HOMOLOGY FOR ONE-DIMENSIONAL SOLENOIDS

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Abstract. Smale spaces are a particular class of hyperbolic topological dynamical systems, defined by David Ruelle. The definition was introduced to give an axiomatic description of the dynamical properties of Smale's Axiom A systems when restricted to a basic set. They include Anosov diffeomeorphisms, shifts of finite type and various solenoids constructed by R.F. Williams. The second author constructed a homology theory for Smale spaces which is based on (and extends) Krieger's dimension group invariant for shifts of finite type. In this paper, we compute this homology for the one-dimensional generalized solenoids of R.F. Williams.

1. Introduction and statement of the results

Smale spaces were defined by David Ruelle as a purely topological version of the basic sets of Axiom A systems which arise in Smale's program for differentiable dynamics [9, 10, 1, 5, 4]. Informally, a pair (X, φ), where X is a compact metric space and φ a homeomorphism of X, is a Smale space if it possesses local coordinates in contracting and expanding directions. Hyperbolic toral automorphisms, one-dimensional generalized solenoids as described by R.F. Williams and shifts of finite type are all examples of Smale spaces.

To be more precise, a Smale space is a compact metric space, (X, d), together with a homeomorphism of X, φ, satisfying certain conditions

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as follows. There exist constants $\varepsilon_X > 0$ and $0 < \lambda < 1$ and a continuous map from

$$\triangle_{\varepsilon_X} = \{(x, y) \in X \times X \quad | \quad d(x, y) \leq \varepsilon_X \quad \}$$

to X (denoted with $[,]$) such that (whenever both sides of an equation are defined):

$B\,1$ $[x, x] = x,$
$B\,2$ $[x, [y, z]] = [x, z],$
$B\,3$ $[[x, y], z] = [x, z],$
$B\,4$ $[\varphi(x), \varphi(y)] = \varphi[x, y],$
$C\,1$ $d(\varphi(x), \varphi(y)) \leq \lambda\, d(x, y),$ whenever $[x, y] = y,$
$C\,2$ $d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda\, d(x, y),$ whenever $[x, y] = x.$

For x in X and $0 < \varepsilon \leq \varepsilon_X$, we define

$$X^s(x, \varepsilon) = \{y \mid \quad d(x, y) \leq \varepsilon\ , [x, y] = y\}$$

$$X^u(x, \varepsilon) = \{y \mid \quad d(x, y) \leq \varepsilon\ , [x, y] = x\}$$

which are called local stable and local unstable sets. The map sending (y, z) in $X^s(x, \varepsilon) \times X^u(x, \varepsilon)$ to $[z, y]$ is then a homeomorphism to a neighbourhood of x in X. This is the local product structure.

In any Smale space X, we say that two points x and y in X are stably (or unstably) equivalent if

$$\lim_{n\to+\infty} d(\varphi^n(x), \varphi^n(y)) = 0 \qquad (or \lim_{n\to-\infty} d(\varphi^n(x), \varphi^n(y)) = 0, \text{resp.}).$$

Let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of x, respectively. As the notation would suggest, there is a close connection between local stable sets and stable equivalence classes (see Chapter 2 of [8]).

In this paper, the Smale spaces of interest will be the one-dimensional generalized solenoids defined by Robert Williams [14, 15], generalized by Inhyeop Yi [17] and later by Klaus Thomsen [11]. The spaces are inverse limits of a single finite graph and a single self-map.

Definition 1.1. Let F be a finite (unoriented), connected graph with vertices F^0 and edges F^1 which we regard as a topological space. Consider a continuous map $f : F \to F$. We say that (F, f) is a pre-solenoid if the following conditions are satisfied for some metric d giving the topology of F:

α) (expansion) there are constants $C > 0$ and $\lambda > 1$ such that $d(f^n(x), f^n(y)) \geq C\lambda^n d(x, y)$ for every $n \in \mathbb{N}$ when $x, y \in e \in F^1$ and there is an edge $e' \in F^1$ with $f^n([x, y]) \in e'$ ($[x, y]$ is the interval in e between x and y),

β) (non-folding) f^n is locally injective on e for each $e \in F^1$ and each $n \in \mathbb{N}$,
γ) (Markov) $f(F^0) \subset F^0$,
δ) (mixing) for every edge $e \in F^1$, there is $m \in \mathbb{N}$ such that $F \subseteq f^m(e)$,
ϵ) (flattening) there is $d \in \mathbb{N}$ such that for all $x \in F$ there is a neighbourhood U_x of x with $f^d(U_x)$ homeomorphic to $(-1, 1)$.

We usually refer to a d satisfying the flattening condition as the *flattening number* of f.

Suppose that (F, f) is a pre-solenoid. Define

$$\overline{F} = \{(x_i)_{i=0}^{\infty} \in F^{\mathbb{N}\cup\{0\}} : f(x_{i+1}) = x_i, i = 0, 1, 2, \cdots\}$$

Then $\overline{F}$ is a compact metric space with the metric

$$D((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i} d(x_i, y_i).$$

We also define $\overline{f} : \overline{F} \to \overline{F}$ by $\overline{f}(x)_i = f(x_i)$ for all $i \in \mathbb{N} \cup \{0\}$. It is a homeomorphism with inverse $\overline{f}^{-1}(x)_i = x_{i+1}, i \geq 0$. Finally, we define the map $\pi : \overline{F} \to F$ by

$$\pi(x_0, x_1, x_2, \ldots) = x_0, \qquad (x_0, x_1, x_2, \ldots) \in \overline{F}.$$

Definition 1.2 ([11])**.** Let (F, f) be a pre-solenoid. The system $(\overline{F}, \overline{f})$ is called a (generalized) one-solenoid.

One-dimensional substitution tiling spaces are examples of generalized one-solenoids.

In [8], the second author introduced a homology theory for Smale spaces. Such a theory was proposed by Bowen to give a homological interpretation of the rationality of the Artin-Mazur zeta function [2] which was proved by Manning. This theory also generalizes Krieger's dimension group invariant for shifts of finite type, which we describe below. To any Smale space (X, φ), there are two sequences of abelian groups $H^s_N(X, \varphi)$ and $H^u_N(X, \varphi)$, for $N \in \mathbb{Z}$. Our aim here is to compute $H^s_*(\overline{F}, \overline{f})$ and $H^u_*(\overline{F}, \overline{f})$, for any generalized one-solenoid constructed as above.

If (X, φ) is a Smale space, then so is (X, φ^n), for every positive integer n. In fact, these two Smale spaces have exactly the same stable and unstable equivalence relations. Somewhat more subtlety, they have naturally isomorphic homology theories in the sense of [8]. If one takes the view that the homology theories produce a sequence of abelian groups together with canonical automorphisms induced by φ (see Chapter 3 of [8]), then, while the groups are the same, the automorphism of the latter is simply the nth power of that of the former.

As our attention will be mainly in computing the groups themselves, we will be quite happy to replace φ by φ^n.

In Lemma 2.4 section 2, we show that we can restrict our attention to pre-solenoids of a particularly nice form. The space F will be a wedge of n circles. That is, the graph has a single vertex, which we denote p. We let $e_1, e_2, \ldots, e_n$ denote the interiors of the edges, that is, the connected, open intervals of $F - \{p\}$, and $E = \{e_1, \ldots, e_n\}$. We will assume that each has a fixed orientation. As $f^{-1}\{p\}$ is finite, for each i, $e_i - f^{-1}\{p\}$ is a union of pairwise disjoint open intervals, which we label as $e_{i,1}, e_{i,2}, \ldots, e_{i,j(i)}$, written in increasing order with respect to the given orientation of e_i. We also assume that $j(i) \geq 3$. Finally, for each (i,j) with $1 \leq i \leq n$ and $1 \leq j \leq j(i)$, there is $1 \leq f(i,j) \leq n$ such that the map f sends $e_{i,j}$ homeomorphically to $e_{f(i,j)}$. In addition, we assume that the flattening number of f is 1.

There is a convenient notation to describe such pre-solenoids. We let E^* denote the set of words on the set E and their inverses. If we are given a function $\widetilde{f} : E \to E^*$, we regard this as a description of the map f as follows. Fix $1 \leq i \leq n$. If

$$\widetilde{f}(e_i) = e_{f(i,1)}^{s(i,1)} e_{f(i,2)}^{s(i,2)} \ldots e_{f(i,j(i))}^{s(i,j(i))}$$

then the interval e_i is divided into $j(i)$ consecutive subintervals, $e_{i,j}, 1 \leq j \leq j(i)$, and the map f carries $e_{i,j}$ homemorphically to $e_{f(i,j)}$, either preserving or reversing the orientation according to whether $s(i,j)$ is 1 or -1.

Pre-solenoids divide into two classes: orientable and non-orientable. We give a precise discussion in section 2, but in our special case, (F, f) is positively orientable if the orientations of the edges may be chosen so that f maps each $e_{i,j}$ (with the relative orientation from e_i) to $e_{f(i,j)}$ is a way that preserves the orientation. That is, $s(i,j) = 1$ for all i, j. On the other hand, it is negatively orientable if the orientations can be chosen so that $s(i,j) = -1$ for all i, j. If it reverses the orientation, for *all* i, j, we simply replace f by f^2, which is positively orientable.

