Type III KMS States on a Class of C^*-Algebras containing O_n and $\mathcal{Q}_{\mathbb{N}}$ and Their Modular Index

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We construct a family of purely infinite, simple, separable, nuclear C^*-algebras , $\mathcal{Q}^\lambda$ for $\lambda \in (0,1)$. These algebras are also in the class $\mathfrak{N}_{nuc}$ and therefore by results of E. Kirchberg and N. C. Phillips they are classified by their K-groups. There is an action of the circle $\mathbb{T}$ with a unique KMS state ψ on each $\mathcal{Q}^\lambda$. For $\lambda = 1/n$, $\mathcal{Q}^{1/n} \cong O_n$, with its usual $\mathbb{T}$ action and KMS state. For $\lambda = p/q$, rational in lowest terms, $\mathcal{Q}^\lambda \cong O_n$ $(n = q-p+1)$ with UHF fixed point algebra of type $(pq)^\infty$. For any $n > 0$, $\mathcal{Q}^\lambda \cong O_n$ for infinitely many λ with distinct KMS states and UHF fixed-point algebras. However, none of the $\mathcal{Q}^\lambda$ are isomorphic to O_∞. For λ irrational the fixed point algebras, are NOT AF and the $\mathcal{Q}^\lambda$ are usually NOT Cuntz algebras. For λ transcendental, $K_1(\mathcal{Q}^\lambda) \cong K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}^\infty$, so that $\mathcal{Q}^\lambda$ is Cuntz' $\mathcal{Q}_{\mathbb{N}}$, [Cu1]. If λ is algebraic (and not rational), then $K_1(\mathcal{Q}^\lambda)$ and $K_0(\mathcal{Q}^\lambda)$ have the same finite rank: $K_1 \otimes \mathbb{Q} \cong K_0 \otimes \mathbb{Q} \cong \mathbb{Q}^k$. If λ and λ^{-1} are both algebraic integers, the **only** O_n which appear are those for which $n \equiv 3 (mod\ 4)$. For each λ, the representation of $\mathcal{Q}^\lambda$ defined by the KMS state ψ generates a type III_λ factor. These algebras fit into the framework of modular index theory / twisted cyclic theory of [CPR2, CRT] and [CNNR].

These new examples were motivated by the 'modular index theory' of [CPR2, CNNR]. We were aiming to find examples of algebras that were not Cuntz-Krieger algebras (or the CAR algebra) and were not previously known in order to explore the possibilities opened by [CNNR]. These algebras, denoted by $\mathcal{Q}^\lambda$ for $0 < \lambda < 1$, are constructed as "corner algebras" of certain crossed product C^*-algebras.

This note is an expanded version of a talk given at the Fields Institute in 2008: complete explanations and proofs will appear elsewhere. I would like to thank Masoud Khalkhali for organizing this conference on Noncommutative Geometry and inviting me to speak. It was a great conference.

1. The Construction of the O^λ

Definition 1.1. *For $0 < \lambda < 1$, let Γ_λ be the countable additive abelian subgroup of $\mathbb{R}$ defined by:*

$$\Gamma_\lambda = \left\{ \sum_{k=-N}^{k=N} n_k \lambda^k \;\middle|\; N \geq 0 \text{ and } n_k \in \mathbb{Z} \right\}.$$

Loosely speaking, Γ_λ consists of Laurent polynomials in λ and λ^{-1} with integer coefficients. It is not only a dense subgroup of $\mathbb{R}$, but is clearly a unital **subring** *of $\mathbb{R}$. To get some idea of the structure of these groups, we prove:*

Proposition 1.2. *Let $0 < \lambda < 1$.*
(1) *If $\lambda = p/q$ where $0 < p < q$ are integers in lowest terms, then $\Gamma_\lambda = \mathbb{Z}[1/n]$, where $n = pq$.*
(2) *If λ and λ^{-1} are both algebraic integers, then $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$ is an internal direct sum where $d \geq 2$ is the degree of the minimal (monic) polynomial in $\mathbb{Z}[x]$ satisfied by λ.*
(3) *If λ is transcendental then, $\Gamma_\lambda = \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}\lambda^k$ is an internal direct sum.*
(4) *If $\lambda = 1/\sqrt{n}$ with $n \geq 2$ a square-free positive integer, then $\Gamma_\lambda = \mathbb{Z}[1/n] + \mathbb{Z}[1/n] \cdot \sqrt{n}$ is an internal direct sum.*

(5) *In general, if* λ *is algebraic with minimal polynomial,* $n\lambda^d + \cdots + m = 0$ *over* $\mathbb{Z}$*, then*

$$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \Gamma_\lambda \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}.$$

Hence, $\operatorname{rank}(\Gamma_\lambda) := \dim_{\mathbb{Q}}(\Gamma_\lambda \otimes_{\mathbb{Z}} \mathbb{Q}) = d$.

Definition 1.3. *Now let* $C_0^\lambda(\mathbb{R})$ *be the separable, commutative (AF)* C^**-subalgebra of* $\mathcal{L}^\infty(\mathbb{R})$ *generated by the countable family of projections* $\mathcal{X}_{[a,b)}$ *where* $a, b \in \Gamma_\lambda$*. That is,*

$$C_0^\lambda(\mathbb{R}) = \text{closure}\left(\left\{ \textstyle\sum_{k=1}^n c_k \mathcal{X}_{[a_k,b_k)} \;\middle|\; c_k \in \mathbb{C}; \quad a_k, b_k \in \Gamma_\lambda \right\}\right).$$

We note that $C_0(\mathbb{R}) \subset C_0^\lambda(\mathbb{R})$, and that $C_0^\lambda(\mathbb{R})$ consists of exactly of those $\mathbb{C}$-valued function on $\mathbb{R}$ which vanish at ∞; are right-continuous at each point of Γ_λ; have finite left-hand limits at each point of Γ_λ; and are continuous at each point not in Γ_λ.

