A homology theory for Smale spaces: a summary

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Abstract

We consider Smale spaces, a particular class of hyperbolic topological dynamical systems, which include the basic sets for Smale's Axiom A systems. We present an algebraic invariant for such systems which is based on Krieger's dimension group for the special case of shifts of finite type. This theory provides a Lefschetz formula relating trace data with the number of periodic points of the system, answering a question posed by R. Bowen. The key ingredient is the existence of Markov partitions with special properties.

1 Introduction

Smale introduced the notion of an Axiom A diffeomorphism of a compact manifold [16]. For such a system, a basic set is a closed invariant subset of the non-wandering set which is irreducible in a certain sense. One of Smale's key observations was that such a set need not be a submanifold. Typically, it is some type of fractal object.

The Artin-Masur zeta function encodes the data of the periodic points for a dynamical system. Manning proved that for any basic set of an Axiom A system, the associated Artin-Masur zeta function is rational. This led Bowen to conjecture the existence of a homology theory for such systems which had a

[*]Supported in part by a grant from NSERC, Canada

Lefschetz-type theorem. Such a theory would immediately imply Manning's rationality result. Such a theory is given in [14]. The aim of this article is to give a short summary of this theory without proofs.

The theory takes, as its starting point, the notion of the dimension group of a shift of finite type introduced by Krieger [9] and the fundamental result of Bowen [2] that every basic set is a factor of a shift of finite type.

In an effort to give a purely topological (i.e. without reference to any smooth structure) description of the dynamics on a basic set, Ruelle introduced the notion of a Smale space [15]: a Smale space is a compact metric space, (X, d), and a homeomorphism, φ, of X, which possesses canonical coordinates of contracting and expanding directions. The precise definition involves the existence of a map $[,]$ giving canonical coordinates. Here, we review only the features necessary for the statements of our results.

There is a constant $\epsilon_X > 0$ and, for each x in X and $0 < \epsilon \leq \epsilon_X$, there are sets $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$, called the local stable and unstable sets, respectively, whose product is homeomorphic to a neighbourhood of x. As ϵ varies, these form a neighbourhood base at x. Moreover, there is a constant $0 < \lambda < 1$ such that

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon_X) \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon_X) \end{aligned}$$

The bracket $[x, y]$ is the unique point in the intersection of $X^s(x, \epsilon_X)$ and $X^u(y, \epsilon_X)$. We say that (X, φ) is non-wandering if every point of X is non-wandering for φ [8].

Stable and unstable equivalence relations are defined by

$$\begin{aligned} R^s &= \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\} \\ R^u &= \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}. \end{aligned}$$

We let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of x in X. However, there are natural, much nicer topologies which are given by using the families $X^s(x', \epsilon), x' \in X^s(x), 0 < \epsilon < \epsilon_X$ and $X^u(x', \epsilon), x' \in X^u(x), 0 < \epsilon < \epsilon_X$, respectively, as bases.

The main examples of such systems are shifts of finite type (of which we will say more in a moment), hyperbolic toral automorphisms, solenoids, substitution tiling spaces (under some hypotheses) and, most importantly, the basic sets for Smale's Axiom A systems [16, 3].

Let (Y, ψ) and (X, φ) be Smale spaces. A *factor map* from (Y, ψ) to (X, φ) is a function $\pi : Y \to X$ which is continuous, surjective and satisfies $\pi \circ \psi = \varphi \circ \pi$. It is clear that, for any y in Y, $\pi(Y^s(y)) \subset X^s(\pi(y))$ and $\pi(Y^u(y)) \subset X^u(\pi(y))$. David Fried [7] defined π to be s-resolving (or u-resolving) if, for every y in Y, the restriction of π to $Y^s(y)$ (or to $Y^u(y)$, respectively) is injective. We say that π is *s-bijective* (or *u-bijective*) if, for every y in Y, π is a bijection from $Y^s(y)$ to $X^s(\pi(y))$ (or from $Y^u(y)$ to $X^u(\pi(y))$, respectively). This actually implies that π is a local homeomorphism from the local stable sets (or unstable sets, respectively) in Y to those in X. In the case that (X, φ) is non-wandering, s-resolving (u-resolving) and s-bijective (u-bijective, respectively) are equivalent.