Following Yi [17], for a pre-solenoid (F, f) as above, we define a graph G_F by setting $G_F^0 = \{e_i \mid 1 \leq i \leq n\}$ and $G_F^1 = \{e_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq j(i)\}$. The initial and terminal maps are given by $i(e_{i,j}) = e_i, t(e_{i,j}) = f(e_{i,j}) = e_{f(i,j)}$, for each suitable i and j.

To any graph G, we associate the dynamical system which is the edge shift of the graph:

$$\Sigma_G = \{ \ (e_m)_{m \in \mathbb{Z}} \mid e_m \in G^1, \ t(e_m) = i(e_{m+1}) \ for \ all \ m \in \mathbb{Z}\},$$

$$(\sigma(e))_m = e_{m+1}.$$

For any e in Σ_G and $K \leq L$, we let $e_{[K,L]} = (e_K, e_{K+1}, \ldots, e_L)$. It is also convenient to define $e_{[K+1,K]} = t(e_K) = i(e_{K+1})$. We use the metric

$$d(e,f) = \inf\{1, 2^{-K-1} \mid K \geq 0, e_{[1-K,K]} = f_{[1-K,K]}\}$$

on Σ_G. It is then easy to see that (Σ_G, σ) is a Smale space with constants $\varepsilon_X = \lambda = \frac{1}{2}$ and

$$[e,f]_k = \begin{cases} f_k & k \leq 0 \\ e_k & k \geq 1. \end{cases}$$

Such a system is a shift of finite type (see [7]). Moreover, every shift of finite type is conjugate to (Σ_G, σ), for some graph G.

For the case when F is a pre-solenoid as above, for notational simplicity, we let (Σ_F, σ) denote the associated shift of finite type for the graph G_F.

In [17], Yi showed the existence of a factor map

$$\overline{\rho} : (\Sigma_F, \sigma) \to (\overline{F}, \overline{f}).$$

We review this construction in section 3. This must be done with some care because we need to describe all pairs I, J in Σ_F such that $\overline{\rho}(I) = \overline{\rho}(J)$. The two important points which emerge (items 3 and 4 from Theorem 3.2) are that $\overline{\rho}$ is at most two-to-one, that is $\#\overline{\rho}^{-1}\{x\} \leq 2$, for all x in $\overline{F}$, and that $\overline{\rho}$ is s-bijective: for every I in Σ_F, $\overline{\rho}$ is a bijection between the stable equivalence class of I in Σ_F, $\Sigma^s(I)$, and the stable equivalence class of $\overline{\rho}(I)$ in $\overline{F}$, $\overline{F}^s(\overline{\rho}(I))$.

The fundamental ingredient in the homology theory of [8] is Krieger's dimension group invariant for shifts of finite type. Let us review the computation briefly. If G is a finite directed graph, we let $\mathbb{Z}G^0$ denote the free abelian group on the vertex set, G^0. The edge data defines two maps

$$\gamma_G^s, \gamma_G^u : \mathbb{Z}G^0 \to \mathbb{Z}G^0$$

by

$$\gamma_G^s(v) = \sum_{e \in G^1, t(e)=v} i(e), \gamma_G^u(v) = \sum_{e \in G^1, i(e)=v} t(e),$$

for each v in G^0. We define $D^s(G)$ to be the inductive limit of the sequence

$$\mathbb{Z}G^0 \xrightarrow{\gamma_G^s} \mathbb{Z}G^0 \xrightarrow{\gamma_G^s} \mathbb{Z}G^0 \xrightarrow{\gamma_G^s} \cdots .$$

We regard this inductive limit as $\mathbb{Z}G^0 \times \mathbb{N}$, modulo the relation $(a,i) \sim (b,j)$ if there is $k \geq 0$ such that $(\gamma^s)^{j+k}(a) = (\gamma^s)^{i+k}(b)$, $a, b \in \mathbb{Z}G^0$, and $i, j \in \mathbb{N}$. We let $[a,i]$ denote the equivalence class of (a,i). It is a group with the operation $[a,i] + [b,j] = [(\gamma^s)^j(a) + (\gamma^s)^i(b), i+j]$. $D^u(G)$ is defined analogously. For any shift of finite type (Σ, σ), Krieger gave a dynamical definition of a group $D^s(\Sigma, \sigma)$ and showed that for any finite graph G, that this invariant could be computed: $D^s(\Sigma_G, \sigma) \cong D^s(G)$. $D^u(\Sigma_G, \sigma) \cong D^u(G)$ is computed in an analogous fashion.

In the fourth section, we describe special features of the dimension groups associated with the graph G_F. We take an open set U_p containing p and small enough so that $f(U_p)$ is contained in $\cup_{i=1}^n e_{i,1} \cup e_{i,j(i)}$. By virtue of the flattening number being 1, we may find $1 \leq a, b \leq n$ (allowing the possibility that $a = b$) and orientations of them such that

$$f(U_p) \subseteq e_{a,1} \cup e_{b,j(b)}.$$

We then define

$$\begin{aligned} A &= \{e_i \mid f(e_{i,1} \cap U_p), f(e_{i,j(i)} \cap U_p) \subseteq e_{a,1}\}, \\ B &= \{e_i \mid f(e_{i,1} \cap U_p), f(e_{i,j(i)} \cap U_p) \subseteq e_{b,j(b)}\}. \end{aligned}$$

These sets are disjoint.

We then let $w = \sum_{e_i \in A} e_i - \sum_{e_j \in B} e_j$ in $\mathbb{Z}G_F^0$ and w^* be the unique group homomorphism $w^* : \mathbb{Z}G_F^0 \to \mathbb{Z}$ which sends the elements of A to 1, those of B to -1 and the elements of $E - A - B$ to 0.

In Theorem 2.6 and Lemma 4.1, we show that $\gamma^s(w) = w$, $w^* \circ \gamma^u = w^*$ and that the following conditions are equivalent:

(1) (F, f) is orientable,
(2) A and B are both empty,
(3) $w = 0$,
(4) $w^* = 0$.

In the fifth section, we compute the homology for one-solenoids. Let us make a brief, simplified comparison between the homology for Smale spaces and the Cech cohomology of a compact manifold. For the latter, one begins with a 'good' open cover: each open set is homeomorphic to a ball in Euclidean space and so is each non-empty intersection of the elements of the cover. The cohomology is then computed (algebraically) from knowing the cohomology of an open ball and the combinatorial data which is the nerve of the cover.

In the Smale space homology, the open cover is replaced by a factor map from a shift of finite type, the intersections of the elements of the cover are replaced with self-products of the factor map (which are also shifts of finite type) and the cohomology of the Euclidean ball is replaced by Krieger's invariant for the shifts.

Using results from [8] and the two particular properties of our factor map $\overline{\rho}$, which we have already described, we are able to greatly simplify the computation. Define

$$\Sigma_1(\overline{\rho}) = \{(I, J) \mid I, J \in \Sigma_F, \overline{\rho}(I) = \overline{\rho}(J)\}.$$

With the map $\sigma \times \sigma$, this is also a shift of finite type. Moreover, the cyclic group of order two acts by permuting the entries: $\alpha(I, J) = (J, I)$. These actions induce actions, denoted α_*, on $D^s(G_F)$ and $D^u(G_F)$.

We define $D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho}))$ to be the quotient of $D^s(\Sigma_1(\overline{\rho}))$ by the subgroup generated by all elements a with $\alpha_*(a) = a$ and all elements of the form $b - \alpha_*(b)$, for all b in $D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho}))$. We also define $D^u_{\mathcal{A}}(\Sigma_1(\overline{\rho}))$ to be the subgroup of $D^u(\Sigma_1(\overline{\rho}))$ by the subgroup generated by all elements a with $\alpha_*(a) = -a$.

In Lemma 5.2, we show that the groups $D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho}))$ and $D^u_{\mathcal{A}}(\Sigma_1(\overline{\rho}))$ are both infinite cyclic groups and we identify their generators in terms of the elements w and w^*.

There are factor maps

$$\delta_0, \delta_1 : \Sigma_1(\overline{\rho}) \to \Sigma_F$$

defined by $\delta_0(I, J) = J, \delta_1(I, J) = I$. Both of these are also s-bijective and, as described in [8], they induce maps, denoted δ^s_0, δ^s_1 on the D^s-invariants and and $\delta^{u*}_0, \delta^{u*}_1$ on the D^u-invariants. However, for s-bijective maps, D^s is covariant, while D^u is contravariant. (This is responsible for the $*$.) Thus we obtain group homomorphisms

$$\begin{array}{rcl} \delta^s_0 - \delta^s_1 : & D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho})) & \to D^s(G_F), \\ \delta^{u*}_0 - \delta^{u*}_1 : & D^u(G_F) & \to D^u_{\mathcal{A}}(\Sigma_1(\overline{\rho})) \end{array}$$

These maps are computed explicitly in Lemmas 5.4 and 5.5.

Using the particular features of our map $\overline{\rho}$, the results from [8], which we summarize in Theorem 5.1 and the discussion which follows, show that

$$\begin{array}{rcl} H^s_0(\overline{F}, \overline{f}) & = & coker(\delta^s_0 - \delta^s_1) \\ H^s_1(\overline{F}, \overline{f}) & = & ker(\delta^s_0 - \delta^s_1) \\ H^u_0(\overline{F}, \overline{f}) & = & ker(\delta^{u*}_0 - \delta^{u*}_1) \\ H^u_1(\overline{F}, \overline{f}) & = & coker(\delta^{u*}_0 - \delta^{u*}_1), \end{array}$$

while $H^s_N(\overline{F}, \overline{f}) = H^u_N(\overline{F}, \overline{f}) = 0$, for all $N \neq 0, 1$.

