphisms of the pre-C^* module $\mathcal{Q}^\lambda_c$ (acting on the left) satisfy this condition, and we denote the algebra of all these endomorphisms by $End^{00}_F(\mathcal{Q}^\lambda_c)$.

Lemma 3.10. *Compare* [PR, Lemma 4.7]. *The following formulas hold in both $\mathcal{L}(X)$ and in $\mathcal{B}(\mathcal{H})$.*
(1) *For $k\geq 0$, we have*

$$\Phi_0 = \Theta^R_{e,e} \ \text{ while for } \ k>0,\ \ \Phi_k = \sum_{m=0}^{m_k}\Theta^R_{S_{k,m},S_{k,m}}.$$

(2) *For $-k<0$, we have*

$$\Phi_{-k} = \Theta^R_{S^*_{k,m},S^*_{k,m}} \ \text{ for any } \ m=0,1,...,m_k-1 \ \text{ and also for } \ m_k \ \text{ if } \ \lambda^{-k} = m_k+1.$$

Proof. Since both Φ_k and the finite rank endomorphisms satisfy the hypotheses of the previous lemma, the first statement of this lemma will follow from calculations done on generators. The following calculations are based on the formulas in Lemma 2.15.
(1) Let $k>0$ and let $x=\sum_l x_l$ be a finite sum of generators, x_l satisfying $\Phi_l(x_l)=x_l$. Then

$$\begin{aligned}\sum_{m=0}^{m_k}\Theta^R_{S_{k,m},S_{k,m}}(x) &= \sum_l\sum_{m=0}^{m_k}\Theta^R_{S_{k,m},S_{k,m}}(x_l) = \sum_l\sum_{m=0}^{m_k}S_{k,m}\Phi(S^*_{k,m}x_l) = \sum_{m=0}^{m_k}S_{k,m}\Phi(S^*_{k,m}x_k)\\ &= \sum_{m=0}^{m_k}S_{k,m}S^*_{k,m}x_k = ex_k = x_k = \Phi_k(x).\end{aligned}$$

For $k=0$ this is a similar but far easier calculation.
(2) Let $-k<0$ and let $x=\sum_l x_l$ be a finite sum of generators as above. Then, for $0\leq m<m_k$

$$\begin{aligned}\Theta^R_{S^*_{k,m},S^*_{k,m}}(x) &= \sum_l\Theta^R_{S^*_{k,m},S^*_{k,m}}(x_l) = \sum_l S^*_{k,m}\Phi(S_{k,m}x_l) = S^*_{k,m}\Phi(S_{k,m}x_{-k})\\ &= S^*_{k,m}S_{k,m}x_{-k} = ex_{-k} = x_{-k} = \Phi_{-k}(x).\end{aligned}$$

□

We recall the following result discussed in Section 3 of [CNNR] (a 'bare hands' proof can be given by the method in [CPR2]).

Proposition 3.11. *Let $\mathcal{N}$ be the von Neumann algebra $\mathcal{N} = (End^{00}_F(\mathcal{Q}^\lambda_c))''$, where we take the commutant inside $\mathcal{B}(\mathcal{H})$. Then $\mathcal{N}$ is semifinite, and there exists a faithful, semifinite, normal trace $\tilde\tau:\mathcal{N}\to\mathbb{C}$ such that for all rank one endomorphisms $\Theta^R_{x,y}$ of $\mathcal{Q}^\lambda_c$,*

$$\tilde\tau(\Theta^R_{x,y}) = (\tau\circ\Phi)(y^*x),\quad x,y\in\mathcal{Q}^\lambda_c.$$

In addition, $\mathcal{D}$ is affiliated to $\mathcal{N}$ and $\pi(\mathcal{Q}^\lambda)$ is a subalgebra of $\mathcal{N}$.

The fact that $\tilde{\tau}(\Phi_k) = \lambda^{-k}$ implies that with respect to the trace $\tilde{\tau}$ we can not expect $\mathcal{D}$ to satisfy a finite summability criterion. We solve this problem exactly as in [CPR2].

Definition 3.12. *We define a new weight on $\mathcal{N}^+$: let $T \in \mathcal{N}^+$ then $\tau_\Delta(T) := \sup_N \tilde{\tau}(\Delta_N T)$ where $\Delta_N = \Delta(\sum_{|k|\leq N} \Phi_k)$.*

Remarks. Since Δ_N is $\tilde{\tau}$-trace-class, we see that $T \mapsto \tilde{\tau}(\Delta_N T)$ is a normal positive linear functional on $\mathcal{N}$ and hence τ_Δ is a normal weight on $\mathcal{N}^+$ which is easily seen to be faithful and semifinite.