2 Shifts of Finite type

Shifts of finite type are described in detail in [10]. We consider a finite directed graph G. This consists of a finite vertex set G^0, a finite edge set G^1 and maps i, t (for initial and terminal) from G^1 to G^0. The associated shift space

$$\Sigma_G = \{e = (e^k)_{k \in \mathbb{Z}} \mid e^k \in G^1, t(e^k) = i(e^{k+1}), k \in \mathbb{Z}\}$$

consists of all bi-infinite paths in G, specified as an edge list. The map σ is the left shift on Σ_G defined by $\sigma(e)^k = e^{k+1}$, for all e in Σ_G and k in $\mathbb{Z}$. By a shift of finite type, we mean any system topologically conjugate to (Σ_G, σ), for some graph G. This is not the usual definition, but is equivalent to it (see Theorem 2.3.2 of [10]). For e in Σ_G, we define $e^{[K,L]} = (e^K, \ldots . e^L)$, for all integers $K \leq L$ and also $e^{[K+1,K]} = t(e^K)$, for convenience. We then use the metric

$$d(e, f) = \inf\{1, 2^{-N-1} \mid e^{[1-N,N]} = f^{[1-N,N]}, N \geq 0\},$$

for e, f in Σ_G. We observe that such systems are Smale spaces by noting that the bracket operation is defined, with $\epsilon_X = 1/2$, as follows. For e, f in Σ_G, $[e, f]$ is defined if $t(e^0) = t(f^0)$ and then it is the sequence, $(\ldots, f^{-1}, f^0, e^1, e^2, \ldots)$. For any $l \geq 1$ and e_0 in Σ_G, the local stable and unstable sets are given by

$$\begin{aligned} \Sigma_G^s(e_0, 2^{-l}) &= \{e \in \Sigma \mid e^k = e_0^k, k \geq -l\}, \\ \Sigma_G^u(e_0, 2^{-l}) &= \{e \in \Sigma \mid e^k = e_0^k, k \leq l\}. \end{aligned}$$

It is a simple matter to see the constant $\lambda = 1/2$ will satisfy the axioms.

Theorem 2.1. *Shifts of finite type are exactly the zero-dimensional (i.e. totally disconnected) Smale spaces.*

The fundamental rôle of shifts of finite type is demonstrated by the following universal property, due to Bowen [2]. This builds on earlier results by many others, including Adler and Weiss and Sinai.

Theorem 2.2 (Bowen)**.** *Let (X, φ) be a non-wandering Smale space. Then there exists a non-wandering shift of finite type (Σ, σ) and a finite-to-one factor map*

$$\pi : (\Sigma, \sigma) \to (X, \varphi).$$

3 Krieger's dimension group invariant

Krieger [9] defined the past and future dimension groups of a shift of finite, (Σ, σ), as follows. Consider $\mathcal{D}^s(\Sigma, \sigma)$ to be the collection of compact open subsets of $\Sigma^s(e, \epsilon)$, as e varies over Σ and $0 < \epsilon \leq \epsilon_\Sigma$. We let $\sim$ denote the smallest equivalence relation on $\mathcal{D}^s(\Sigma, \sigma)$ such that

1. If E, F are in $\mathcal{D}^s(\Sigma, \sigma)$ with $[E, F] = F$ and $[F, E] = E$ (meaning both are defined), then $E \sim F$,
2. If $E, F, \varphi(E)$ and $\varphi(F)$ are all in $\mathcal{D}^s(\Sigma, \sigma)$, then $E \sim F$ if and only if $\varphi(E) \sim \varphi(F)$.

As an example, it is a simple matter to check that for any e_0, f_0 and integer $l \geq 1$, the sets $\Sigma_G^s(e_0, 2^{-l})$ and $\Sigma_G^s(f_0, 2^{-l})$ (as described just prior to Theorem 2.2) are equivalent if $i(e_0^{-l}) = i(f_0^{-l})$.

We generate a free abelian group on the equivalence classes of elements, E, of $\mathcal{D}^s(\Sigma, \sigma)$ (denoted $[E]$) subject to the additional relation that $[E \cup F] = [E] + [F]$, if $E \cup F$ is in $\mathcal{D}^s(\Sigma, \sigma)$ and E and F are disjoint. The result is denoted by $D^s(\Sigma, \sigma)$. There is an analogous definition of $D^u(\Sigma, \sigma)$.

