CONSTRUCTING MINIMAL HOMEOMORPHISMS ON POINT-LIKE SPACES AND A DYNAMICAL PRESENTATION OF THE JIANG–SU ALGEBRA

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Abstract. The principal aim of this paper is to give a dynamical presentation of the Jiang–Su algebra. Originally constructed as an inductive limit of prime dimension drop algebras, the Jiang–Su algebra has gone from being a poorly understood oddity to having a prominent positive role in George Elliott's classification programme for separable, nuclear C*-algebras. Here, we exhibit an étale equivalence relation whose groupoid C*-algebra is isomorphic to the Jiang–Su algebra. The main ingredient is the construction of minimal homeomorphisms on infinite, compact metric spaces, each having the same cohomology as a point. This construction is also of interest in dynamical systems. Any self-map of an infinite, compact space with the same cohomology as a point has Lefschetz number one. Thus, if such a space were also to satisfy some regularity hypothesis (which our examples do not), then the Lefschetz–Hopf Theorem would imply that it does not admit a minimal homeomorphism.

0. Introduction

The fields of operator algebras and dynamical systems have a long history of mutual influence. On the one hand, dynamical systems provide interesting examples of operator algebras and have often provided techniques which are successfully imported into the operator algebra framework. On the other hand, results in operator algebras are often of interest to those in dynamical systems. In ideal situations, significant information is retained when passing from dynamics to operator algebras, and vice versa.

This relationship has been particularly interesting for the classification of C*-algebras. An extraordinary result in this setting is the classification, up to strong orbit equivalence, of the minimal dynamical systems on a Cantor set and the corresponding K-theoretical classification of the associated crossed product C*-algebras [8, 20]. Classification for separable, simple, nuclear C*-algebras remains an interesting open problem. To every simple separable nuclear C*-algebra one assigns a computable set of invariants involving K-theory, tracial state spaces, and the pairing between these objects. George Elliott conjectured that for all such C*-algebras,

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an isomorphism at the level of invariants, now known as Elliott invariants, might be lifted to a *-isomorphism at the level of C*-algebras.

A remarkable number of positive results have been obtained. However, examples—including examples of some crossed product C*-algebras arising from minimal dynamical systems—have also shown pathologies undetectable by the Elliott invariant [7, 24, 28, 32]. One such algebra, constructed by Xinhui Jiang and Hongbing Su, gives a C*-algebra $\mathcal{Z}$ with invariant isomorphic to $\mathbb{C}$ [10]. The importance of the Jiang–Su algebra for the classification programme cannot be understated: in the case that a C*-algebra A has weakly unperforated K_0-group (for a definition, see for example [2, Definition 6.7.1]), its Elliott invariant is isomorphic to the Elliott invariant of $A \otimes \mathcal{Z}$. The original Elliott conjecture then predicts that for any simple separable nuclear C*-algebra, $A \cong A \otimes \mathcal{Z}$. In such a case A is said to be a $\mathcal{Z}$-stable C*-algebra. So far, each counterexample to Elliott's conjecture involves two C*-algebras, one of which is not $\mathcal{Z}$-stable. This leads to the following revised conjecture:

0.1 CONJECTURE: *Let A and B be simple separable unital nuclear* C**-algebras. Suppose that A and B are $\mathcal{Z}$-stable and have isomorphic Elliott invariants. Then $A \cong B$.*

More recently, there has been significant interest in transferring C*-algebraic regularity properties to the language of topological dynamics with the aim of showing that the appropriate properties pass from dynamical system to associated crossed product C*-algebra. Much of the motivation for this has come from the theory of von Neumann algebras, which has close ties to ergodic theory. Currently, the classification programme for C*-algebras is seeing rapid advancement by adapting results for von Neumann algebras to the setting of C*-algebras. For a good discussion on this interplay, we refer the reader to [25]. Of interest here are the comparisons between $\mathcal{Z}$ and its von Neumann counterpart, the hyperfinite II$_1$-factor, $\mathcal{R}$.

Like the Jiang–Su algebras, $\mathcal{R}$ is strongly self-absorbing (see [29]), absorbed (after taking tensor products) by certain factors with a particularly nice structure, and can be characterised uniquely in various abstract ways. Francis Murray and John von Neumann realise $\mathcal{R}$ using their group measure space construction, the von Neumann algebra version of the crossed product of a commutative C*-algebra by the integers [18]. Its ties to ergodic theory were deepened in [5], where Alain Connes proves that $\mathcal{R}$ can be realised dynamically by any measure space (X, μ) with a probability measure preserving action of a discrete amenable group G. Such a dynamical presentation for $\mathcal{Z}$ has so far been missing from the C*-algebraic theory.

In light of this, it has become increasingly important to find a suitable dynamical interpretation of $\mathcal{Z}$. In this paper, we construct such a presentation of the Jiang–Su algebra via a minimal étale equivalence relation (that is, an equivalence relation with countable dense equivalence classes.) See [22] for more about these groupoids. Along the way, we tackle an old question in dynamical systems: which compact, metric spaces admit minimal homeomorphisms?

Of course, many well-known systems provide positive answers (Cantor sets from odometers, the circle from irrational rotations). Perhaps the most famous positive result is that of Albert Fathi and Michael Herman who exhibited minimal, uniquely ergodic homeomorphisms on all odd-dimensional spheres of dimension greater than one [6] and Alistair Windsor's subsequent generalization to arbitrary numbers of ergodic measures [33]. In fact, we will use these results in a crucial way in our construction.

There are also negative results. Perhaps the most famous, and the most relevant for our discussion, is the Lefschetz–Hopf theorem (see for example [4]) which asserts that for "nice" spaces (for example, absolute neighbourhood retracts), the cohomology of a space may contain enough information to conclude that any continuous self-map of the space has a periodic point. If we also ask that the space be infinite, then it does not admit a minimal homeomorphism. An example where this holds is any even-dimensional sphere. The same conclusion holds for any contractible absolute neighbourhood retract (ANR). More generally, it also applies to any ANR whose cohomology is the same as a point.

Here we build minimal homeomorphisms ζ on "point-like" spaces: infinite, compact metric spaces with the same cohomology and K-theory as a point. In fact, our spaces are inverse limit of ANR's so while our results are positive, they sit perilously close to the Lefschetz–Hopf trap. For a survey on fixed point properties, see [1].

We construct such minimal dynamical systems with any prescribed number of ergodic Borel probability measures. If the system (Z, ζ) constructed is uniquely ergodic case, the resulting C*-algebra $C(Z) \rtimes_\zeta \mathbb{Z}$ then has the same invariant as $\mathcal{Z}$, except for its nontrivial K_1-group. For such a crossed product, nontrivial K_1 is unavoidable (the class of the unitary implementing the action is nontrivial). However, upon breaking the orbit equivalence relation across a single point, K_1 disappears while the rest of the invariant remains the same. Now the C*-algebra arising from this equivalence relation does in fact have the correct invariant and we are able to use classification theory to conclude that it must be $\mathcal{Z}$.

Our construction itself is more general, and in fact we are able to produce C*-algebras isomorphic to any simple inductive limit of prime dimension drop algebras with an arbitrary number of extreme tracial states, as constructed in [10]. From the C*-algebraic perspective such a construction is interesting: even the range of the invariant for such C*-algebras remains unknown. In the uniquely ergodic case, classification for the resulting crossed product follows from [30, 31] (which uses the main result in [27]). Without assuming unique ergodicity, we may appeal to Lin's generalisation [13] of the third author's classification result for products with Cantor systems [26], to show all our minimal dynamical systems result in classifiable crossed products. We note that as we were finishing this paper, Lin posted a classification theorem for all crossed product C*-algebras associated to minimal dynamical systems with mean dimension zero [14], but our results do not rely on his proof.

