$\mathbb{Z}^d$-odometers and cohomology

Thierry Giordano[*]
Department of Mathematics and Statistics,
University of Ottawa,
585 King Edward, Ottawa, Ontario, Canada K1N 6N5

Ian F. Putnam[†]
Department of Mathematics and Statistics,
University of Victoria,
Victoria, B.C., Canada V8W 3R4

Christian F. Skau[‡]
Department of Mathematical Sciences,
Norwegian University of Science and Technology (NTNU),
N-7034 Trondheim, Norway

September 26, 2017

Abstract

Cohomology for actions of free abelian groups on the Cantor set has (when endowed with an order structure) provided a complete invariant for orbit equivalence. In this paper, we study a particular class of actions of such groups called odometers (or profinite actions) and investigate their cohomology. We show that for a free, minimal $\mathbb{Z}^d$-odometer, the first cohomology group provides a complete invariant for the action up to conjugacy. This is in contrast with the situation for orbit equivalence where it is the cohomology in dimension d

[*]Supported in part by a grant from NSERC, Canada
[†]Supported in part by a grant from NSERC, Canada
[‡]Supported in part by the Norwegian Research Council

which provides the invariant. We also consider classification up to isomorphism and continuous orbit equivalence.

1 Introduction

We recall some basic definitions from dynamical systems (see [14], [17] or [20]).

We say that (X, φ) is an action of a group G if X is a topological space and, for every g in G, $\varphi^g : X \to X$ is a homeomorphism satisfying the condition that, for all g, h in G, we have $\varphi^g \circ \varphi^h = \varphi^{gh}$. We will only consider groups G which are countable.

An action (X, φ) of G is *free*, if whenever g is in G and x in X satisfy $\varphi^g(x) = x$, then $g = e$. We say that the action is *minimal* if the only closed subsets Y of X such that $\varphi^g(Y) = Y$, for all g in G, are the empty set and X.

If (X_1, φ_1) and (X_2, φ_2) are actions of the groups G_1 and G_2, respectively, then $(X_1 \times X_2, \varphi_1 \times \varphi_2)$ is an action of $G_1 \times G_2$.

We also recall that if (X, φ) and (Y, ψ) are actions of the group G, a factor map $\pi : (X, \varphi) \to (Y, \psi)$ is a continuous surjection, $\pi : X \to Y$ such that $\pi \circ \varphi^g = \psi^g \circ \pi$, for every g in G.

An interesting class of systems where the space X is compact and totally disconnected arises if the group is assumed to be *residually finite*: there is a decreasing sequence of subgroups $G \supseteq G_1 \supseteq G_2 \supseteq \cdots$, each having finite index , $[G : G_n] < \infty$ and whose intersection is just the identity. Then the group G acts in an obvious way on each finite quotient space G/G_n and on the inverse limit of the system

$$G/G_1 \leftarrow G/G_2 \leftarrow \cdots$$

Such systems have a long and rather complex history, particularly in ergodic theory. We refer the reader to [15] and [19] for a full discussion. In the topological category, we refer the reader to [5] and [7]. Such systems often appear under the category of profinite completions, but we shall refer to them as G-odometers, simply because the terminology of 'odometer' is standard for the $G = \mathbb{Z}$ case.

Our main interest here will be in $\mathbb{Z}^d$-odometers, $d \geq 1$. We use $e_1, \ldots, e_d$ for the standard set of generators of $\mathbb{Z}^d$. We let $\varepsilon_1, \ldots, \varepsilon_d$ denote the dual basis; that is, these are group homomorphisms from $\mathbb{Z}^d$ to $\mathbb{Z}$. We use $< \cdot, \cdot >$ to denote the usual inner product on $\mathbb{R}^d$.

Definition 1.1. *Let (X, φ) be an action of the group G and (Y, ψ) be an action of the group H.*

1. *If $G = H$, a* conjugacy, *$h : (X, \varphi) \to (Y, \psi)$, is a homeomorphism $h : X \to Y$ such that $h \circ \varphi^g = \psi^g \circ h$, for all g in G. If such a map exists, we say that (X, φ) and (Y, ψ) are* conjugate.