Theorem 3.1 (Krieger [9])**.** *Let G be a finite directed graph and (Σ_G, σ) be the associated shift of finite type. Then $D^s(\Sigma_G, \sigma)$ (or $D^u(\Sigma_G, \sigma)$, respectively) is isomorphic to the inductive limit of the sequence*

$$\mathbb{Z}G^0 \xrightarrow{\gamma^s} \mathbb{Z}G^0 \xrightarrow{\gamma^s} \cdots$$

where $\mathbb{Z}G^0$ denotes the free abelian group on the vertex set G^0 and the map $\gamma^s(v) = \sum_{t(e)=v} i(e)$, for any v in G^0 (or respectively, replacing γ^s by γ^u, whose definition is the same, interchanging the rôles of i and t).

The first crucial result for the development of our theory is the following functorial property of D^s and D^u, which can be found in [4].

Theorem 3.2. *Let (Σ, σ) and (Σ', σ) be shifts of finite type and let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map.*

1. *If π is s-bijective, then there are natural homomorphisms*

$$\begin{array}{rcccc} \pi^s & : & D^s(\Sigma,\sigma) & \to & D^s(\Sigma',\sigma), \\ \pi^{u*} & : & D^u(\Sigma',\sigma) & \to & D^u(\Sigma,\sigma). \end{array}$$

2. *If π is u-bijective, then there are natural homomorphisms*

$$\begin{array}{rcccc} \pi^u & : & D^u(\Sigma,\sigma) & \to & D^u(\Sigma',\sigma), \\ \pi^{s*} & : & D^s(\Sigma',\sigma) & \to & D^s(\Sigma,\sigma). \end{array}$$

The idea is simple enough: in the covariant case, the induced map sends the class of a set E in $\mathcal{D}^s(\Sigma, \sigma)$ to the class of $\pi(E)$, while in the contravariant case, the map sends the class of E' in $\mathcal{D}^s(\Sigma', \sigma)$ to the class of $\pi^{-1}(E')$. The latter is not correct since $\pi^{-1}(E')$ may not even be contained in a single stable equivalence class, but it suffices that it may be written as a finite union of elements of $\mathcal{D}^s(\Sigma, \sigma)$. These ideas can be made precise under the stated hypotheses.

4 s/u-bijective pairs

The key ingredient in our construction is the following notion.

Definition 4.1. *Let (X, φ) be a Smale space. An s/u-*bijective pair*, π, for (X, φ) consists of Smale spaces (Y, ϕ) and (Z, ζ) and factor maps*

$$\pi_s : (Y, \psi) \to (X, \varphi), \quad \pi_u : (Z, \zeta) \to (X, \varphi)$$

such that

1. *$Y^u(y, \epsilon)$ is totally disconnected, for all y in Y and $0 < \epsilon \leq \epsilon_Y$,*
2. *π_s is s-bijective,*
3. *$Z^s(z, \epsilon)$ is totally disconnected, for all z in Z and $0 < \epsilon \leq \epsilon_Z$,*

4. π_u is u-bijective.

To summarize the idea in an informal way, the space Y is an extension of X, where the local unstable sets are totally disconnected, while the local stable sets are homeomorphic to those in X. The existence of such s/u-bijective pairs, at least for non-wandering (X, φ), can be deduced from the results of [13] or [6]. It can be viewed as a coordinate-wise version of Bowen's theorem.

If $(Y, \psi, \pi_u, Z, \zeta, \pi_s)$ is an s/u-bijective pair for (X, φ), then we form the fibred product $\Sigma(\pi) = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$ with the map $\sigma = \psi \times \zeta|_{\Sigma(\pi)}$ and the two canonical projections $\rho_u : \Sigma(\pi) \to Y$ and $\rho_s : \Sigma(\pi) \to Z$. It is an easy matter to check that the former is u-bijective while the latter is s-bijective. It follows that both local stable and local unstable sets in $\Sigma(\pi)$ are totally disconnected and, hence, that $(\Sigma(\pi), \sigma)$ is a shift of finite type. Thus, we recover Bowen's result 2.2 with the additional condition that π may be factored as $\pi = \pi_s \circ \rho_u = \pi_u \circ \rho_s$, with maps with the additional properties described. We remark that it is not true that an arbitrary map π as in Theorem 2.2 has such a decomposition.