In Section 1, we start with a minimal diffeomorphism, φ, on an d-sphere, for odd $d > 1$, which we denote by S^d. This is a logical place to begin since the cohomology of the sphere differs from that of a point only in dimension d. From this, we construct a space, Z, together with a minimal homeomorphism, ζ. This system is an extension of (S^d, φ); that is, there a a factor map from (Z, ζ) onto (S^d, φ). The space Z is an infinite compact finite-dimensional metric space with the same cohomology and K-theory as a single point. From our minimal dynamical system we show in Section 2 that the C*-algebra of the associated orbit-breaking equivalence relation is isomorphic to the Jiang–Su algebra, assuming we have begun with uniquely ergodic (S^d, φ). In Section 3 we show that, with any number of ergodic measures, the associated transformation group C*-algebras and their orbit-breaking subalgebras can be distinguished by their tracial states spaces and are all isomorphic to direct limits of dimension-drop algebras. Finally, in Section 4 we make some comments on further questions.

1. Constructing the system

In this section we fix $d > 1$ odd and a minimal diffeomorphism $\varphi : S^d \to S^d$, but we remind the reader that the space constructed does in fact depend on which minimal dynamical system (S^d, φ) we use. In particular, we may choose (S^d, φ) to have any number of ergodic probability measures [33]. We fix an orientation on S^d and note that φ is orientation-preserving; otherwise the system would have a fixed point.

1.1 Lemma: *Let x be any point of S^d and v any non-zero tangent vector at x. There exists*

$$\lambda : [0,1] \to S^d$$

satisfying the following:

(i) $\lambda(0) = x, \lambda'(0) = v, \lambda(1) = \varphi(x), \lambda'(1) = D\varphi(v)$,
(ii) *λ is a a C^1-embedding, in particular $\lambda'(t) \neq 0$, for all t in* $[0,1]$,
(iii) *For all $n \neq 0$,*

$$\varphi^n(\lambda([0,1)) \cap \lambda([0,1)) = \emptyset.$$

Proof: The space of all C^1-maps from $[0,1]$ into S^d which satisfy the first condition is a non-empty, complete metric space with the metric from the C^1-norm. Let Λ be the subset of these also satisfying the second condition of the conclusion. This is clearly non-empty and open (see for example [9]). Thus, Λ is a Baire space.

We need to establish the existence of a map satisfying the third condition as well. We treat the end points of $[0,1]$ separately.

For each integer $n \neq 0, 1$, let Λ_n^0 be those elements of Λ such that $\varphi^n(x) \notin \lambda[0,1]$. This is clearly an open dense subset of Λ. In particular, the intersection over all $n \neq 0, 1$, which we denote Λ_∞^0, is a dense G_δ in Λ, and hence also a Baire space.

Fix λ in Λ. For each $n \geq 1, k \geq 2$, define

$$\begin{aligned} R_{n,k}(\lambda) &= \{(s_1, s_2, \ldots, s_k) \mid s_i \in [0,1], \varphi^n(\lambda(s_i)) = \lambda(s_{i+1}), 1 \leq i < k\}, \\ X_{n,k}(\lambda) &= \{s_1 \mid (s_1, s_2, \ldots, s_k) \in R_{n,k}(\lambda)\}. \end{aligned}$$

Let us start with some simple observations.

(i) $X_{n,k}(\lambda) \supseteq X_{n,k+1}(\lambda)$, for all $n \geq 1, k \geq 2$.
(ii) It follows from the first condition that $(0,1) \in R_{1,2}(\lambda)$ and is an isolated point; $0 \in X_{1,2}(\lambda)$ and is an isolated point.
(iii) It follows from the preceding two properties that, for any $k \geq 2$, if 0 is in $X_{1,k}(\lambda)$, then it is an isolated point.
(iv) For all n, k, $R_{n,k}(\lambda)$ is closed in $[0,1]^k$ and $X_{n,k}(\lambda)$ is closed in $[0,1]$.

(v) For all $n \geq 1, k \geq 2$, we have

$$X_{n,k}(\lambda) = \emptyset \Leftrightarrow R_{n,k}(\lambda) = \emptyset \Rightarrow R_{n,k+1}(\lambda) = \emptyset \Leftrightarrow X_{n,k+1}(\lambda) = \emptyset.$$

(vi) $\{\lambda \in \Lambda \mid X_{1,2}(\lambda) = \{0\}\}$ is open in Λ.

(vii) For any $n \geq 1, k \geq 2$, $\{\lambda \in \Lambda \mid X_{n,k}(\lambda) = \emptyset\}$ is open in Λ.

(viii) If $X_{1,2}(\lambda) = \{0\}$ and $X_{n,2}(\lambda) = \emptyset$, for all $n \neq 0, 1$, then λ satisfies the last condition of the conclusion of the lemma.

Our first important claim is that, for any $n \geq 1$, there exists $k > 1$ with $R_{n,k}(\lambda) = \emptyset$. Suppose the contrary. We note there is an obvious map from $R_{n,k}$ to $R_{n,k-1}$ and we may form the inductive limit of this system. Using the compactness of $R_{n,k}$, we see that if each $R_{n,k}$ is non-empty, we may find a sequence $s_1, s_2, \ldots$ such that $\varphi^n(\lambda(s_i)) = \lambda(s_{i+1})$, for all $i \geq 1$. But this means that the forward orbit of $\lambda(s_1)$ under φ is contained in $\cup_{i=0}^{n-1} \varphi^i(\lambda[0,1])$, which is a closed subset of S^d. It is non-empty and cannot be all of S^d on dimensionality grounds. This then contradicts the minimality of φ.

Our second claim is the following. Suppose that λ is in Λ, $n \geq 1, k \geq 2, (n,k) \neq (1,2)$ are such that $X_{n,k+1}(\lambda) = \emptyset$. Then for any $\epsilon > 0$, there is μ in Λ with $\|\lambda - \mu\|_1 < \epsilon$ and $X_{n,k}(\mu) = \emptyset$.

In view of the first claim above and the fact that $X_{n,k+1}(\lambda) = \emptyset$ is an open property, there is no loss of generality if we assume that λ is in Λ^0_∞. The immediate consequence of this is that $0, 1$ are not in $X_{n,k}(\lambda)$.

Suppose that s is in $[0,1]$ with $\varphi^n(\lambda(s))$ in $\lambda(X_{n,k}(\lambda)))$, then $\varphi^n(\lambda(s)) = s_1$ with $(s_1, s_2, \ldots, s_k)$ in $R_{n,k}(\lambda)$ and it follows that $(s, s_1, s_2, \ldots, s_k)$ is in $R_{n,k+1}(\lambda)$ which contradicts our hypothesis that $X_{n,k+1}(\lambda) = \emptyset$. We conclude that the sets $\lambda(X_{n,k}(\lambda)))$ and $\varphi^n(\lambda[0,1])$ must be disjoint. Without loss of generality assume that ϵ is strictly less than half the distance between these two compact sets.

For each s in $X_{n,k}(\lambda)$, select $0 < a_s < s < b_s < 1$ such that $\lambda(a_s, b_s)$ is contained in the ball of radius $\epsilon/2$ about $\lambda(s)$. These open intervals cover $X_{n,k}(\lambda)$. We may extract a finite subcover. If these intervals overlap, we may replace them with their unions to obtain $0 < a_1 < b_1 < a_2 < \cdots < b_n < 1$ with the union of the (a_i, b_i), which we denote by U, covering $X_{n,k}(\lambda)$. Observe that this means the points a_i, b_i are not in $X_{n,k}(\lambda)$.

Based on dimensionality, we may make an arbitrarily small C^1-perturbation of λ on U, which we call μ, not changing the value or derivative at the endpoints, so that the image is disjoint from $\varphi^{-n} \circ \lambda([0,1] - U)$. To be slightly more precise, the μ can be chosen from Λ so that $\|\lambda - \mu\|_1 < \epsilon/2$ and $\|\varphi^n \circ \lambda - \varphi^n \circ \mu\|_1 < \epsilon$.