We are now ready to state our main results.

Theorem 1.3. *Let (F, f) be a pre-solenoid and $(\overline{F}, \overline{f})$ be its associated one-solenoid. If (F, f) is orientable, then*

$$H^s_N(\overline{F}, \overline{f}) = \begin{cases} D^s(G_F) & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

If (F, f) is not orientable, then

$$H^s_N(\overline{F}, \overline{f}) = \begin{cases} D^s(G_F)/< 2[w,1] > & N = 0, \\ 0 & N \neq 0. \end{cases}$$

(Here, $< 2[w,1] >$ denotes the cyclic subgroup generated by $2[w,1]$.)

Theorem 1.4. *Let (F, f) be a pre-solenoid and $(\overline{F}, \overline{f})$ be its associated one-solenoid. If (F, f) is orientable, then*

$$H^u_N(\overline{F}, \overline{f}) = \begin{cases} D^u(G_F) & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

If (F, f) is not orientable, then

$$H^u_N(\overline{F}, \overline{f}) = \begin{cases} Ker(w^*) & N = 0, \\ \mathbb{Z}_2 & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

Corollary 1.5. *Let (F, f) be a pre-solenoid and $(\overline{F}, \overline{f})$ be its associated one-solenoid. If (F, f) is orientable then all of the homology groups of $(\overline{F}, \overline{f})$ are torsion free.*

If (F, f) is not orientable, then we have

$$Tor(H^s_0(\overline{F}, \overline{f})) \cong Tor(H^u_1(\overline{F}, \overline{f})) \cong \mathbb{Z}_2,$$

where $Tor(H)$ denotes the torsion subgroup of H, and the remaining homology groups are torsion free.

Notice, in particular, that this means that the notion of orientability is independent of the choice of (F, f). This provides a new proof of this fact, shown by Thomsen [11]. (See Theorem 2.2.)

We remark that it seems these groups are the same as the K-theory groups computed by Thomsen [11] for the C^*-algebras associated with the two heteroclinic relations on the solenoid. (The description in [11] looks rather different from ours.) This is not unexpected in a low-dimensional example. More generally, one expects a spectral sequence to compute the K-theory of the C^*-algebras from the homology.

The Cech cohomology of the space $\overline{F}$ may be computed as follows. First, $\overline{F}$ is written as the inverse limit of the stationary system with space F and map f and so its cohomology is the direct limit of the stationary system of groups $\check{H}^*(F)$ with maps f^*. Since F is the wedge of n circles, its cohomology is $\mathbb{Z}$ in dimension zero, $\mathbb{Z}^n$ in dimension one and zero in all other dimensions. Moreover, a simple direct computation shows that in the special case that (F, f) is orientable, then the map f^* is the identity in dimension zero and agrees with γ^s_G in dimension one if we identify $\mathbb{Z}G^0$ with $\mathbb{Z}^n$ in an obvious way. Therefore, we have the following.

Corollary 1.6. *If $(\overline{F}, \overline{f})$ is an orientable one-solenoid then*

$$\begin{aligned} H^s_0(\overline{F}, \overline{f}) &\cong \check{H}^1(\overline{F}) \\ H^s_1(\overline{F}, \overline{f}) &\cong \check{H}^0(\overline{F}). \end{aligned}$$

Presumably, this type of result holds in much greater generality. It seems reasonable to think that the homology for Smale spaces coincides with the Cech cohomology of the underlying space when the stable or unstable sets are contractible, although with some dimension shift. Notice however some kind of orientability hypothesis is necessary, as follows.

If (F, f) is not orientable, then the map on homology, f^*, notices the difference in orientations while γ^s_G does not. In particular, the map h of Example 2.5 induces an isomorphism on $\check{H}^*(F)$ and so we have $\check{H}^1(\overline{F}) \cong \mathbb{Z}^2$ and hence this group is not isomorphic to $H^s_0(\overline{F}, \overline{h})$.

2. One-solenoids

We first note the following. A proof can be given using the techniques in Wieler [13].

Theorem 2.1. [12] *One-solenoids are Smale spaces.*

Considering a pre-solenoid (F, f), an orientation of F is defined to be a collection of homeomorphisms $\psi_e : (0, 1) \to e$, $e \in E$. We say that f is positively (respectively, negatively) oriented with respect to the orientation $\psi_e, e \in E$, when the function

$$\psi^{-1}_{e'} o f o \psi_e : \psi^{-1}_e(e \cap f^{-1}(e')) \to [0, 1]$$

is increasing (respectively, decreasing) for each $e, e' \in E$. A pre-solenoid (F, f) is positively (respectively, negatively) oriented when there is an orientation of the edges in F such that f is positively (respectively, negatively) oriented with respect to that orientation. (F, f) is oriented when it is either positively or negatively oriented. Notice that if (F, f) is oriented, then (F, f^2) is positively oriented. We say that (F, f) is orientable if F has an orientation making (F, f) oriented.

The one-solenoid $(\overline{F}, \overline{f})$ is orientable when there is an oriented pre-solenoid (F_1, f_1) such that $(\overline{F}, \overline{f})$ is conjugate to $(\overline{F_1}, \overline{f_1})$. When (F_1, f_1) can be chosen to be positively (resp, negatively) oriented, we say that $(\overline{F}, \overline{f})$ is positively (resp, negatively) orientable[11]. Thomsen showed that the orientability of the one-solenoid is independent of its presentation.

Theorem 2.2. [11] *Let (F, f) be a pre-solenoid. Then $(\overline{F}, \overline{f})$ is positively (resp, negatively) orientable if and only if (F, f) is positively (resp, negatively) oriented.*

We observe that if (F, f) is a pre-solenoid. Then for $n \in \mathbb{N}$, (F, f^n) is also a pre-solenoid. Moreover, if d is a flattening number, then (F, f^n), $n \geq d$ is a pre-solenoid whose flatting number is one. Also observe that $(\overline{F}, \overline{f^n})$ is the same as $(\overline{F}, \overline{f}^n)$.

Theorem 2.3. [15, §5] *Let $(\overline{F}, \overline{f})$ be a one-solenoid. Then there is an integer n and pre-solenoid (F', f') such that $(\overline{F}, \overline{f^n})$ is conjugate to $(\overline{F'}, \overline{f'})$ and F' has a single vertex. That is, F' is a wedge of circles.*

The pre-solenoid (F', f') is usually called an elementary presentation for the solenoid $(\overline{F}, \overline{f^n})$. We will usually denote the single vertex by p.

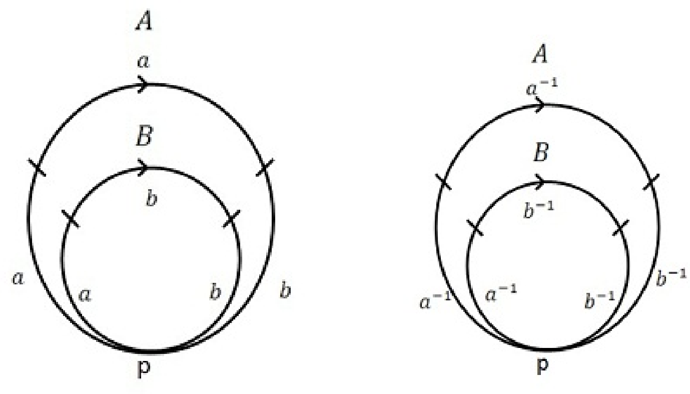
We begin our analysis of a pre-solenoid, (F, f), having a single vertex by observing that $f^{-1}\{p\}$ is a finite subset of F and removing these points then divides the edges of F into a finite collection of edges.

Lemma 2.4. [17] *Suppose that (F, f) is a pre-solenoid with a single vertex p. Let $E = \{e_1, ...e_n\}$ be the edge set of F with a given orientation. For each edge $e_i \in E$, we can give $e_i - f^{-1}\{p\}$ the partition $\{e_{i,j}\}, 1 \leq j \leq j(i)$, satisfying the following*

1. *the initial point of $e_{i,1}$ is the initial point of e_i.*
2. *the terminal point of $e_{i,j}$ is the initial point of $e_{i,j+1}$ for $1 \leq j < j(i)$,*
3. *the terminal point of $e_{i,j(i)}$ is the terminal point of e_i, and*
4. *there is (with a small abuse of notation) $1 \leq f(i,j) \leq n$ such that $f\mid_{e_{i,j}}$ maps $e_{i,j}$ homeomorphically to $e_{f(i,j)}$. We also set $s(i,j)$ to be ± 1 according to whether $f\mid_{e_{i,j}}$ preserves or reverses orientation.*

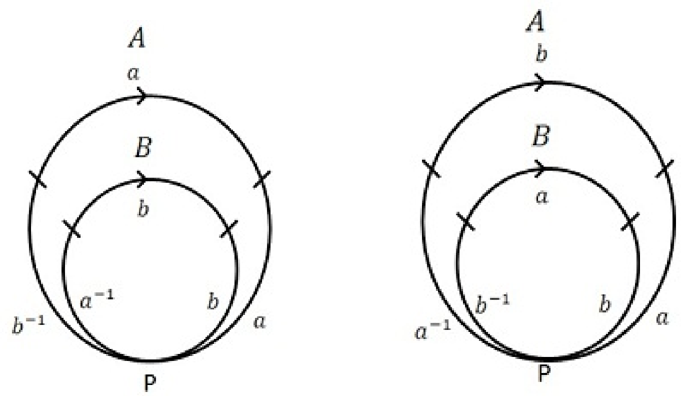
Example 2.5. Let F be a wedge of two clockwise circles a, b with a unique vertex p and f, g, k, h given by the wrapping rules: $f : a \to aab, b \to abb$, $g : a \to a^{-1}a^{-1}b^{-1}, b \to a^{-1}b^{-1}b^{-1}$, $k : a \to b^{-1}aa, b \to a^{-1}bb$ and $h : a \to a^{-1}ba, b \to b^{-1}ab$. Then (F, f) and (F, k) are positively oriented pre-solenoids, (F, g) is a negatively oriented pre-solenoid and (F, h) is not an oriented pre-solenoid. Figures 1 and 2 show these pre-solenoids.