Theorem 4.2. *If (X, φ) is non-wandering, then there exists an s/u-bijective pair for (X, φ).*

Definition 4.3. *Let $\pi = (Y, \psi, \pi_u, Z, \zeta, \pi_s)$ be an s/u-bijective pair for (X, φ). For each $L, M \geq 0$, we define*

$$\begin{aligned} \Sigma_{L,M}(\pi) &= \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid y_l \in Y, z_m \in Z, \\ &\qquad \pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L, 0 \leq m \leq M\} \end{aligned}$$

For simplicity, we also denote $\Sigma_{0,0}(\pi)$ by $\Sigma(\pi)$.

We define $\sigma_{L,M} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M}(\pi)$ by

$$\sigma_{L,M}(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\psi(y_0), \ldots, \psi(y_L), \zeta(z_0), \ldots, \zeta(z_M)).$$

For $L \geq 1$ and $0 \leq l \leq L$, we let $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ be the map which deletes entry y_l. Similarly, the map $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ deletes entry z_m, for $M \geq 1, 0 \leq m \leq M$.

The important properties of these systems and maps is summarized as follows.

Theorem 4.4. *1. For every $L, M \geq 0$, $(\Sigma_{L,M}(\pi), \sigma_{L,M})$ is a shift of finite type.*

2. For $L \geq 1$, $0 \leq l \leq L$, the map $\delta_{l,}$ is an s-bijective factor map.

3. For $M \geq 1$, $0 \leq m \leq M$, the map $\delta_{,m}$ is a u-bijective factor map.

5 Homology

There are actually two homology theories here. One, based on the dimension group D^s will be denoted by H^s_* and the other, based on D^u, will be denoted by H^u_*. We will concentrate on the former for the remainder of this note.

It is worth noting that, if (X, φ) is a Smale space, then so is (X, φ^{-1}), although 'stable' in the former is the same as 'unstable' in the latter. Following this idea carefully through the theory, one can check that $H^u(X, \varphi)$ is naturally isomorphic to $H^s(X, \varphi^{-1})$.

Let (X, φ) be a Smale space and suppose that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-bijective pair for (X, φ). By applying Krieger's invariant to each of the shifts of finite type, $\Sigma_{L,M}(\pi)$, and the maps between them induced by the maps $\delta_{l,}, \delta_{,m}, 0 \leq l \leq L, 0 \leq m \leq M$, one obtains a double complex. In a standard way, we can take the homology of this double complex as follows.

We define

$$C^s(\pi)_N = \bigoplus_{L-M=N} D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$$

for every N in $\mathbb{Z}$ and a boundary map $d^s(\pi)_N : C^s_N(\pi) \to C^s_{N-1}(\pi)$ by

$$d^s(\pi)_N | D^s(\Sigma_{L,M}, \sigma_{L,M}) = \sum_{l=0}^{L} (-1)^l \delta^s_{l,} + \sum_{m=0}^{M+1} (-1)^{m+L} \delta^{s*}_{,m}.$$

where, in the special case $L = 0$, we set $\delta^s_{0,} = 0$.

Unfortunately, this complex is rather large and we will replace it with others that are more manageable. In more technical terms, the fact that Krieger's invariant is covariant for the maps $\delta_{l,}$ but contravariant for the maps $\delta_{,m}$ mean that the the double complex is a 'second quadrant' double complex, despite our choice of indexing $L, M \geq 0$. In simplicial homology, there are two complexes which may be used. One is called the ordered complex (which is analogous to one which we have above) and the second is called the alternating complex. In the former, the N-chains are constructed from $N+1$-tuples of vertices, all in a single simplex. In the latter, the N-chains are constructed from the N-simplices. The difference is that the former allows for repetition of vertices (i.e. degenerate simplices) and distinguishes between permutations of the vertices. The latter does not consider the order, except for orientation.

In our case, we have actions of the group permutation group $S_{L+1} \times S_{M+1}$ on $(\Sigma_{L,M}(\pi), \sigma_{L,M})$ and hence on the invariant $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$. We write these actions on the right. We form $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ which the quotient of $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$ by the subgroup of all elements which are fixed under some transposition and all elements of the form $a - sgn(\alpha)a \cdot (\alpha, 1)$, where α is in S_{L+1}. We also form a subgroup $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ of $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$ consisting of those elements a such that $a \cdot (1, \beta) = sgn(\beta)a$, for all β in S_{M+1}. Finally, we form a subgroup of the former, also a quotient of the latter, $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$, which incorporates the action of $S_{L+1} \times S_{M+1}$. We define complexes $(C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))$, $(C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi))$ and $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$ using these groups instead.

