Markov partitions and homology for $\frac{n}{m}$-solenoids

Nigel D. Burke,
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada V8W 2Y2,
nburke@uvic.ca,

Ian F. Putnam,*
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada V8W 2Y2,
ifputnam@uvic.ca

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Abstract

Given a relatively prime pair of integers, $n \geq m > 1$, there is associated a topological dynamical system which we refer to as an $\frac{n}{m}$-solenoid. It is also a Smale space, as defined by David Ruelle, meaning that it has local coordinates of contracting and expanding directions. In this case, these are locally products of the real and various p-adic numbers. In the special case, $m = 2, n = 3$ and for $n > 3m$, we construct Markov partitions for such systems. The second author has developed a homology theory for Smale spaces and we compute this in these examples, using the given Markov partitions, for all values of $n \geq m > 1$ and relatively prime.

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1 Introduction

This paper is concerned with certain topological dynamical systems of an algebraic nature. Moreover, they are defined on metric spaces and have natural co-ordinates of contracting and expanding directions. In short, they are examples of Smale spaces, as defined by David Ruelle [8] to provide an axiomatic framework for the dynamics that appear on the basic sets (or the non-wandering set) in Smale's Axiom A systems [9]. In [7], the second author introduced a kind of homology theory for Smale spaces and the principal objective of this paper is to compute this invariant for these algebraic examples. This computation requires finding Markov partitions with particularly nice properties for the systems. In [13], A.M. Wilson claimed to construct Markov partitions for these systems, but there appears to be an error in the argument. We discuss this problem and give valid Markov partitions (in many, although not all, cases).

We describe the dynamical systems of interest. Some of the material on p-adic numbers is very basic, but we include it for completeness. We refer the reader to [4] for a complete treatment. If p is any prime, we define a norm on the set of rational numbers, $\mathbb{Q}$, by $|0|_p = 0$ and

$$|p^k \frac{i}{j}|_p = p^{-k},$$

where k is any integer and i, j are non-zero integers relatively prime to p. The formula $|a - b|_p$ then defines a metric (if fact, an ultrametric) on $\mathbb{Q}$ and we let $\mathbb{Q}_p$ denote its completion. It is a field called the p-adic numbers. Topologically, $\mathbb{Q}_p$ is a locally compact and totally disconnected ultrametric space. We let $\mathbb{Z}_p$ denote the closure of the usual integers, which is a compact, open subset. It is also a subring and any non-zero integer relatively prime to p has an inverse in $\mathbb{Z}_p$. The most interesting dynamical feature of $\mathbb{Q}_p$ is that multiplication by p is a contraction (by the factor p^{-1}). Multiplication by any non-zero integer relatively prime to p is an isometry.

If $p < q$ are prime numbers, we consider $\mathbb{Z}[(pq)^{-1}]$, the subgroup of the rationals additively generated by all numbers of the form $(pq)^{-k}, k \geq 1$. The map sending a in $\mathbb{Z}[(pq)^{-1}]$ to (a, a, a) in $\mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q$ embeds the former as a lattice: its image is discrete and the quotient is compact. We let X denote this quotient and ρ denote the quotient map.

We also define $\varphi : X \to X$ by

$$\varphi \circ \rho(a, r, b) = \rho\left(p^{-1}qa, p^{-1}qr, p^{-1}qb\right),$$

for a in $\mathbb{Q}_p$, r in $\mathbb{R}$ and b in $\mathbb{Q}_q$. Locally, our space X looks like $\mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q$ and multiplication by $p^{-1}q$ expands in the first two factors and contracts in the third. Without giving a precise definition, this means that (X, φ) is a Smale space with local unstable sets that are homeomorphic to opens subsets of $\mathbb{Q}_p \times \mathbb{R}$ and local stable sets that are homeomorphic to open sets in $\mathbb{Q}_q$. We refer the reader to [7, 8] for a more complete treatment.

We can extend the construction above to the case where $2 \leq m < n$ are relatively prime integers. We first define $|a|_m = \sum_{p|m} |a|_p$, where a is a rational number. We let $\mathbb{Q}_m$ and $\mathbb{Z}_m$ be the completions of $\mathbb{Q}$ and $\mathbb{Z}$ respectively in the associated metric. It is a consequence of the Chinese Remainder Theorem that

$$\mathbb{Q}_m \cong \Pi_{p|m} \mathbb{Q}_p,$$

with the rational numbers embedded diagonally on the right. Both $\mathbb{Q}_m$ and $\mathbb{Z}_m$ are rings and, while the former is not a field, the latter contains an inverse for every non-zero integer relatively prime to m. Again in the natural metric, multiplication by m on $\mathbb{Q}_m$ contracts; specifically, we have

$$|ma - mb|_m \leq 2^{-1}|a - b|_m,$$

for all a, b in $\mathbb{Q}_m$. Multiplication by any non-zero integer relatively prime to m is an isometry.

We define X to be the quotient of $\mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n$ by the lattice $\mathbb{Z}[(mn)^{-1}]$ and define φ with the same formula as earlier, replacing $p^{-1}q$ by $m^{-1}n$. The same comments we made in the p, q-case on the stable and unstable sets are valid here. We refer to (X, φ) as the $\frac{n}{m}$-solenoid.

In [7], the second author defined a homology theory for Smale spaces and the main objective in this paper is to compute this invariant on these algebraic examples. To any Smale space (X, φ), at least with every point non-wandering, there are associated countable abelian groups $H^s_N(X, \varphi)$ and $H^u_N(X, \varphi)$, for all integers N. In addition, each comes with a canonical automorphism induced by φ, denoted by φ^s_* and φ^u_*, respectively. These intend to quantify the topology of the space and the dynamics in an analogous way to what Cech cohomology does for topological spaces. The exact relationship with more conventional cohomology theories is not understood, but they are certainly different. For shifts of finite type, these groups are finite rank, while that is not the case for the Cech cohomology of the underlying spaces which are typically Cantor sets.

Their definition relies in a critical way on Krieger's dimension group invariants for a shift of finite type [5]. In fact, if (X, φ) is a shift of finite type, these homology groups recover Krieger's invariant at $N = 0$ and are zero in all other degrees. This homology theory also provides a Lefschetz Theorem which computes the periodic point data for (X, φ) in terms the trace of the automorphism induced by φ. The existence of such a theory was first conjectured by Bowen [3].

We also mention that this homology has been computed on rather different solenoids of a more topological nature first constructed by Williams [12] and later studied by Yi [14] and Thomsen [10] as inverse limits of branched one-manifolds by the second author with M. Amini and S. Gholikandi Saeidi [1].

We state the main results of this paper as follows.

Theorem 1.1. *Let $n > m \geq 2$ and assume that m, n are relatively prime. Let (X, φ) be the associated $\frac{n}{m}$-solenoid. We have*

$$\begin{array}{ccccc} H^s_0(X,\varphi) & \cong & H^u_0(X,\varphi) & \cong & \mathbb{Z}[1/n], \\ H^s_1(X,\varphi) & \cong & H^u_1(X,\varphi) & \cong & \mathbb{Z}[1/m] \end{array}$$

and $H^s_N(X, \varphi) = H^u_N(X, \varphi) = 0$, for all $N \neq 0, 1$. Moreover, under these identifications, the canonical automorphisms induced by φ on the $H^s_N(X, \varphi)$ groups are

$$\begin{array}{ccl} \varphi^s_*(a) & = & n^{-1}a, \quad a \in H^s_0(X,\varphi) \\ \varphi^u_*(b) & = & nb, \quad b \in H^u_0(X,\varphi) \\ \varphi^s_*(c) & = & m^{-1}c, \quad c \in H^s_1(X,\varphi) \\ \varphi^u_*(d) & = & md, \quad d \in H^u_1(X,\varphi). \end{array}$$

Let us note the following consequence, which is already well-known.

Corollary 1.2. *Let $n > m \geq 2$ and assume that m, n are relatively prime. Let (X, φ) be the associated $\frac{n}{m}$-solenoid.*

For each integer $p \geq 1$, we have

$$\#\{x \in X \mid \varphi^p(x) = x\} = n^p - m^p.$$

It is also probably worth stating explicitly the computations for the inverse systems. Of course, if (X, φ) is the $\frac{n}{m}$-solenoid, then (X, φ^{-1}) is the $\frac{m}{n}$-solenoid.

Corollary 1.3. *Let $n > m \geq 2$ and assume that m, n are relatively prime. Let (X, φ) be the associated $\frac{n}{m}$-solenoid. We have*

$$\begin{array}{rcccl} H_0^s(X,\varphi^{-1}) & \cong & H_0^u(X,\varphi^{-1}) & \cong & \mathbb{Z}[1/n], \\ H_{-1}^s(X,\varphi^{-1}) & \cong & H_{-1}^u(X,\varphi^{-1}) & \cong & \mathbb{Z}[1/m] \end{array}$$

and $H_N^s(X,\varphi^{-1}) = H_N^u(X,\varphi^{-1}) = 0$, for all $N \neq 0, -1$. Moreover, under these identifications, the canonical automorphisms induced by φ on the $H_N^s(X,\varphi^{-1})$ groups are

$$\begin{array}{rcll} \varphi_*^s(a) & = & na, & a \in H_0^s(X,\varphi^{-1}) \\ \varphi_*^u(b) & = & n^{-1}b, & b \in H_0^u(X,\varphi^{-1}) \\ \varphi_*^s(a) & = & mc, & c \in H_{-1}^s(X,\varphi) \\ \varphi_*^u(d) & = & m^{-1}d, & d \in H_{-1}^u(X,\varphi^{-1}). \end{array}$$

Before getting into the details, let us discuss the computation of the homology theory in general terms. Our Smale space, (X, φ), has the nice feature that its local stable sets are totally disconnected. In this situation, we must find a shift of finite type [6], (Σ, σ), with a factor map

$$\pi : (\Sigma, \sigma) \to (X, \varphi)$$

which is s-bijective; that is, each stable equivalence class in (Σ, σ) is mapped bijectively to a stable equivalence class in (X, φ). (More specifically, we need an s/u-bijective pair, in the language of [7]. The reader will not need to understand this terminology for most of the paper.) The computation of the homology then relies on a careful analysis on when the map π is N-to-one, for various values of N. It is a fundamental result of [7] that the results of these computations depend on (X, φ), but not on the choice of π or (Σ, σ).