As in [CPR2], we now give another way to define τ_Δ which is not only conceptually useful but also makes a number of important properties straightforward to verify. Many proofs require only trivial notation changes and the substitution of $n^{\pm}$ with $\lambda^{\mp}$.

Notation. Let $\mathcal{M}$ be the relative commutant in $\mathcal{N}$ of the operator Δ. Equivalently, $\mathcal{M}$ is the relative commutant of the set of spectral projections $\{\Phi_k | k \in \mathbb{Z}\}$ of $\mathcal{D}$. Clearly, $\mathcal{M} = \sum_{k\in\mathbb{Z}} \Phi_k \mathcal{N} \Phi_k$.

Definition 3.13. *As $\tilde{\tau}$ restricted to each $\Phi_k \mathcal{N} \Phi_k$ is a faithful finite trace with $\tilde{\tau}(\Phi_k) = \lambda^{-k}$ we define $\widehat{\tau}_k$ on $\Phi_k \mathcal{N} \Phi_k$ to be λ^k times the restriction of $\tilde{\tau}$. Then, $\widehat{\tau} := \sum_k \widehat{\tau}_k$ on $\mathcal{M} = \sum_{k\in\mathbb{Z}} \Phi_k \mathcal{N} \Phi_k$ is a faithful normal semifinite trace $\widehat{\tau}$ with $\widehat{\tau}(\Phi_k) = 1$ for all k.*

We use $\widehat{\tau}$ to give an alternative expression for τ_Δ below

Lemma 3.14. *An element $m \in \mathcal{N}$ is in $\mathcal{M}$ if and only if it is in the fixed point algebra of the action, $\sigma_t^{\tau_\Delta}$ on $\mathcal{N}$ defined for $T \in \mathcal{N}$ by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it} T \Delta^{-it}$. Both $\pi(F^\lambda)$ and the projections Φ_k belong to $\mathcal{M}$. The map $\Psi : \mathcal{N} \to \mathcal{M}$ defined by $\Psi(T) = \sum_k \Phi_k T \Phi_k$ is a conditional expectation onto $\mathcal{M}$ and $\tau_\Delta(T) = \widehat{\tau}(\Psi(T))$ for all $T \in \mathcal{N}^+$. That is, $\tau_\Delta = \widehat{\tau} \circ \Psi$ so that $\widehat{\tau}(T) = \tau_\Delta(T)$ for all $T \in \mathcal{M}^+$. Finally, if one of $A, B \in \mathcal{M}$ is $\widehat{\tau}$-trace-class and $T \in \mathcal{N}$ then $\tau_\Delta(ATB) = \tau_\Delta(A\Psi(T)B) = \widehat{\tau}(A\Psi(T)B)$.*

Proof. The proof is the same as the proof of [CPR2, Lemma 3.9] with λ^k in place of n^{-k}. □

Lemma 3.15. *The modular automorphism group $\sigma_t^{\tau_\Delta}$ of τ_Δ is inner and given by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it} T \Delta^{-it}$. The weight τ_Δ is a* KMS *weight for the group $\sigma_t^{\tau_\Delta}$, and $\sigma_t^{\tau_\Delta}|_{\mathcal{Q}^\lambda} = \sigma_t^{\tau\circ\Phi}$.*

Proof. This follows from: [KR, Thm 9.2.38], which gives us the KMS properties of τ_Δ: the modular group is inner since Δ is affiliated to $\mathcal{N}$. The final statement about the restriction of the modular group to $\mathcal{Q}^\lambda$ is clear. □

We now have the key lemma:

Lemma 3.16. *Suppose g is a function on $\mathbb{R}$ such that $g(\mathcal{D})$ is τ_Δ trace-class in $\mathcal{M}$, then for all $f \in F^\lambda$ we have*

$$\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(D))\tau(f) = \tau(f) \sum_{k\in\mathbb{Z}} g(k).$$

Proof. First note that $\tau_\Delta(g(\mathcal{D})) = \widehat{\tau}(\sum_{k\in\mathbb{Z}} g(k)\Phi_k) = \sum_{k\in\mathbb{Z}} g(k)\widehat{\tau}(\Phi_k) = \sum_{k\in\mathbb{Z}} g(k)$. We first do the computation for $f \in F_c^\lambda$ so that all the sums are finite. Now,

$$\tau_\Delta(\pi(f)g(\mathcal{D})) = \widehat{\tau}(\pi(f) \sum_{k\in\mathbb{Z}} g(k)\Phi_k) = \sum_{k\in\mathbb{Z}} g(k)\widehat{\tau}(\pi(f)\Phi_k)$$

$$= \sum_{k\in\mathbb{Z}} g(k)\widehat{\tau}_k(\pi(f)\Phi_k) = \sum_{k\in\mathbb{Z}} g(k)\lambda^k \tilde{\tau}(\pi(f)\Phi_k).$$

So it suffices to see for each $k \in \mathbb{Z}$, we have $\tilde{\tau}(\pi(f)\Phi_k) = \lambda^{-k}\tau(f)$.