We claim this μ satisfies $X_{n,k}(\mu) = \emptyset$. Suppose to the contrary that $(s_1, s_2, \ldots, s_k)$ is in $X_{n,k}(\mu)$. If, for some $j < k$, s_j is in U, then $\mu(s_j)$ is not in $\varphi^{-n} \circ \lambda([0,1] - U)$ or equivalently, $\varphi^n(\mu(s_j))$ is not in $\lambda([0,1] - U) = \mu([0,1] - U)$. On the other hand, $\varphi^n(\mu(s_j))$ is within ϵ of $\varphi^n(\lambda[0,1])$ and hence outside the ball of radius ϵ of $\lambda(X_{n,k}(\lambda))$, which contains $\mu(U)$. Between the two cases, we have shown that if s_j is in U, then $\varphi^n(\mu(s_j))$ is not in $\mu([0,1])$. This contradicts $\varphi^n(\mu(s_j)) = \mu(s_{j+1})$.

The only remaining case is that $s_1, s_2, \ldots, s_{k-1}$ all lie in $[0,1] - U$. But on this set, $\mu = \lambda$ and so we have $(s_1, s_2, \ldots, s_k)$ is in $X_{n,k}(\mu) = X_{n,k}(\lambda)$ and hence s_1 is in $X_{n,k}(\lambda) \subseteq U$, a contradiction.

Our third claim is a minor variation of the second to deal with the special case $n = 1, k = 2$, since 0 always lies in $X_{1,2}(\lambda)$. Suppose that λ is in Λ such that $X_{1,3}(\lambda) = \emptyset$. Then for any $\epsilon > 0$, there is μ in Λ with $\|\lambda - \mu\|_1 < \epsilon$ and $X_{1,2}(\mu) = \{0\}$. The idea is that 0 will be an isolated point of $X_{1,2}(\mu)$ and we can simply repeat the rest of the argument above replacing $X_{n,k}(\mu)$ by $X_{1,2}(\mu) - \{0\}$.

Our fourth claim is that, for any $n \neq 0, 1$, $\{\lambda \mid X_{n,2}(\lambda) = \emptyset\}$ is dense in Λ. Let λ be in Λ and let $\epsilon > 0$. From our first claim, we may find k with $X_{n,k}(\lambda) = \emptyset$. Next, use the second claim to find λ_1 within $\epsilon/2$ of λ with $X_{n,k-1}(\lambda_1) = \emptyset$. Apply the second claim again to find λ_2 within $\epsilon/4$ of λ_1 with $X_{n,k-2}(\lambda_2) = \emptyset$. Continuing in this way, we will end up with $X_{n,2}(\lambda_{k-2}) = \emptyset$ and

$$\|\lambda - \lambda_{k-2}\|_1 \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \ldots + \frac{\epsilon}{2^{k-2}} < \epsilon.$$

The fifth claim is a minor variation of the third: $\{\lambda \mid X_{1,2}(\lambda) = \{0\}\}$ is dense in Λ. The proof is the same as that of the fourth claim, using the third claim in place of the second.

As a result of all of this, the set

$$\{\lambda \mid X_{1,2}(\lambda) = \{0\}\} \cap (\cap_{n \neq 0,1} \{\lambda \mid X_{n,2}(\lambda) = \emptyset\})$$

is dense in Λ and hence non-empty. These maps satisfy all the desired conditions. ∎

Given a map $\lambda : [0, 1] \to S^d$ satisfying Lemma 1.1, we define a second map

$$\lambda_{\mathbb{R}} : \mathbb{R} \to S^d, \quad \lambda_{\mathbb{R}}(s) = \phi^{\lfloor s \rfloor}(\lambda(s \bmod 1))$$

where $\lfloor s \rfloor$ denotes the floor function applied to the real number s. The conditions in Lemma 1.1 ensure $\lambda_{\mathbb{R}}$ is also a C^1-embedding.

1.2 Lemma: *Let $F : [1, 2] \times \mathbb{R}^{d-1} \to \mathbb{R}^d$ be an orientation preserving C^1-embedding such that, for each $s \in [1, 2]$,*

$$F(s, 0) = (s, 0).$$

Then, there exists $G : [-1, 2] \times \mathbb{R}^{d-1} \to \mathbb{R}^d$ a continuous function satisfying the following

(i) *for $s \in [-1, 2]$, $G(s, 0) = (s, 0)$;*
(ii) *for $x \in \mathbb{R}^{d-1}$ and $s \in [-1, 0]$, $G(s, x) = (s, x)$;*
(iii) *for $x \in \mathbb{R}^{d-1}$ and $s \in [1, 2]$, $G(s, x) = F(s, x)$;*
(iv) *there exists $\delta > 0$ such that $G|_{[-1,2] \times \overline{B^{d-1}(\delta)}}$ is injective.*

Proof: Note that we have

$$F(s, x) = (s + F_1(s, x), F_2(s, x))$$

where $F_1(s, 0) = 0$ and $F_2(s, 0) = 0$. Observe that

$$DF(s, x) = \begin{pmatrix} 1 + \frac{\partial F_1}{\partial s}(s, x) & \frac{\partial F_1}{\partial x}(s, x) \\ \frac{\partial F_2}{\partial s}(s, x) & [DF_2(s, x)]_{i,j=2}^{d-1}(s, x) \end{pmatrix}$$

where by abuse of notation the partial derivative of F_1 with respect to x, $\frac{\partial F_1}{\partial x}$, where $x = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, denotes $(\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_{d-1}})$ and $[DF_2(s, x)]_{ij=2}^{d-1}$ is the minor of the Jacobian matrix of $F_2(s, x)$ obtained by removing the first row and column. The conditions $F_1(s, 0) = 0$, respectively $F_2(s, 0) = 0$ imply that $\frac{\partial F_1}{\partial s}(s,0) = 0$, respectively $\frac{\partial F_2}{\partial s}(s,0) = 0$. Moreover, the fact that F is an orientation-preserving C^1-embedding implies

$$\det(DF(s,x)) > 0.$$

At $x = 0$ we have

$$\det(DF(s,0)) = \det([DF_2(s,0)]_{i,j=2}^{d-1}),$$

so continuity implies that there is some $\delta > 0$ such that

$$\det([DF_2(s,x)]_{i,j=2}^{d-1}) > 0 \tag{1}$$

for every $x \in B^{d-1}(\delta)$.

We define

$$G(s,x) = \begin{cases} F(s,x) & : \; s \in [1,2] \\ (s + F_1(1,x), F_2(1,x)) & : \; s \in [3/4,1) \\ (s + 4(s-1/2)F_1(1,x), F_2(1,x)) & : \; s \in [1/2,3/4) \\ (s, (4s-1)^{-1}F_2(1,(4s-1)x)) & : \; s \in (1/4,1/2) \\ (1/4, [DF_2(1,0)]_{i,j=2}^{d}\; x) & : \; s = 1/4 \\ (s, [a_{i,j}(s)]_{i,j=1}^{d-1}\; x) & : \; s \in [0,1/4) \\ (s,x) & : \; s \in [-1,0) \end{cases}$$

where $[a_{i,j}(s)]_{i,j=1}^{d-1}$ is a smooth arc of matrices with positive determinant with

$$[a_{i,j}(1/4)]_{i,j=1}^{d-1} = [DF_2(1,0)]_{i,j=2}^{d}, \quad [a_{i,j}(0)]_{i,j=1}^{d-1} = I.$$

(Recall that $\det([DF_2(1,0)]_{i,j=2}^{d}) > 0$.) It is easy to check that $G(s,x)$ is a continuous function satisfying (i) – (iii).

We show G satisfies (iv). On the interval $[1,2]$ injectivity is clear since F is an embedding.

For the next interval, $[3/4,1)$ we have that $(1+F_1(1,x), F_2(1,x))$ is an embedding from $\{1\}\times\mathbb{R}^{d-1} \hookrightarrow \mathbb{R}^d$, by the same reasoning. Suppose that $(s+F_1(1,x), F_2(1,x)) = (s' + F_1(1,x'), F_2(1,x'))$. From equation (1) we have $\det([DF_2(1,x)]_{i,j=2}^{d}) > 0$, so using the inverse function theorem (and possibly decreasing δ) $F_2(1,\cdot)$ is invertible in $B^{d-1}(\delta)$. Hence $x = x'$ and then also $s = s'$, so G is injective on $[3/4,1)\times B^{d-1}(\delta)$.