The shift of finite type which we will use is the full shift on n symbols. That is, we have $\Sigma = \{1, 2, \ldots, n\}^{\mathbb{Z}}$ with the usual left shift map σ. To find a factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$, it suffices for us to find subsets $R_1, \ldots, R_n$ of X which are pairwise disjoint, the union of their interiors is dense in X and such that, for every ι in $\Sigma = \{1, 2, \ldots, n\}^{\mathbb{Z}}$,

$$\cap_{K=1}^{\infty} \overline{\cap_{k=-K}^{K} \varphi^{-k}(Int(R_{\iota(k)}))}$$

is a single point. The value $\pi(\iota)$ is then defined to be the single point of this intersection. We will refer to the collection $R_1, \ldots, R_n$ as a Markov partition. The sets we construct (or rather their closures) will actually satisfy the conditions for Markov partitions as described in [3], although we will not need this fact nor prove it. Our only interest in them is in obtaining the map π as described.

In [13], Wilson investigated these systems. He also considered the case of endomorphisms of the solenoid, which we do not. The main result of [13], for the automorphisms, is that they are Bernoulli. To prove this, a measurable partition, denoted P, was defined. Using our notation, and in the case $m = 2, n = 3$, P consists of the three sets

$$R_i = \rho\left(\mathbb{Z}_2 \times \left[3^{-1}(i-1), 3^{-1}i\right] \times \mathbb{Z}_3\right), i = 1, 2, 3.$$

Part of the proof of the main result of [13] is showing that $\vee_{n=-\infty}^{\infty}\varphi^{-n}(P)$ is the partition into single points. This is false, as follows. Let $x = \rho\,(5^{-1}2, 5^{-1}2, 5^{-1}2)$. First, observe that $\varphi(x) = \rho\,(5^{-1}3, 5^{-1}3, 5^{-1}3)$ and that

$$\begin{aligned} \varphi^2(x) &= \rho\left(10^{-1}9, 10^{-1}9, 10^{-1}9\right) \\ &= \rho(\left(\left(10^{-1}9, 10^{-1}9, 10^{-1}9\right) - \left(2^{-1}, 2^{-1}, 2^{-1}\right)\right) \\ &= x. \end{aligned}$$

Second, since $3^{-1} < 5^{-1}2 < 5^{-1}3 < 3^{-1}2$, x and $\varphi(x)$ are both in R_2 and neither is in R_1 or R_3. In other words, $\{x, \varphi(x)\}$ is invariant under φ and is contained in a single element of P and is disjoint from the others. Hence, the same statement holds for each partition $\varphi^{-n}(P), n \in \mathbb{Z}$, and it follows from this that $\{x, \varphi(x)\}$ is contained in a single element of $\vee_{n=-\infty}^{\infty}\varphi^{-n}(P)$. It appears to the authors that the element of $\vee_{n=-\infty}^{\infty}\varphi^{-n}(P)$ containing x is actually a Cantor set. In our notation, with R_1, R_2, R_3 as above and $\iota(k) = 2, k \in \mathbb{Z}$, we have

$$\{x, \varphi(x)\} \subseteq \cap_{K=1}^{\infty}\overline{\cap_{k=-K}^{K}\varphi^{-k}(Int(R_{\iota(k)}))}$$

The error in [13] would appear to be in the proof of Proposition 2.3 on the top of page 73. In the notation given there, letting $x = (5^{-1}2, 5^{-1}2, \ldots)$ and $y = (5^{-1}3, 5^{-1}3, \ldots)$, then x_r and y_r lie in the same element (in fact, in the interior of the same element) of the partition $\omega_{C_r}^{-1} \vee_{i=-r}^{r} \omega_2^{-r-i}\omega_3^{-r+i}S(3)$. Wilson's argument that $x_r = y_r$ is obviously incorrect.

In [11], the same partition is used and denoted ξ and the claim, in the final case of the proof, that

$$diam\left(\vee_{j=-n}^{n}\alpha^j(\xi)\right) \to 0, \text{ as } n \to \infty,$$

follows from the arguments of [13], is also false.

Here, we will give $R_1, R_2, \ldots, R_n$ which *do* satisfy these conditions, but are obviously different from the ones given in [13]. It is interesting to note that, locally, our space is the product of $\mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n$. One nice feature of Wilson's partition (if we ignore for the moment that it fails to prove the desired conclusion) is that the elements are actually of the form $A \times B \times C$ in this local picture. Ours will not be; they will be of the form $D \times C$, where $D \subseteq \mathbb{Q}_m \times \mathbb{R}$. It seems unlikely that partitions with the desired properties and elements of the stronger product form exist.

While the proof contains an error, the main result of [13], that the automorphisms are Bernoulli, is presumably valid. Indeed, Weiss [] claimed to have a proof before [13], although the details did not appear.

This could also be deduced from our results here. We give a very explicit description in Lemma 2.19 of exactly where our map π fails to be one-to-one and from this it can be shown that it is one-to-one on a dense set of full measure. Hence, our map π can be shown to be an isomorphism at the level of measurable systems (with suitable measures). Of course, π is not an isomorphism in the topological category and, indeed, this argument precisely ignores our main interest in understanding where π is many-to-one.

There is yet one more possible proof that the automorphisms are Bernoulli, using Wilson's original partitions. Curiously, it seems likely that the failure of Wilson's argument happens only on a set of measure zero. More precisely, the union of all elements of the partition $\vee_{n=-\infty}^{\infty} \varphi^{-n}(P)$ which are *not* single points appear to be measure zero. This means that one can define a map π (different from ours) on a dense set of full measure and show that it is a measurable isomorphism. This is speculative; the arguments presented in [13] are not correct, since it is claimed they work everywhere. Of course, even if all this works, this map will not be useful in the computation of the homology.

In the second section, we construct our Markov partitions. We do this first in the case of $m = 2, n = 3$, since there seems to be particular interest in that case as being the most basic in an obvious sense. We then consider the general case with the added hypothesis that $3m < n$. The general idea is the same, but there are some subtle differences.

Let us take a moment to discuss why the hypothesis $3m < n$ is appropriate or even relevant. Since our space, locally, is the product of two totally disconnected ones and the real line, its covering dimension is one. Therefore, it seems reasonable to try to find a cover by a shift of finite type where the map is either one-to-one or two-to-one at every point. This is exactly the case in [1]. As we indicated above, understanding where our factor map is N-to-one, for various values of N, is key to the computation of our homology theory. The situation of an at-most-two-to-one map has a greatly simplifying effect.

The natural candidate for the covering shift of finite type is the full shift on n symbols. Of course, this system has n fixed points while our solenoid has $n - m$ fixed points. Therefore, a necessary condition for the existence of an at-most-two-to-one map from the full n-shift onto the $\frac{n}{m}$-solenoid is that $2(n-m) \geq n$ or $n \geq 2m$. (One can already see this problem surfacing in our construction for the case $m = 2, n = 3$.) In fact, for certain technical reasons, it actually helps to make the stronger hypothesis that $3m < n$. Moreover, it turns out that this special case will actually suffice for the computations of homology in the general case due to a somewhat sneaky argument.

In the third section, we make use of our factor map from the shift of finite type obtained in the second section to compute the homology. This contains the final proofs of our three main results above.

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2 Markov partitions

The aim of this section is to find Markov partitions for an $\frac{n}{m}$-solenoid. We begin with the special case $n = 3, m = 2$.

2.1 The case $m = 2, n = 3$

Define $A = \mathbb{Z}_2 \times (0, 1]$ and $\overline{A} = \mathbb{Z}_2 \times [0, 1]$. The notation is to suggest the closure in $\mathbb{Q}_2 \times \mathbb{R}$. The following is proved in [11].

Lemma 2.1. *The set $A \times \mathbb{Z}_3$ is a fundamental domain for $\mathbb{Z}[6^{-1}]$ in $\mathbb{Q}_2 \times \mathbb{R} \times \mathbb{Q}_3$. That is, each coset of $\mathbb{Z}[6^{-1}]$ in $\mathbb{Q}_2 \times \mathbb{R} \times \mathbb{Q}_3$ intersects $A \times \mathbb{Z}_3$ in exactly one point.*

In finding our Markov partition, the first step is to construct a subset of $\overline{A}$ which will essentially describe the boundaries of the rectangles.

For each $i = 0, 1$, define $\alpha_i : \overline{A} \to \overline{A}$ by

$$\alpha_i(a, r) = \left(3^{-1}2a + 3^{-1}i, 3^{-1}2r + 3^{-1}i\right),$$

for (a, r) in $\overline{A}$.