Now, by Theorem 2.35 $\pi(F^\lambda)''$ is a type II_1 factor on $\mathcal{H}$ whose unique trace say Tr (with norm one) extends the trace τ on F^λ in the sense that $Tr(\pi(f)) = \tau(f)$. Since the projection Φ_k is in the commutant of the factor $\pi(F^\lambda)''$ the map

$$T \in \pi(F^\lambda)'' \mapsto T\Phi_k = \Phi_k T \Phi_k$$

is a normal isomorphism by [D, Chapter 1, section 2, Prop. 2] and so it has a unique normalised trace also given by $Trace(T\Phi_k) = Tr(T)$. But $\tilde{\tau}(T\Phi_k)$ is a trace on $\Phi_k\pi(F^\lambda)''\Phi_k = \pi(F^\lambda)''\Phi_k$ and so must be $\tilde{\tau}(\Phi_k) = \lambda^{-k}$ times the unique norm one trace. That is, we get the required formula:

$$\tilde{\tau}(\pi(f)\Phi_k) = \lambda^{-k}Trace(\pi(f)\Phi_k) = \lambda^{-k}Tr(\pi(f)) = \lambda^{-k}\tau(f).$$

So for $f \in F_c^\lambda$, we have the formula:

$$\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(D))\tau(f) = \sum_{k\in\mathbb{Z}} g(k)\tau(f).$$

Now, the right hand side is a norm-continuous function of f. To see that the left side is norm-continuous we do it in more generality. Let $T \in \mathcal{N}$, then since $\widehat{\tau}$ is a trace on $\mathcal{M}$ we get:

$$|\tau_\Delta(Tg(\mathcal{D}))| = |\widehat{\tau}(\Psi(Tg(\mathcal{D}))| = |\widehat{\tau}(\Psi(T)g(\mathcal{D}))| \le \|\Psi(T)\|\widehat{\tau}(|g(\mathcal{D})|) \le \|T\|\widehat{\tau}((|g(\mathcal{D})|) = \|T\|\tau_\Delta(|g(\mathcal{D})|).$$

That is the left hand side is norm-continuous in T and so we have the formula:

$$\tau_\Delta(\pi(f)g(\mathcal{D})) = \tau_\Delta(g(\mathcal{D}))\tau(f) = \sum_{k\in\mathbb{Z}} g(k)\tau(f)$$

for all $f \in F^\lambda$. □

Proposition 3.17. *(i) We have* $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \tau_\Delta)$*. That is,* $\tau_\Delta((1+\mathcal{D}^2)^{-s/2}) < \infty$ *for all* $s > 1$*. Moreover, for all* $f \in F^\lambda$

$$\lim_{s\to 1^+}(s-1)\tau_\Delta(\pi(f)(1+\mathcal{D}^2)^{-s/2}) = 2\tau(f)$$

so that $\pi(f)(1+\mathcal{D}^2)^{-1/2}$ *is a measurable operator in the sense of* [**?**].

(ii) For $\pi(a) \in \pi(\mathcal{Q}^\lambda) \subset \mathcal{N}$ *the following (ordinary) limit exists and*

$$\widehat{\tau}_\omega(\pi(a)) = \frac{1}{2}\lim_{s\to 1^+}(s-1)\tau_\Delta(\pi(a)(1+\mathcal{D}^2)^{-s/2}) = \tau \circ \Phi(a),$$

the original KMS *state* $\psi = \tau \circ \Phi$ *on* $\mathcal{Q}^\lambda$*.*

Proof. (i) This proof is identical to [CPR2, Proposition 3.12].
(ii) This proof is the same as [CPR2, Proposition 3.14] with $\mathcal{Q}^\lambda, F^\lambda$ replacing O_n, F. □