On $[1/2,3/4)$, we again have that $F_2(1,x) = F_2(1,x')$ implies that $x = x'$ if $x, x' \in B^{d-1}(\delta)$. Moreover, if $G(s,x) = G(s',x)$, then $s = s'$, so again G is injective (this time on $[1/2,3/4) \times B^{d-1}(\delta)$).

In $(1/4,1/2)$ if $G(s,x) = G(s',x')$ it is automatic that $s = s'$, hence the result follows again since $F_2(1,x)$ is injective on $B^{d-1}(\delta)$.

At $s = 1/4$ the result follows from the Jacobian condition on F_2 (equation (1)).

For the interval $(0,1/4)$, the result follows from the fact that each $[a_{i,j}(s)]_{i,j=1}^{d-1}$ have determinant greater than zero, hence $G(s,x)$ is injective on $[-1,2] \times B^{d-1}(\delta)$. ∎

We remark that $G(s,x)$ is C^1 except for when $s \in \{1/2, 3/4, 1\}$. With a bit more care, one can show that $G(s,x)$ could be made C^1, but we will not need this.

1.3 Lemma: *There exists $\epsilon > 0$ and a continuous injection*

$$\tau_0 : [-\epsilon, 1+\epsilon] \times \overline{B^{d-1}(1)} \to S^d$$

satisfying the following.

(i) *For* $|s| \leq \epsilon$ *and* x *in* $\overline{B^{d-1}(1)}$,

$$\tau_0(s+1, x) = \varphi(\tau_0(s, x)).$$

(ii) *For all* $n \neq 0$,

$$\varphi^n(\tau_0([0,1) \times \{0\})) \cap \tau_0([0,1) \times \{0\}) = \emptyset.$$

PROOF: Let $\lambda_{\mathbb{R}} : \mathbb{R} \to S^d$ be as defined following the proof of Lemma 1.1. By the tubular neighbourhood theorem there exists a C^1 embedding

$$\alpha : [-1/2, 1+1/2] \times \mathbb{R}^{d-1} \to S^d$$

such that

$$\alpha(s, 0) = \lambda_{\mathbb{R}}|_{[-1/2,1+1/2]}. \tag{2}$$

There exist $\epsilon, \gamma > 0$ such that

$$\varphi \circ \alpha([-\epsilon, \epsilon] \times B^{d-1}(\gamma)) \subset \mathrm{Im}(\alpha).$$

Let $\beta : \mathbb{R} \times B^{d-1}(\gamma) \to \mathbb{R} \times B^{d-1}(\gamma)$ be given by $\beta(s, x) = (s-1, x)$. Define $F : [1-\epsilon, 1+\epsilon] \times B^{d-1}(\gamma) \to \mathbb{R}^d$ by

$$F = \alpha^{-1} \circ \varphi \circ \alpha \circ \beta.$$

By construction, F is an orientation-preserving C^1-embedding satisfying $F(s, 0) = (s, 0)$ for every $s \in [1-\epsilon, 1+\epsilon]$. Hence we may apply the previous lemma to get $G : [-\epsilon, 1+\epsilon] \times B^{d-1}(\gamma) \to [-1/2, 1+1/2] \times \mathbb{R}^{d-1}$. (Note that the intervals are slightly different but this does not matter). From the lemma we have that

$$G(s, 0) = (s, 0)$$

for every $s \in [-\epsilon, 1+\epsilon]$ and that G is injective by possibly shrinking γ.

Define $\tau_0 : [-\epsilon, 1+\epsilon] \times \overline{B^{d-1}(1)} \to S^d$ by $\tau_0 = \alpha \circ G$, (where we are tacitly using the fact that $\overline{B^{d-1}(1)}$ can be identified with $\overline{B^{d-1}(\gamma/2)}$.)

We now show that τ_0 has properties (i) and (ii). Property (ii) follows immediately from the fact that $\tau_0(s, 0) = \lambda_{\mathbb{R}}(s)$ and Lemma 1.1 (iv). For property (i) we have, for $s \in [-\epsilon, \epsilon]$, that

$$\tau_0(s+1, x) = (\alpha \circ G)(s+1, x) = \varphi \circ \alpha \circ \beta(s+1, x) = \varphi(\alpha(s, x)),$$

and

$$\varphi(\tau_0(s, x)) = \varphi(\alpha \circ G(s, x)) = \varphi(\alpha(s, x)).$$

∎

Given τ_0 as in the previous lemma, we define $\tau : \mathbb{R} \times \overline{B^{d-1}(1)} \to S^d$ by

$$\tau(s, x) = \varphi^{\lfloor s \rfloor} \tau_0(s \bmod 1, x). \tag{3}$$

1.4 LEMMA: *There exists a sequence of positive numbers* $1 > \rho_1 > \rho_2 > \cdots > 0$ *such that* $\tau|_{[-n-2,n+2] \times \overline{B^{d-1}(2\rho_n)}}$ *is injective.*

Define functions $r_n : \mathbb{R} \to [0, 1]$ by

$$r_n(s) = \begin{cases} 0, & |s| > n, \\ n - |s|, & n - \rho_n \leq |s| \leq n, \\ \rho_n, & |s| \leq n - \rho_n. \end{cases}$$

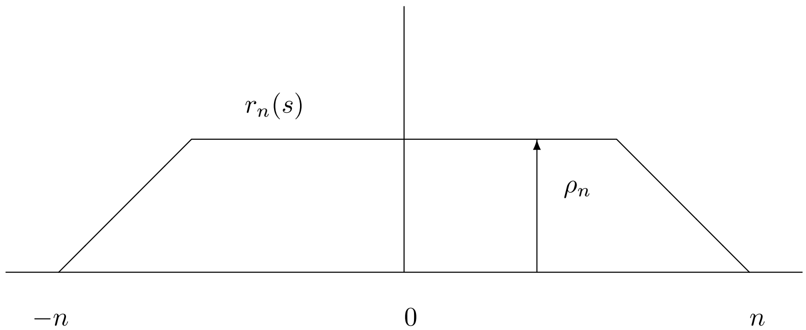

FIGURE 1. Graph of $r_n(s)$

For $n \geq 1$, define

$$L_n = \tau([-n, n] \times \{0\}), \text{ and } L_\infty = \tau(\mathbb{R} \times \{0\}).$$

Also define, for each $n \geq 1$,

$$X_n = S^d - \tau\{(s, x) \mid -n \leq s \leq n, |x| \leq r_n(s)\}. \tag{4}$$

1.5 LEMMA: *The closure of X_n, $\overline{X_n}$, is contractible.*

PROOF: It is clear that $\tau(\{(s, x) \mid -n < s < n, |x| < r_n(s)\})$ is the complement of $\overline{X_n}$, it is an open set in S^d homeomorphic to an open ball. Removing an open set of this form in S^d yields a space homeomorphic to a closed ball in $\mathbb{R}^d$, which is contractible. ∎

We define a function $\beta_n : S^d - L_n \to S^d$ as follows. Let

$$R_n = \tau\{(s, x) \mid -n \leq s \leq n, 0 < |x| \leq 2r_n(s), x \in \mathbb{R}^d\}. \tag{5}$$

Observe that any point in R_n may be written uniquely as $\tau(s, tx)$, with $-n \leq s \leq n$, $0 < t \leq 2r_n(s)$ and x in S^{d-2}. We then define

$$\beta_n(\tau(s, tx)) = \tau\left(s, \left(\frac{t}{2} + r_n(s)\right) x\right). \tag{6}$$

Observe that β_n fixes any point with $t = 2r_n(s)$. We set β_n to be the identity on $S^d - L_n - R_n$.

1.6 LEMMA: *For $\beta_n : S^d - L_n \to S^d$ defined as in (6), the following hold:*

(i) *β_n is continuous.*
(ii) *The image $\beta_n(S^d - L_n)$ is X_n.*
(iii) *β_n is injective.*
(iv) *$\beta_n^{-1} : X_n \to S^d - L_n$ is continuous.*

PROOF: Each of (i) – (iv) is a straightforward calculation. ∎

Let us add one more useful observation: at this point, β_n is not defined on $\tau(\pm n, 0)$, but if we extend the definition to leave these fixed, it is still continuous.