Lemma 2.2. *1. We have*

$$\begin{aligned} \alpha_0(\overline{A}) &= 2\mathbb{Z}_2 \times [0, 3^{-1}2], \\ \alpha_1(\overline{A}) &= (1 + 2\mathbb{Z}_2) \times [3^{-1}, 1]. \end{aligned}$$

2. The sets $\alpha_0(\overline{A})$ and $\alpha_1(\overline{A})$ are disjoint.

3. For $i = 0, 1$, b in $\mathbb{Z}_3$ and (a, r) in A, we have

$$\varphi^{-1}(\rho(a, r, i + 3b)) = \rho(\alpha_i(a, r), i + 2b).$$

Moreover, $(\alpha_i(a, r), i + 2b)$ is in $A \times \mathbb{Z}_3$.

Proof. The first statement is obvious from the definitions. The second follows easily from the first.

For the last part, we consider the cases $i = 0, 1$ separately. First observe that in both cases, if (a, r) is in A, then $3^{-1}2r$ is in $(0, 3^{-1}2]$ and $3^{-1}2a$ is in $\mathbb{Z}_2$.

If $i = 0$, then $2 \cdot 3^{-1}3b$ is in $\mathbb{Z}_3$. It follows that $(3^{-1}2a, 3^{-1}2r, 2b)$ is in $A \times \mathbb{Z}_3$ and we are done.

If $i = 1$, then $3^{-1}2(1 + 3b)$ is in $\mathbb{Z}_3 + 3^{-1}2$. Moreover, we have

$$\begin{aligned} 3^{-1}2a + 3^{-1} &\in 2\mathbb{Z}_2 + 1 \\ 3^{-1}2r + 3^{-1} &\in (0, 3^{-1}2] + 3^{-1} \subset (3^{-1}, 1], \\ 3^{-1}2(1 + 3b) + 3^{-1} &\in \mathbb{Z}_3 + 3^{-1}2 + 3^{-1} = \mathbb{Z}_3. \end{aligned}$$

□

Lemma 2.3. *Let $A_0 = \overline{A}$ and, for each $k \geq 1$, define*

$$A_k = \alpha_0(A_{k-1}) \cup \alpha_1(A_{k-1}) \subseteq \overline{A}.$$

1. For all $k \geq 0$, the sets $\alpha_0(A_k)$ and $\alpha_1(A_k)$ are disjoint.

2. A_k is closed and $A_k \subseteq A_{k-1}$, for all $k \geq 1$.

3. *For any a in $\mathbb{Z}_2$ and $k \geq 1$, we have*

$$A_k \cap (\{a\} \times [0,1]) = \{a\} \times I,$$

where I is a closed interval of length $(3^{-1}2)^k$.

4. *There exists a continuous function $\partial : \mathbb{Z}_2 \to [0,1]$ such that*

$$A_\infty = \cap_{k=1}^{\infty} A_k = \{(a, \partial(a)) \mid a \in \mathbb{Z}_2\}.$$

5. *The function ∂ satisfies*

$$\partial\left(3^{-1}2a\right) = 3^{-1}2\partial(a), \quad \partial\left(3^{-1}2a + 3^{-1}\right) = 3^{-1}2\partial(a) + 3^{-1},$$

for all a in $\mathbb{Z}_2$.

6. *For a in $\mathbb{Z}_2$, $\partial(a) = 0$ if and only if $a = 0$ and $\partial(a) = 1$ if and only if $a = 1$.*

Proof. The first three parts are all elementary and we omit the proofs. For the fourth, it follows from the third that

$$A_\infty \cap (\{a\} \times [0,1]) = \{a\} \times \{r\},$$

for some r in $[0,1]$ and setting $\partial(a) = r$ yields the desired function. Its continuity follows from the fact that A_∞ is closed.

It follows from part 2 and the definition that

$$\alpha_0(A_k) \subseteq \alpha_0(A_{k-1}) \subseteq A_k.$$

and hence $\alpha_0(A_\infty) \subseteq A_\infty$. According to the definitions of ∂ and α_0, this means that

$$\partial\left(3^{-1}2a\right) = 3^{-1}2\partial(a),$$

for all a in $\mathbb{Z}_2$. Replacing α_0 with α_1 proves the second equality of part 5.

It is a simple matter to see that $(0,0), (1,1)$ are in $\overline{A}$ and $\alpha_0(0,0) = (0,0), \alpha_1(1,1) = (1,1)$. It follows that $(0,0), (1,1)$ are in A_∞, or $\partial(0) = 0, \partial(1) = 1$.

From the fact that

$$A_1 \cap [0,1] \times (1 + 2\mathbb{Z}_2) \subseteq [1/3, 1] \times (1 + 2\mathbb{Z}_2),$$

we see that $\partial(a) \geq 3^{-1}$, for a in $(1 + 2\mathbb{Z}_2)$. If $k \geq 1$ and a is in $2^k\mathbb{Z}_2$ but not in $2^{k+1}\mathbb{Z}_2$, then by induction, we have

$$\partial(a) \geq \left(3^{-1}2\right)^k 3^{-1} > 0.$$

We conclude that $\partial(a) = 0$ implies a is in $2^k\mathbb{Z}_2$ for all k, and so $a = 0$. A similar argument proves that $\partial(a) = 1$ implies the $a = 1$. □

We define the set $B \subseteq \mathbb{Z}_2 \times [-1, 1]$ by

$$B = \{(a, r) \in \mathbb{Z}_2 \times \mathbb{R} \mid \partial(a + 1) - 1 < r \leq \partial(a)\}.$$

Lemma 2.4. *The set $B \times \mathbb{Z}_3$ is a fundamental domain for $\mathbb{Z}[6^{-1}]$ in $\mathbb{Q}_2 \times \mathbb{R} \times \mathbb{Q}_3$.*

Proof. Define

$$\begin{aligned} B^- &= \{(a, r) \in A \mid r \leq \partial(a)\} \\ B^+ &= \{(a, r) \in A \mid \partial(a) < r\}. \end{aligned}$$

First write

$$A \times \mathbb{Z}_3 = B^+ \times \mathbb{Z}_3 \cup B^- \times \mathbb{Z}_3$$

and the sets on the right are disjoint. It follows that

$$(B^+ \times \mathbb{Z}_3 - (1, 1, 1)) \cup B^- \times \mathbb{Z}_3$$

is also a fundamental domain. Next observe that

$$\begin{aligned} B^+ \times \mathbb{Z}_3 - (1, 1, 1) &= \{(a, r) \in \mathbb{Z}_2 \times [-1, 1] \mid \partial(a + 1) < r + 1 \leq 1\} \times \mathbb{Z}_3 \\ &= \{(a, r) \in \mathbb{Z}_2 \times [-1, 1] \mid \partial(a + 1) - 1 < r \leq 0\} \times \mathbb{Z}_3 \end{aligned}$$

It follows that

$$(B^+ \times \mathbb{Z}_3 - (1, 1, 1)) \cup B^- \times \mathbb{Z}_3 = B \times \mathbb{Z}_3.$$

and we are done. □

Define maps $\beta_0, \beta_1, \beta_2 : B \to \mathbb{Z}_2 \times \mathbb{R}$ by

$$\begin{aligned}\beta_0(a,r) &= \left(3^{-1}2a, 3^{-1}2r\right)\\ \beta_1(a,r) &= \left(3^{-1}2a + 3^{-1}, 3^{-1}2r + 3^{-1}\right)\\ \beta_2(a,r) &= \left(3^{-1}2a - 3^{-1}, 3^{-1}2r - 3^{-1}\right),\end{aligned}$$

for (a,r) in B. (Of course, β_0, β_1 are defined by the same formula as α_0, α_1, respectively, but they have different domains.)

Lemma 2.5. *1. The sets $\beta_0(B), \beta_1(B)$ and $\beta_2(B)$ are all contained in B, they are pairwise disjoint and their union is B.*

2. For any (a,r) in B, b in $\mathbb{Z}_3$ and $j = 0,1,2$, $(\beta_j(a,r), 2b)$ is in $B \times \mathbb{Z}_3$ and

$$\varphi^{-1}(\rho(a,r,j+3b)) = \rho(\beta_j(a,r), j+2b).$$

3. For any $j = 0,1,2$ and $X \subseteq B$, we have

$$diam(\beta_j(X)) \leq 3^{-1}2 diam X.$$

Proof. Suppose that (a,r) is in B. First, we observe that

$$\partial\left(3^{-1}2a\right) = 3^{-1}2\partial\left(a\right) > 3^{-1}2r.$$

This provides the correct upper bound on $\beta_0(B)$. Secondly, we check that

$$\partial\left(3^{-1}2a + 3^{-1}\right) = 3^{-1}2\partial\left(a\right) + 3^{-1} > 3^{-1}2r + 3^{-1}.$$

This provides the correct upper bound on $\beta_1(B)$. Thirdly, we check that

$$\begin{aligned}\partial\left(3^{-1}2a + 1\right) - 1 &= \partial\left(3^{-1}2(a+1) + 3^{-1}\right) - 1\\ &= 3^{-1}2\partial\left(a+1\right) + 3^{-1} - 1\\ &= 3^{-1}2(\partial\left(a+1\right) - 1)\\ &\leq 3^{-1}2r.\end{aligned}$$

This provides the correct lower bound on $\beta_0(B)$. Fourthly, we check that

$$\begin{aligned}\partial\left(3^{-1}2a - 3^{-1} + 1\right) - 1 &= \partial\left(3^{-1}2(a+1)\right) - 1\\ &= 3^{-1}2\partial\left(a+1\right) - 1\\ &= 3^{-1}2\left(\partial\left(a+1\right) - 1\right) - 3^{-1}\\ &\leq 3^{-1}2r - 3^{-1}\end{aligned}$$

This provides the correct lower bound for $\beta_2(B)$.