2. *An* isomorphism *between the actions is a pair (h, α), where $h : X \to Y$ is a homeomorphism and $\alpha : G \to H$ is a group isomorphism such that $h \circ \varphi^g = \psi^{\alpha(g)} \circ h$, for all g in G. If such a pair exists, we say that (X, φ) and (Y, ψ) are* isomorphic.

Of course, in the second definition, even when $G = H$, we allow α to be non-trivial.

We will also be considering orbit equivalence between our systems [12]. The terminology is a reflection of the fact that in a G-action, (X, φ), for any point x in X, the set $\{\varphi^g(x) \mid g \in G\}$ is called the orbit of x under φ.

Definition 1.2. *Let (X, φ) be an action of the group G and (Y, ψ) be an action of the group H. We say they are* orbit equivalent *if there is a homeomorphism $h : X \to Y$ such*

$$h\{\varphi^g(x) \mid g \in G\} = \{\psi^{g'}(h(x)) \mid g' \in H\},$$

for all x in X. The function h is called an orbit equivalence.

Note that in the definitions of conjugacy and isomorphism, the groups must be isomorphic, which is not the case here.

Suppose that (X, φ) and (Y, ψ) are orbit equivalent and h is a map as in the definition. If we assume that both actions are free, then there are unique functions $\alpha : X \times G \to H$ and $\beta : Y \times H \to G$ such that

$$h(\varphi^g(x)) = \psi^{\alpha(x,g)}(h(x)),$$

for all x in X, g in G and

$$h^{-1}(\psi^{g'}(y)) = \varphi^{\beta(y,g')}(h^{-1}(y)),$$

for all y in Y and g' in H. These functions are usually called the *orbit cocycles* associated with h. It is important to note that despite the continuity properties of φ, ψ and h, these functions need not be continuous. It is then a natural notion to require some type of continuity.

Definition 1.3. *[18] If (X, φ) is a free action of G and (Y, ψ) is a free action of H, we say they are* continuously orbit equivalent *if there is an orbit equivalence, h, between them, whose cocycles, α and β, are both continuous (with the usual topologies on X, Y, the discrete topologies on G, H and the product topologies on $X \times G$ and $Y \times H$).*

It is also worth noting that for minimal actions of $\mathbb{Z}$ on the Cantor set, the somewhat peculiar property that $\alpha(\cdot, 1)$ and $\beta(\cdot, 1)$ each have at most one point of discontinuity is called *strong* orbit equivalence. (For more information, see [12].)

It is probably worth noting for the record the fairly obvious facts: conjugacy implies isomorphism, which implies continuous orbit equivalence, which implies orbit equivalence.

Our main results will be a classification of $\mathbb{Z}^d$-odometers up to conjugacy, isomorphism, orbit equivalence and continuous orbit equivalence. In fact, some results along these lines have already been obtained in [5], [4] and [18]. What is new in this paper is three features. The first is an alternate description of the construction. Instead of starting with a sequence

$$\mathcal{G} : \mathbb{Z}^d = G_1 \supseteq G_2 \supseteq \cdots$$

we start with a single group $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. The passage between the two is obtained by duality, either Pontryagin or in the sense of dual lattice. More specifically, H is obtained from $\mathcal{G}$ as $H = \cup_{n=1}^{\infty} G_n^*$, where G^* denotes the dual lattice of G, but we explore this in more detail in the next section. Our associated $\mathbb{Z}^d$-odometer is denoted by (Y_H, ψ_H). There is a small benefit here in that we replace the data of a sequence, $\mathcal{G}$, by a single object, namely H.