Definition 3.18. *The triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *along with* γ, ψ, $\mathcal{N}$, τ_Δ *satisfying properties (0) to (3) below is called the* **modular spectral triple** *of the dynamical system* $(\mathcal{Q}^\lambda, \gamma, \psi)$
0) The $*$*-subalgebra* $\mathcal{A} = \mathcal{Q}_c^\lambda$ *of the algebra* $\mathcal{Q}^\lambda$ *is faithfully represented in* $\mathcal{N}$ *with the latter acting on the Hilbert space* $\mathcal{H} = \mathcal{L}^2(\mathcal{Q}^\lambda, \psi)$*,*
1) there is a faithful normal semifinite weight τ_Δ *on* $\mathcal{N}$ *such that the modular automorphism group of* τ_Δ *is an inner automorphism group* σ_t *(for* $t \in \mathbb{C}$*) of (the Tomita algebra of)* $\mathcal{N}$ *with* $\sigma_i|_{\mathcal{A}} = \sigma$ *in the sense that* $\sigma_i(\pi(a)) = \pi(\sigma(a))$*, where* σ *is the automorphism* $\sigma(a) = \Delta^{-1}(a)$ *on* $\mathcal{A}$*,*
2) τ_Δ *restricts to a faithful semifinite trace* $\widehat{\tau}$ *on* $\mathcal{M} = \mathcal{N}^\sigma$*, with a faithful normal projection* $\Psi : \mathcal{N} \to \mathcal{M}$ *satisfying* $\tau_\Delta = \widehat{\tau} \circ \Psi$ *on* $\mathcal{N}$*,*
3) with $\mathcal{D}$ *the generator of the one parameter group implementing the gauge action of* $\mathbb{T}$ *on* $\mathcal{H}$ *we have:* $[\mathcal{D}, \pi(a)]$ *extends to a bounded operator (in* $\mathcal{N}$*) for all* $a \in \mathcal{A}$ *and for* λ *in the resolvent set of* $\mathcal{D}$*,* $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{M}, \tau_\Delta)$*, where* $\mathcal{K}(\mathcal{M}, \tau_\Delta)$ *is the ideal of compact operators in* $\mathcal{M}$ *relative to* τ_Δ*. In particular,* $\mathcal{D}$ *is affiliated to* $\mathcal{M}$*.*

For matrix algebras $\mathcal{A} = \mathcal{Q}_c^\lambda \otimes M_k$ over $\mathcal{Q}_c^\lambda$, $(\mathcal{Q}_c^\lambda \otimes M_k, \mathcal{H} \otimes M_k, \mathcal{D} \otimes Id_k)$ is also a modular spectral triple in the obvious fashion.

We need some technical lemmas for the discussion in the next Section. A function f from a complex domain Ω into a Banach space X is called **holomorphic** if it is complex differentiable in norm on Ω. The following is proved in [CPR2, Lemma 3.15].

Lemma 3.19. *(1) Let $\mathcal{B}$ be a C^*-algebra and let $T \in \mathcal{B}^+$. The mapping $z \mapsto T^z$ is holomorphic (in operator norm) in the half-plane $Re(z) > 0$.*
(2) Let $\mathcal{B}$ be a von Neumann algebra with faithful normal semifinite trace ϕ and let $T \in \mathcal{B}^+$ be in $\mathcal{L}^{(1,\infty)}(\mathcal{B}, \phi)$. Then, the mapping $z \mapsto T^z$ is holomorphic (in trace norm) in the half-plane $Re(z) > 1$.
(3) Let $\mathcal{B}$, and T be as in item (2) and let $A \in \mathcal{B}$ then the mapping $z \mapsto \phi(AT^z)$ is holomorphic for $Re(z) > 1$.

Lemma 3.20. *In these modular spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for matrices over the algebras $\mathcal{Q}^\lambda$ we have $(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{M}, \tau_\Delta)$ for all $s > 1$ and for $x \in \mathcal{N}$, $\tau_\Delta(x(1+\mathcal{D}^2)^{-r/2})$ is holomorphic for $Re(r) > 1$ and we have for $a \in \mathcal{Q}_c^\lambda$, $\tau_\Delta([\mathcal{D}, \pi(a)](1+\mathcal{D}^2)^{-r/2}) = 0$, for $Re(r) > 1$.*

Proof. We include a brief proof since there are some small but important differences from [CPR2, Lemma 3.16]. Since the eigenvalues for $\mathcal{D}$ are precisely the set of integers, and the projection Φ_k on the eigenspace with eigenvalue k satisfies $\tau_\Delta(\Phi_k) = 1$, it is clear that $(1+\mathcal{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{M}, \tau_\Delta)$. Now, $\tau_\Delta(x(1+\mathcal{D}^2)^{-r/2}) = \widehat{\tau}(\Psi(x)(1+\mathcal{D}^2)^{-r/2})$ is holomorphic for $Re(r) > 1$ by item (3) of the previous lemma.