To complete the proof it suffices to observe that all three maps preserve order in the $\mathbb{R}$ component. Suppose (a, r) is in the lower boundary of B, meaning that $r = \partial(a+1) - 1$, then

$$\begin{aligned} \beta_1(a, r) &= \left(3^{-1}2a + 3^{-1}, 3^{-1}2r + 3^{-1}\right) \\ &= \left(3^{-1}2(a+1) - 3^{-1}, 3^{-1}2(r+1) - 3^{-1}\right) \\ &= \beta_2(a+1, r+1) \end{aligned}$$

where $\partial(a+1) = r+1$. That is, $(a+1, r+1)$ is in the upper boundary of $\beta_2(B)$. By reversing the argument we see that the upper boundary of $\beta_2(B)$ coincides with the lower boundary of $\beta_1(B)$.

The proof of part 2 is analogous to that of Part 3 of Lemma 2.2 and the third part is clear. □

We are now ready to define our Markov partition.

Definition 2.6. *1. For $j = 0, 1, 2$, define*

$$R_j = \rho(B \times (j + 3\mathbb{Z}_3)).$$

2. For any $K \leq 0 \leq L$ and function $\iota : \{K, K+1, \ldots, L-1, L\} \to \{0, 1, 2\}$, we define

$$R_\iota = \cap_{k=K}^{L} \varphi^{-k}(R_{\iota(k)}).$$

The following result essentially establishes the Markov property for our rectangles, but gives even more information.

Lemma 2.7. *1. For $0 \leq L$ and $\iota : \{0, 1, \ldots, L-1, L\} \to \{0, 1, 2\}$, we have*

$$R_\iota = \rho(\beta_{\iota(1)} \circ \cdots \beta_{\iota(L)}(B) \times (\iota(0) + 3\mathbb{Z}_3)).$$

2. For $K \leq 0$ and $\iota : \{K, K+1, \ldots, -1, 0\} \to \{0, 1, 2\}$, we have

$$R_\iota = \rho\left(B \times \left(\sum_{k=K}^{0} \left(2^{-1}3\right)^{-k} \iota(k) + 3^{1-K}\mathbb{Z}_3\right)\right).$$

Proof. We argue by induction on L. The case $L = 0$ is trivial. Next suppose that $L > 0$ and the statement holds for $L - 1$. Write

$$\cap_{l=0}^{L} \varphi^{-l}(R_{\iota(l)}) = R_{\iota(0)} \cap \varphi^{-1}(\cap_{l=0}^{L-1} \varphi^{-l}(R_{\iota(l+1)})).$$

We may apply the induction hypothesis to the function $l \to \iota(l+1)$ to see that

$$\cap_{l=0}^{L-1} \varphi^{-l}(R_{\iota(l+1)}) = \rho\left(\beta_{\iota(2)} \circ \cdots \beta_{\iota(L)}(B) \times (\iota(1) + 3\mathbb{Z}_3)\right).$$

Applying part 2 of Lemma 2.5, we see that

$$\begin{aligned} \varphi^{-1}(\cap_{l=0}^{L-1} \varphi^{-l}(R_{\iota(l+1)})) &= \varphi^{-1} \circ \rho\left(\beta_{\iota(2)} \circ \cdots \beta_{\iota(L)}(B)\right. \\ &\quad \left.\times (\iota(1) + 3\mathbb{Z}_3)\right) \\ &= \rho\left(\beta_{\iota(1)} \circ \cdots \beta_{\iota(L)}(B) \times \mathbb{Z}_3\right). \end{aligned}$$

Since all of our sets are contained in the fundamental domain $B \times \mathbb{Z}_3$, we have

$$\begin{aligned} \cap_{l=0}^{L} \varphi^{-l}(R_{\iota(l)}) &= \rho\left(\beta_{\iota(1)} \circ \cdots \beta_{\iota(L)}(B) \times \mathbb{Z}_3\right. \\ &\quad \left.\cap B \times (\iota(0) + \mathbb{Z}_3)\right) \\ &= \rho(\beta_{\iota(1)} \circ \cdots \beta_{\iota(L)}(B) \times (\iota(0) + 3\mathbb{Z}_3)). \end{aligned}$$

The second statement is also proved by induction, on $-K$ with the case $K = 0$ being clear. Now suppose $K < 0$ and that the statement holds for $K + 1$. We write

$$\begin{aligned} R_\iota &= \cap_{k=K}^{0} \varphi^{-k}(R_{\iota(k)}) \\ &= R_{\iota(0)} \cap \varphi(\cap_{k=K}^{-1} \varphi^{-1-k}(R_{\iota(k)})) \\ &= R_{\iota(0)} \cap \varphi \circ \rho\left(B \times \left(\sum_{k=K+1}^{0} \left(2^{-1}3\right)^{-k} \iota(k-1) + 3^{-K}\mathbb{Z}_3\right)\right) \\ &= \varphi \circ \rho\left((\beta_{\iota(0)}(B) \times \mathbb{Z}_3) \cap \left(B \times \left(\sum_{k=K}^{-1} \left(2^{-1}3\right)^{-1-k} \iota(k) + 3^{-K}\mathbb{Z}_3\right)\right)\right) \\ &= \varphi \circ \rho\left(\beta_{\iota(0)}(B) \times \left(\sum_{k=K}^{-1} \left(2^{-1}3\right)^{-1-k} \iota(k) + 3^{-K}\mathbb{Z}_3\right)\right) \\ &= \rho\left(B \times \left(\sum_{k=K}^{0} \left(2^{-1}3\right)^{-k} \iota(k) + 3^{1-K}\mathbb{Z}_3\right)\right) \end{aligned}$$

where we have used the induction hypothesis in the second line and the fact that all our sets are contained in the fundamental domain $B \times \mathbb{Z}_3$. □

Theorem 2.8. *For each* $0 \leq i, j \leq 2$, $\varphi^{-1}(Int(R_i)) \cap Int(R_j) \neq \emptyset$. *Moreover, for any function* ι *in* $\{0,1,2\}^{\mathbb{Z}}$, *the intersection*

$$\cap_{K=0}^{\infty} \overline{\cap_{k=-K}^{K} \varphi^k(R_{\iota(k)})}$$

is a single point and the map sending ι *to this point defines a factor map from* $(\{0,1,2\}^{\mathbb{Z}}, \sigma)$ *to* (X, φ).

Proof. The set $B \times \mathbb{Z}_3$ has finite diameter. It follows from Lemma 2.7 and the final part of Lemma 2.5 that

$$diam\left(\cap_{k=-K}^{K} \varphi^k(R_{\iota(k)})\right) \leq 3^{-K} 2^K diam(B) + 3^{-K}.$$

The closure has the same diameter which tends to zero as K tends toward infinity. Hence the intersection is at most a single point, but since the sets are compact, it is also non-empty. The remaining parts of the proof are standard. □

2.2 The case $3m < n$

As explained in the introduction, the hypothesis $2m < n$ is necessary to obtain a factor map from a shift of finite type which is at most two-to-one. We strengthen that slightly to $3m < n$ for technical reasons. This condition will be assumed throughout the rest of the section.

Begin by letting $A = \mathbb{Z}_m \times (-1, 1]$ and $\overline{A} = \mathbb{Z}_m \times [-1, 1]$. For each $1 \leq i \leq m$, define $\alpha_i : \overline{A} \to \mathbb{Z}_m \times \mathbb{R}$ by

$$\alpha_i(a, r) = \left(n^{-1}ma - n^{-1}i, n^{-1}mr - n^{-1}i\right),$$

for (a, r) in $\overline{A}$.

Lemma 2.9. *1. For* $1 \leq i \leq m$, *we have*

$$\alpha_i(\overline{A}) \subseteq \left(-n^{-1}i + m\mathbb{Z}_m\right) \times \left[-2n^{-1}m, n^{-1}m\right] \subseteq \overline{A}.$$

2. The sets $\alpha_i(\overline{A}), 1 \leq i \leq m$, *are pairwise disjoint.*

Proof. First observe that

$$\begin{aligned} n^{-1}m[-1,1] - n^{-1}i &= \left[-n^{-1}(m+i), n^{-1}(m-i)\right] \\ &\subseteq \left[-2n^{-1}m, n^{-1}m\right] \\ &\subseteq [-1,1], \end{aligned}$$

since $3m < n$. The first part follows easily from this.

The second part follows from the fact that the sets $-n^{-1}i + m\mathbb{Z}_m$ are pairwise disjoint in $\mathbb{Z}_m$ since n is relatively prime to m. □

We now repeat the construction of a subset of $\overline{A}$ which will essentially describe the boundaries of the rectangles.

Lemma 2.10. *Let $A_0 = \overline{A}$ and, for each $k \geq 1$, define*

$$A_k = \cup_{i=1}^{m} \alpha_i(A_{k-1}) \subseteq \overline{A}.$$

1. *For each fixed $k \geq 0$, the sets $\alpha_i(A_k), 1 \leq i \leq m$, are pairwise disjoint.*

2. *A_k is closed and $A_k \subseteq A_{k-1}$, for all $k \geq 1$.*

3. *For any a in $\mathbb{Z}_m$ and $k \geq 1$, we have*

$$A_k \cap (\{a\} \times [-1,1]) = \{a\} \times I,$$

where I is a closed interval of length $2\left(n^{-1}m\right)^k$.