As a consequence of the expanding condition, and by replacing (F, f) by (F, f^n), for some $n \geq 1$ if necessary, we may assume that $j(i) \geq 3$, for all i.



(A) Pre-solenoid (F,f) (B) Pre-solenoid (F,g)

Figure 1



(A) Pre-solenoid (F,k) (B) Pre-solenoid (F,h)

Figure 2

We next let U_p be a neighbourhood of p, sufficiently small so that $f(U_p)$ is contained in $\cup_i(e_{i,1}\cup e_{i,j(i)})$, which is a neighbourhood of p, also. From the fact that the flattening number is one, $f(U_p)$ is contained in the union of exactly two of these intervals (and no fewer). First suppose, these two are contained in e_a and e_b, with $a \neq b$, then by simply reversing the orientations on these intervals as needed, we may assume that the two intervals are $e_{a,1}$ and $e_{b,j(b)}$. The other case to consider is when the two are both in the same e_a, but then they must be $e_{a,1}$ and $e_{a,j(a)}$. In any case, we have a and b such that $f(U_p) \subset e_{a,1} \cup e_{b,j(b)}$, allowing the possibility that $a = b$.

It follows for every i that the two sets $f(e_{i,1} \cap U_p)$ and $f(e_{i,j(i)} \cap U_p)$ are contained in one of $e_{a,1}$ or $e_{b,j(b)}$. It also follows from the non-flattening condition that $f(e_{a,1} \cap U_p)$ and $f(e_{b,j(b)} \cap U_p)$ cannot be contained in the same one. Replacing f by f^2 if necessary, we can assume that $f(e_{a,1} \cap U_p)) \subset e_{a,1}$ and $f(e_{b,j(b)} \cap U_p) \subset e_{b,j(b)}$.

Notice that if $f(e_{i,j} \cap U_p) \subset e_{a,1}$, then we have $f(e_{i,j}) = e_a$ and an analogous result holds for $e_{b,j(b)}$. In particular, we have $f(e_{a,1}) = e_a$ and $f(e_{b,j(b)}) = e_b$.

For each $i, 1 \leq j < j(i)$, we let $x_{i,j}$ denote the unique terminal point of $e_{i,j}$ which is also the initial point of $e_{i,j+1}$. Notice that the set of all such $x_{i,j}$ is exactly $f^{-1}\{p\} - \{p\}$. Let $U_{i,j}$ be a neighbourhood of this point such that $f(U_{i,j}) \subset U_p$. Notice then that $f^2(e_{i,j} \cap U_{i,j})$ and $f^2(e_{i,j} \cap U_{i,j+1})$ are each contained in $e_{a,1}$ or in $e_{b,j(b)}$. Moreover, due to the flattening condition, they cannot both be contained in the same one.

Consider the following sets:

$$
\begin{aligned}
A &= \{e_i \in E \mid f(e_{i,1} \cap U_p), f(e_{i,j(i)} \cap U_p) \subset e_{a,1}\} \\
B &= \{e_i \in E \mid f(e_{i,1} \cap U_p), f(e_{i,j(i)} \cap U_p) \subset e_{b,j(b)}\}
\end{aligned}
$$

Theorem 2.6. *Let (F, f) be a pre-solenoid with a single vertex and $e_{a,1}, e_{b,j(b)}$ as above. The sets A and B are disjoint. Moreover, (F, f) is orientable if and only if the set $A \cup B$ is empty.*

Proof. The first statement is clear since $j(b) \geq 3$ means that $e_{a,1} \neq e_{b,j(b)}$. First suppose that $A \cup B$ is empty. The edges A and B have already been given orientations when we define $e_{a,1}$ and $e_{b,j(b)}$ (and these are consistent when $a = b$). Consider any e_i in E and we assume it has been given some orientation (so that $e_{i,1}$ and $e_{i,j(i)}$ are defined). By hypothesis, $A \cup B$ is empty and this means that e_i is not in $A \cup B$. If we consider the two sets $f(e_{i,1} \cap U_p), f(e_{i,j(i)} \cap U_p)$, one is contained in $e_{a,1}$ and the other in $e_{b,j(b)}$. By reversing the orientation of e_i if necessary, we may assume that $f(e_{i,1} \cap U_p) \subset e_{a,1}$, while $f(e_{i,j(i)} \cap U_p) \subset e_{b,j(b)}$.

We claim that the pre-solenoid is now positively oriented. We will show that each $e_{i,j}, 1 \leq j < j(i)$ is mapped in an orientation preserving way to $f(e_{i,j})$. We proceed by induction on j. The case of $j = 1$ is true simply by our choice of orientation on e_i; $e_{i,1} \cap U_p$ is mapped into $e_{a,1}$, which is the initial segment of $a = f(e_{i,1})$. Assume the statement is true for some given i, j and let $e_{i'} = f(e_{i,j})$. Let V be a neighbourhood of the boundary point between $e_{i,j}$ and $e_{i,j+1}$ sufficiently small so that $f(V) \subset U_p$. Then $e_{i,j} \cap V$ is being mapped by f into $e_{i',j(i')} \cap U_p$, by induction hypothesis. By the choice of orientation on $e_{i'}$, we know that $f(e_{i',1)} \cap U_p)$ is contained in $e_{a,1}$ and as $e_{i'}$ is in $E - A - B$, $f(e_{i',j(i'))} \cap U_p)$

is contained in $e_{b,j(b)}$. By the non-folding hypothesis, $f^2(e_{i,j+1} \cap U_{i,j})$ cannot be contained in $e_{b,j(b)}$. It follows then that it is contained in $e_{a,1}$. This then implies, letting $f(e_{i,j+1}) = e_{i''}$, that $f(e_{i,j+1} \cap V)$ cannot be contained in $e_{i'',j(i'')}$. The conclusion of the induction statement follows.

Conversely suppose that the 1-solenoid (F, f) is oriented. It follows immediately that for all i, $f(e_{i,1} \cap U_p) \subset e_{a,1}$ while $f(e_{i,j(i)} \cap U_p) \subset e_{b,j(b)}$. But this means that e_i is not in $A \cup B$. $\square$

3. The factor map

As in Yi [17], we consider two maps

$$\begin{aligned} \rho : (\Sigma_F, \sigma) &\rightarrow (F, f), \\ \overline{\rho} : (\Sigma_F, \sigma) &\rightarrow (\overline{F}, \overline{f}) \end{aligned}$$

as follows. For each point $I = (I_m)_{m \in \mathbb{Z}} \in \Sigma_F$, let

$$\{\rho(I)\} = \bigcap_{M=0}^{\infty} \overline{\cap_{m=0}^{M} f^{-m}(I_m)} \tag{3.1}$$

$$\overline{\rho}(I) = (\rho(I), \rho \circ \sigma^{-1}(I), \rho \circ \sigma^{-2}(I), \ldots) \tag{3.2}$$

To see that the set in the definition of $\rho(I)$ is a singleton, observe that the intersection $\cap_{m=0}^{M} f^{-m}(I_m)$ is an open interval in I_0 and its closure is a closed interval. Moreover, the lengths of these intervals decrease geometrically with M. It follows that the intersection is a single point. (The definition looks a little unusual. This is essentially due to the fact that we are trying to use the sets $\pi^{-1}(e_i)$ as a Markov partition, but they are slightly too large; the ends wrap around and meet at the vertex.) It is easy to verify that $\overline{\rho}$ is a factor map, that $\rho \circ \sigma = f \circ \rho$, and obviously $\rho = \pi \circ \overline{\rho}$.

We need to describe specific properties of the map $\overline{\rho} : \Sigma_F \rightarrow \overline{F}$. We begin with the following rather technical lemma. The proof is essentially found in Yi [17], but we provide a proof here for completeness and because we will require some more precise information about ρ. The point is that our homology computations to be done later will require precise knowledge about the points $I \neq J$ in Σ_F with $\overline{\rho}(I) = \overline{\rho}(J)$. We will let $\mathcal{A}_0 = \{e_{i,1}, e_{i,j(i)} \mid 1 \leq i \leq n\}$.

Lemma 3.1. *Let (F, f) be a pre-solenoid with unique vertex p.*

(1) *Suppose that $I = (I_m)_{m \in \mathbb{Z}}$ and $J = (J_m)_{m \in \mathbb{Z}}$ in Σ_F are such that*
  (a) $\overline{\rho}(I) = \overline{\rho}(J)$,
  (b) $I_0 \neq J_0$, *and*
  (c) $\rho(I) = \rho(J) \neq p$.