To see the last statement, we observe that $\tau_\Delta([\mathcal{D}, \pi(a)](1+\mathcal{D}^2)^{-r/2}) = \tau_\Delta(\Psi([\mathcal{D}, \pi(a)])(1+\mathcal{D}^2)^{-r/2})$, so it suffices to see that $\Psi([\mathcal{D}, \pi(a)]) = 0$ for $a \in \mathcal{A} = \mathcal{Q}_c^\lambda$. To this end, let $a = f \cdot \delta_g$ where $det(g) = \lambda^n$ is one of the linear generators of $\mathcal{Q}_c^\lambda$. Then by calculating the action of the operator $\mathcal{D}\pi(f \cdot \delta_g)$ on the linear generators $f_i \cdot \delta_{h_i}$ of the Hilbert space, $\mathcal{H}$, we obtain:

$$\mathcal{D}\pi(f \cdot \delta_g) = n\pi(f \cdot \delta_g) + \pi(f \cdot \delta_g)\mathcal{D} \quad \text{that is} \quad [\mathcal{D}, \pi(f \cdot \delta_g)] = \log_\lambda(|g|)\pi(f \cdot \delta_g).$$

More generally,

$$[\mathcal{D}, \pi(\sum_{i=1}^m c_i f_i \cdot \delta_{h_i})] = \sum_{i=1}^m c_i(\log_\lambda(|h_i|))\pi(f_i \cdot \delta_{h_i}).$$

If we apply Ψ to this equation, we see that $\Psi(\pi(f_i \cdot \delta_{h_i})) = \pi(\Phi(f_i \cdot \delta_{h_i})) = 0$ whenever $\log_\lambda(|h_i|) \neq 0$, and so the whole sum is 0. We also observe that $[\mathcal{D}, \pi(a)] \in \pi(\mathcal{Q}_c^\lambda)$ for all $a \in \mathcal{Q}_c^\lambda$. This is not too surprising since $\mathcal{D}$ is the generator of the action γ of $\mathbb{T}$ on $\mathcal{Q}^\lambda$. $\square$

3.3. **Modular K_1.** We now make appropriate modifications to [CPR2, Section 4]) using [CNNR] introducing elements of these modular spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (where $\mathcal{A}$ is a matrix algebra over $\mathcal{Q}_c^\lambda$) that will have a well defined pairing with our Dixmier functional $\widehat{\tau}_\omega$. Let $A = \mathcal{Q}^\lambda$. Following [HR] we say that a unitary (invertible, projection,...) in the $n \times n$ matrices over $\mathcal{Q}^\lambda$ for some n is a unitary (invertible, projection,...) over $\mathcal{Q}^\lambda$. We write σ_t for the automorphism $\sigma_t \otimes Id_n$ of $\mathcal{A}$.

Definition 3.21. *Let v be a partial isometry in the $*$-algebra $\mathcal{A}$. We say that v satisfies the* **modular condition** *with respect to σ if the operators $v\sigma_t(v^*)$ and $v^*\sigma_t(v)$ are in the fixed point algebra $F \subset \mathcal{A}$ for all $t \in \mathbb{R}$. Of course, any partial isometry in F is a modular partial isometry.*

Lemma 3.22. *(*[CPR2, Lemma 4.8]*) Let $v \in \mathcal{A}$ be a modular partial isometry. Then we have*

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

is a modular unitary over $\mathcal{A}$. Moreover there is a modular homotopy $u_v \sim u_{v^*}$.

Note that in [CPR2] we used a different approach which is implied by the one given here. In [CPR2] we defined modular unitaries in terms of the regular automorphism:

$$\pi(\sigma(a)) = \pi(\Delta^{-1}(a)) = \Delta^{-1}\pi(a)\Delta = \sigma_i(\pi(a)).$$

That is we said that a unitary in $\mathcal{A}$ is modular if $u\sigma(u^*)$ and $u^*\sigma(u)$ are in the fixed point algebra.

Examples.
(1) For $k, j > 0$ recall $S_{k,m} \in \mathcal{Q}_c^\lambda$ with $m < m_k$ (see Definition 2.14) we write $P_{k,m} = S_{k,m}S_{k,m}^* = \mathcal{X}_{[m\lambda^k,(m+1)\lambda^k)} \cdot \delta_1$ which is in clearly F^λ. Then for each $\{k, m\}, \{j, n\}$ we have a unitary

$$u_{\{k,m\},\{j,n\}} = \begin{pmatrix} 1 - P_{k,m} & S_{k,m}S_{j,n}^* \\ S_{j,n}S_{k,m}^* & 1 - P_{j,n} \end{pmatrix}.$$

It is simple to check that this a self-adjoint unitary satisfying the modular condition, and that $\tau(P_{k,m}) = \lambda^k$ and $\tau(P_{j,n}) = \lambda^j$. These examples behave very much like the $S_\mu S_\nu^*$ examples of [CPR2].