The most obvious advantage of these new complexes is the following. Each s-bijective or u-bijective factor map is finite-to-one. Indeed, there are constants L_0 and M_0 such that

$$\#\pi_s^{-1}\{x\} \leq L_0, \#\pi_u^{-1}\{x\} \leq M_0$$

for all x in X. It follows that $D^s_{\mathcal{Q},}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if $L \geq L_0$, $D^s_{,\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if $M \geq M_0$ and $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if either $L \geq L_0$ or $M \geq M_0$.

Standard techniques in homological algebra yield the following.

Theorem 5.1. *Let π be an s/u-bijective pair for the Smale space (X, φ). We have a commutative diagram of chain complexes and chain maps as shown:*

$$\begin{array}{ccc}
(C^s_{,\mathcal{A}}(\pi), d^s_{,\mathcal{A}}(\pi)) & \xrightarrow{J} & (C^s(\pi), d^s(\pi)) \\
\downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q} \\
(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi)) & \xrightarrow{J_{\mathcal{Q}}} & (C^s_{\mathcal{Q},}(\pi), d^s_{\mathcal{Q},}(\pi))
\end{array}$$

Moreover, the maps $Q_{\mathcal{A}}$ and $J_{\mathcal{Q}}$ both induce isomorphisms on homology.

So the three new complexes all have the same homology. It seems likely that the original complex also has the same homology, but this does not seem to follow easily from standard techniques.

Definition 5.2. *Let (X, φ) be a Smale space and π be an s/u-bijective pair for (X, φ). We define $H^s(\pi)$ to be the homology of the complex $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$. There is an analogous definition of $H^u(\pi)$.*

We remark that the definition is valid for any Smale space which has an s/u-bijective pair, which includes all non-wandering Smale spaces.

6 Properties

We want to establish some basic properties of our theory. The first crucial result is the following. It is stated in a slightly informal manner, but one which conveys the main idea.

Theorem 6.1. *Let (X, φ) be a Smale space and π be an s/u-bijective pair for (X, φ). $H^s_*(\pi)$ is independent of the s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ and depends only on (X, φ).*

Henceforth, we denote $H^s_*(\pi)$ by $H^s_*(X, \varphi)$ instead.

Theorem 6.2. *The homology theory H^s_* is functorial in the following sense. If $\rho : (X, \varphi) \to (X', \varphi')$ is an s-bijective factor map, then there are induced group homomorphisms*

$$\rho^s : H^s_N(X, \varphi) \to H^s_N(X', \varphi'),$$

for all N in $\mathbb{Z}$. If the map ρ is a u-bijective factor, then there are induced group homomorphisms

$$\rho^{s*} : H^s_N(X', \varphi') \to H^s_N(X, \varphi),$$

for all N in $\mathbb{Z}$.

The following is an easy consequence of our earlier observation that $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ is non-zero for only finitely many values of L and M and the fact that Krieger's invariant is always a finite rank abelian group.

Theorem 6.3. *For any Smale space (X, φ) which has an s/u-bijective pair, the groups $H^s_N(X, \varphi)$ are finite rank and are non-zero for only finite many values of N.*

Finally, we have the following analogue of the Lefschetz formula. Given (X, φ), we can regard φ^{-1} as a factor map from this system to itself. It is both s-bijective and u-bijective and so, by Theorem 6.2, induces an automorphism of our invariant, denoted $(\varphi^{-1})^s$. The proof of the following result, already known in the case of shifts of finite type, uses ideas of Manning [11].

Theorem 6.4. *For any non-wandering Smale space (X, φ) and $p \geq 1$, we have*

$$\sum_{N\in\mathbb{Z}}(-1)^N Tr[((\varphi^{-1})^s \otimes 1_{\mathbb{R}})^p : H^s_N(X,\varphi)\otimes\mathbb{Q} \to H^s_N(X,\varphi)\otimes\mathbb{Q}]$$
$$= \#\{x \in X \mid \varphi^p(x) = x\}.$$

7 Examples

We present four examples where the computations above may be carried out quite explicitly. All of the examples are computed using the double complex $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$.