Definition 1.4. *Let* $G_\lambda \supset G_\lambda^0$ *be the following countable discrete groups of matrices:*

$$G_\lambda = \left\{ \begin{pmatrix} \lambda^n & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \Gamma_\lambda,\ n \in \mathbb{Z} \right\} \supset G_\lambda^0 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \Gamma_\lambda \right\}.$$

Of course, G_λ^0 *is isomorphic to the additive group* Γ_λ*, and* G_λ *is semidirect product of* $\mathbb{Z}$ *acting on* $G_\lambda^0 \cong \Gamma_\lambda$*. We let* G_λ *act on* $\mathbb{R}$ *as an "ax+b" group, noting that the action leaves* Γ_λ *invariant. That is,*

$$\text{for } t \in \mathbb{R} \text{ and } g = \begin{pmatrix} \lambda^n & a \\ 0 & 1 \end{pmatrix} \in G_\lambda \text{ define } g \cdot t := \lambda^n t + a.$$

We use this action on $\mathbb{R}$ *to define the transpose action* α *of* G_λ *on* $C_0^\lambda(\mathbb{R})$:

$$\alpha_g(f)(t) = f(g^{-1}t) \text{ for } f \in C_0^\lambda(\mathbb{R}) \text{ and } t \in \mathbb{R}.$$

Definition 1.5. *We define the separable* C^**-algebras* $A^\lambda \supset A_0^\lambda$ *as the crossed products:*

$$A^\lambda := G_\lambda \rtimes_\alpha C_0^\lambda(\mathbb{R}) = \mathbb{Z} \rtimes (G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R})) \supset A_0^\lambda := G_\lambda^0 \rtimes_\alpha C_0^\lambda(\mathbb{R}).$$

Proposition 1.6. *The algebras* A^λ*, and* A_0^λ *are simple, separable, nuclear and stable* C^**-algebras in the bootstrap class* $\mathfrak{N}_{nuc}$*. Moreover,* A^λ *is purely infinite.*

In order to obtain unital C^*-algebras we cut-down these algebras by the projection, e corresponding to $\mathcal{X}_{[0,1)} \in C_0^\lambda(\mathbb{R}) \subset A_0^\lambda \subset A^\lambda$. Explicitly, we define:

Definition 1.7.

$$\mathcal{Q}^\lambda := eA^\lambda e \qquad F^\lambda := eA_0^\lambda e.$$

We will also have occasion to use the dense $*$*-subalgebras* $\mathcal{Q}_c^\lambda$ *and* F_c^λ *consisting of finitely supported functions* $x : G^\lambda \to C_0^\lambda(\mathbb{R})$ *(respectively,* $x : G_0^\lambda \to C_0^\lambda(\mathbb{R})$*) where the values* $x(g)$ *are either* 0 *or finite linear combinations of characteristic functions* $\mathcal{X}_{[a,b)}$ *with* $a, b \in \Gamma_\lambda$*.*

Corollary 1.8. *The algebras* $\mathcal{Q}^\lambda$*, and* F^λ *are simple, separable, unital, nuclear* C^**-algebras in the bootstrap class* $\mathfrak{N}_{nuc}$*. Moreover,* $\mathcal{Q}^\lambda$ *is purely infinite, and The orthogonal family of projections* $e_n = \mathcal{X}_{[n,n+1)} \in C_0^\lambda(\mathbb{R})$ *for* $n \in \mathbb{Z}$ *are mutually equivalent by partial isometries in* A_0^λ *and the finite sums* $E_N := \sum_{n=-N}^{N-1} e_n = \mathcal{X}_{[-N,N)}$ *form an approximate identity for* A^λ *(and for* A_0^λ*) so that*

$$A^\lambda \cong \mathcal{Q}^\lambda \otimes \mathcal{K}(l^2(\mathbb{Z})) \text{ and } A_0^\lambda \cong F^\lambda \otimes \mathcal{K}(l^2(\mathbb{Z})).$$

2. K-Theory of the algebras $C^*(\Gamma_\lambda)$, $\mathcal{Q}^\lambda$, and F^λ.

We first compute the K-Theory of the algebras $C^*(\Gamma_\lambda) \cong C(\hat{\Gamma}_\lambda)$ using Proposition 1.2.

Proposition 2.1. *Let* $0 < \lambda < 1$.
(1) *If* $\lambda = p/q$ *is rational in lowest terms so that* $\Gamma_\lambda = \mathbb{Z}[1/n]$, *where* $n = pq$, *then*

$$K_0(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1_{\hat{\Gamma}_\lambda}] \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \mathbb{Z}[1/n].$$

(2) *If* λ *and* λ^{-1} *are both algebraic integers, so that* $\Gamma_\lambda = \mathbb{Z} + \mathbb{Z}\lambda + \cdots + \mathbb{Z}\lambda^{d-1}$ *is an internal direct sum as above, then*

$$K_0(C(\hat{\Gamma}_\lambda)) = \bigwedge^{even}(\Gamma_\lambda) = \bigoplus_{k=0, k\ even}^{d} \bigwedge^{k}(\Gamma_\lambda) \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^{odd}(\Gamma_\lambda) = \bigoplus_{k=1, k\ odd}^{d} \bigwedge^{k}(\Gamma_\lambda).$$

(3) *If* λ *is transcendental then,*

$$K_0(C(\hat{\Gamma}_\lambda)) = \bigwedge^{even}(\Gamma_\lambda) = \bigoplus_{k=0, k\ even}^{\infty} \bigwedge^{k}(\Gamma_\lambda) \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) = \bigwedge^{odd}(\Gamma_\lambda) = \bigoplus_{k=1, k\ odd}^{\infty} \bigwedge^{k}(\Gamma_\lambda).$$

(4) *If* $\lambda = 1/\sqrt{n}$ *with* $n \geq 2$ *a square-free positive integer, then*

$$K_0(C(\hat{\Gamma}_\lambda)) \cong \mathbb{Z} \oplus \mathbb{Z}[1/n] \quad \text{and} \quad K_1(C(\hat{\Gamma}_\lambda)) \cong \mathbb{Z}[1/n] \oplus \mathbb{Z}[1/n]$$

(5) *In general, if* λ *is algebraic with* $n\lambda^d + \cdots + m = 0$ *over* $\mathbb{Z}$ *then the composition of the inclusions*

$$\mathbb{Z} \oplus \mathbb{Z}\lambda \oplus \cdots \oplus \mathbb{Z}\lambda^{d-1} \subseteq \Gamma_\lambda \subseteq \mathbb{Z}[\frac{1}{mn}] \oplus \mathbb{Z}[\frac{1}{mn}]\lambda \oplus \cdots \oplus \mathbb{Z}[\frac{1}{mn}]\lambda^{d-1}$$

induces an inclusion on K*-Theory, so that both of the following maps are one-to-one*

$$\bigwedge^{even}(\mathbb{Z}^d) \cong K_0(C^*(\mathbb{Z}\oplus \cdots \mathbb{Z}\lambda^{d-1})) \hookrightarrow K_0(C(\hat{\Gamma}_\lambda)) \text{ and } \bigwedge^{odd}(\mathbb{Z}^d) \cong K_1(C^*(\mathbb{Z}\oplus \cdots \mathbb{Z}\lambda^{d-1})) \hookrightarrow K_1(C(\hat{\Gamma}_\lambda)).$$

Item (1) is proved directly, while (2) and (3) follow from the known K-Theory of (commutative) tori. Items (4) and (5) are deduced from results of C. Schochet [Sc] on the K-Theory of tensor products.