(2) For $k, j > 0$ consider the "leftover" partial isometries S_{k,m_k} and S_{j,m_j} of Definition 3.13 which we will denote by S_k and S_j to lighten the notation. We let $v_{j,k} = S_jS_k^*$ and calculate its range and initial projections which are both in F^λ:

$$P_j = S_jS_k^*S_kS_j^* = \mathcal{X}_{[m_j\lambda^j, m_j\lambda^j+\lambda^j(\lambda^{-k}-m_k))} \cdot \delta_1, \text{ and}$$

$$P_k = S_kS_j^*S_jS_k^* = \mathcal{X}_{[m_k\lambda^k, m_k\lambda^k+\lambda^k(\lambda^{-j}-m_j))} \cdot \delta_1.$$

We note for future reference that:

$$\tau(P_j) = \lambda^j(\lambda^{-k} - m_k) \text{ and } \tau(P_k) = \lambda^k(\lambda^{-j} - m_j).$$

We also note that we have a modular unitary $u_{j,k}$:

$$u_{j,k} = \begin{pmatrix} 1 - P_k & S_kS_j^* \\ S_jS_k^* & 1 - P_j \end{pmatrix}.$$

Define the modular K_1 group as follows.

Definition 3.23. *Let $K_1(A, \sigma)$ be the abelian group with one generator $[v]$ for each partial isometry v over A satisfying the modular condition and with the following relations:*

1) $[v] = 0$ *if v is over F,*
2) $[v] + [w] = [v \oplus w]$,
3) *if v_t, $t \in [0, 1]$, is a continuous path of modular partial isometries in some matrix algebra over A then $[v_0] = [v_1]$.*

One could use modular unitaries as in [CPR2] in place of these modular partial isometries.

The following can now be seen to hold.

Lemma 3.24. *(Compare* [CPR2, Lemma 4.9]*) Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be our modular spectral triple relative to $(\mathcal{N}, \tau_\Delta)$ and set $F = \mathcal{A}^\sigma$ and $\sigma : \mathcal{A} \to \mathcal{A}$. Let $L^\infty(\Delta) = L^\infty(\mathcal{D})$ be the von Neumann algebra generated by the spectral projections of Δ then $L^\infty(\Delta) \subset \mathcal{Z}(\mathcal{M})$. Let $v \in \mathcal{A}$ be a partial isometry with vv^*, $v^*v \in F$. Then $\pi(v)Q\pi(v^*) \in \mathcal{M}$ and $\pi(v^*)Q\pi(v) \in \mathcal{M}$ for all spectral projections Q of $\mathcal{D}$, if and only if v is modular. That is, $\pi(v)\Delta\pi(v^*)$ and $\pi(v^*)\Delta\pi(v)$ (or $\pi(v)\mathcal{D}\pi(v^*)$ and $\pi(v^*)\mathcal{D}\pi(v)$) are both affiliated to $\mathcal{M}$ if and only if v is modular.*

Thus we see that modular partial isometries conjugate Δ to an operator affiliated to $\mathcal{M}$, and so $v\Delta v^*$ commutes with Δ (and $v\mathcal{D}v^*$ commutes with $\mathcal{D}$).

We will next show that there is an analytic pairing between (part of) modular K_1 and modular spectral triples. To do this, we are going to use the analytic formulae for spectral flow in [CP2].

3.4. **The mapping cone algebra.** Our aim in the remainder is to calculate an index pairing explicitly for the matrix algebras $\mathcal{A}$ over the smooth subalgebra $\mathcal{Q}_c^\lambda$ of $\mathcal{Q}^\lambda$. In the following few pages we will sometimes abuse notation and write a in place of $\pi(a)$ for $a \in \mathcal{A}$ in order to make our formulae more readable. Whenever we do this, however, we will use $\sigma_i(\cdot) = \Delta^{-1}(\cdot)\Delta$ the spatial version of the algebra homomorphism, σ. We will generally use the spatial version σ_i when in the presence of operators not in $\pi(\mathcal{A})$.

We briefly review some results from [CNNR], that provide an interpretation of the modular index pairing given by the spectral flow.