The second novelty is the computation of the first cohomology group, $H^1(Y_H, \psi_H)$. Specifically, we show in Theorem 4.3 that this is isomorphic to H. Even better, using the fact that the odometer has a unique invariant probability measure μ, we describe a natural map $\tau_\mu^1 : H^1(Y_H, \psi_H) \to \mathbb{R}^d$. (It is worth noting that the map is coordinate dependent, however.) In Theorem 4.4, we show that, for $d = 1, 2$, the range is exactly H and the map is an isomorphism.

The third novelty is that, building on the first two, we are able to give a quite simple classification of $\mathbb{Z}^d$-odometers up to conjugacy, isomorphism, continuous orbit equivalence and orbit equivalence in terms of the group H. We summarize the results of Corollaries 5.1 and 5.5 and Theorem 5.7 in the following theorem, Theorem 1.5. Before stating this, we need a bit of notation. It must be clear that that in cases of interest, $\mathbb{Z}^d$ has infinite index in H. Never-the-less, we find it useful to describe the index data in the following form.

Definition 1.4. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. We define the* superindex *of $\mathbb{Z}^d$ in H by*

$$[[H : \mathbb{Z}^d]] = \{[H' : \mathbb{Z}^d] \mid \mathbb{Z}^d \subseteq H' \subseteq H, [H' : \mathbb{Z}^d] < \infty\}.$$

Recall that $GL_d(\mathbb{Q})$ consists of the invertible $d \times d$ matrices with rational entries, while $GL_d(\mathbb{Z})$ consists of the $d \times d$ matrices with integer entries and determinant ± 1. Also recall that $SL_d(\mathbb{Q})$ and $SL_d(\mathbb{Z})$ consist of the respective subgroups with matrices of determinant one.

Theorem 1.5. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ and $\mathbb{Z}^{d'} \subseteq H' \subseteq \mathbb{Q}^{d'}$ be two groups and assume each is dense in $\mathbb{Q}^d$ and $\mathbb{Q}^{d'}$, respectively.*

1. *If $d = 1, 2$, then the $\mathbb{Z}^d$-actions (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are conjugate if and only if $H = H'$.*
2. *If $d, d' = 1, 2$, then the $\mathbb{Z}^d$-action (Y_H, ψ_H) and the $\mathbb{Z}^{d'}$-action $(Y_{H'}, \psi_{H'})$ are isomorphic if and only if $d = d'$ and there is α in $GL_d(\mathbb{Z})$ such that $\alpha H = H'$.*
3. *If $d, d' = 1, 2$, then the $\mathbb{Z}^d$-action (Y_H, ψ_H) and the $\mathbb{Z}^{d'}$-action $(Y_{H'}, \psi_{H'})$ are continuously orbit equivalent if and only if $d = d'$ and there is α in $GL_d(\mathbb{Q})$ with $\det(\alpha) = \pm 1$ such that $\alpha H = H'$.*
4. *The $\mathbb{Z}^d$-action (Y_H, ψ_H) and the $\mathbb{Z}^{d'}$-action $(Y_{H'}, \psi_{H'})$ are orbit equivalent if and only if $[[H : \mathbb{Z}^d]] = [[H' : \mathbb{Z}^{d'}]]$.*

We note in Corollary 5.9 that in the case $d = d' = 1$, the four conditions are all equivalent. In Example 5.10, we also give examples to show that, aside from the obvious implications we noted earlier, the four conditions are distinct, although all of these were provided initially by others.

2 $\mathbb{Z}^d$-odometers

Let d be a positive integer. We say that $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a decreasing sequence of finite-index subgroups of $\mathbb{Z}^d$ if we have

1. $\mathbb{Z}^d = G_1 \supseteq G_2 \supseteq \cdots,$

2. $[\mathbb{Z}^d : G_n] < \infty$, for all $n \geq 1$.