Proposition 2.2. *If* $\lambda = p/q$ *is rational in lowest terms, we show directly that* F^λ *is a UHF algebra of type* n^∞ *where* $n = pq$, *and so:*

$$K_0(F^\lambda) \cong \mathbb{Z}[1/n] \text{ and } K_1(F^\lambda) = \{0\}.$$

Since $A^\lambda = \mathbb{Z} \rtimes A_0^\lambda$ *and* $A_0^\lambda \cong F^\lambda \otimes \mathcal{K}$ *we use the Pimsner-Voiculescu exact sequence to compute:*

$$K_1(\mathcal{Q}^\lambda) = \{0\}, \text{ and } K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}[1/(pq)]/(1-\lambda)\mathbb{Z}[1/(pq)] \cong \mathbb{Z}_{(q-p)}.$$

Remarks. Since the identity in $\mathcal{Q}^\lambda$ is a generator for $\mathbb{Z}_{(q-p)}$ results of Kirchberg-Phillips imply that $\mathcal{Q}^\lambda \cong O_m$ where $m = (q-p)+1$. If $\lambda = 1/n$ we can explicitly write down the n generators for $\mathcal{Q}^{1/n} \cong O_n$, and $F^{1/n}$ is (isomorphic to) the usual UHF subalgebra of O_n. However, for general rational numbers $\lambda = p/q$ the situation remains a little mysterious.

For λ **irrational** we again use the Pimsner-Voiculescu exact sequence to reduce to the K-Theory of A_0^λ. However, in this case these algebras are never AF, and we must use the theory of groupoid C^*-algebras. In particular we must use Ian Putnam's exact sequence for the K-Theory of certain principal groupoid C^*-algebras.

First we establish some notation. Let $\mathbb{R}_\lambda$ be the locally compact Gelfand spectrum of the commutative C^*-algebra $C_0^\lambda(\mathbb{R})$ so that $C_0^\lambda(\mathbb{R}) \cong C_0(\mathbb{R}_\lambda)$. Now, $C_0(\mathbb{R}) \subset C_0(\mathbb{R}_\lambda)$ and the induced proper surjection: $\mathbb{R}_\lambda \to \mathbb{R}$ has the property that points in $\mathbb{R}$ that are not in Γ_λ have a single pre-image while points $\gamma \in \Gamma_\lambda$ have exactly two pre-images in $\mathbb{R}_\lambda$ denoted by γ^- and γ^+. To simplify notation, we let $\Gamma = \Gamma_\lambda \cong G^0_\lambda$. Thus, $\Gamma \subset \mathbb{R}$ is a countable dense subgroup of $\mathbb{R}$ which acts on $\mathbb{R}$ by translations. Before looking at the crossed product of Γ acting on $C^\lambda_0(\mathbb{R}) = C_0(\mathbb{R}_\lambda)$ (which gives us A^λ_0) we first consider the crossed product of Γ acting on $C_0(\mathbb{R})$. Since Γ acts on $\mathbb{R}$ by translation we can Fourier transform to get an isomorphism:

$$\Gamma \rtimes C_0(\mathbb{R}) \cong \hat{\mathbb{R}} \rtimes C(\hat{\Gamma}).$$

Then, by Connes' Thom isomorphism we get for $i = 0, 1$:

$$K_i(\Gamma \rtimes C_0(\mathbb{R})) \cong K_i(\hat{\mathbb{R}} \rtimes C(\hat{\Gamma})) \cong K_{i+1}(C(\hat{\Gamma})).$$

We are able to identify the image of the generator of $K_1(C_0(\mathbb{R})) \cong \mathbb{Z}$ with the class of the identity function [1] in $K_0(C(\hat{\Gamma}))$.

In the notation of [Put2] we define the transformation groupoids:

$$G := \mathbb{R}_\lambda \rtimes \Gamma, \quad G' := \mathbb{R} \rtimes \Gamma, \quad \text{and} \quad H := \Gamma \rtimes \Gamma.$$

Then, $A^\lambda_0 = C^*_r(G)$ is the reduced C^*-algebra of G; $\Gamma \rtimes C_0(\mathbb{R}) = C^*_r(G')$ is the reduced C^*-algebra of G'; and $\mathcal{K}(l^2(\Gamma)) = C^*_r(H)$ is the reduced C^*-algebra of H.
Thus, there are two disjoint embeddings of Γ in $\mathbb{R}_\lambda$:

$$i_0, i_1 : \Gamma \to \mathbb{R}_\lambda \ : \quad i_0(\gamma) = \gamma^-, \quad i_1(\gamma) = \gamma^+.$$

Now in order to mesh with the notation of [Put2], we let $Y := \Gamma$ with the equivalence relation, "="; $X := \mathbb{R}_\lambda$, with the equivalence relation $(i_0(\gamma) \sim i_1(\gamma))$; and quotient $\pi : X \to X' := \mathbb{R}$ where $X' = X/(i_0(\gamma) \sim i_1(\gamma)) = \mathbb{R}$; while the "factor groupoid" of $G = \mathbb{R}_\lambda \times \Gamma = X \times \Gamma$ is $G' := \mathbb{R} \times \Gamma = X' \times \Gamma$.

In a natural way we represent each of these three C^*-algebras (faithfully) on $\mathcal{H} := l^2(\Gamma^+) \oplus l^2(\Gamma^-)$ where $\Gamma^\pm = \{\gamma^\pm \mid \gamma \in \Gamma\}$. Then from the six-term exact sequence of [Put2] we get:

$$\begin{array}{ccccc} K_1(C^*_r(H)) & \longrightarrow & K_0(C^*_r(G')) & \longrightarrow & K_0(C^*_r(G)) \\ \uparrow & & & & \downarrow \\ K_1(C^*_r(G)) & \longleftarrow & K_1(C^*_r(G')) & \longleftarrow & K_0(C^*_r(H)) \end{array}$$

In our set-up this becomes:

$$\begin{array}{ccccc} \{0\} & \longrightarrow & K_0(\Gamma \rtimes C_0(\mathbb{R})) & \longrightarrow & K_0(\Gamma \rtimes C_0(\mathbb{R}_\lambda)) \\ \uparrow & & & & \downarrow \\ K_1(\Gamma \rtimes C_0(\mathbb{R}_\lambda)) & \longleftarrow & K_1(\Gamma \rtimes C_0(\mathbb{R})) & \longleftarrow & \mathbb{Z} \end{array}$$

Which by Connes' Thom isomorphism becomes:

$$\begin{array}{ccccc} \{0\} & \longrightarrow & K_1(C(\hat{\Gamma})) & \longrightarrow & K_0(A^\lambda_0) \\ \uparrow & & & & \downarrow \\ K_1(A^\lambda_0) & \longleftarrow & K_0(C(\hat{\Gamma})) & \longleftarrow & \mathbb{Z} \end{array}$$

With a little more effort we obtain the following result on the K-Theory of A^λ_0 and hence of F^λ.