If $F \subset A$ is a sub-C^*-algebra of the C^*-algebra A, then the mapping cone algebra for the inclusion is

$$M(F,A) = \{f : \mathbb{R}_+ = [0,\infty) \to A : f \text{ is continuous and vanishes at infinity, } f(0) \in F\}.$$

When F is an ideal in A it is known that $K_0(M(F,A)) \cong K_0(A/F)$, [Put1]. In general, $K_0(M(F,A))$ is the set of homotopy classes of partial isometries $v \in M_k(A)$ with range and source projections vv^*, v^*v in $M_k(F)$, with operation the direct sum and inverse $-[v] = [v^*]$. All this is proved in [Put1].

It is shown in [CNNR] that there is a natural map that injects $K_1(A,\sigma)$ into $K_0^{\mathbb{T}}(M,F)$, the equivariant K-theory of the mapping cone algebra. Note that the $\mathbb{T}$ action on A lifts in the obvious way to the mapping cone. Now, it was shown in [CPR1] that certain Kasparov A,F-modules extend to Kasparov $M(F,A),F$-modules, and this was extended to the equivariant case in [CNNR]. Importantly the theory applies to the equivariant Kasparov module coming from a circle action. The extension is explicit, namely there is a pair $(\hat{X},\hat{\mathcal{D}})$ which is a graded unbounded Kasparov module for the mapping cone algebra $M(F,A)$ constructed using a generalised APS construction, [APS3].

If v is a partial isometry in $M_k(\mathcal{A})$, setting

$$e_v(t) = \begin{pmatrix} 1 - \frac{vv^*}{1+t^2} & -iv\frac{t}{1+t^2} \\ iv^*\frac{t}{1+t^2} & \frac{v^*v}{1+t^2} \end{pmatrix},$$

defines e_v as a projection over $M(F,A)$.

Then in [CNNR] we showed that if $v \in \mathcal{A}$ is a modular partial isometry we have

$$\begin{aligned} \langle [e_v] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right], [(\hat{X},\hat{\mathcal{D}})] \rangle &= -\mathrm{Index}(PvP : v^*vP(X) \to vv^*P(X)) \in K_0(F) \\ &= \mathrm{Index}(Pv^*P : vv^*P(X) \to v^*vP(X)) \in K_0^{\mathbb{T}}(F). \end{aligned} \tag{4}$$

We thus obtain an index map $K_1(A,\sigma) \to K_0^{\mathbb{T}}(F)$. The latter may be thought of as the ring of Laurent polynomials $K_0(F)(\chi,\chi^{-1})$ where we think of χ,χ^{-1} as generating the representation ring of $\mathbb{T}$. We may obtain a real valued invariant from this map by evaluating χ at $e^{-\beta}$ where β is the inverse temperature of our KMS state and applying the trace to the resultant element of $K_0(F)$. Then one of the main results of [CNNR] is that the real valued invariant so obtained is identical with the spectral flow invariant of the next subsection. However the general theory of [CNNR] does not tell us the range of this index map and it is the latter that is of interest for these explicit calculations.

3.5. **A local index formula for the algebras $\mathcal{Q}^\lambda$.** Using the fact that we have full spectral subspaces we know from [CNNR] that there is a formula for spectral flow which is analogous to the local index formula in noncommutative geometry. We remind the reader that $\tau_\Delta = \widehat{\tau} \circ \Psi$ where $\Psi : \mathcal{N} \to \mathcal{M}$ is the canonical expectation, so that τ_Δ **restricted to** $\mathcal{M}$ is $\widehat{\tau}$.

Theorem 3.25. *(Compare* [CPR2, Theorem 5.5]*) Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be the $(1,\infty)$-summable, modular spectral triple for the algebra $\mathcal{Q}^\lambda$ we have constructed previously. Then for any modular partial isometry v and for any Dixmier trace $\widehat{\tau}_{\tilde{\omega}}$ associated to $\widehat{\tau}$, we have spectral flow as an actual limit*

$$sf_{\widehat{\tau}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \frac{1}{2}\lim_{s\to 1+}(s-1)\widehat{\tau}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-s/2}) = \frac{1}{2}\widehat{\tau}_{\tilde{\omega}}(v[\mathcal{D}, v^*](1+\mathcal{D}^2)^{-1/2}) = \tau\circ\Phi(v[\mathcal{D}, v^*]).$$

The functional on $\mathcal{A}\otimes\mathcal{A}$ defined by $a_0\otimes a_1 \mapsto \frac{1}{2}\lim_{s\to 1+}(s-1)\tau_\Delta(a_0[\mathcal{D}, a_1](1+\mathcal{D}^2)^{-s/2})$ is a σ-twisted b, B-cocycle (see the proof below for the definition).