Let d_0 be any fixed metric on S^d which yields the usual topology. Of course, since S^d is compact, d_0 is bounded.

We define a sequence of metrics, d_n on $S^d - L_n$, by

$$d_n(x, y) = d_0(\beta_n(x), \beta_n(y)), \quad x, y \in S^d - L_n. \tag{7}$$

That is, β_n is an isometry from $(S^d - L_n, d_n)$ to (X_n, d_0).

1.7 Definition: *Define Z_n to be the completion of $S^d - L_n$ in d_n.*

Given these definitions, the following is obvious since the completion of X_n in the metric d_0 is simply its closure.

1.8 Lemma: *The map β_n extends to a homeomorphism from (Z_n, d_n) to $(\overline{X_n}, d_0)$. In particular, Z_n is connected, contractible, compact, $\dim(Z_n) \leq d$ and is an absolute neighbourhood retract.*

Now we establish the following important relations between our metrics.

1.9 Lemma: *Let $(z_k)_{n\in\mathbb{N}}$ be a sequence in $S^d - L_n$, $n \geq 1$. Then*

(i) *if $(z_k)_{n\in\mathbb{N}}$ is Cauchy in d_n then it is also Cauchy in d_0,*
(ii) *if $(z_k)_{n\in\mathbb{N}}$ is Cauchy in d_n then it is also Cauchy in d_{n-1}.*

Proof: Let us consider the first part. It suffices to prove that if $(z_k)_{k\in\mathbb{N}}$ is any sequence in $S^d - L_n$ such that $\beta_n(z_k)$ is Cauchy in S^d in d_0, then $(z_k)_{k\in\mathbb{N}}$ itself is Cauchy in d_0.

Let us first suppose that $(z_k)_{k\in\mathbb{N}}$ lies entirely in the complement of R_n. Then β_n is the identity on z_k and the conclusion is clear. Now let us assume the sequence lies entirely inside R_n. We write $z_k = \tau((s_k, t_k x_k))$, with $s_k \in [-n, n]$, $0 < t_k \leq 2r_n(s_k)$ and x_k in S^{d-2}. The fact $(\beta_n(z_k))_{k\in\mathbb{N}}$ is Cauchy in d_0 and that τ is uniformly continuous means that $\left(s_k, \left(\frac{t_k}{2} + r_n(s_k)\right) x_k\right)$ is Cauchy in the usual metric of $[-n, n] \times \mathbb{R}^{d-1}$. It follows that $(s_k)_{k\in\mathbb{N}}$ converges to some s in $[-n, n]$, while $\left(\frac{t_k}{2} + r_n(s_k)\right) x_k$ converges to y. Taking norms and recalling that x_k is a unit vector, we see that $\frac{t_k}{2} + r_n(s_k)$ converges to $|y|$. Putting these together we have

$$2|y| = \lim_k t_k + 2r_n(s_k) = 2r_n(s) + \lim_k t_k$$

and so t_k converges to $2(|y| - r_n(s))$.

If $|y| - r_n(s) = 0$, then $t_k x_k$ converges to the zero vector. If $|y| - r_n(s) \neq 0$, then $x_k, k \geq 1$ converges to $(2(|y| - r_n(s)))^{-1} y$ and again $t_k x_k$ is convergent.

Finally, we need to consider the case where $(z_k)_{k\in\mathbb{N}}$ contains infinitely many terms outside the region and infinitely many terms inside the region. From the arguments above, the two subsequence each converge in the usual topology to two points, say z_{out} and z_{in}, respectively. But we also have $\beta_n(z_{out}) = z_{out}$. So the entire sequence $\beta_n(z_k)$ is converging to the point z_{out} which lies in the range of β_n. Then the conclusion follows from the fact that β_n^{-1} is continuous. This completes the proof of the first statement.

For the second part, it suffices to prove that if $(z_k)_{k\in\mathbb{N}}$ is any sequence in $S^d - L_n$ such that $(\beta_n(z_k))_{k\in\mathbb{N}}$ is Cauchy in S^d in the usual metric, then $(\beta_{n-1}(z_k))_{k\in\mathbb{N}}$ also Cauchy.

Let us consider the case when the sequence $(z_k)_{k\in\mathbb{N}}$ is outside R_n. In this case, we have $\beta_n(z_k) = z_k$. Then $(z_k)_{k\in\mathbb{N}}$ is Cauchy and hence convergent. We note that the limit point lies outside of R_n or on its boundary, and hence is not in L_{n-1}. It follows since β_{n-1} is continuous that $\beta_{n-1}(z_k)$ is convergent and hence Cauchy.

Now suppose that z_k is in R_n and write $z_k = \tau(s_k, t_k x_k)$ as before. As in the first case, we know that s_n converges to some s in $[-n, n]$, while $\frac{t_k}{2} + r_n(s_k)x_k$ converges to y. If $|y| - r_n(s) \neq 0$, then as before, z_k is converging to a point in $S^d - L_n$, which is a subset of $S^d - L_{n-1}$. We use the fact that β_{n-1} is continuous to conclude that $\beta_{n-1}(z_k)$ is convergent and hence Cauchy.

Now suppose that $|y| - r_n(s) = 0$. In this case, t_k is converging to zero and this means that

$$y = \lim_k \left(\frac{t_k}{2} + r_n(s_k)\right) x_k = \lim_k r_n(s_k)x_k.$$

First suppose that $1 - n < s < n - 1$. This implies that $r_n(s) = r_n \neq 0$ and so x_k is actually convergent. It follows that

$$\lim_k \beta_{n-1}(z_k) = \lim_k \tau\left(s_k, \left(\frac{t_k}{2} + r_{n-1}(s_k)\right) x_k\right) = \tau\left(s, r_{n-1}(s) \lim_k x_k\right)$$

and so $\beta_{n-1}(z_k), k \geq 1$ is Cauchy.

Next, suppose that $n - 1 \leq |s| \leq n$. Here we have $r_{n-1}(s) = 0$ and so

$$\lim_k \beta_{n-1}(z_k) = \lim_k \tau\left(s_k, \left(\frac{t_k}{2} + r_{n-1}(s_k)\right) x_k\right) = \tau(s, 0)$$

and again $(\beta_{n-1}(z_k))_{k\in\mathbb{N}}$ is Cauchy.

Finally, suppose $(\beta_{n-1}(z_k))_{k\in\mathbb{N}}$ contains infinitely many terms outside the region and infinitely many terms inside the region. As in the proof of (i), we can find two subsequences $(\beta_{n-1}(z_k))_{\text{in}}$ and $(\beta_{n-1}(z_k)_{\text{out}})$. We know $(\beta_{n-1}(z_k)_{\text{out}})$ converges to some x_{out} by the above. One checks that x_{out} lies outside R_n so $\beta_n(x_{\text{out}}) = x_{\text{out}}$. By the same reasoning as in part (i), we have that $(\beta_n(z_k))_{k\in\mathbb{N}}$ converges to x_{out}. Then by continuity of β_n, β_n^{-1} and β_{n-1} we have that

$$\beta_{n-1}(z_k) = \beta_{n-1} \circ \beta_n^{-1} \circ \beta_n(z_k) \to \beta_{n-1}(x_{\text{out}}), \text{ as } k \to \infty$$

so $(\beta_{n-1}(z_k))_{k\in\mathbb{N}}$ is also Cauchy. ∎

Observe that, for every $n \geq 1$, S^d is the completion of $S^d - L_n$ in d_0. It is also the completion of $S^d - L_\infty$ in d_0.

The immediate consequence of the last lemma is that the identity map on $S^d - L_\infty$ extends to well-defined, continuous maps $\pi_n : Z_n \to Z_{n-1}$ and $q_n : Z_n \to S^d$. Furthermore, both π_n and q_n are surjective since in both cases $\mathrm{Im}(Z_n)$ is compact and dense.

Define a metric d_∞ on $S^d - L_\infty$ by

$$d_\infty(x, y) = \sum_{n\geq 1} 2^{-n} d_n(x, y).$$

This uses the fact that each d_n is bounded by the same number which bounds d_0. We observe that a sequence $(z_k)_{k\in\mathbb{N}}$ is Cauchy in d_∞ if and only if it is Cauchy in each d_n.