Proposition 2.3. *For $0 < \lambda < 1$,*

$$K_0(F^\lambda) = K_0(A^\lambda_0) \cong K_1(C(\hat{\Gamma}_\lambda)) \quad \text{and} \quad K_1(F^\lambda) = K_1(A^\lambda_0) \cong K_0(C(\hat{\Gamma}_\lambda))/[1]\mathbb{Z}.$$

In particular, if λ is irrational, then $K_1(A^\lambda_0) \neq \{0\}$ so that A^λ_0 is not an AF-algebra.

Remarks. The last statement of the Proposition follows from items (3) and (5) of Proposition 2.1 since $K_0(C(\hat{\Gamma}_\lambda))$ is not singly generated when λ is irrational.

Now for λ **irrational** we can use the Pimsner-Voiculescu exact sequence to calculate the K-Theory of $\mathcal{Q}^\lambda$ in cases where we can effectively compute the action of λ on the K-Theory of A_0^λ. For example:

If λ is transcendental, then $K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}^\infty \cong K_1(\mathcal{Q}^\lambda)$. Then by classification theory $\mathcal{Q}^\lambda$ is isomorphic to Cuntz's algebra, $\mathcal{Q}_\mathbb{N}$.

If $\lambda = 1/\sqrt{n}$ where n is square-free, then $K_0(\mathcal{Q}^\lambda) \cong \mathbb{Z}/(n-1)\mathbb{Z} \cong K_1(\mathcal{Q}^\lambda)$. In particular, if $n = 2$ then $\mathcal{Q}^{1/\sqrt{2}} \cong O_2$, but if $n \geq 3$ then $\mathcal{Q}^{1/\sqrt{n}}$ is not even a Cuntz-Krieger algebra.

If λ and λ^{-1} are both algebraic with minimal polynomial of degree d, then by Proposition 2.1 $K_0(F^\lambda) \cong \mathbb{Z}^{2^{d-1}}$ and $K_1(F^\lambda) \cong \mathbb{Z}^{(2^{d-1}-1)}$ with natural bases so that one can write down explicit integer matrices for the actions of λ. By row and column reducing these matrices to obtain the Smith Normal Form one can exactly compute the K-Theory of $\mathcal{Q}^\lambda$ in this case.

For example, if $\lambda^2 + a\lambda - 1 = 0$ with $a > 0$ so that $\lambda = (1/2)(\sqrt{a^2+4} - a)$ then $K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/a\mathbb{Z}$ and $K_1(\mathcal{Q}^\lambda) = \mathbb{Z}/2\mathbb{Z}$. None of these algebras are Cuntz-Krieger algebras. In particular, if $a = 1$ then λ is the inverse of the golden mean and $K_0(\mathcal{Q}^\lambda) = \{0\}$ and $K_1(\mathcal{Q}^\lambda) = \mathbb{Z}/2\mathbb{Z}$.

As another example, suppose $\lambda^2 + a\lambda + 1 = 0$ where $a \leq -3$. Then, $K_0(\mathcal{Q}^\lambda) = \mathbb{Z} \oplus (\mathbb{Z}/(a+2)\mathbb{Z})$ and $K_1(\mathcal{Q}^\lambda) = \mathbb{Z}$. For these algebras, $\mathcal{Q}^\lambda$ has the correct K-theory to be a Cuntz-Krieger algebra, and is therefore stably isomorphic to one. When $a = -3$ so that $\lambda = (1/2)(3 - \sqrt{5}))$ we have $K_0(\mathcal{Q}^\lambda) = \mathbb{Z} = K_1(\mathcal{Q}^\lambda)$.

As a final specific example, suppose $\lambda^3 + m\lambda^2 + (m-1)\lambda + 1 = 0$ with $m \geq 0$. Then $K_0(\mathcal{Q}^\lambda) = \mathbb{Z}/(4m+2)\mathbb{Z}$ and $K_1(\mathcal{Q}^\lambda) = \{0\}$. Then, $\mathcal{Q}^\lambda \cong O_{(4m+3)}$. This example is not completely random as the next result shows.

Proposition 2.4. *If λ and λ^{-1} are both algebraic integers and $\mathcal{Q}^\lambda$ is stably isomorphic to a Cuntz algebra O_n then the minimal polynomial of λ has odd degree and constant term* $+1$; $n \equiv 3(\text{mod}4)$; *and all such Cuntz algebras appear this way.*

3. The Action of $\mathbb{T}$ on $\mathcal{Q}^\lambda$ and Its Unique KMS State

Now $C_0^\lambda(\mathbb{R})$ has a faithful semifinite (norm) lower semicontinuous trace given by integration on positive elements. This trace is clearly invariant under $G_0^\lambda \cong \Gamma_\lambda$ which acts by translations. Therefore $A_0^\lambda = G_0^\lambda \rtimes_\alpha C_0^\lambda(\mathbb{R})$ also has a faithful semifinite (norm) lower semicontinuous trace obtained by composing the trace on $C_0^\lambda(\mathbb{R})$ with the conditional expectation: $A_0^\lambda \to C_0^\lambda(\mathbb{R})$. Finally we obtain a faithful semifinite (norm) lower semicontinuous weight on $\mathcal{Q}^\lambda$ by writing $A^\lambda = \mathbb{Z} \rtimes A_0^\lambda$ and composing the trace on A_0^λ with the conditional expectation $A^\lambda \to A_0^\lambda$. Now the projection $e \in A_0^\lambda$ obtained from the element $\mathcal{X}_{[0,1]} \in C_0^\lambda(\mathbb{R})$ clearly has trace $= 1$ and therefore has weight $= 1$ as an element of A^λ. Therefore the weight on A^λ restricted to $\mathcal{Q}^\lambda = eA^\lambda e$ defines a state ψ on $\mathcal{Q}^\lambda$.