4. *There exists a continuous function $\partial : \mathbb{Z}_m \to [-2n^{-1}m, n^{-1}m]$ such that*

$$A_\infty = \cap_{k=1}^{\infty} A_k = \{(a, \partial(a)) \mid a \in \mathbb{Z}_m\}.$$

5. *A_∞ and $A_\infty + (1,1)$ are disjoint.*

6. *The function ∂ satisfies*

$$\partial\left(n^{-1}ma - n^{-1}i\right) = n^{-1}m\partial(a) - n^{-1}i,$$

for all a in $\mathbb{Z}_m$ and $1 \leq i \leq m$.

Proof. The first three parts are all obvious. For the fourth, it follows from the third that

$$A_\infty \cap (\{a\} \times [-1, 1]) = \{a\} \times \{r\},$$

for some r in $[-1, 1]$ and setting $\partial(a) = r$ yields the desired function. Its continuity follows from the fact that A_∞ is closed.

We note that $\partial(\mathbb{Z}_m) \subseteq [-2n^{-1}m, n^{-1}m] \subseteq (-\frac{2}{3}, \frac{1}{3})$, because of our hypothesis on m, n. It follows that $\partial(\mathbb{Z}_m) + 1$ and $\partial(\mathbb{Z}_m)$ are disjoint and part 5 follows.

It follows from part 2 and the definition that

$$\alpha_i(A_n) \subseteq \alpha_i(A_{n-1}) \subseteq A_n.$$

and hence $\alpha_i(A_\infty) \subseteq A_\infty$. So if $(a, \partial(a))$ is a point in A_∞, then so is $(n^{-1}ma - m^{-1}i, n^{-1}m\partial(a) - m^{-1}i)$. The conclusion for part 6 follows from the definition of ∂. □

We point out that condition 5 represents a substantial improvement on what we had in the previous subsection where this failed.

We define the set $B \subseteq \mathbb{Z}_m \times [-1, 1]$ by

$$B = \{(a, r) \in \mathbb{Z}_m \times \mathbb{R} \mid \partial(a + 1) - 1 < r \leq \partial(a)\}.$$

The following result has the same proof as Lemma 2.4 and we omit the details.

Lemma 2.11. *The set $B \times \mathbb{Z}_n$ is a fundamental domain for $\mathbb{Z}[(mn)^{-1}]$ in $\mathbb{Q}_m \times \mathbb{R} \times \mathbb{Q}_n$.*

The following result is quite obvious, so we omit the proof, but it will be convenient to have on record.

Lemma 2.12. *We have*

$$\begin{aligned} B \setminus Int(B) &= A_\infty, \\ \overline{B} \setminus B &= A_\infty - (1, 1) \end{aligned}$$

Define maps $\beta_j : \overline{B} \to \mathbb{Z}_m \times \mathbb{R}, 1 \leq j \leq n$, by

$$\beta_j(a, r) = \left(n^{-1}ma - n^{-1}j, n^{-1}mr - n^{-1}j\right),$$

for (a, r) in B and $1 \leq j \leq n$. (Of course, the first m of these maps agree with the α_i, but do not have the same domain.)

Lemma 2.13. *1. Each of the maps $\beta_j, 1 \leq j \leq n$ preserves the obvious order in the $\mathbb{R}$-component.*

2. For each $n - m < j \leq n$, we have

$$\beta_j((a,r) - (1,1)) = \beta_{j-m}(a,r),$$

for all (a,r) in A_∞.

3. For each $1 \leq j \leq m$, we have

$$\beta_j(A_\infty) \subseteq A_\infty \cap \left(-n^{-1}j + m\mathbb{Z}_m\right) \times [-1,1].$$

We also have

$$\beta_j(a,r) - (1,1) = \beta_{n-m+j}((a,r) - (1,1)),$$

for all (a,r) in A_∞. In particular,

$$\beta_{n-m+j}(A_\infty - (1,1)) \subseteq A_\infty - (1,1).$$

Proof. The first statement is obvious. For the second, let (a,r) be in A_∞; i.e. $r = \partial(a)$. Then we have

$$\begin{aligned}
\beta_j(a-1, \partial(a)-1) &= \left(n^{-1}m(a-1) - n^{-1}j, n^{-1}m(\partial(a)-1) - n^{-1}j\right) \\
&= \left(n^{-1}ma - n^{-1}(m+j), n^{-1}m\partial(a) - n^{-1}(m+j)\right) \\
&= \beta_{j+m}(a, \partial(a)).
\end{aligned}$$

and the conclusion follows.

For the third part, we can repeat the argument above, but noting that for $1 \leq j \leq m$, we may apply part 6 of Lemma 2.10 to obtain

$$\begin{aligned}
\beta_j(a, \partial(a)) &= \left(n^{-1}ma - n^{-1}j, n^{-1}m\partial(a) - n^{-1}j\right) \\
&= \left(n^{-1}ma - n^{-1}j, \partial\left(n^{-1}ma - n^{-1}j\right)\right)
\end{aligned}$$

From this we see that β_j maps A_∞ into itself and also into $(-n^{-1}j + m\mathbb{Z}_m) \times [-1,1]$. We also have

$$\begin{aligned}
\beta_{n-m+j}(a-1, \partial(a)-1) &= \big(n^{-1}m(a-1) - n^{-1}(n-m+j), \\
&\quad n^{-1}m(\partial(a)-1) - n^{-1}(n-m+j)\big) \\
&= \big(n^{-1}ma - n^{-1}(n+j), \\
&\quad n^{-1}m\partial(a) - n^{-1}(n+j)\big)
\end{aligned}$$

and hence

$$\begin{aligned}
\beta_{n-m+j}(a-1,\partial(a)-1)+(1,1) &= \big(n^{-1}ma-n^{-1}(n+j),\\
&\quad n^{-1}m\partial(a)-n^{-1}(n+j)\big)+(1,1)\\
&= \big(n^{-1}ma-n^{-1}j,\\
&\quad n^{-1}m\partial(a)-n^{-1}j\big)\\
&= \beta_j(a,\partial(a)).
\end{aligned}$$

□

Lemma 2.14. *1. For $1 \le j \le n$, the sets $\beta_j(B)$ are all contained in B, are pairwise disjoint and their union is B.*

2. For any $1 \le j \le j' \le n$, if $\overline{\beta_j(B)}$ and $\overline{\beta_{j'}(B)}$ meet, then $j' = j + m$ and the intersection is

$$\beta_j(A_\infty - (1,1)) = \beta_{j+m}(A_\infty).$$

3. If (a, r) is in B, $1 \le j \le n$ and b is in $\mathbb{Z}_n$, then $(\beta_j(a,r), mb)$ is in $B \times \mathbb{Z}_n$ and

$$\varphi^{-1} \circ \rho(a, r, m^{-1}j + nb) = \rho(\beta_j(a,r), mb)$$

4. For any $1 \le j \le n$, we have

$$\varphi^{-1} \circ \rho(B \times (j + n\mathbb{Z}_n)) = \beta_{mj}(B) \times \mathbb{Z}_n,$$

where mj is understood modulo n.

5. For $0 \le j < n$, if X is any subset of B, then

$$diam(\beta_j(X)) \le 2^{-1} diam(X).$$

Proof. First observe that $\beta_j(B) \subseteq -n^{-1}j + m\mathbb{Z} \times [-1,1]$, which means that $\beta_j(B)$ and $\beta_{j'}(B)$ have disjoint closures if j, j' are not congruent modulo m. We fix $1 \le j \le m$, and list the elements of its equivalence class modulo m up to n: $j = j_1 < j_2 < \cdots < j_L$, where $n - m < j_L \le n$. The top boundary of $\beta_{j_0}(B)$ lies in A_∞ by part 3 of Lemma 2.13. Moreover, for each $1 \le l < L$, the bottom boundary of $\beta_{j_l}(B)$ coincides with the top boundary of $\beta_{j_{l+1}}(B)$ by Lemma 2.12 and part 2 of Lemma 2.14. Finally, Lemma 2.13

shows that the bottom boundary of $\beta_{j_L}(B)$ lies in $A_\infty - (1,1)$ and is part of the bottom boundary of B. The first two parts of the Lemma follow from these observations.

For the third part, we see that $\beta_j(a,r)$ does indeed lie in B from the first two parts. Moreover, we have

$$\begin{aligned}
\varphi^{-1} \circ \rho(a,r,m^{-1}j+nb) &= \rho\left(n^{-1}ma, n^{-1}mr, n^{-1}m\left(m^{-1}j+nb\right)\right) \\
&= \rho\left(n^{-1}ma, n^{-1}mr, \left(n^{-1}j\right)+mb\right) \\
&= \rho\left(n^{-1}ma - n^{-1}j, n^{-1}mr - n^{-1}j, mb\right) \\
&= \rho(\beta_j(a,r), mb).
\end{aligned}$$

The fourth statement follows immediately from the third.

For the last statement, in the $\mathbb{Z}_m$-coordinate, each β_j multiplies by $n^{-1}m$ and translates. Translation is an isometry, as is multiplication by n^{-1}, while multiplication by m contracts by a factor 2^{-1} as mentioned in the introduction. In the $\mathbb{R}$-coordinate, the map multiplies by $n^{-1}m$ and translates and $n^{-1}m < 2^{-1}$. □

We are now ready to define the rectangles in our Markov partition.

Definition 2.15. *1. For $1 \le j \le n$, define*

$$R_j = \rho(B \times (j + n\mathbb{Z}_n)).$$

2. For any $K \le 0 \le L$ and function $\iota : \{K, K+1, \ldots, L-1, L\} \to \{1,2,\ldots,n\}$, we define

$$R_\iota = \cap_{k=K}^{L} \varphi^{-k}(R_{\iota(k)}).$$

Lemma 2.16. *1. For $0 \le L$ and $\iota : \{0,1,\ldots,L-1,L\} \to \{1,\ldots,n\}$, we have*

$$R_\iota = \rho(\beta_{m\iota(1)} \circ \cdots \circ \beta_{m\iota(L)}(B) \times (\iota(0) + n\mathbb{Z}_n)),$$

where $m\iota(i)$ is understood modulo n.