Remark. Spectral flow in this setting is independent of the path joining the endpoints of unbounded self adjoint operators affiliated to $\mathcal{M}$ however it is not obvious that this is enough to show that it is constant on homotopy classes of modular unitaries. This latter fact is true but the proof is lengthy so we refer to [CNNR].

Theorem 3.26. *We let $(\mathcal{Q}_c^\lambda \otimes M_2, \mathcal{H}\otimes\mathbb{C}^2, \mathcal{D}\otimes 1_2)$ be the modular spectral triple of $(\mathcal{Q}_c^\lambda\otimes M_2)$.*
(1) *Let u be a modular unitary defined in Section 5 of the form*

$$u_{\{k,m\},\{j,n\}} = \begin{pmatrix} 1-P_{k,m} & S_{k,m}S_{j,n}^* \\ S_{j,n}S_{k,m}^* & 1-P_{j,n}\end{pmatrix}.$$

Then the spectral flow is positive being given by

$$sf_{\tau_\Delta}(\mathcal{D}, u\mathcal{D}u^*) \quad = \quad (k-j)(\lambda^j - \lambda^k) \in \mathbb{Z}[\lambda] \subset \Gamma_\lambda.$$

(2) *Let u be a modular unitary defined in Section 5 of the form:*

$$u_{j,k} = \begin{pmatrix} 1-P_k & S_kS_j^* \\ S_jS_k^* & 1-P_j\end{pmatrix},$$

where $S_kS_j^* = S_{k,m_k}S_{j,m_j}^*$ and P_k and P_j are its range and initial projections, respectively. Then the spectral flow is given by

$$sf_{\tau_\Delta}(\mathcal{D}, u\mathcal{D}u^*) \quad = \quad (k-j)[\lambda^j(\lambda^{-k}-m_k) - \lambda^k(\lambda^{-j}-m_j)] \in \Gamma_\lambda.$$

Proof. We have already observed that these are, in fact modular unitaries. For the computations we use a calculation from the proof of Lemma 3.20 to get in example (1):

$$u[\mathcal{D}\otimes 1_2, u] = \begin{pmatrix} 1-P_{k,m} & S_{k,m}S_{j,n}^* \\ S_{j,n}S_{k,m}^* & 1-P_{j,n}\end{pmatrix}\begin{pmatrix} 0 & [\mathcal{D}, S_{k,m}S_{j,n}^*] \\ [\mathcal{D}, S_{j,n}S_{k,m}^*] & 0\end{pmatrix}$$

$$= \begin{pmatrix} 1-P_{k,m} & S_{k,m}S_{j,n}^* \\ S_{j,n}S_{k,m}^* & 1-P_{j,n}\end{pmatrix}\begin{pmatrix} 0 & (k-j)S_{k,m}S_{j,n}^* \\ (j-k)S_{j,n}S_{k,m}^* & 0\end{pmatrix} = (k-j)\begin{pmatrix} -P_{k,m} & 0 \\ 0 & P_{j,n}\end{pmatrix}.$$

So using Theorem 3.25 and our previous computation of the Dixmier trace, Proposition 3.17, and the fact that $P_{k,m} = S_{k,m}S_{k,m}^* = \mathcal{X}_{[m\lambda^k,(m+1)\lambda^k)}\cdot\delta_1$ so that $\tau(P_{k,m}) = \lambda^k$ we have

$$sf_{\tau_\Delta}(\mathcal{D}, u_{k,m}\mathcal{D}u_{k,m}) = (k-j)\tau(P_{j,n}-P_{k,m}) = (k-j)(\lambda^j-\lambda^k).$$

This number is always positive as the reader may check, and is contained in $\mathbb{Z}[\lambda]$.

The computations in example (2) are similar and use the fact that $P_k = \mathcal{X}_{[m_k\lambda^k, m_k\lambda^k+\lambda^k(\lambda^{-j}-m_j))}\cdot\delta_1$, so that $\tau(P_k) = \lambda^k(\lambda^{-j}-m_j) \in \Gamma_\lambda$. In these examples, the spectral flow is **not** contained in the smaller polynomial ring, $\mathbb{Z}[\lambda]$. $\square$

Remarks. The observation of [CPR2] that the twisted residue cocycle formula for spectral flow is calculating Araki's relative entropy of two KMS states [Ar] also applies to the examples in this subsection.

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E-mail address: `acarey@maths.anu.edu.au`, `phillips@math.uvic.ca`, `putnam@math.uvic.ca`, `rennie@maths.anu.edu.au`