Since $A^\lambda = \mathbb{Z} \rtimes A_0^\lambda$ there is a natural dual action of $\mathbb{T}$ on A^λ and the fixed point algebra under this action of $\mathbb{T}$ is A_0^λ. Since $e \in A_0^\lambda$, it is fixed by the action of $\mathbb{T}$ and therefore the action restricts to an action of $\mathbb{T}$ on $\mathcal{Q}^\lambda = eA^\lambda e$. We call this the **gauge action** of $\mathbb{T}$ on $\mathcal{Q}^\lambda$. The fixed-point subalgebra of $\mathcal{Q}^\lambda$ under this action is exactly $F^\lambda = eA_0^\lambda e$, and ψ restricts to a (faithful, finite) trace τ on F^λ. We have the following theorem.

Theorem 3.1. *The state ψ on $\mathcal{Q}^\lambda$ is invariant under the gauge action of $\mathbb{T}$. Moreover, considered as an action of $\mathbb{R}$, the state ψ is a* KMS_β *state for $\beta = log(\lambda^{-1})$. Furthermore, ψ is the unique* KMS *state for this action regardless of β. The GNS representation $\pi : \mathcal{Q}^\lambda \to \mathcal{B}(\mathcal{H}_\psi)$ afforded by ψ generates a type III_λ factor in $\mathcal{B}(\mathcal{H}_\psi)$.*

Remarks. It is not difficult to show that the conditional expectation, $A^\lambda \to A_0^\lambda$ restricts to a conditional expectation $\Phi : \mathcal{Q}^\lambda \to F^\lambda$ and that the KMS state $\psi = \tau \circ \Phi$. We let $\mathcal{D}$ denote the generator of the one parameter unitary group which implements the action of $\mathbb{T}$ on $\mathcal{H}_\psi$ and let Δ denote the modular operator on $\mathcal{H}_\psi$ for state the ψ. We have the following result.

Proposition 3.2. *With the left action of $\mathcal{Q}^\lambda$ on $\mathcal{H}_\psi$, the action of $\mathbb{T}$ is unitarily implemented on $\mathcal{H}_\psi$ and the generator $\mathcal{D}$ of this unitary group is related to the modular operator Δ for ψ by:*

$$\Delta = \lambda^{\mathcal{D}} \quad \text{or} \quad e^{it\mathcal{D}} = \Delta^{it/log(\lambda)}.$$

4. The Modular Spectral Triple for $\mathcal{Q}^\lambda$

Definition 4.1. *To simplify notation let $\mathcal{Q} = \mathcal{Q}^\lambda$ and $F = F^\lambda$. Then $\mathcal{Q}$ becomes a right pre-Hilbert F-module with F-valued inner product $(a|b) := \Phi(a^*b)$. We let X denote the Hilbert F-module obtained by completing $\mathcal{Q}$ in the module norm $||x||_X = ||\Phi(x^*x)||^{1/2}$.*

Now the left action of $\mathcal{Q}$ on itself extends uniquely to an action of $\mathcal{Q}$ as adjointable operators on X. Moreover the action of $\mathbb{T}$ on $\mathcal{Q}$ extends uniquely to a representation of $\mathbb{T}$ as unitaries on X. If we denote this representation by $z \mapsto u_z$, then for each $k \in \mathbb{Z}$ we define the k-th spectral subspace of this representation to be

$$X_k := \{x \in X \mid u_z(x) = z^k x \ \text{ for all } \ z \in \mathbb{T}\}.$$

The projection operator, Φ_k from X onto X_k exists, is adjointable and defined by:

$$\Phi_k(x) = \frac{1}{2\pi}\int_{\mathbb{T}} z^{-k} u_z(x) d\theta, \ \ z = e^{i\theta}, \ \ x \in X.$$

For $x \in X$ we use the notation $x_k = \Phi_k(x)$ and then define $X_{\mathcal{D}} \subset X$ to be the linear subspace

$$X_{\mathcal{D}} = \{x = \sum_{k\in\mathbb{Z}} x_k \in X \mid \ \| \sum_{k\in\mathbb{Z}} k^2 (x_k|x_k) \| < \infty\}.$$

For $x \in X_{\mathcal{D}}$ we define $\mathcal{D}(x) = \sum_{k\in\mathbb{Z}} k x_k$.

Proposition 4.2. *The operator $\mathcal{D} : X_{\mathcal{D}} \to X$ is a self-adjoint, regular operator on X with discrete spectrum $= \mathbb{Z}$ and spectral projections Φ_k for each $k \in \mathbb{Z}$. Moreover, the finite sums $\sum_{|k|\le N} \Phi_k$ converge strictly to the identity operator on X.*

Now, if $a, b \in \mathcal{Q}$ then the "rank one" operator $\theta_{a,b} : \mathcal{Q} \to \mathcal{Q}$ defined by $\theta_{a,b}(c) = a(b|c) = a\Phi(b^*c)$ is in fact a bounded linear operator in the Hilbert space norm on $\mathcal{H}_\psi$, with bound at most $\| a \| \| b \|$. This is a straightforward calculation using the inequalities $S^*T^*TS \le \| T \|^2 S^*S$, and $\Phi(x^*x) \le \Phi(x^*)\Phi(x)$, since Φ is 2-positive. So we regard these operators as operators in $\mathcal{B}(\mathcal{H}_\psi)$.

Definition 4.3. *We define $\mathcal{N}$ to be the von Neumann algebra on $\mathcal{H}_\psi$, generated by all the rank one operators $\theta_{a,b}$ with a, b in $\mathcal{Q}$.*

Proposition 4.4. *There is a faithful, normal, semifinite trace $\tilde{\tau}$ on $\mathcal{N}$ such that all rank one operators are in the ideal of definition of $\tilde{\tau}$ and*

$$\tilde{\tau}(\theta_{a,b}) = \tau \circ \Phi(b^*a) = \psi(b^*a), \quad a, b \in \mathcal{Q}.$$

Moreover, $\pi(\mathcal{Q}) \subset \mathcal{N}$; $\mathcal{D}$ is affiliated to $\mathcal{N}$; and $\tilde{\tau}(\Phi_k) = \lambda^{-k}$.