Then, up to interchanging I and J, we have
(a) $\rho(I) = \rho(J) \in f^{-1}\{p\} - \{p\}$,
(b) $I_m = J_m$, *for all* $m < 0$,
(c) $\{I_0, J_0\} = \{e_{i,j}, e_{i,j+1}\}$, *for some* $i, 1 \leq j < j(i)$,
(d) I_1 *and* J_1 *are both in* $\mathcal{A}_0$ *with* $f(I_1 \cap U_p) \subset e_{a,1}$ *and* $f(J_1 \cap U_p) \subset e_{b,j(b)}$,
(e) $I_m = e_{a,1}, J_m = e_{b,j(b)}$, *for all* $m \geq 2$.

(2) *If* $e \neq e_{a,1}, e_{b,j(b)}$ *is any element of* $\mathcal{A}_0$ *with* $f(e \cap U_p) \subset e_{a,1}$, *then there exists* $I \neq J$ *satisfying the conclusion of the first part and* $I_1 = e$. *If* $e \neq e_{a,1}, e_{b,j(b)}$ *is any element of* $\mathcal{A}_0$ *with* $f(e \cap U_p) \subset e_{b,j(b)}$, *then there exists* $I \neq J$ *satisfying the conclusion of the first part and* $J_1 = e$.

Proof. We know that $\rho(I) = \rho(J)$ is in the closure of the sets I_0 and J_0. On the other hand, I_0 and J_0 are disjoint since they are unequal. It follows that $\rho(I)$ is a boundary point of each and is therefore in $f^{-1}\{p\}$. Since we assume that $\rho(I) \neq p$, it is in the closure of exactly two elements of E. More specifically, there exist a unique $i, 1 \leq j < j(i)$ with $\{I_0, J_0\} = \{e_{i,j}, e_{i,j+1}\}$.

We now want to show that $I_m = J_m$, for all $m < 0$. We know that $f^{-m}(\rho(\sigma^m(I))) = \rho(I) \neq p$ and so $\rho(\sigma^m(I))$ is in the interior of a unique element of E. On the other hand, it is also in the closure of

$$\cap_{i=0}^{-m} f^{-i}(\sigma^m(I)_i) = \cap_{i=0}^{-m} f^{-i}(I_{i+m}) \subset I_m.$$

This means that I_m is the unique element of E containing $\rho(\sigma_m(I))$. The same argument applies to J and using $\rho(\sigma^m(I)) = \rho(\sigma^m(J))$, we conclude that $I_m = J_m$.

Next, we claim that for any integer $M \geq 2$, we have $\{I_M, J_M\} = \{e_{a,1}, e_{b,j(b)}\}$ and $I_{M+1} = I_M, J_{M+1} = J_M$. Let V be a neighbourhood of $\rho(I)$ such that $f^m(V) \subset U_p$ for all $1 \leq m \leq M+2$ and so that f^{M+2} is injective on V. We know that for $2 \leq m \leq M+2$, $f^m(I_0 \cap V)$ and $f^m(J_0 \cap V)$ are each contained in a set of the form $e_{a,1} \cap U_p$ or $e_{b,j(b)} \cap U_p$. They cannot be contained in the same one, because of the non-folding axiom. Therefore, up to switching I and J, we have $f^M(I_0 \cap V) \subset e_{a,1} \cap U_p$ and $f^M(J_0 \cap V) \subset e_{b,j(b)} \cap U_p$. These imply $f^{M+1}(I_0 \cap V) \subset e_{a,1}$ and $f^{M+1}(J_0 \cap V) \subset e_{b,j(b)}$. Since $\rho(I)$ is defined as an intersection, we may choose an integer $L > M+1$ such that $\cap_{n=0}^{L} f^{-n}(I_n) \subset V$. This implies that $f^M(I_0 \cap V) \subset I_M$ and $f^{M+1}(I_0 \cap V) \subset I_{M+1}$. But since the intervals of E are pairwise disjoint and the set $f^M(I_0 \cap V)$ is contained in both I_M and $e_{a,1}$, we conclude these must be equal. Similarly, we find $I_{M+1} = e_{a,1}$, $J_M = J_{M+1} = e_{b,j(b)}$.

For the proof of the second part, we consider the first statement only. We know that f is surjective. Moreover, since the elements of $\mathcal{A}_0$ all map to $e_{a,1}$ or $e_{b,j(b)}$, we may find some i, j where $f(U_{i,j})$ meets $e \cap U_p$. One of $f(e_{i,j} \cap U_{i,j})$ and $f(e_{i,j+1} \cap U_{i,j})$ meets e. Let us assume it is the former. Then $f^2(e_{i,j} \cap U_{i,j})$ is contained in $e_{a,1}$ and, by the non-folding axiom, $f^2(e_{i,j+1} \cap U_{i,j})$ must be contained in $e_{b,j(b)}$. Let $I_0 = e_{i,j}$, $J_0 = e_{i,j+1}$, $I_1 = e$ and J_1 be the unique element of $\mathcal{A}_0$ containing $f(e_{i,j+1} \cap U_{i,j})$ and $I_m = e_{a,1}, J_m = e_{b,j(b)}$, for all $m \geq 2$. The elements $I_m = J_m, n < 0$, may be chosen arbitrarily so that $f(I_{m-1}) \supset I_m$, for all m. □

Theorem 3.2. *Suppose (F, f) be a pre-solenoid with unique vertex p.*

(1) *If $I = (I_m)_{m\in\mathbb{Z}}$ and $J = (J_m)_{m\in\mathbb{Z}}$ in Σ_F are such that $\overline{\rho}(I) = \overline{\rho}(J)$, then, up to interchanging I and J, one of the following holds.*
   (a) *$I = J$.*
   (b) *$I_m = e_{a,1}, J_m = e_{b,j(b)}$, for all integers m.*
   (c) *There is a unique M such that $I_M \neq J_M$, $I_m = J_m$, for all $m < M$ and $I_m = e_{a,1}, J_m = e_{b,j(b)}$, for all $m \geq M+2$. Moreover, M is characterized by the condition $\overline{\rho} \circ \sigma^M(I) \in f^{-1}\{p\} - \{p\}$.*
(2) *The map $\overline{\rho}$ is one-to-one on $\Sigma_F - \bigcup_{m=0}^{\infty} \overline{\rho}^{-1} \circ \pi^{-1} \circ f^{-m-1}\{p\}$.*
(3) *The map $\overline{\rho} : (\Sigma_F, \sigma) \to (\overline{F}, \overline{f})$ is at most two-to-one.*
(4) *The map $\overline{\rho} : (\Sigma_F, \sigma) \to (\overline{F}, \overline{f})$ is an s-bijective map.*

Proof. For the first part, Lemma 3.1 proves that conclusion (c) holds under the conditions $I \neq J$ and $\overline{\rho} \circ \sigma^M(I) \neq p$, for some M. It remains to consider the case $I \neq J$ and $\overline{\rho} \circ \sigma^M(I) = p$, for all M.

But then for any M, $I_M \cap f^{-1}(I_{M+1}) \cap f^{-2}(I_{M+2})$ must contain p in its closure. It follows that this set is contained in either $e_{i,1}$ or $e_{i,j(i)}$ for some i. In the first case, $f(U_p \cap I_M \cap f^{-1}(I_{M+1}) \cap f^{-2}(I_{M+2}))$ is a non-empty subset of $f(U_p \cap e_{i,1}) \subset e_{a,1}$ and also of I_{M+1}. This implies that $I_{M+1} = e_{a,1}$. In addition, $f^2(U_p \cap I_M \cap f^{-1}(I_{M+1}) \cap f^{-2}(I_{M+2}))$ is a non-empty subset of $f^2(U_p \cap e_{i,1}) \subset e_{a,1}$ and also of I_{M+2}. This implies that $I_{M+2} = e_{a,1}$. In the latter case, the same argument proves that $I_{M+1} = I_{M+2} = e_{b,j(b)}$. What we have shown is that, for any M, I_{M+1} is either $e_{a,1}$ or $e_{b,j(b)}$ and $I_{M+1} = I_{M+2}$. The conclusion follows since this holds for all M.