If G is any subgroup of $\mathbb{Z}^d$, we let φ_G denote the $\mathbb{Z}^d$-action on $\mathbb{Z}^d/G$ given by

$$\varphi_G(k)(l+G) = k+l+G, \text{ for all } k, l \in \mathbb{Z}^d.$$

Given the decreasing sequence above, the obvious quotient map $q_n : \mathbb{Z}^d/G_{n+1} \to \mathbb{Z}^d/G_n$ is then a factor map and we define $(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ to be the inverse limit of the systems

$$(\mathbb{Z}^d/G_1, \varphi_{G_1}) \xleftarrow{q_1} (\mathbb{Z}^d/G_2, \varphi_{G_2}) \xleftarrow{q_2} \cdots$$

The natural map from $(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ to $(\mathbb{Z}^d/G_n, \varphi_{G_n})$ is denoted π_n, for $n \geq 1$.

Definition 2.1. *[3] A $\mathbb{Z}^d$-odometer is any system $(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$, where $\mathcal{G}$ is a decreasing sequence of finite-index subgroups of $\mathbb{Z}^d$.*

The proof of the following is direct and we leave it to the reader.

Theorem 2.2. *Let $\mathcal{G}$ be a decreasing sequence of finite-index subgroups of $\mathbb{Z}^d$.*

1. *If $G_n \neq G_{n+1}$ for infinitely many n, then $X_{\mathcal{G}}$ is a Cantor set.*

2. *$(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ is minimal.*

3. *The action is free if and only if $\cap_{n=1}^{\infty} G_n = \{0\}$.*

4. *There is a unique $\varphi_{\mathcal{G}}$-invariant probability measure $\mu_{\mathcal{G}}$ on $X_{\mathcal{G}}$ which satisfies*
$$\mu_{\mathcal{G}}(\pi_n^{-1}\{k+G_n\}) = [\mathbb{Z}^d : G_n]^{-1},$$
for all $n \geq 1$, k in $\mathbb{Z}^d$.

5. *The formula*
$$d_{\mathcal{G}}(x,y) = \sup\{0, n^{-1} \mid n \geq 1, \pi_n(x) \neq \pi_n(y)\}$$
for x, y in $X_{\mathcal{G}}$ defines a metric in which $\varphi_{\mathcal{G}}$ is isometric.

Our aim here is to present $\mathbb{Z}^d$-odometers in a slightly different fashion, although the difference is rather cosmetic. A crucial feature is that the group $\mathbb{Z}^d$ is abelian, as we will use Pontryagin duality in an essential way.

We begin with a group $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. It follows that

$$H/\mathbb{Z}^d \subseteq \mathbb{Q}^d/\mathbb{Z}^d \subseteq \mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{T}^d.$$

We let ρ denote the inclusion map of H/Z^d in $\mathbb{T}^d$; that is,

$$\rho((r_1, \ldots, r_d) + \mathbb{Z}^d) = (e^{2\pi i r_1}, \ldots, e^{2\pi i r_d}), r \in H.$$

For any locally compact abelian group K, we let $\widehat{K}$ denote its Pontryagin dual [16]. We let $Y_H = \widehat{H/\mathbb{Z}^d}$. The groups H and $H/\mathbb{Z}^d$ are given the discrete topology so that Y_H is compact. Since $H \subseteq \mathbb{Q}^d$, the quotient $H/\mathbb{Z}^d$ is torsion, so Y_H is totally disconnected.

We suppress the natural isomorphism $\widehat{\mathbb{T}^d} \cong \mathbb{Z}^d$ and consider

$$\widehat{\rho} : \mathbb{Z}^d \to \widehat{H/\mathbb{Z}^d}.$$

We then obtain an action of $\mathbb{Z}^d$ on Y_H, which we denote by ψ_H, by

$$\psi_H^n(x) = x + \widehat{\rho}(n), n \in \mathbb{Z}^d, x \in Y_H.$$

More specifically, if $x : H/\mathbb{Z}^d \to \mathbb{T}$ is a group homomorphism, then

$$\psi_H^n(x)(h + \mathbb{Z}^d) = x(h)e^{2\pi i <h,n>}, h \in H, n \in \mathbb{Z}^d.$$