Since $\tilde{\tau}(\Phi_k) = \lambda^{-k}$ we see that $\mathcal{D}$ does not satisfy any finite summability criterion. We use $\Delta = \lambda^{\mathcal{D}}$ to define a new weight on $\mathcal{N}^+$. For $T \in \mathcal{N}^+$ let $\tau_\Delta(T) = sup_N \tilde{\tau}(\Delta_N T)$ where $\Delta_N T = \Delta(\sum_{|k|\leq N} \Phi_k)$.

Proposition 4.5. *τ_Δ is a faithful, normal, semifinite weight on $\mathcal{N}^+$.*

We now give another way to define τ_Δ which is not only conceptually useful but also makes a number of important properties straightforward to verify.

Notation. Let $\mathcal{M}$ be the relative commutant in $\mathcal{N}$ of the operator Δ. Equivalently, $\mathcal{M}$ is the relative commutant of the set of spectral projections $\{\Phi_k | k \in \mathbb{Z}\}$ of $\mathcal{D}$. Clearly, $\mathcal{M} = \sum_{k\in\mathbb{Z}} \Phi_k \mathcal{N} \Phi_k$.

Definition 4.6. *As $\tilde{\tau}$ restricted to each $\Phi_k \mathcal{N} \Phi_k$ is a faithful finite trace with $\tilde{\tau}(\Phi_k) = \lambda^{-k}$ we define $\widehat{\tau}_k$ on $\Phi_k \mathcal{N} \Phi_k$ to be λ^k times the restriction of $\tilde{\tau}$. Then, $\widehat{\tau} := \sum_k \widehat{\tau}_k$ on $\mathcal{M} = \sum_{k\in\mathbb{Z}} \Phi_k \mathcal{N} \Phi_k$ is a faithful normal semifinite trace $\widehat{\tau}$ with $\widehat{\tau}(\Phi_k) = 1$ for all k.*

We now use $\widehat{\tau}$ to give an alternative expression for τ_Δ :

Lemma 4.7. *An element $m \in \mathcal{N}$ is in $\mathcal{M}$ if and only if it is in the fixed point algebra of the action, $\sigma_t^{\tau_\Delta}$ on $\mathcal{N}$ defined for $T \in \mathcal{N}$ by $\sigma_t^{\tau_\Delta}(T) = \Delta^{it} T \Delta^{-it}$. Both $\pi(F)$ and the projections Φ_k belong to $\mathcal{M}$. The map $\Psi : \mathcal{N} \to \mathcal{M}$ defined by $\Psi(T) = \sum_k \Phi_k T \Phi_k$ is a conditional expectation onto $\mathcal{M}$ and $\tau_\Delta(T) = \widehat{\tau}(\Psi(T))$ for all $T \in \mathcal{N}^+$. That is, $\tau_\Delta = \widehat{\tau} \circ \Psi$ so that $\widehat{\tau}(T) = \tau_\Delta(T)$ for all $T \in \mathcal{M}^+$. Finally, if one of $A, B \in \mathcal{M}$ is $\widehat{\tau}$-trace-class and $T \in \mathcal{N}$ then $\tau_\Delta(ATB) = \tau_\Delta(A\Psi(T)B) = \widehat{\tau}(A\Psi(T)B)$.*

We now have the key lemma, but we omit the proof:

Lemma 4.8. *Suppose g is a function on $\mathbb{R}$ such that $g(\mathcal{D})$ is τ_Δ trace-class in $\mathcal{M}$, then for all $f \in F$ we have*

$$\tau_\Delta(\pi(f) g(\mathcal{D})) = \tau_\Delta(g(D))\tau(f) = \tau(f) \sum_{k\in\mathbb{Z}} g(k).$$

Proposition 4.9. *(i) We have $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \tau_\Delta)$ in the sense that, $\tau_\Delta((1+\mathcal{D}^2)^{-s/2}) < \infty$ for all $s > 1$. Moreover, for all $f \in F^\lambda$*

$$\lim_{s\to 1^+} (s-1)\tau_\Delta(\pi(f)(1+\mathcal{D}^2)^{-s/2}) = 2\tau(f)$$

so that $\pi(f)(1+\mathcal{D}^2)^{-1/2}$ is a measurable operator in the sense of [C].

(ii) For $\pi(a) \in \pi(\mathcal{Q}^\lambda) \subset \mathcal{N}$ the following (ordinary) limit exists and

$$\widehat{\tau}_\omega(\pi(a)) = \frac{1}{2} \lim_{s\to 1^+} (s-1)\tau_\Delta(\pi(a)(1+\mathcal{D}^2)^{-s/2}) = \tau \circ \Phi(a) = \psi(a),$$

the original KMS *state $\psi = \tau \circ \Phi$ on $\mathcal{Q}^\lambda$.*

Proof. (i) This proof is identical to [CPR2, Proposition 3.12].
(ii) This proof is the same as [CPR2, Proposition 3.14] with $\mathcal{Q}^\lambda, F^\lambda$ replacing O_n, F. □

Definition 4.10. *Let $\mathcal{A} = \mathcal{Q}_c^\lambda$ and let γ be the gauge represntation of $\mathbb{T}$ as automorphism of $\mathcal{Q}^\lambda$. The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ along with γ, ψ, $\mathcal{N}$, τ_Δ satisfying properties (0) to (3) below is called the* **modular spectral triple** *of the dynamical system $(\mathcal{Q}^\lambda, \gamma, \psi)$*
0) The $*$-subalgebra $\mathcal{A}$ of $\mathcal{Q}^\lambda$ is faithfully represented in $\mathcal{N}$ acting on the Hilbert space $\mathcal{H}_\psi$,
1) there is a faithful normal semifinite weight τ_Δ on $\mathcal{N}$ such that the modular automorphism group of τ_Δ is an inner automorphism group $\sigma_t = Ad(\Delta^{it})$ (for $t \in \mathbb{C}$) of (the Tomita algebra of) $\mathcal{N}$ with $\sigma_i|_\mathcal{A} = \sigma$ in the sense that $\sigma_i(\pi(a)) = \pi(\sigma(a))$, where σ is the automorphism $\sigma(a) = \Delta^{-1}(a)$ on $\mathcal{A}$,
2) τ_Δ restricts to a faithful semifinite trace $\widehat{\tau}$ on $\mathcal{M} = \mathcal{N}^\sigma$, with a faithful normal projection $\Psi : \mathcal{N} \to \mathcal{M}$ satisfying $\tau_\Delta = \widehat{\tau} \circ \Psi$ on $\mathcal{N}$,
3) with $\mathcal{D}$ the generator of the one parameter group implementing the gauge action of $\mathbb{T}$ on $\mathcal{H}$ we have: $[\mathcal{D}, \pi(a)]$ extends to a bounded operator (in $\mathcal{N}$) for all $a \in \mathcal{A}$ and for λ in the resolvent set of $\mathcal{D}$, $(\lambda - \mathcal{D})^{-1} \in \mathcal{K}(\mathcal{M}, \tau_\Delta)$, where $\mathcal{K}(\mathcal{M}, \tau_\Delta)$ is the ideal of compact operators in $\mathcal{M}$ relative to τ_Δ. In particular, $\mathcal{D}$ is affiliated to $\mathcal{M}$.