2. For $K \le 0$ and $\iota : \{K, K+1, \ldots, -1, 0\} \to \{1,\ldots,n\}$, we have

$$R_\iota = \rho\left(B \times \left(\sum_{k=K}^{0} \left(m^{-1}n\right)^{-k} \iota(k) + n^{1-K}\mathbb{Z}_n\right)\right).$$

Proof. Of course, the proof is similar to the $m = 2, n = 3$ case. We argue by induction on L. The case $L = 0$ is trivial. Next suppose that $L > 0$ and the statement holds for $L - 1$. Write

$$\cap_{l=0}^{L}\varphi^{-l}(R_{\iota(l)}) = R_{\iota(0)} \cap \varphi^{-1}(\cap_{l=0}^{L-1}\varphi^{-l}(R_{\iota(l+1)})).$$

We may apply the induction hypothesis to the function $l \to \iota(l+1)$ to see that

$$\cap_{l=0}^{L-1}\varphi^{-l}(R_{\iota(l+1)}) = \rho(\beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(L)}(B) \times (\iota(1) + n\mathbb{Z}_n))$$

Applying part 4 of Lemma 2.14, we see that

$$\begin{aligned}\varphi^{-1}(\cap_{l=0}^{L-1}\varphi^{-l}(R_{\iota(l+1)})) &= \varphi^{-1} \circ q(\beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(L)}(B) \\ &\quad \times(\iota(1) + n\mathbb{Z}_n)) \\ &= q(\beta_{m\iota(1)} \circ \cdots \circ \beta_{m\iota(L)}(B) \times \mathbb{Z}_n).\end{aligned}$$

Since all of our sets are contained in the fundamental domain $B \times \mathbb{Z}_n$, we have

$$\begin{aligned}\cap_{l=0}^{L}\varphi^{-l}(R_{\iota(l)}) &= q(\beta_{m\iota(1)} \circ \cdots \circ \beta_{m\iota(L)}(B) \times \mathbb{Z}_n \\ &\quad \cap B \times (\iota(0) + \mathbb{Z}_n)) \\ &= q(\beta_{m\iota(1)} \circ \cdots \circ \beta_{m\iota(L)}(B) \times (\iota(0) + n\mathbb{Z}_n))\end{aligned}$$

The proof of the second statement is exactly the same as that in Lemma 2.7 and we omit the details. □

The following is proved in exactly the same way as Theorem 2.8, with part 5 of Lemma 2.14 and Lemma 2.16 replacing Lemmas 2.5 and 2.7, respectively, and we omit the details.

Theorem 2.17. *For any function ι in $\{1, \ldots, n\}^{\mathbb{Z}}$, the intersection*

$$\cap_{K=0}^{\infty}\overline{\cap_{k=-K}^{K}\varphi^{-k}(R_{\iota(k)})}$$

is a single point and the map sending ι to this point defines a factor map, which we denote by π, from $(\{1, \ldots, n\}^{\mathbb{Z}}, \sigma)$ to (X, φ).

We will use Σ to denote $\{1, \ldots, n\}^{\mathbb{Z}}$.

Our next task is to carefully analyze where π maps two points to one and the first step is to understand the intersection of the elements of the Markov partition.

Lemma 2.18. *Let $j, j' : \{0,1\} \to \{1, \ldots, n\}$ satisfy $j \neq j'$ and*

$$\overline{R_j} \cap \overline{R_{j'}} \neq \emptyset.$$

Then, after possibly interchanging j and j', we have exactly one of the following:

1. *$j(0) = j'(0)$, $j(1) = j'(1) + 1 (\mathrm{mod}\, n)$ and $m < mj(1) \leq n$,*
2. *$j(0) = j'(0) + 1 (\mathrm{mod}\, n)$, $j(1) = j'(1) + 1 (\mathrm{mod}\, n)$ and $1 \leq mj(1) \leq m$,*

where $mj(1)$ is interpreted as the unique element of its $\mathrm{mod}\, n$-equivalence class which is between 1 and n.

Proof. From the results of Lemma 2.12, Lemma 2.13 and Lemma 2.14, we see that for $1 \leq mj(1) \leq n - m$, we have

$$\begin{aligned} \overline{R_j} &= \rho\left(\overline{\beta_{mj(1)}(B)} \times (j(0) + n\mathbb{Z}_n)\right), \\ &= \rho\left(\beta_{mj(1)}(B \cup (A_\infty - (1,1))) \times (j(0) + n\mathbb{Z}_n)\right) \\ &= \rho\left((\beta_{mj(1)}(B) \cup \beta_{m(j(1)+1)}(A_\infty)) \times (j(0) + n\mathbb{Z}_n))\right). \end{aligned}$$

and that the set

$$(\beta_{mj(1)}(B) \cup \beta_{m(j(1)+1)}(A_\infty)) \times (j(0) + n\mathbb{Z}_n))$$

is contained in our fundamental domain $B \times \mathbb{Z}_n$.

On the other hand, for $n - m < mj(1) \leq n$, we have

$$\begin{aligned} \overline{R_j} &= \rho\left(\overline{\beta_{mj(1)}(B)} \times j(0) + n\mathbb{Z}_n\right), \\ &= \rho\left(\beta_{mj(1)}(B \cup (A_\infty - (1,1))) \times (j(0) + n\mathbb{Z}_n)\right) \\ &= \rho\left(\beta_{mj(1)}(B) \times (j(0) + n\mathbb{Z}_n)\right. \\ &\quad \left. \cup(\beta_{m(j(1)+1)}(A_\infty) \times (j(0) + 1 + n\mathbb{Z}_n))\right). \end{aligned}$$

and that the set

$$\beta_{mj(1)}(B) \times (j(0) + n\mathbb{Z}_n) \cup \beta_{m(j(1)+1)}(A_\infty) \times (j(0) + 1 + n\mathbb{Z}_n)$$

is contained in our fundamental domain $B \times \mathbb{Z}_n$. Similar results hold for j', of course. Recall that the sets $\beta_{mj}(B)$ are pairwise disjoint for different values of j. The same holds for $j + n\mathbb{Z}_n$.

First, we suppose that $1 \leq mj(1), mj'(1) \leq n - m$ and the sets

$$\begin{array}{ccc} (\beta_{mj(1)}(B) \cup \beta_{m(j(1)+1)}(A_\infty)) & \times & (j(0) + n\mathbb{Z}_n)) \\ (\beta_{mj'(1)}(B) \cup \beta_{m(j'(1)+1)}(A_\infty)) & \times & (j'(0) + n\mathbb{Z}_n)) \end{array}$$

have non-trivial intersection. Simply comparing the $\mathbb{Z}_n$-components, we see that $j(0) = j'(0)$. We also assume that $j \neq j'$, which means that $j(1) \neq j'(1)$, so we must have $mj(1) = m(j'(1) + 1)$ (or the other way around) and so $j(1) = j'(1) + 1$ as claimed. Since $1 \leq mj'(1) \leq n - m$, we have $m < mj(1) \leq n$. This is covered by the first case in the conclusion.

Next, suppose that exactly one of $mj(1), mj'(1)$ is greater than $n - m$, say, $1 \leq mj(1) \leq n - m < mj'(1) \leq n$. We assume that the sets

$$\begin{array}{ccc} (\beta_{mj(1)}(B) \cup \beta_{m(j(1)+1)}(A_\infty)) & \times & (j(0) + n\mathbb{Z}_n)) \\ \beta_{mj'(1)}(B) \times (j'(0) + n\mathbb{Z}_n) & \cup & \beta_{m(j'(1)+1)}(A_\infty) \times (j'(0) + 1 + n\mathbb{Z}_n) \end{array}$$

have non-trivial intersection. Again, we first compare the $\mathbb{Z}_n$ components and see that either $j(0) = j'(0)$ or $j(0) = j'(0) + 1$. If $j(0) = j'(0)$, since $j \neq j'$, $mj'(1)$ must equal $m(j(1) + 1)$, or $j'(1) = j(1) + 1$. After reversing the roles of j and j', this is also covered by the first case of the conclusion. If $j(0) = j'(0) + 1$, we must have $m(j'(1) + 1)$ equals $mj(1)$ or $m(j(1) + 1)$. As $n - m < mj'(1) \leq n$, we have $1 \leq m(j'(1) + 1) \leq m < m(j(1) + 1)$ and so the second is not possible. We conclude that $j(0) = j'(0) + 1$ and $m(j'(1) + 1) = mj(1)$, or $j(1) = j'(1) + 1$. This is covered by the second case in the conclusion.

Finally, we consider the case that $n - m < mj(1), mj'(1) \leq n$. We assume that the sets

$$\begin{array}{ccc} \beta_{mj(1)}(B) \times (j(0) + n\mathbb{Z}_n) & \cup & \beta_{m(j(1)+1)}(A_\infty) \times (j(0) + 1 + n\mathbb{Z}_n) \\ \beta_{mj'(1)}(B) \times (j'(0) + n\mathbb{Z}_n) & \cup & \beta_{m(j'(1)+1)}(A_\infty) \times (j'(0) + 1 + n\mathbb{Z}_n) \end{array}$$

have non-trivial intersection. Since we assume that $j \neq j'$, the first set of each cannot intersect the first of the other, and similarly for the second of each. We are left to consider $mj'(1) = m(j(1)+1)$. But as $n - m < mj(1), mj'(1) \leq n$, this is impossible. $\square$

The last Lemma then provides the following analogue for infinite sequences.