1.10 Definition: *Define Z to be the completion of $S^d - L_\infty$ in the metric d_∞.*

Based on the relationship between the (Z_n, d_n), $n \geq 1$, we also have the following description of (Z, d_∞):

1.11 Lemma: *The space Z is homeomorphic to the inverse limit of the system*

$$Z_1 \stackrel{\pi_2}{\leftarrow} Z_2 \stackrel{\pi_3}{\leftarrow} \cdots .$$

1.12 Many of the properties of $Z_n, n \geq 1$ which we observed in Lemma 1.8 are preserved under inverse limits: compactness, connectedness and $\dim(Z_n) \leq d$. Of course, contractibility is not preserved, but both cohomology and K-theory are continuous. Therefore we have the following:

Corollary: *The space Z is an infinite connected, compact metric space with finite covering dimension. It has the same cohomology and K-theory as a point; in fact, $C(Z)$ is KK-equivalent to $\mathbb{C}$.*

Proof: The only thing we have not shown is the KK-equivalence, but this follows from the UCT and $K_*(C(Z)) \cong \mathbb{Z} \oplus 0$. ∎

The identity map on $S^d - L_\infty$ extends to a well-defined, continuous, surjective map $q : Z \to S^d$.

1.13 Lemma: *For each point x in $S^d - L_\infty$, $q^{-1}\{x\}$ is a single point.*

Proof: We treat the points of Z and S^d as equivalence classes of Cauchy sequences in $S^d - L_\infty$ in the metrics d_∞ and d_0, respectively. Let $(w_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$ be two Cauchy sequences $S^d - L_\infty$ in d_∞ and suppose they map to the same point under q, That is, they are equivalent in the metric d_0. Suppose also that they converge to a point z in $S^d - L_\infty$ in d_0. Fix $n \geq 1$. As z is in $S^d - L_\infty$ it is also in $S^d - L_n$. The latter is open in S^d in the metric d_0, so we may find an open ball B whose closure is contained in $S^d - L_n$. For k sufficiently large, both z_k and w_k are in B. The function β_n is defined and continuous on B and so we conclude that both sequences $\beta_n(z_k)$ and $\beta_n(w_k)$ are converging to $\beta_n(z)$ in d_0. Hence, the sequences z_k and w_k are equivalent in the d_n metric. As this is true for every n, these sequences are also equivalent in the d_∞ metric and we are done. ∎

We now turn to the problem of defining our minimal dynamical system.

1.14 Lemma: *If a sequence $(z_k)_{k\in\mathbb{N}}$ in $S^d - L_\infty$ is Cauchy in d_n, with $n \geq 2$, then both sequences $(\varphi(z_k))_{k\in\mathbb{N}}$ and $(\varphi^{-1}(z_k))_{k\in\mathbb{N}}$ are Cauchy in d_{n-1}.*

Proof: We consider only the case $(\varphi(z_k))_{k\in\mathbb{N}}$, the other being similar. Once again it suffices to assume that $(\beta_n(z_k))_{k\in\mathbb{N}}$ is Cauchy in d_0 and show the same is true for $(\beta_{n-1} \circ \varphi(z_k))_{k\in\mathbb{N}}$.

If the entire sequence lies outside of R_n, then $z_k = \beta_n(z_k)$ converges (in the d_0 metric) to some point, say y, in the complement of R_n or on its boundary. If y is not in L_n, then $\varphi(y)$ is not in L_{n-1} and then same argument using the continuity of β_{n-1} gives the desired result. The only point in the closure of the complement of R_n with $\varphi(y)$ in L_{n-1} is $\tau(-n, 0)$ with $\varphi(y) = \tau(1-n, 0)$. We noted earlier that β_{n-1} extends continuously to this point by fixing it and so the same argument works here.

Now assume that z_k lies in R_n but the sequence $\varphi(z_k)$ lies outside R_{n-1}. In this case, $\beta_{n-1}(\varphi(z_k)) = \varphi(z_k)$. But we know from the first part of Lemma 1.9 that $(z_k)_{k\in\mathbb{N}}$ is Cauchy with respect to d_0 and hence so is $\varphi(z_k)$ and we are done.

Now assume z_k lies in R_n and $\varphi(z_k)$ lies in R_{n-1}. As before we write

$$z_k = \tau(s, tx),$$

with $s \in [-n, n]$, $0 < t \leq 2r_n(s)$ and $x \in S^{d-2}$. In this case, using the definition of τ given in (3), we have

$$\beta_{n-1} \circ \varphi(z_k) = \tau\left(s_k + 1, \left(\frac{t_k}{2} + r_{n-1}(s_k+1)\right) x_k\right).$$

The argument proceeds exactly as before. We know s_k converges to s and t_k converges to $2(|y| - r_n)$.

If this is positive then x_k also converges and the desired result follows easily from the formula above for $\beta_{n-1} \circ \varphi(z_k)$.

If $|y| = r_n(s)$ then t_k converges to zero. We break this up into three cases.

Case 1: $s > n-2$. Then $\varphi(z_k)$ is not in R_{n-1} and this case is already done.

Case 2: $-n < s \leq n-2$. Here $r_n(s) > 0$ and it follows from the fact that $\left(\frac{t_k}{2} + r_n(s_k)\right) x_k$ is converging to y that the sequence x_k itself is convergent. The convergence of $\beta_{n-1} \circ \varphi(z_k)$ follows from the formula above.

Case 3: $s = -n$. In this case, both t_k and $r_{n-1}(s_k + 1)$ are converging to 0 and the convergence of $\beta_{n-1} \circ \varphi(z_k)$ again follows from the formula above.

Finally, when $(z_k)_{k\in\mathbb{N}}$ has infinitely many terms lying both outside and inside the region, the proof is similar to previous calculations. ∎

1.15 Corollary: *The map φ on $S^d - L_\infty$ extends to a homeomorphism of (Z, d_∞), denoted by ζ. Moreover, we have $q \circ \zeta = \varphi \circ q$; that is q is a factor map from (Z, ζ) to (S^d, φ).*

1.16 Theorem: *The homeomorphism ζ of Z is minimal.*

Proof: Let Y be a non-empty, closed (hence compact), ζ-invariant subset of Z. It follows that $q(Y)$ is a non-empty, compact (hence closed) φ-invariant subset of S^d and hence $q(Y) = S^d$. As the quotient map q is injective on $S^d - L_\infty \subseteq Z$, it follows that $S^d - L_\infty \subseteq Y$. As Y is closed and Z is defined as the completion of $S^d - L_\infty$, we conclude that $Y = Z$ and so ζ is minimal. ∎

1.17 Theorem: *The factor map q defined in Corollary 1.15 induces an affine bijection between the ζ-invariant Borel probability measures on Z and the φ-invariant Borel probability measures on S^d.*

Proof: Let μ be a ζ-invariant measure. Then $q^*(\mu)$ is φ-invariant. The set L_∞ is a Borel subset of S^d. It is also φ-invariant. Moreover, the system φ, restricted to L_∞ is conjugate to the map $x \to x+1$ on $\mathbb{R}$, which has no nonzero finite invariant measures. This implies that $q^*(\mu)(L_\infty) = 0$. This means that $S^d - L_\infty$ has full measure under μ and as q is a bijection on this set, the conclusion follows. ∎

1.18 We remark that one may modify the construction of the embedding of $\mathbb{R}$ into S^d to an embedding of two disjoint copies of $\mathbb{R}$. Proceeding in an analogous fashion, the space Z_n is homeomorphic to the sphere with two open balls removed. This can easily seen to be homeomorphic to $[0,1] \times S^{d-1}$. Continuing, the space Z can be seen to have the same cohomology as the even sphere S^{d-1} and admits a minimal homeomorphism (which can be arranged to be uniquely ergodic) while S^{d-1} does not by the Lefschetz–Hopf Theorem.