For matrix algebras $\mathcal{A} = \mathcal{Q}_c^\lambda \otimes M_k$ over $\mathcal{Q}_c^\lambda$, $(\mathcal{Q}_c^\lambda \otimes M_k, \mathcal{H} \otimes M_k, \mathcal{D} \otimes Id_k)$ is also a modular spectral triple in the obvious fashion.

4.1. **Modular K_1.** We now make appropriate modifications to [CPR2, Section 4]) using [CNNR] introducing elements of these modular spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (where $\mathcal{A}$ is a matrix algebra of analytic elements over $\mathcal{Q}^\lambda$) that will have a well defined pairing with our Dixmier functional $\widehat{\tau}_\omega$. Let $A = \mathcal{Q}^\lambda$. Following [HR] we say that a unitary (invertible, projection,...) in the $n \times n$ matrices over $\mathcal{Q}^\lambda$ for some n is a unitary (invertible, projection,...) over $\mathcal{Q}^\lambda$. We will also use σ_t for the inflated automorphism $\sigma_t \otimes Id_n$ of $\mathcal{A}$.

Definition 4.11. *Let v be a partial isometry in the $*$-algebra $\mathcal{A}$. We say that v satisfies the* **modular condition** *with respect to σ if the operators $v\sigma_t(v^*)$ and $v^*\sigma_t(v)$ are in the fixed point algebra $F \subset \mathcal{A}$ for all $t \in \mathbb{R}$. Of course, any partial isometry in F is a modular partial isometry.*

Lemma 4.12. *(*[CPR2, Lemma 4.8]*) Let $v \in \mathcal{A}$ be a modular partial isometry. Then we have*

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}$$

is a modular unitary over $\mathcal{A}$. Moreover there is a modular homotopy $u_v \sim u_{v^*}$.

Note that in [CPR2] we used a different approach which is implied by the one given here. In [CPR2] we defined modular unitaries in terms of the regular automorphism:

$$\pi(\sigma(a)) = \pi(\Delta^{-1}(a)) = \Delta^{-1}\pi(a)\Delta = \sigma_i(\pi(a)).$$

That is we said that a unitary in $\mathcal{A}$ is modular if $u\sigma(u^*)$ and $u^*\sigma(u)$ are in the fixed point algebra Define the modular K_1 group as follows.

Definition 4.13. *Let $K_1(A, \sigma)$ be the abelian group with one generator $[v]$ for each partial isometry v over A satisfying the modular condition and with the following relations:*

1) $[v] = 0$ *if v is over F,*
2) $[v] + [w] = [v \oplus w]$,
3) *if v_t, $t \in [0, 1]$, is a continuous path of modular partial isometries in some matrix algebra over A then $[v_0] = [v_1]$.*

One could use modular unitaries as in [CPR2] in place of these modular partial isometries.

The following can now be seen to hold.

Lemma 4.14. *(Compare* [CPR2, Lemma 4.9]*) Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be our modular spectral triple relative to $(\mathcal{N},\tau_\Delta)$ and set $F=\mathcal{A}^\sigma$ and $\sigma:\mathcal{A}\to\mathcal{A}$. Let $L^\infty(\Delta)=L^\infty(\mathcal{D})$ be the von Neumann algebra generated by the spectral projections of Δ then $L^\infty(\Delta)\subset\mathcal{Z}(\mathcal{M})$. Let $v\in\mathcal{A}$ be a partial isometry with vv^*, $v^*v\in F$. Then $\pi(v)Q\pi(v^*)\in\mathcal{M}$ and $\pi(v^*)Q\pi(v)\in\mathcal{M}$ for all spectral projections Q of $\mathcal{D}$, if and only if v is modular. That is, $\pi(v)\Delta\pi(v^*)$ and $\pi(v^*)\Delta\pi(v)$ (or $\pi(v)\mathcal{D}\pi(v^*)$ and $\pi(v^*)\mathcal{D}\pi(v)$) are both affiliated to $\mathcal{M}$ if and only if v is modular.*

Thus we see that modular partial isometries conjugate Δ to an operator affiliated to $\mathcal{M}$, and so $v\Delta v^*$ commutes with Δ (and $v\mathcal{D}v^*$ commutes with $\mathcal{D}$).

We will next show that there is an analytic pairing between (part of) modular K_1 and modular spectral triples. To do this, we are going to use the analytic formulae for spectral flow in [CP2].

4.2. **A local index formula for the algebras $\mathcal{Q}^\lambda$.** Using the fact that we have full spectral subspaces we know from [CNNR] that there is a formula for spectral flow which is analogous to the local index formula in noncommutative geometry. We remind the reader that $\tau_\Delta=\hat{\tau}\circ\Psi$ where $\Psi:\mathcal{N}\to\mathcal{M}$ is the canonical expectation, so that τ_Δ **restricted to $\mathcal{M}$** is $\hat{\tau}$.

Theorem 4.15. *(Compare* [CPR2, Theorem 5.5]*) Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be the $(1,\infty)$-summable, modular spectral triple for the algebra $\mathcal{Q}^\lambda$ we have constructed previously. Then for any modular partial isometry v and for any Dixmier trace $\hat{\tau}_{\tilde{\omega}}$ associated to $\hat{\tau}$, we have spectral flow as an actual limit*

$$sf_{\hat{\tau}}(vv^*\mathcal{D},v\mathcal{D}v^*)=\frac{1}{2}\lim_{s\to1+}(s-1)\hat{\tau}(v[\mathcal{D},v^*](1+\mathcal{D}^2)^{-s/2})=\frac{1}{2}\hat{\tau}_{\tilde{\omega}}(v[\mathcal{D},v^*](1+\mathcal{D}^2)^{-1/2})=\tau\circ\Phi(v[\mathcal{D},v^*]).$$

The functional on $\mathcal{A}\otimes\mathcal{A}$ defined by $a_0\otimes a_1\mapsto\frac{1}{2}\lim_{s\to1^+}(s-1)\tau_\Delta(a_0[\mathcal{D},a_1](1+\mathcal{D}^2)^{-s/2})$ is a σ-twisted b,B-cocycle (see the proof below for the definition).