Example 7.1. *Suppose (Σ, σ) is a shift of finite type. In this case, an s/u-bijective pair is just $(Y, \psi) = (Z, \zeta) = (\Sigma, \sigma)$. Only the $0,0$-term in the double complex is non-zero and it is just $D^s(\Sigma, \sigma)$. Hence, $H^s_N(\Sigma, \sigma)$ is just $D^s(\Sigma, \sigma)$, for $N = 0$, and zero otherwise.*

Example 7.2. *For $m \geq 2$, let (X, φ) be the m^∞-solenoid. More specifically, we let*

$$X = \{(z_0, z_1, \ldots) \mid z_n \in \mathbb{T}, z_n = z^m_{n+1}, n \geq 0\},$$

with the map

$$\varphi(z_0, z_1, \ldots) = (z^m_0, z^m_1, z^m_2, \ldots),$$

for $(z_0, z_1, \ldots)$ in X. In this case, there is an s-bijective factor map onto (X, φ) from the full m-shift (i.e. G is the graph with one vertex and m edges). The simplest s/u-bijective pair here is $(Y, \psi) = (\Sigma_G, \sigma)$ and $(Z, \zeta) = (X, \varphi)$. The only non-zero groups in the double complex occur for (L, M) equal to $(0, 0)$ and $(1, 0)$ and these are $\mathbb{Z}[m^{-1}]$ and $\mathbb{Z}$, respectively. The boundary maps are all zero (only one needs to be computed) and $H^s_N(X, \varphi)$ is isomorphic to $\mathbb{Z}[m^{-1}]$, for $N = 0$, $\mathbb{Z}$, for $N = 1$ and zero for all other N.

This is a special of a one-dimensional solenoid. The general case is described and analyzed in [1].

Example 7.3. *Let $n > m > 1$ be relatively prime. Let X be the mn-solenoid as above and define*

$$\varphi(z_0, z_1, \ldots) = (z_1^{n^2}, z_2^{n^2}, z_3^{n^2}, \ldots),$$

Note that X is the dual of the discrete group $\mathbb{Z}[\frac{1}{nm}]$ and φ is the dual of the automorphism which is multiplication by $\frac{n}{m}$. We refer to (X, φ) as an $\frac{n}{m}$-solenoid. Here, the local stable sets are totally disconnected (in fact, they are open sets in the field of n-adic numbers) while the local unstable sets are of the form $(-t, t) \times C$, where C is totally disconnected. Here, (Y, σ) is the full shift on n symbols, while $Z = X$.

We have $H_N^s(X, \varphi)$ is isomorphic to $\mathbb{Z}[1/n]$ for $N = 0$, $\mathbb{Z}[1/m]$, for $N = 1$, and 0 for all other values of N. For complete details, see [5].

Example 7.4. *Let X be the 2-torus, $\mathbb{T}^2$, and φ be the hyperbolic automorphism determined by the matrix*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The systems (Y, ψ) and (Z, ζ) are both DA (or derived from Anosov) systems. Moreover, the shift of finite type $(\Sigma_{0,0}(\pi), \sigma_{0,0})$ arises from the Markov partition with three rectangles which appears in many dynamics texts, for example [10]. The only non-zero terms in the double complex are in positions $(L, M) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$. The calculation yields $H_N^s(X, \varphi)$ is $\mathbb{Z}$ for $N = 1$ and $N = -1$ and is $\mathbb{Z}^2$ for $N = 0$. Notice that the homology coincides with that of the torus, except with a dimension shift.

Example 7.5. *There is an example via inverse limits roughly based on the Sierpinski gasket. Its local stable sets are totally disconnected while its local unstable sets look like the Sierpinski gasket. Its homology is the same as that of the full 3-shift. The example is given in [17], although the homology calculations have not appeared as yet.*

8 Concluding remarks

Remark 8.1. *It is certainly a natural question to ask whether this theory can be computed from other (already existing) machinery. A more specific question would be to relate our homology to, say, the Cech cohomology of the* *classifying space of the topological equivalence relation R^s. (For a discussion of the topology, see [12].) There are examples, such as the first three above, where they are different, but only up to a dimension shift (depending on the space under consideration).*

Remark 8.2. *An important motivation in the construction of this theory was to compute the K-theory of certain C^*-algebras associated with the Smale space (X, φ). See [12] for a discussion of these C^*-algebras. At present, there seems to be a spectral sequence which relates the two; this work is still in progress.*

Remark 8.3. *I would like to thank the referee for a number of helpful comments.*

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