Lemma 2.19. *Consider the following three (mutually exclusive) conditions on ι, ι' in $\{1, \ldots, n\}^{\mathbb{Z}}$*

1. $\iota = \iota'$,
2. *there is a unique integer k such that*
$$\begin{array}{rll} \iota(l) = \iota'(l), & \text{if} & l < k, \\ \iota(l) = \iota'(l) + 1, & \text{if} & l \geq k, \\ m < m\iota(k) \leq n, & & \\ 1 \leq m\iota(l) \leq m, & \text{if} & l > k, \end{array}$$
3. $\iota(l) = \iota'(l) + 1$, *and* $1 \leq m\iota(l) \leq m$, *for every integer* l.

We have $\pi(\iota) = \pi(\iota')$ if and only if, after possibly interchanging ι and ι', one of the three conditions holds.

Proof. Let us first consider the 'if' implication. If $\iota = \iota'$, then the conclusion is clear. Suppose that ι and ι' satisfy the second condition. Since $\pi \circ \sigma = \varphi \circ \pi$, by replacing ι, ι' by $\sigma^{1-k}(\iota), \sigma^{1-k}(\iota')$, we may assume that $k = 1$. First, we apply Lemma 2.16 in order to write
$$\overline{\cap_{-K}^{K} \varphi^{-k}(R_{\iota(k)})} = \rho(C \times D),$$
where
$$C = \beta_{m\iota(1)} \circ \cdots \beta_{m\iota(K)}(\overline{B}),$$
and
$$D = \sum_{k=-K}^{0} (m^{-1}n)^{-k} \iota(k) + n^{1-K} \mathbb{Z}_n.$$
We have a similar expression for
$$\overline{\cap_{-K}^{K} \varphi^{-k}(R_{\iota'(k)})} = \rho(C' \times D').$$

Notice that since $\iota(l) = \iota'(l)$, for $l < 1$, we have $D = D'$.

We use Lemma 2.12 and the fact that $1 \leq j = m\iota(l) \leq m$, $\iota(l) = \iota'(l) - 1$, for all $1 < l \leq K$ and the second part of Lemma 2.13:
$$\begin{array}{rcl} C' & = & \beta_{m\iota'(1)} \circ \cdots \beta_{m\iota'(K)}(\overline{B}) \\ & \supseteq & \beta_{m\iota'(1)} \circ \cdots \beta_{m\iota'(K)}(A_\infty - (1,1)) \\ & = & \beta_{m\iota'(1)} \circ \cdots \beta_{m\iota'(K-1)}(\beta_{m\iota(K)}(A_\infty) - (1,1)) \\ & = & \beta_{m\iota'(1)} \circ \cdots \beta_{m\iota'(K-2)}(\beta_{m\iota(K-1)} \circ \beta_{m\iota(K)}(A_\infty) - (1,1)) \\ & = & \cdots \\ & = & \beta_{m\iota'(1)} \left(\beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(K)}(A_\infty) - (1,1) \right). \end{array}$$

At this point, we use $m < m\iota(1) \leq n$, $\iota(1) = \iota'(1) + 1$ and the first part of Lemma 2.13 to conclude

$$C' \supseteq \beta_{m\iota(1)} \circ \beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(K)}(A_\infty).$$

On the other hand, Lemma 2.12 gives us

$$\beta_{m\iota(1)} \circ \beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(K)}(A_\infty) \subseteq \beta_{m\iota(1)} \circ \beta_{m\iota(2)} \circ \cdots \circ \beta_{m\iota(K)}(B) = C.$$

From these calculations, we conclude that $C \times D \cap C' \cap D'$ is non-empty. Since this holds for every K, we see that $\{\pi(\iota)\} \cap \{\pi(\iota')\}$ is also non-empty, so $\pi(\iota) = \pi(\iota')$. This establishes the result in the case the second condition holds.

In the third case, for each k in $\mathbb{Z}$, define ι_k, ι'_k in $\{1, 2, \ldots, n\}^{\mathbb{Z}}$ as follows. First, $\iota_k(l) = \iota'_k(l) = 1$, if $l < k$. Then set $\iota_k(k) = m^{-1}(m+1)$ and $\iota'_k(k) = \iota_k(k) - 1 = m^{-1}$. Finally, set $\iota_k(l) = \iota'_k(l)$, for all $l > k$.

Then for every k, ι_k and ι'_k satisfy the second condition so $\pi(\iota_k) = \pi(\iota'_k)$. On the other hand ι_k and ι'_k converge to ι and ι', respectively as k goes to $-\infty$. Hence by continuity, we have $\pi(\iota) = \pi(\iota')$.

Now, we consider the 'only if' implication. Fix an integer k for a moment. We know that

$$\pi(\iota) \in \overline{\varphi^{-k}(R_{\iota(k)}) \cap \varphi^{-k-1}(R_{\iota(k+1)})},$$

and

$$\pi(\iota') \in \overline{\varphi^{-k}(R_{\iota'(k)}) \cap \varphi^{-k-1}(R_{\iota'(k+1)})}.$$

Since $\pi(\iota) = \pi(\iota')$, we know that the intersection of $\overline{R_{\iota(k)}) \cap \varphi^{-1}(R_{\iota(k+1)})}$ and $\overline{R_{\iota'(k)}) \cap \varphi^{-1}(R_{\iota'(k+1)})}$ is non-empty. We may apply Lemma 2.18 to the functions sending $i = 0, 1$ to $\iota(k+i)$ and to $\iota'(k+i)$. This tells us that, if $\iota(k) \neq \iota'(k)$, then we have, after possibly interchanging ι and ι',

$$\iota(k) = \iota'(k) + 1, \iota(k+1) = \iota'(k+1) + 1, 1 \leq m\iota(k) \leq m.$$

So if $\iota(k) \neq \iota'(k)$, the same holds for all $l > k$. By negation, if $\iota(k) = \iota'(k)$, the same holds for $l < k$. Now if we have k where $\iota(k) = \iota'(k)$ and $\iota(k+1) \neq \iota'(k+1)$, then the same argument using Lemma 2.18 shows that $1m < m\iota(k+1) \leq n$ (assuming $\iota(k) = \iota'(k) + 1$). This completes the proof. $\square$

This last result establishes two properties of our factor map π which will be very important for the computation of our homology.

Theorem 2.20. *The map π is s-bijective and $\#\pi^{-1}\{x\} \leq 2$ for all x in X.*

Proof. We have identified all pairs ι, ι' with $\pi(\iota) = \pi(\iota')$ and it is clear that no two are right tail equivalent, or stably equivalent. (Recall that ι, ι' are right tail equivalent if there is an integer L such that $\iota(l) = \iota'(l)$, for all $l \geq L$.) Thus, π is s-resolving, meaning that it is injective when restricted to a stable equivalence class. On the other hand, the full shift is obviously mixing and so the same is true of the solenoid. It follows from Theorem 2.5.8 of [7] that π is also s-bijective.

Next, suppose that $\iota_1, \iota_2, \iota_3$ are distinct, but all have the same image under π. Then, for all sufficiently large l, we have

$$\iota_1(l) = \iota_2(l) \pm 1, \iota_1(l) = \iota_3(l) \pm 1, \iota_2(l) = \iota_3(l) \pm 1.$$

But the first two imply that $\iota_2(l) = \iota_3(l)$ or $\iota_2(l) = \iota_3(l) \pm 2$ so this is inconsistent with the third. Hence, π is at most two-to-one. □

3 Homology

The objective of this section is to prove our three main results.

3.1 Proof of Theorem 1.1: the case $3m < n$

We now begin our proof of Theorem 1.1 under the additional hypothesis that $3m < n$. We will only concern ourselves with the computations of the groups $H^s_N(X, \varphi)$; the groups $H^u_N(X, \varphi)$ are done similarly. We use the notation of [7].

Let us recall a few basic concepts from [7]. From our s-bijective factor map $\pi : (\Sigma, \sigma) \to (X, \varphi)$, we define, for $N \geq 0$,

$$\Sigma_N(\pi) = \{(\iota_0, \iota_1, \ldots, \iota_N) \mid \pi(\iota_0) = \pi(\iota_1) = \cdots = \pi(\iota_N)\}.$$

Each is a shift of finite type and $\Sigma_0(\pi) = \Sigma$.

For any shift of finite type, (Σ, σ), Krieger defined past and future dimension groups, which we denote by $D^s(\Sigma, \sigma)$ and $D^u(\Sigma, \sigma)$. We will be mainly interested in $D^s(\Sigma_N(\pi), \sigma)$, when $N = 0, 1$. In the case $N = 1$, there is an obvious action of the permutation group on two symbols, S_2, on $\Sigma_1(\pi)$ and hence also on its dimension group. We write this action on the right. The group $D^s_{\mathcal{Q}}(\Sigma_1(\pi), \sigma)$ is the quotient of $D^s(\Sigma_N(\pi), \sigma)$ by the subgroup generated by all elements a with $a \cdot \alpha = sgn(\alpha)a$, for all α in S_2 and all elements of the form $a \cdot \alpha - sgn(\alpha)a$, a in $D^s(\Sigma_N(\pi), \sigma)$.

First, Theorem 7.2.1 of [7] asserts that our homology groups $H^s_*(X, \varphi)$ can be computed as the homology of the complex $D^s_{\mathcal{Q}}(\Sigma_*(\pi), d^s_{\mathcal{Q}}(\pi))$. (Rather curiously, it will not be necessary for us to compute the boundary maps, so we do not give the definition here.)