1.19 The map $q : Z \to S^d$ factors through Z_n:

$$Z \stackrel{p_n}{\to} Z_n \stackrel{q_n}{\to} S^d,$$

where p_n is the obvious map from Z to Z_n, $n \geq 1$. The sets $q_n^{-1}(\lambda(0)) \subseteq Z_n$ and $q^{-1}(\lambda(0)) \subseteq Z$ are both seen to be homeomorphic to S^{d-1}. If we regard these as embeddings of S^{d-1} into Z_n and Z, respectively, the former is clearly homotopic to a point since Z_n is contractible. On the other hand, it would seem that the latter is non-trivial and suggests that the homotopy group $\pi_{d-1}(Z)$ is non-trivial. In particular, if this is correct, Z is not contractible.

2. A dynamical presentation of the Jiang–Su algebra

We begin this section by recalling some facts about the Jiang–Su algebra (see [10]). In what follows, for any $n \in \mathbb{N}$, we let M_n denote the C*-algebra of $n \times n$ matrices over $\mathbb{C}$.

2.1 Definition: *Let $p, q \in \mathbb{N}$. The (p,q)-dimension drop algebra $A_{p,q}$ is the defined to be*

$$A_{p,q} = \{f \in C([0,1], M_p \otimes M_q \mid f(0) \in M_p \otimes 1_q, f(0) \in 1_p \otimes M_q\}.$$

Note that when p and q are relatively prime, $A_{p,q}$ is projectionless, that is, its only projections are 0 and 1.

2.2 Theorem: [10, Theorem 4.5] *Let G be an inductive limit of a sequence of finite cyclic groups and Ω a nonempty metrizable Choquet simplex. Then there exists a simple unital infinite-dimensional projectionless* C**-algebra A which is isomorphic to an inductive limit of dimension drop algebras and satisfies*

$$((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A)) \cong ((\mathbb{Z}, \mathbb{Z}_+, 1), G, \Omega).$$

In the same paper, Jiang and Su showed that any two such simple inductive limits of finite direct sums of dimension drop algebras are isomorphic if and only if their Elliott invariants are isomorphic [10, Theorem 6.2]. Moreover, the isomorphism of C*-algebras can be chosen to induce the isomorphism at the level of the invariant.

2.3 Definition: *The Jiang–Su algebra $\mathcal{Z}$ is the unique simple unital infinite-dimensional inductive limit of finite direct sums of dimension drop algebras satisfying*

$$\begin{aligned} ((K_0(\mathcal{Z}), K_0(\mathcal{Z})_+, [1_{\mathcal{Z}}]), K_1(\mathcal{Z}), T(\mathcal{Z})) &\cong ((\mathbb{Z}, \mathbb{Z}_+, 1), 0, \{\text{pt}\}) \\ &\cong ((K_0(\mathbb{C}), K_0(\mathbb{C})_+, [1_{\mathbb{C}}]), K_1(\mathbb{C}), T(\mathbb{C})) \end{aligned}$$

The goal of this section is to exhibit the Jiang–Su algebra $\mathcal{Z}$ as the C*-algebra of a minimal étale equivalence relation. As described in the introduction, this should be seen in analogy to the von Neumann algebra–measurable dynamical setting where the hyperfinite II_1 factor $\mathcal{R}$ is shown to be the von Neumann algebra an amenable measurable equivalence relation.

2.4 Definition: *Let X be a compact metrizable space. An equivalence relation $\mathcal{E} \subset X \times X$ with countable equivalence classes is called minimal if every equivalence class is dense in X.*

Let (X, α) be a minimal dynamical system of an infinite compact metric space. Let $\mathcal{E} \subset X \times X$ denote the orbit equivalence relation of (X, α). As described in [22], it is equipped with a natural topology in which it is étale. Note that the orbit equivalence relation from a minimal dynamical system is a minimal equivalence relation.

2.5 Definition: *For $y \in X$ the orbit-breaking equivalence relation $\mathcal{E}_y$ is defined as follows: If $(x, x') \in \mathcal{E}$ then $(x, x') \in \mathcal{E}_y$ if $\alpha^n(x) \neq y$ for any $n \in \mathbb{Z}$ or there are $n, m \geq 0$ such that $\alpha^n(x) = \alpha^m(x') = y$ or there are $n, m < 0$ such that $\alpha^n(x) = \alpha^m(x') = y$.*

Note that this splits any equivalence class in $\mathcal{E}$ containing the point y into two equivalence classes: one consisting of the forward orbit, the other of the backwards orbit. It is easily seen to be an open subset of $\mathcal{E}$ in the relative topology and, with that topology, is also étale.

This next result is well-known, but we rephrase it in terms of equivalence relations and give a proof for completeness.

2.6 Proposition: *Let (X, α) be a minimal dynamical system on an infinite compact metrizable space. For any $y \in X$, $\mathcal{E}_y$ is minimal and $\mathrm{C}^*(\mathcal{E}_y)$ is simple.*

Proof: Since α is minimal, for any point $x \in X$ both the forward orbit and backwards orbit are dense in X. It follows that every equivalence classes of $\mathcal{E}_y$ is dense in X. Since $\mathcal{E}_y$ is minimal the associated C*-algebra $\mathrm{C}^*(\mathcal{E}_y)$ is simple. (That $\mathrm{C}^*(\mathcal{E}_y)$ is simple also shown in [16, Proposition 2.5].) ∎

2.7 In what follows, $\mathcal{Q}$ denotes the universal UHF algebra, that is, the UHF algebra with $K_0(\mathcal{Q}) = \mathbb{Q}$.

Proposition: *Let Z be an infinite compact metrizable space satisfying $K^0(Z) \cong \mathbb{Z}$ and $K^1(Z) = 0$. Let $\zeta : Z \to Z$ be a minimal, uniquely ergodic homeomorphism. Then for any $z \in Z$ we have*

$$C^*(\mathcal{E}_z) \cong \mathcal{Z}.$$

Proof: First, we claim that the class of the trivial line bundle is the generator of $K^0(Z)$. Taking any map from the one-point space into Z and composing with the only map from Z onto a point, the composition (in that order) is clearly the identity. It then follows from our hypothesis on $K^*(Z)$ that these two maps actually induce isomorphisms at the level of K-theory and the claim follows.

Then, by the Pimsner–Voiculescu exact sequence, one calculates that

$$K_0(C(Z) \rtimes_\zeta \mathbb{Z}) \cong \mathbb{Z} \cong K_1(C(Z) \rtimes_\zeta \mathbb{Z}).$$

Next, we use the six-term exact sequence in [21, Theorem 2.4] (see also [21, Example 2.6]) to calculate that

$$K_0(\mathrm{C}^*(\mathcal{E}_z)) \cong \mathbb{Z}, \qquad K_1(\mathrm{C}^*(\mathcal{E}_z)) = 0.$$

Furthermore, we have that $T(C(Z) \rtimes_\zeta \mathbb{Z}) \cong T(\mathrm{C}^*(\mathcal{E}_z))$ [17, Theorem 1.2]. Thus $\mathrm{C}^*(\mathcal{E}_z)$ has the same invariant as $\mathcal{Z}$. By [17, Section 3], $\mathrm{C}^*(\mathcal{E}_z)$ is a simple approximately subhomogeneous algebra with no dimension growth. Since there is only one tracial state, projections separate traces and it follows from [3, Theorem 1.4] that $\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}$ has real rank zero whence $\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}$ is tracially approximately finite [34, Theorem 2.1]. Now $\mathcal{Z} \otimes \mathcal{Q}$ is also TAF. Since both these C^*-algebras are in the UCT class, we may apply [15, Theorem 5.4] to get that $\mathrm{C}^*(\mathcal{E}_z) \cong \mathcal{Z}$. ∎

2.8 Theorem: *There is a compact metric space Z with minimal, étale equivalence relation $\mathcal{E} \subset Z \times Z$ such that $\mathrm{C}^*(\mathcal{E}) \cong \mathcal{Z}$.*

Proof: For any $d > 1$ odd, there is a uniquely ergodic diffeomorphism $\varphi : S^d \to S^d$. Following the construction Section 1, there is a minimal dynamical system (Z, ζ) where Z satisfies the hypotheses of Theorem 2.7 by 1.12. Hence for any $z \in Z$ we have $\mathrm{C}^*(\mathcal{E}_z) \cong \mathcal{Z}$. ∎