Remark. Spectral flow in this setting is independent of the path joining the endpoints of unbounded self adjoint operators affiliated to $\mathcal{M}$ however it is not obvious that this is enough to show that it is constant on homotopy classes of modular unitaries. This latter fact is true but the proof is lengthy so we refer to [CNNR].

Theorem 4.16. *We let $(\mathcal{Q}_c^\lambda\otimes M_2,\mathcal{H}\otimes\mathbb{C}^2,\mathcal{D}\otimes1_2)$ be the modular spectral triple of $(\mathcal{Q}_c^\lambda\otimes M_2)$.*
(1) *For any $k,j\in\mathbb{Z}$ we can define a modular unitary u of the form in Lemma 4.12 so that the spectral flow is positive being given by*

$$sf_{\tau_\Delta}(\mathcal{D},u\mathcal{D}u^*)\;=\;(k-j)(\lambda^j-\lambda^k)\in\mathbb{Z}[\lambda]\subset\Gamma_\lambda.$$

(2) *If λ^{-k} and λ^{-j} are not integers we let $m_k=[\lambda^{-k}]$ and $m_j=[\lambda^{-j}]$ we can define modular unitaries u of the form in Lemma 4.12 where the spectral flow is given by*

$$sf_{\tau_\Delta}(\mathcal{D},u\mathcal{D}u^*)\;=\;(k-j)[\lambda^j(\lambda^{-k}-m_k)-\lambda^k(\lambda^{-j}-m_j)]\in\Gamma_\lambda.$$

Remarks. The actual form of the modular unitaries in the previous theorem is a little complicated so we omit it.

Remarks. The observation of [CPR2] that the twisted residue cocycle formula for spectral flow is calculating Araki's relative entropy of two KMS states [Ar] also applies to the examples in this subsection.

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References

[Ar] H. Araki, *Relative entropy of states of von Neumann algebras*, Publ. RIMS, Kyoto Univ., **11** (1976) 809–833 and *Relative entropy for states of von Neumann algebras II*, Publ. RIMS, Kyoto Univ., **13** (1977) 173–192.

[APS3] M.F. Atiyah, V.K. Patodi, I.M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Camb. Phil. Soc. **79** (1976) 71–99.

[CNNR] A.L. Carey, R. Nest, S. Neshveyev, A. Rennie, *Twisted cyclic theory, equivariant KK-theory and KMS states*, to appear in J. reine angew. Math.

[CP1] A. L. Carey, J. Phillips, *Unbounded Fredholm modules and spectral flow*, Canadian J. Math., **50** (4) (1998) 673–718.

[CP2] A. L. Carey, J. Phillips, *Spectral flow in θ-summable Fredholm modules, eta invariants and the JLO cocycle*, K-Theory, **31** (2004) 135–194.

[CPR1] A.L. Carey, J. Phillips, A. Rennie, *A noncommutative Atiyah-Patodi-Singer index theorem in KK-theory*, to appear in J. reine angew. Math.

[CPR2] A.L. Carey, J. Phillips, A. Rennie, *Twisted cyclic theory and an index theory for the gauge invariant* KMS *state on Cuntz algebras*, to appear in Journal of K-Theory.

[CPS2] A.L. Carey, J. Phillips, F. Sukochev, *Spectral flow and Dixmier traces*, Adv. Math, **173** (2003) 68–113.

[CPRS2] A.L. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index formula in semifinite von Neumann algebras I: Spectral Flow*, Adv. in Math. **202** (2006) 451–516.

[CRSS] A. L. Carey, A. Rennie, A. Sedaev, F. Sukochev, *Dixmier traces and asymptotics of zeta functions*, **somewhere**

[CRT] A.L. Carey, A. Rennie, K. Tong, *Spectral flow invariants and twisted cyclic theory from the Haar state on* $SU_q(2)$, to appear in J. Geom. Phys.

[C0] A. Connes, *Une classification des facteurs de type III*, Annales Scientifiques de le'Ecole Norm. Sup., 4em serie t. 6 (1973) 18–252.

[C] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[Cu] J. Cuntz, *Simple C^*-algebras generated by isometries*, Commun. Math. Phys, **57** (1977) 173–189.

[Cu1] J. Cuntz, *C^*-algebras associated with the $ax+b$-semigroup over* $\mathbb{N}$ in*K-Theory and Noncommutative Geometry*, Eds, G. Cortinas, J. Cuntz, M Karoubi, R. Nest, C.A. Weibel, EMS Series of Congress Reports, Volume 2.

[D] J. Dixmier, *von Neumann algebras*, North-Holland, 1981.

[E] G. Elliott, *Some simple C^*-algebras constructed as crossed products with discrete outer automorphism groups*, Publ. RIMS Kyoto Univ., **16** (1980) 299-311.

[HR] N. Higson, J. Roe, *Analytic K-homology*, Oxford University Press, 2000.

[KNR] J. Kaad, R. Nest, A. Rennie, *KK-theory and spectral flow in von Neumann algebras*, arXive:math.OA/0701326.

[L] E. C. Lance, *Hilbert C^*-modules*, Cambridge University Press, Cambridge, 1995.

[PR] D. Pask, A. Rennie, *The noncommutative geometry of graph C^*-algebras I: The index theorem*, Journal of Functional Analysis, **233** (2006) 92–134.

[PhR] J. Phillips, I. Raeburn, *Semigroups of isometries, Toeplitz algebras and twisted crossed products*, J. Int. Equat. and Op. Th., **17** (1993) 579-602.

[Ped] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Math. Soc. monographs **14**, Academic Press, London 1979.

[Put1] I. Putnam, *An excision theorem for the K-theory of C^*-algebras*, J. Operator Theory, **38** (1997) 151–171.

[Put2] I. Putnam, *On the K-theory of C^*-algebras of principal groupoids*, Rocky Mountain J. Math., **28** no. 4 (1998) 1483–1518.

[RS] M. Rørdam and E. Størmer, *Classification of nuclear C^*-algebras. Entropy in operator algebras*, Encyclopedia of Mathematical Sciences, **126** (2002), Springer, Berlin.

[Sc] C. Schochet, *Topological methods for C^*-algebras II: geometric resolutions and the Künneth formula*, Pac. J. Math., **98** No. 2 (1982) 443–458.

[Ta] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Mathematics, **128** (1970), Springer, Berlin.