The second and third statements are immediate. The fourth follows from the first: no non-trivial pair I and J with $\overline{\rho}(I) = \overline{\rho}(J)$ are stably equivalent. Thus, $\overline{\rho}$ is s-resolving. The shift Σ_F is irreducible from the hypotheses on (F, f), so $\overline{\rho}$ is s-bijective by Theorem 2.5.8 of [8]. □

4. Dimension groups for the shifts of finite type

Next, we must discuss some basic facts about the dimension groups $D^s(G_F)$ and $D^u(G_F)$,

For any finite directed graph, G, and $K \geq 2$, a path of length K in G is a sequence $(e_1, e_2, ..., e_K)$ where e_k is in G^1, for each $1 \leq k \leq K$ and $t(e_k) = i(e_{k+1})$, for $1 \leq k < K$. We let G^K denote the set of all paths of length K in G and, simultaneously, the graph whose vertex set is G^{K-1} and whose edge set is G^K with initial and terminal maps

$$i(e_1, e_2, ..., e_K) = (e_1, e_2, ..., e_{K-1}), \quad t(e_1, e_2, ..., e_K) = (e_2, e_3, ..., e_K).$$

We let $\mathbb{Z}G^K$ denote the free abelian group of the set G^K, for any $K \geq 0$. If A is some subset of G^K, we let $Sum(A) = \Sigma_{a \in A} a$. The initial and terminal maps $i, t : G^K \to G^{K-1}$ induce group homomorphisms, also denoted i, t from $\mathbb{Z}G^K$ to $\mathbb{Z}G^{K-1}$. In addition, we have a group homomorphism $t^* : \mathbb{Z}G^{K-1} \to \mathbb{Z}G^K$ defined by $t^*(e) = Sum(t^{-1}\{e\})$. We define the map $\gamma^s_G = i \circ t^*$ and $D^s(G^K)$ is defined to be the inductive limit of the sequence

$$\mathbb{Z}G^{K-1} \stackrel{\gamma^s_G}{\to} \mathbb{Z}G^{K-1} \stackrel{\gamma^s_G}{\to} \cdots$$

As explained in [8], the results for different values of K are all naturally isomorphic. In fact, the map i induces an isomorphism from $D^s(G^K)$ to $D^s(G^{K-1})$. These groups are all isomorphic to $D^s(\Sigma_G, \sigma)$. There are analogous definitions of i^*, $\gamma^u_G = t \circ i^*$ and $D^u(G^K)$.

Occasionally, we will write γ^s and γ^u instead of $\gamma^s_{G_F}$ and $\gamma^u_{G_F}$ if no confusion can arise.

The four examples of pre-solenoids in Section 2 each have the same graph G with two vertices, two loops at each vertex and one edge in each direction between the vertices. For this graph, we have

$$D^s(G^0_F) = D^u(G^0_F) = \{(i, i+j) : \ i \in \mathbb{Z}[1/3], j \in \mathbb{Z}\}$$

by

$$\begin{cases} D^s(G^0_F) \to \{(i, i+j) : i \in \mathbb{Z}[1/3], j \in \mathbb{Z}\}, \\ [\upsilon_1 + \upsilon_2, k] \to (3^{-(k-1)}, 3^{-(k-1)}), \\ [\upsilon_1 - \upsilon_2, l] \to (-1, 1). \end{cases}$$

The sets A and B we defined in Section 2 provide us with specific elements of the dimension groups $D^s(G_F)$ and $D^u(G_F)$ associated with (Σ_F, σ).

Lemma 4.1. *Let (F, f) be a pre-solenoid with a single vertex. Define w in $\mathbb{Z}G^0_F$ by $w = Sum(A) - Sum(B)$. Also, let $w^* : \mathbb{Z}G^0_F \to \mathbb{Z}$ be the group homomorphism which sends each element of A to 1, each* *element of B to -1 and the elements of $E - A - B$ to zero. We have the following:*

(1) *$w = 0$ if and only if (F, f) is orientable,*
(2) *$\gamma_G^s(w) = w$,*
(3) *$\mathbb{Z}G_F^0 / < w >$ is torsion free, where $< w >$ denotes the cyclic subgroup generated by w,*
(4) *$w^* = 0$ if and only if (F, f) is orientable,*
(5) *$w^* \circ \gamma_G^u = w^*$.*
(6) *If w^* is non-zero, then it is surjective.*

Proof. The first and fourth statements follow immediately from Theorem 2.6.

We compute $\gamma_{G_F}^s(w)$. Write $\gamma_{G_F}^s = \sum_i k_i e_i$, for some choice of integers k_i. For a fixed k_i, it follows from the definitions of γ_G^s and G that k_i is the number of $1 \leq j \leq j(i)$ with $f(e_{i,j}) \in A$ minus the number of such j with $f(e_{i,j}) = B$.

Construct α in $\{a, b\}^{2j(i)}$ by first considering the sequence of open sets

$$e_{i,1} \cap U_p, e_{i,1} \cap U_{i,1}, e_{i,2} \cap U_{i,1} \ldots, e_{i,j(i)} \cap U_{i,j(i)-1}, e_{i,j(i)} \cap U_p.$$

To each we apply f^2 and obtain a sequence of sets, each being contained in either $e_{a,1}$ or $e_{b,j(b)}$. The sequence of values of a or b are obtained accordingly. It follows from the non-folding condition that for every j, $\alpha_{2j} \neq \alpha_{2j+1}$. Also, $f(e_{i,j})$ is in A if and only if $\alpha_{2j-1} = \alpha_{2j} = a$ and is in B if and only if $\alpha_{2j-1} = \alpha_{2j} = b$. So k_i is the number of consecutive a's minus the number of consecutive b's.

We claim that if $(\alpha_{2j}, \alpha_{2j+1}) = (b, a)$ for some j, we may change it to (a, b) without altering the value k_i. There are four cases to consider, depending on the values of α_{2j-1} and α_{2j+2}. First, suppose they are both a. In this case, $(\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}) = (a, b, a, a)$ and contains exactly one pair of adjacent a's and no adjacent b's. Switching the places of the two central entries does not change this fact and so does not alter k_i. Next, we suppose that $(\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}) = (a, b, a, b)$. Here, we have no adjacent a's or b's. Switching the two central entries results in a pair of adjacent a's and a pair of adjacent b's, but the difference is still zero. There are two other cases which are done similarly and we leave to the reader.

Now we may assume that $(\alpha_{2j}, \alpha_{2j+1}) = (a, b)$ for all j. Now we consider the possible values of α_1 and $\alpha_{2j(i)}$. If both are a, then there is exactly one pair of adjacent a's (α_1, α_2) and no adjacent b's. This means that $k_i = 1$. On the other hand, it also follows that e_i is in A in this case. If $\alpha_1 = a$ and $\alpha_{2j(i)} = b$, then there is one pair of adjacent

a's and one pair of adjacent b's. If $\alpha_1 = b$ and $\alpha_{2j(i)} = a$, then there are adjacent a's or adjacent b's. In both cases, we have $k_i = 0$. But also in these cases, we see that e_i is in $E - A - B$. Finally, in the case that $\alpha_1 = \alpha_{2j(i)} = b$, we see that $k_i = -1$ and that e_i is in B. We have now proved that $\gamma^s_{G_F}(w) = w$.

The third statement is clear. For the fifth, we can regard w as a group homomorphism from $\mathbb{Z}$ to $\mathbb{Z}G^0_F$. Then the map w^* is simply the dual of this map and $\gamma^u_{G_F}$ is simply the dual of $\gamma^s_{G_F}$. By dual, we mean to replace a group H by $Hom(H, \mathbb{Z})$. Moreover, we identify $\mathbb{Z}A$ and $Hom(\mathbb{Z}A, \mathbb{Z})$, for any set A, by the canonical isomorphism. In this way, the fifth statement simply follows from the first.

For the last statement, if w^* is non-zero, then $A \cup B$ is non-empty. The conclusion follows since the value of w^* on $A \cup B$ is contained in $\{1, -1\}$. □

In the case where the elements w and w^* are non-zero, they also provide elements having analogous properties at the level of the inductive limit groups. The following is an immediate consequence of the last lemma and we omit the proof.

Lemma 4.2. *Suppose that (F, f) is a non-orientable pre-solenoid with a single vertex.*

(1) *For all n, $[w, n] = [w, 1] \neq 0$ in $D^s(G_F)$ and the quotient group $D^s(G_F)/ < [w, 1] >$ is torsion free.*
(2) *The map $w^* : D^u(G_F) \to \mathbb{Z}$ defined by $w^*[a, n] = w^*(a)$, for $a \in \mathbb{Z}G^0$ is well-defined and surjective.*

5. Homology

In general, the computation of the homology groups $H^s_N(X, \varphi), H^s_N(X, \varphi), N \in \mathbb{Z}$, for a Smale space (X, φ) is a rather complicated business involving double complexes. However, we may appeal to two special features in our case. The first is that our solenoid is the image of a shift of finite type under an s-bijective factor map ρ. A general investigation of such Smale spaces can be found in Wieler [13]. This reduces the double complexes to usual complexes indexed by the integers. The second feature is that the map ρ is at most two-to-one. This means that there are only two non-zero entries in the complex. Specifically, Theorem 4.2.12 and Theorem 7.2.1 of [8] provide us with the following description which we state in some generality.