Next, Theorem 2.20 and Theorem 4.2.12 of [7] tells us that the groups $D^s_{\mathcal{Q}}(\Sigma_N(\pi), \sigma)$ are zero for $N \geq 2$. Hence we are left with the computation for $N = 0, 1$ and the map $d^s_{\mathcal{Q}}(\pi)$ between them.

The easiest calculation is the group for $N = 0$, for then we are simply calculating Krieger's dimension group invariant for the full n-shift. As the underlying graph has n-vertices and exactly one edge between each pair, $D^s(\Sigma, \sigma)$ is the inductively limit

$$\lim \mathbb{Z}^n \to \mathbb{Z}^n \to \cdots$$

where the maps are multiplication by the $n \times n$ matrix whose entries are all ones. The result is $\mathbb{Z}[1/n]$ and the canonical automorphism induced by σ is multiplication by n^{-1}.

Next, we turn to $D^s_{\mathcal{Q}}(\Sigma_1(\pi), \sigma)$. The first step is to find a graph which presents the shift $(\Sigma_1(\pi), \sigma)$. We construct a graph, G_1, as follows. The vertex set is

$$G^0_1 = \{(i,i), (i, i+1), (i+1, i) \mid 1 \leq i \leq n\},$$

where $n + 1 = 1$. The edge set consists of ordered pairs of vertices:

$$\begin{aligned} G^1_1 \quad = \quad & \{((i,i),(j,j)) \mid 1 \leq i, j \leq n\} \\ & \cup \{((i,i),(j,j+1)),((i,i),(j+1,j)) \mid 1 \leq i \leq n, m < mj \leq n\} \\ & \cup \{((i,i+1),(j,j+1)),((i+1,i),(j+1,j)) \mid 1 \leq mj \leq m\}. \end{aligned}$$

The initial and terminal maps from G^1_1 to G^0_1 are simply the projection onto the first and second ordered pairs, respectively. We let (Σ_{G_1}, σ) denote the edge shift associated with the graph G_1.

The following is an immediate consequence of Lemma 2.19 and our definition of G_1.

Lemma 3.1. *The vertex shift (Σ_{G_1}, σ) is explicitly conjugate to $(\Sigma_1(\pi), \sigma)$ by suppressing the distinction that elements of the former set are infinite* *sequences of ordered pairs, while elements of the latter are ordered pairs of infinite sequences.*

The computation of $D^s_{\mathcal{Q}}(\Sigma_1(\pi), \sigma)$ is given at the end of 4.2 of [7] and we follow the notation there. In the permutation group on two symbols, S_2, we let ε denote the identity. We need to find a subset B^0_1 of G^0_1 which meets each orbit with trivial isotropy exactly once and does not meet the others. The obvious candidate is

$$B^0_1 = \{(i, i+1) \mid 1 \leq i \leq n\},$$

where again $n + 1 = 1$. For $(i, i+1)$ in B^0_1, it is a simple matter to see that

$$\begin{aligned} t^*_{\mathcal{A}}((i, i+1), 1) &= \{(q, \alpha) \in G^1_1 \times S_2 \mid t(q) = (i, i+1), i(q) \cdot \alpha \in B^0_1\} \\ &= \{(((j, j+1), (i, i+1)), \varepsilon) \mid 1 \leq j \leq n\} \end{aligned}$$

if $1 \leq mi \leq m$ and is empty for $m < mi \leq n$. Then we have $\gamma^s_B : \mathbb{Z}B^0_1 \to \mathbb{Z}B^0_1$ is given by

$$\begin{aligned} \gamma^s_B((i, i+1)) &= \sum_{(q,\alpha) \in t^*((i,i+1),1)} sgn(\alpha) i(q) \cdot \alpha \\ &= \sum_{1 \leq j \leq n} (j, j+1) \end{aligned}$$

if $1 \leq mi \leq m$ and is zero otherwise. Combined with an easy computation of the inductive limit

$$\lim \mathbb{Z}B^0_1 \stackrel{\gamma^s_B}{\to} \mathbb{Z}B^0_1 \stackrel{\gamma^s_B}{\to} \cdots$$

Corollary 4.2.14 of [7] gives the following.

Lemma 3.2. *We have*

$$D^s_{\mathcal{Q}}(\Sigma_1(\pi), \sigma) \cong \mathbb{Z}[m^{-1}],$$

and under this isomorphism, the map induced by σ is multiplication by m^{-1}.

The last piece of information needed is the boundary map

$$d^s_{\mathcal{Q}}(\pi) : D^s_{\mathcal{Q}}(\Sigma_1(\pi), \sigma) \to D^s_{\mathcal{Q}}(\Sigma_0(\pi), \sigma).$$

It is a group homomorphism and we know the domain is isomorphic to $\mathbb{Z}[m^{-1}]$ and the range is isomorphic to $\mathbb{Z}[n^{-1}]$. We escape having to do any computations by simply observing that the only group homomorphism between these groups is the zero homomorphism. We conclude that

$$H^s_1(X,\varphi) = \ker(d^s_{\mathcal{Q}}(\pi))_1) = D^s_{\mathcal{Q}}(\Sigma_1(\pi),\sigma) \cong \mathbb{Z}[m^{-1}],$$

while

$$H^s_0(X,\varphi) = D^s_{\mathcal{Q}}(\Sigma_0(\pi),\sigma)/Im(d^s_{\mathcal{Q}}(\pi))_1) = D^s_{\mathcal{Q}}(\Sigma_0(\pi),\sigma) \cong \mathbb{Z}[n^{-1}],$$

and the maps induced by φ are as claimed.

The completes the proof in the special case that $3m < n$.

3.2 Proof of Theorem 1.1: the general case

For the general case, $m < n$, we choose a positive, odd integer l such that $3m^l < n^l$. The computations we have done hold for the solenoid (X, φ^l). But this system has exactly the same homology groups as (X, φ) so

$$H^s_0(X,\varphi) = H^s_0(X,\varphi^l) \cong \mathbb{Z}[(n^l)^{-1}] = \mathbb{Z}[n^{-1}].$$

The map induced by φ^l is multiplication by n^{-l}. We can view $\varphi : (X, \varphi^l) \to (X, \varphi^l)$ as an s-bijective factor map, so it induces a map on homology. Moreover, its l-th power is the map induced by φ^l which is multiplication by n^{-l}. But as l was odd, we conclude that the map induced by φ must be multiplication by n^{-1}. A similar argument takes care of the first homology group.

3.3 The remaining proofs

The proof of Corollary 1.2 is an immediate consequence of Theorem 1.1 and Theorem 6.1.1 of [7].

For the proof of Corollary 1.3, we adopt the terminology of [7]. For the proof of 1.1, we have used the fact that, with a small abuse of notation, $\pi = (\Sigma, \sigma, \pi, X, \varphi, id_X)$ is an s/u-bijective pair for (X, φ). From this, we we have computed the groups in the complex $C^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}, L, M \geq 0$ and the maps $d^s_{\mathcal{Q},\mathcal{A}}(\pi)_{L,M}, L, M \geq 0$ between them.

We need to do the same for the Smale space (X, φ^{-1}). Of course, the main point is that the stable sets for φ^{-1} are precisely the unstable sets for

φ, and vice verse. In short, $\pi' = (X, \varphi^{-1}, id_X, \Sigma, \sigma^{-1}, \pi)$ is an s/u-bijective pair for (X, φ^{-1}).

The second key observation is that, for any shift of finite type (Σ, σ), there is a natural isomorphism

$$D^s(\Sigma, \sigma^{-1}) \cong D^u(\Sigma, \sigma)$$

and the isomorphism conjugates the canonical automorphism of the former to that of the latter.

It follows from the definitions that, for all $L, M \geq 0$,

$$C^s_{\mathcal{Q},\mathcal{A}}(\pi')_{L,M} \cong C^u_{\mathcal{Q},\mathcal{A}}(\pi)_{M,L},$$

for all $L, M \geq 0$, and this isomorphism conjugates the canonical automorphism of the former to that of the latter. The conclusion of 1.3 follows at once.

References

[1] M Amini, S. Gholikandi and I.F. Putnam, *Homology for one-dimensional solenoids*, preprint.

[2] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 725-747.

[3] R. Bowen, *On Axiom A diffeomorphisms*, AMS-CBMS Reg. Conf. **135**, Providence, 1978.

[4] F.Q. Gouvea, *p-adic Numbers: An Introduction* 2nd ed., Springer, New York, 1997.

[5] W. Krieger, *On dimension functions and topological Markov chains*, Invent. Math. **56** (1980), 239-250.

[6] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.

[7] I. F. Putnam, *A homology theory for Smale spaces*, *Mem. A.M.S.*, to appear.

[8] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math. and its Appl. 5, Addison-Wesley, Reading, 1978.

[9] S. Smale, *Differentiable dynamical systems*,Bull. Amer. Math. Soc. **73** (1967), 747-817.

[10] K. Thomsen, *The homoclinic and heteroclinic C^*-algebra of a generalized one-dimensional solenoid*, Ergod. Th. & Dynam. Sys. **30** (2010), 263-308.

[11] T. Ward and Y. Yamama, *Markov partitions reflecting the geometry of* $\times 2 \times 3$, Discrete and Continuous Dynamical Systems **24** 2 (2009), 613-624.

[12] R.F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 37-487.

[13] A.M. Wilson, *On endomorphisms of a solenoid*, Proc. A.M.S. **55**(1976), 69-74.

[14] I. Yi, *Canonical symbolic dynamics for one-dimensional generalized solenoids*, Trans. A.M.S. **353** (2001), 37413767.