3. Classification in the non-uniquely ergodic case

At present, few examples of crossed product C^*-algebras associated to minimal dynamical systems without real rank zero are known. This is largely due to a lack of examples of minimal dynamical systems (X, α) with $\dim X > 0$ and more than one ergodic measure. In general, classification for C^*-algebras without real rank zero is much more difficult. Real rank zero implies a plentiful supply of projections. Not only does this suggest more information is available in the invariant (in particular, the K_0-group), but it also makes the C^*-algebras easier to manipulate into a particular form, for example, to show it is tracially approximately finite (TAF) as defined in [11]. For a long time, the minimal diffeomorphisms of odd dimensional spheres were the main example of minimal dynamical systems leading to C^*-algebras which were not at least rationally TAF (that is, tracially approximately finite after tensoring with the universal UHF algebra). Their classification remained elusive for quite some time. By Theorem 1.16, our construction gives further examples lying outside the real rank zero case and we are able to use the classification techniques from the setting of the spheres to classify these crossed products. Furthermore, by using Winter's classification by embedding result [36, Theorem 4.2], we also classify the projectionless C^*-algebras obtained from the corresponding orbit-breaking sub equivalence relations.

For a given minimal homeomorphism $\varphi : S^d \to S^d$ let Z_φ denote the space constructed in Section 1, and denote by ζ the resulting minimal homeomorphism $\zeta : Z_\varphi \to Z_\varphi$ as in Theorem 1.16.

3.1 Proposition: $T(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \cong T(C(S^d) \rtimes_\varphi \mathbb{Z})$.

Proof: This follows immediately from Theorem 1.17, since tracial states on the crossed product are in one-to-one correspondence with invariant Borel probability measures of the dynamical system. ∎

3.2 Theorem: *As above, $\mathcal{Q}$ denotes the universal UHF algebra. Let $\mathcal{A}$ be the class of simple separable unital nuclear C^*-algebras given by*

$$\mathcal{A} = \{C(Z_\varphi) \rtimes_\zeta \mathbb{Z} \mid \varphi : S^d \to S^d, d > 1 \textit{ odd, is a minimal diffeomorphism } \}.$$

Then for any $A \in \mathcal{A}$, $A \otimes \mathcal{Q}$ is tracially approximately an interval algebra (TAI).

PROOF: Z_φ is infinite, compact, connected, finite-dimensional and $U(C(Z_\varphi)) = U_0(C(Z_\varphi))$. Since ζ and $\mathrm{id}_{C(Z_\varphi)}$ induce the same map on K-theory of $C(Z_\varphi)$ it follows from the UCT that $[(\cdot)\circ\zeta^{-1}] = [\mathrm{id}_{C(Z_\varphi)}]$ in $KK(C(Z_\varphi), C(Z_\varphi))$ and hence in $KL(C(Z_\varphi), C(Z_\varphi))$ [23, 2.4.8]. It follows from [13, Theorem 6.1] that $A \otimes \mathcal{Q}$ is TAI. ∎

3.3 COROLLARY: *If $A, B \in \mathcal{A}$ then $A \cong B$ if and only if $T(A) \cong T(B)$.*

PROOF: It follows from the previous theorem and [12, Corollary 11.9] that $A\otimes\mathcal{Z} \cong B\otimes\mathcal{Z}$ if and only if $\mathrm{Ell}(A\otimes\mathcal{Z}) \cong \mathrm{Ell}(B\otimes\mathcal{Z})$. By [30, Theorem B] (or [31, Theorem 0.2]) A and B are both $\mathcal{Z}$-stable. For any $\varphi : S^d \to S^d$ minimal homeomorphism we have

$$K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \cong \mathbb{Z}, \quad K_1(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \cong \mathbb{Z},$$

which follows immediately from the Pimsner–Voiculescu exact sequence and the fact that ζ_* is the identity on K-theory. Thus Elliott invariants are the same up to the tracial state space and we see $A \cong B$ if and only if $T(A) \cong T(B)$. ∎

Let $z \in Z_\varphi$ and let $\mathcal{E}_z$ denote the equivalence relation given by breaking the orbit of ζ at the point z.

3.4 THEOREM: *For every $z \in Z$, the C^*-algebra $\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}$ is TAI.*

PROOF: By [17, Section 3] $\mathrm{C}^*(\mathcal{E}_z)$ is an inductive limit of recursive subhomogeneous C^*-algebras with base spaces of dimension less than or equal to $\dim(Z_\varphi)$. Since $\dim(Z_\varphi) < \infty$ it follows that $\mathrm{C}^*(\mathcal{E}_z)$ has finite nuclear dimension. By Proposition 2.6, $\mathrm{C}^*(\mathcal{E}_z)$ is simple.

Let $\iota : \mathrm{C}^*(\mathcal{E}_z)\otimes\mathcal{Q} \hookrightarrow A\otimes\mathcal{Q}$ denote the unital embedding. Then the map induced on the tracial state spaces $T(\iota) : T(\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}) \to T((C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q})$ is a homeomorphism by Proposition 3.1 and the fact that $\mathcal{Q}$ has a unique tracial state $\tau_{\mathcal{Q}}$. Moreover,

$$\iota_0 : K_0(\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}) \to K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q})$$

is an ordered group isomorphism [27, Lemma 4.3], (see also [19, Theorem 4.1 (5)]). Since $K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z})) \cong \mathbb{Z}$, we have that $S(K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z})))$ is a point. Thus $\tau_* = \tau'_* \in S(K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z})))$ for any $\tau, \tau' \in T(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}))$ and since any tracial state on $C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q}$ is of the form $\tau \otimes \tau_{\mathcal{Q}}$, it follows that $\tau_* = \tau'_* \in S(K_0(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q}))$ for any $\tau, \tau' \in T(C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q})$.

By Theorem 3.2, $C(Z_\varphi) \rtimes_\zeta \mathbb{Z}) \otimes \mathcal{Q}$ is TAI. It now follows from [36, Theorem 4.2] that $\mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q} \otimes \mathcal{Q} \cong \mathrm{C}^*(\mathcal{E}_z) \otimes \mathcal{Q}$ is TAI. ∎

3.5 COROLLARY: *Let*

$$\mathcal{B} = \{\mathrm{C}^*(\mathcal{E}_z) \mid z \in Z_\varphi, \varphi : S^d \to S^d, d > 1 \textit{ odd, is a minimal diffeomorphism}\}.$$

Then $A, B \in \mathcal{B}$ are isomorphic to projectionless inductive limits of prime dimension drop algebras, and $A \cong B$ if and only if $T(A) \cong T(B)$.

PROOF: If $A \in \mathcal{B}$ then A is $\mathcal{Z}$-stable by [35]. After noting this, the proof that $A \cong B$ if and only if $T(A) \cong T(B)$ is as in Corollary 3.3. For any $z \in Z_\varphi$ we have

$$K_0(\mathrm{C}^*(\mathcal{E}_z)) \cong \mathbb{Z}, \qquad K_1(\mathrm{C}^*(\mathcal{E}_z)) = 0, \qquad T(\mathrm{C}^*(\mathcal{E}_z)) \cong T(C(S^d) \rtimes_\varphi \mathbb{Z}),$$

which follows from [21, Example 2.6], [19, Theorem 4.1] and Proposition 3.1. Now it follows from [10, Theorem 4.5] that $A, B \in \mathcal{B}$ are isomorphic to projectionless inductive limits of prime dimension drop algebras. ∎

4. Outlook

Although our construction shows that $\mathcal{Z}$ can be realized as a minimal étale equivalence relation, it is certainly not unique (we can start with any odd dimensional sphere, for example). It would be interesting to further investigate the possibility of realizing various properties of $\mathcal{Z}$ at the dynamical level. For example, could there be a suitable notion of "strongly self-absorbing" at the level of equivalence relations? Could we see regularity properties such as mean dimension zero (which may be equivalent to $\mathcal{Z}$-stability of the crossed product C*-algebra) after appropriately taking a product with our system?

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