Theorem 5.1. [8] *Let (X, φ) be a Smale space, (Σ, σ) be a shift of finite type and $\overline{\rho} : (\Sigma, \sigma) \to (X, \varphi)$ be a factor map. Assume that $\overline{\rho}$ is s-bijective and that $\#\overline{\rho}^{-1}\{x\} \leq 2$, for all x in X.*

(1) *The homology $H^s_N(X,\varphi)$ is naturally isomorphic to the homology of the complex $(D^s_{\mathcal{Q}}(\Sigma_*(\overline{\rho})), d^s(\overline{\rho}))$. Similarly, the homology $H^u_N(X,\varphi)$ is naturally isomorphic to the homology of the complex $(D^u_{\mathcal{A}}(\Sigma_*(\overline{\rho})), d^{u*}(\overline{\rho}))$.*
(2) *The only non-zero terms in the complexes $(D^s_{\mathcal{Q}}(\Sigma_*(\overline{\rho})), d^s(\overline{\rho}))$ and $(D^u_{\mathcal{A}}(\Sigma_*(\overline{\rho})), d^{u*}(\overline{\rho}))$ occur in entries $N=0$ and $N=1$.*
(3) *For $N \neq 0,1$, we have*
$$H^s_N(X,\varphi) = H^u_N(X,\varphi) = 0.$$
(4) *We have*
$$\begin{array}{rcl} H^s_0(X,\varphi) & = & coker\left(d^s(\overline{\rho})_1 : D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho})) \to D^s_{\mathcal{Q}}(\Sigma_0(\rho))\right) \\ H^s_1(X,\varphi) & = & ker\left(d^s(\overline{\rho})_1 : D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho})) \to D^s_{\mathcal{Q}}(\Sigma_0(\overline{\rho}))\right) \\ H^u_0(X,\varphi) & = & ker\left(d^{u*}(\overline{\rho})_1 : D^s_{\mathcal{A}}(\Sigma_0(\overline{\rho})) \to D^s_{\mathcal{A}}(\Sigma_1(\overline{\rho}))\right) \\ H^u_1(X,\varphi) & = & coker\left(d^{u*}(\overline{\rho})_1 : D^s_{\mathcal{A}}(\Sigma_0(\overline{\rho})) \to D^s_{\mathcal{A}}(\Sigma_1(\overline{\rho}))\right) \end{array}$$

For definitions of complexes $D^s_{\mathcal{Q}}(\Sigma_i(\overline{\rho}))$, $D^s_{\mathcal{A}}(\Sigma_i(\overline{\rho}))$, $i=0,1$, see Section 4.1 of [8].

Our first task is to identify the two groups involved in the two complexes. It is a general fact that the groups in position 0 are the simplest. We have
$$\begin{array}{rcccl} D^s_{\mathcal{Q}}(\Sigma_0(\overline{\rho})) & = & D^s(\Sigma_F,\sigma) & = & D^s(G^L_F), \\ D^u_{\mathcal{A}}(\Sigma_0(\overline{\rho})) & = & D^u(\Sigma_F,\sigma) & = & D^u(G^L_F), \end{array}$$
for any $L \geq 1$.

Next, we turn to the groups in position 1 of our complexes. Notice that we write $e^L_{a,1}$ for $(e_{a,1}, e_{a,1}, \ldots, e_{a,1})$ and $e^L_{b,j(b)}$ for $(e_{b,j(b)}, e_{b,j(b)}, \ldots, e_{b,j(b)})$ in G^L_F, for $L \geq 1$. Implicit here is the fact that for each $L \geq 1$, (Σ_{G^L},σ) is conjugate to (Σ_F,σ). Henceforth, we will drop the subscript from G_F and write G.

For any integer $L \geq 1$, we define the graph G^L_1 as follows. The vertices are obtained by taking any (I,J) in $\Sigma_1(\overline{\rho})$ and restricting the sequences to $[1, L-1]$ and the edges are obtained by restricting to $[1,L]$. In [8], the notion of a factor map $\overline{\rho} : \Sigma_{G^L} \to \overline{F}$ is given. The point is that this condition is always satisfied for some $L \geq 1$ and when it is, then Σ_{G^L} is conjugate to $\Sigma_1(\overline{\rho})$.

Lemma 5.2. *Let L be any integer such that*
$$\overline{\rho} : (\Sigma_{G^L_F}, \sigma) \to (\overline{F}, \overline{f})$$
is regular (see [8]*).*

(1) *Then $D^s_{\mathcal{Q}}(\Sigma_1(\overline{\rho})) = D^s_Q(G^L_1)$ is an infinite cyclic group with generator*
$$Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), 1] = Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), m],$$

for all $m \geq 1$.

(2) *Then* $D^u_{\mathcal{A}}(\Sigma_1(\overline{\rho})) = D^u_{\mathcal{A}}(G^L_1)$ *is an infinite cyclic group with generator*

$$[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) - (e^{L-1}_{b,j(b)}, e^{L-1}_{a,1}), 1] = [(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) - (e^{L-1}_{b,j(b)}, e^{L-1}_{a,1}), m],$$

for all $m \geq 1$.

Proof. Theorem 3.2 gives us a complete description of all such pairs (I, J) with $\overline{\rho}(I) = \overline{\rho}(J)$. The hypothesis on L guarantees that the graph G^L_1 is a presentation of $\Sigma_1(\overline{\rho})$.

Let (I, J) be in G^{L-1}_1. If $I = J$, then this is the zero element of $D^s_{\mathcal{Q}}(G^{L-1}_1)$. If $I \neq J$, then (I, J) and (J, I) represent the same element of $\mathcal{Q}(\mathbb{Z}G^{L-1}_1, S_2)$, which is the quotient $\mathbb{Z}G^{L-1}_1$ when considering the action by the permutation group S_2.

First consider the case $\{I, J\} \neq \{e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}\}$. This implies that $\{I_1, J_1\} \neq \{e_{a,1}, e_{b,j(b)}\}$. It follows from Theorem 3.2 that if (I', J') is in G^{2L-1}_1 with $t^L(I', J') = (I, J)$, then the first $L - 1$ entries of I' and J' are equal. This means that

$$(\gamma^s)^L(I, J) = i^L \circ (t^L)^*(I, J) = 0 \in \mathcal{Q}(\mathbb{Z}G^{L-1}_1, S_2).$$

This in turn implies $Q[(I, J), m] = 0$ in $D^s_{\mathcal{Q}}(G^L_1)$, for any positive integer m.

Finally, we consider $(I, J) = (e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$. It follows that $i \circ t^*(I, J)$ is the sum of $(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$ and other terms, all of the type considered above.This means that $\gamma^s(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) - (e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$ is zero in the limit under γ^s. We conclude that $Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), 1] = Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), m] \neq 0$, for any positive integer m and is a generator for $D^s_{\mathcal{Q}}(G^{L-1}_1)$.

For the other case, the group $D_{\mathcal{A}}(\mathbb{Z}G^{L-1}_1, S_2)$ is generated by elements of the form $(I, J) - (J, I)$, where $I \neq J$. It follows immediately from Theorem 3.2 that for any such (I, J), if (I', J') is in G^{2L-1}_1 and satisfies $i^L(I', J')$, then the last $L - 1$ entries of (I', J') are $(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$ or $(e^{L-1}_{b,j(b)}, e^{L-1}_{a,1})$. This means that $(\gamma^u)^L(I, J) = t^L \circ (i^L)^*(I, J)$ is a multiple of $(e^L_{a,1}, e^{L-1}_{b,j(b)})$ or $(e^{L-1}_{b,j(b)}, e^{L-1}_{a,1})$. We also note that if $(I, J) = (e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$ then the (I', J') above is unique and we deduce $\gamma^u(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) = (e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$. The conclusion follows. □

Having identified the groups involved, our next task is to compute the boundary maps between them. The first step is the following technical lemma.

Lemma 5.3. *The number* $K = 1$ *satisfies Lemma 2.7.1 of* [8] *for* $\overline{\rho} : (\Sigma_F, \sigma) \to (\overline{F}, \overline{f})$.

Proof. Let I, J, I', J' be in Σ_F and satisfy $\overline{\rho}(I) = \overline{\rho}(J)$, $\overline{\rho}(I') = \overline{\rho}(J')$, $I_m = I'_m$, for all $m \geq 0$, and J, J' are stably equivalent. We need to prove that $J_m = J'_m$, for all $m \geq 1$.

First observe that $I_m = I'_m$, for all $m \geq 0$ implies that $\rho(I) = \rho(I')$ and hence we have $\rho(J) = \rho(I) = \rho(I') = \rho(J')$.

We proceed by considering the three cases given in the first statement of Theorem 3.2. If $I = J$, then since I' and J' are stably equivalent, we must have $I' = J'$ also. If both I, J and I', J' satisfy the second condition, then $I = I'$ and $J = J'$ and the conclusion holds.

We consider the case that I, J satisfies the third condition. Let M be the unique integer given in the condition for I, J. First suppose that $M \geq 0$. Then $f^M(\rho(I)) = \rho(\sigma^M(I)) \in f^{-1}\{p\} - \{p\}$. It follows then that $f^M(\rho(I')) \in f^{-1}\{p\} - \{p\}$. From this fact, it follows that the pair I', J' is also of the third type and the unique N' from that condition is equal to M. We apply Lemma 3.1 to the pairs $\sigma^M(I), \sigma^M(J)$ and $\sigma^M(I'), \sigma^M(J')$. Then $J_m = J'_m$ for all $m \geq M$ follows from the uniqueness statement in the conclusion of Lemma 3.1. On the other hand, for $0 \leq m < M$, we have $J_m = I_m = I'_m = J'_m$ and we are done.

Next, we consider the case $M < 0$. Here we have $J_m = e_{b,j(b)}$, for all $m \geq 1$. If I', J' is also of third type with $M' \geq 0$, the the same argument above, reversing the roles of I, J and I', J' would imply $M = M' \geq 0$ which is a contradiction. So either I', J' is of the third type with $M' < 0$ or it is of the second type. In either case, we have $J'_m = e_{b,j(b)}$ for all $m \geq 1$ and we are done. □

We are now ready to compute the boundary map for the first complex.

Lemma 5.4. *Let L be any integer such that*

$$\overline{\rho} : (\Sigma_{G^L}, \sigma) \to (\overline{F}, \overline{f})$$

is regular (see [8]). Then we have

$$d^{s,1}_{\mathcal{Q}}(\overline{\rho})_1(Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), 1]) = 2[w, 1].$$

(Recall that $w = 0$ if and only if (X, f) is orientable.)

Proof. In view of Lemma 5.3, we may use $K = 1$ to compute $\delta^{s,1}_0$ and $\delta^{s,1}_1$ from $D^s_{\mathcal{Q}}(G^L_1, S_2)$ to $D^s(G^{L+1})$. The former group is cyclic and is generated by $Q[(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}), 1]$. Moreover, the the map i^L induces an isomorphism between $D^s(G)$. It suffices to prove that

$$i^L \circ \delta_0(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) - i^L \circ \delta_1(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)}) = 2w.$$

and we are done.

We must find all I in G^L such that there exists J with (I, J) in G^L_1 and $t(I, J) = (e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$. An almost complete description is given by Lemma 3.1: this is $I = (I_1, \ldots, I_L)$ with I_1 in $\mathcal{A}_0$ with $f(I_1 \cap U_p) \subset e_{a,1}$. If e_i is in A, then both $(e_{i,1}, e_{a,1}, \ldots, e_{a,1})$ and $(e_{i,j(i)}, e_{a,1}, \ldots, e_{a,1})$ appear in this sum. After applying i^L, we obtain $2e_i$. If e_i is in $E - A - B$, then one of these two appears in the sum and the other does not. After applying i^L we get e_i. Finally, if e_i is in B, then neither appears in the list and e_i does not appear in the sum after applying i^L.

The reason this is not quite all, is that in Lemma 3.1, it is possible that $I_1 = e_{a,1}$ and $J_1 = e_{b,j(b)}$. Let us assume in such a case that $f(e_{i,j} \cap U_{i,j}) \subset e_{a,1}$ and $f(e_{i,j+1} \cap U_{i,j}) \subset e_{b,j(b)}$ for the other case is similar. Then we have $(e_{i,j}e^{L-1}_{a,1}, e_{i,j+1}e^{L-1}_{b,j(b)})$. Applying $i^L \circ \delta_0$ to such an element gives

$$i^L(e_{i,j}e^{L-1}_{a,1}) = i(e_{i,j}) = e_i.$$

Now let us compute $i^L \circ \delta_1(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$. First consider the extra elements we found at the end of the last paragraph. Here we obtain the element

$$i^L(e_{i,j+1}e^{L-1}_{b,j(b)}) = i(e_{i,j+1}) = e_i.$$

When we take the difference, these terms cancel. By an argument exactly analogous to the one above, what we are left with in our computation of $i^L \circ \delta_1(e^{L-1}_{a,1}, e^{L-1}_{b,j(b)})$ is that each element of A does not appear, each element of $E - A - B$ appears with coefficient 1 and each element of B appears with coefficient 2. Taking the difference we get

$$2Sum(A) - 2Sum(B) = 2w,$$

and this completes the proof. $\square$

We move on to the other boundary map.

Lemma 5.5. *Let L be any integer such that*

$$\overline{\rho} : (\Sigma_{G^L}, \sigma) \to (\overline{F}, \overline{f})$$

is regular (see [8]). Then we have

$$d^{u*,1}_{\mathcal{A}}(\overline{\rho})_0(a) = 2w^*(a)[(e^L_{a,1}, e^L_{b,j(b)}) - (e^L_{b,j(b)}, e^L_{a,1}), 1],$$

for all $a \in D^u(G)$. (Recall that $w^* = 0$ if and only if (X, f) is orientable.)

Proof. It suffices to consider $a = [e_i, m]$, for some e_i in E and $m \geq 1$. We first consider the case that e_i is in A. Recall that the canonical isomorphism from $D^s(G)$ to $D^s(G^{L+1})$ is induced by the map i^{L*} which sends e_i to the sum of all paths $I = (I_1, \ldots, I_L)$ with $i(I_1) = e_i$. To this element we wish to apply $d^{u*,1}_{\mathcal{A}}(\overline{\rho})_0(a)$. Let B denote the set of all pairs (I,J) in G^L_1 with I fixed as above so that

$$d^{u*,1}_{\mathcal{A}}(\overline{\rho})_0(a) = Sum\{t(I) \mid (I,J) \in B\} - Sum\{t(J) \mid (I,J) \in B\}.$$

Then we must sum over I, $i^L(I) = i(I_0) = e_i$. Divide B into two subsets: those with $i(J_0) = i(I_0) = e_i$ and B_0, the remaining ones. The first group contributes nothing, since for each pair (I,J), both I and J appear when summing over $i^L(I) = e_i$ and their contribution to the sum is exactly opposite and cancel. We are left to compute

$$Sum\{t(I) \mid (I,J) \in B_0\} - Sum\{t(J) \mid (I,J) \in B_0\}.$$

It follows from Lemma 5.3 and the fact that we assume e_i is in A that B_0 is empty unless $I = e_{i,1}e^{L-1}_{a,1}$ or $I = e_{i,j(i)}e^{L-1}_{a,1}$. In each case, taking $t(I)-t(J)$ and summing over B_0, we obtain $e^L_{a,1} - e^L_{b,j(b)}$. Now summing over the two values of I, we get $2(e^L_{a,1} - e^L_{b,j(b)})$. We have verified the conclusion for e_i in A. The case that e_i is in B is done in a similar way.

In the case that e_i is in $E - A - B$, there are again two I's consider, but they are $I = e_{i,1}e^{L-1}_{a,1}$ and $I = e_{i,j(i)}e^{L-1}_{b,j(b)}$ (or reversing the first entries). The terms $t(I) - t(J)$ are then opposite for these two I's and the total contribution is zero. That is, we have shown the conclusion holds for e_i in $E - A - B$. □

We finally remark that the proof of Theorem 1.3 follows easily from Theorem 5.1, Lemmas 5.2 and 5.4. The proof of Theorem 1.4 follows easily from Theorem 5.1, Lemmas 5.2 and 5.5.

Example 5.6. Suppose $(X,f),(X,g)$ *and* (X,k) be the pre-solenoids defined in Example 2.5, then $H^s_N(\overline{F},\overline{f}) = H^s_N(\overline{F},\overline{g}) = H^s_N(\overline{F},\overline{k})$ for each $N \geq 0$

$$H^s_N(\overline{F},\overline{f}) = \begin{cases} \{(i, i+j) : i \in \mathbb{Z}[1/3], j \in \mathbb{Z}\} & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

Also for the 1-solenoid $(\overline{F},\overline{h})$ defined in that example:

$$H^s_N(\overline{F},\overline{h}) = \begin{cases} \{(i, i+j) : i \in \mathbb{Z}[1/3], j \in \mathbb{Z}\}/2\mathbb{Z}(-1,1) & N = 0, \\ 0 & N \neq 0. \end{cases}$$

Example 5.7. Suppose $(X,f),(X,g)$ and (X,k) are the pre-solenoids defined in Example 2.5, then $H^u_N(\overline{F},\overline{f}) = H^u_N(\overline{F},\overline{g}) = H^u_N(\overline{F},\overline{k})$

$$H^u_N(\overline{F},\overline{f}) = \begin{cases} \{(i, i+j) : i \in \mathbb{Z}[1/3], j \in \mathbb{Z}\} & N = 0, \\ \mathbb{Z} & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

And also for the pre-solenoid (X, h) defined in that example:

$$H_N^u(\overline{F}, \overline{h}) = \begin{cases} \mathbb{Z}[1/3] & N = 0, \\ \mathbb{Z}_2 & N = 1, \\ 0 & N \neq 0, 1. \end{cases}$$

References

[1] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems: Recent Advances*, North-Holland, Amsterdam-London-New York-Tokyo, 1994.

[2] R. Bowen, Markov partitions for Axiom A diffeomorphisms, *Amer. J. Math.* 92 (1970), 725-747.

[3] R. Bowen, *On Axiom A diffeomorphisms*, AMS-CBMS Reg. Conf. 135, Providence, 1978.

[4] T. Fisher, Resolving extensions of finitely presented systems, *Acta. Appl. Math.*, to appear.

[5] D. Fried, Finitely presented dynamical systems, *Ergod. Th. & Dynam. Sys.* 7 (1987), 489- 507.

[6] W. Krieger, On dimension functions and topological Markov chains, *Invent. Math.* 56 (1980), 239-250.

[7] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.

[8] I. F. Putnam, A homology theory for Smale spaces, *Mem. A.M.S.*, to appear.

[9] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math. and its Appl. 5, Addison-Wesley, Reading, 1978.

[10] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* 73 (1967), 747-817.

[11] K. Thomsen, The homoclinic and heteroclinic C^*-algebra of a generalized one-dimensional solenoid, *Ergod. Th. & Dynam. Sys.*, 30 (2010), 263-308.

[12] K. Thomsen, *C^*-algebras of homoclinic and heteroclinic structure in expansive dynamics*, Mem. A.M.S. 206 (2010).

[13] S. Wieler, Smale spaces via inverse limits, *Ergod. Th. & Dynam. Sys.*, to appear.

[14] R.F. Williams, One-dimensional non-wandering sets, *Topology* 6 (1967), 37-487.

[15] R.F. Williams, Classification of 1-dimensional attractors, *Proc. Symp. Pymp. Pure Math.* 14 (1970), 341-361.

[16] R.F. Williams, Expanding attractors, *IHES Publ. Math.* 43 (1974), 169-203.

[17] I. Yi, Canonical symbolic dynamics for one-dimensional generalized solenoids, *Trans. A.M.S.* 353 (2001), 37413767.