If we rewrite this, using $<< \cdot, \cdot >>$ to denote the pairing between $H/\mathbb{Z}^d$ and its dual, then we have

$$<< h + \mathbb{Z}^d, \psi_H^n(x) >>= e^{2\pi <h,n>} << h + \mathbb{Z}^d, x >>,$$

for every h in H, n in $\mathbb{Z}^d$ and x in Y_H. In other words, the function $<< h + \mathbb{Z}^d, \cdot >>$ is a continuous eigenfunction for the action with eigenvalue $(e^{2\pi h_1}, \ldots, e^{2\pi h_d})$. In particular, the spectrum [20] of the action is the set $\{(e^{2\pi h_1}, \ldots, e^{2\pi h_d}) \mid h \in H\}$.

We make a few simple observations on this construction.

Proposition 2.3. *1. If $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$, then the action (Y_H, ψ_H) is free if and only if H is dense in $\mathbb{Q}^d$.*

2. If $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$, then $\#Y_H = [H : \mathbb{Z}^d]$.

3. If $\mathbb{Z}^d \subseteq H \subseteq H' \subseteq \mathbb{Q}^d$, then there is a natural factor map from $(Y_{H'}, \psi_{H'})$ to (Y_H, ψ_H).

4. If $\mathbb{Z}^d \subseteq H_1 \subseteq H_2 \subseteq \cdots \mathbb{Q}^d$ then the inverse limit of

$$\lim (Y_{H_1}, \psi_{H_1}) \xleftarrow{\widehat{i_1}} (Y_{H_2}, \psi_{H_2}) \xleftarrow{\widehat{i_2}} \cdots$$

is conjugate to (Y_H, ψ_H), where $H = \cup_n H_n$.

To analyze the system, (Y_H, ψ_H), we say that a sequence of subgroups, $H_1, H_2, \ldots$ is an increasing sequence of finite-index extensions of $\mathbb{Z}^d$ if

1. $\mathbb{Z}^d = H_1 \subseteq H_2 \subseteq \cdots$,
2. $[H_n : \mathbb{Z}^d] < \infty$, for all $n \geq 1$.

Observe that, for any group $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$, there exists such a sequence with union H by simply taking $H_n = (\frac{1}{n!}\mathbb{Z})^d \cap H, n \geq 1$. From this point, the last proposition gives us a rather complete description of the action (Y_H, ψ_H).

The link between the two constructions which we have described, $\mathbb{Z}^d = G_1 \supseteq G_2 \supseteq \cdots$ and $\mathbb{Z}^d = H_1 \subseteq H_2 \subseteq \cdots$ is given by duality. Recall that a subgroup $K \subseteq \mathbb{R}^d$ is a *lattice* if it is discrete and $\mathbb{R}^d/K$ is compact. In particular, this holds if $\mathbb{Z}^d$ is a finite index subgroup of K. The dual lattice K^* is defined by

$$K^* = \{g \in \mathbb{R}^d \mid < k, g > \in \mathbb{Z}, \text{ for all } k \in K\}.$$

It is a simple matter to check that if $H \subseteq \mathbb{Q}^d$ is a lattice, then $H^* \subseteq \mathbb{Q}^d$ also.

Lemma 2.4. *Let $\mathbb{Z}^d \subseteq K \subseteq \mathbb{Q}^d$ be a subgroup with $[K : \mathbb{Z}^d] < \infty$. Then there is a conjugacy*

$$h_K : (Y_K, \psi_K) \to (\mathbb{Z}^d/K^*, \varphi_{K^*}).$$

Moreover, if $\mathbb{Z}^d \subseteq K_1 \subseteq K_2 \subseteq \mathbb{Q}^d$ are subgroups with $[K_2 : \mathbb{Z}^d] < \infty$, let i denote the inclusion of $K_1/\mathbb{Z}^d$ in $K_2/\mathbb{Z}^d$. It is clear that $K_1^* \supseteq K_2^*$ and we let q denote the natural quotient map from $\mathbb{Z}^d/K_2^*$ to $\mathbb{Z}^d/K_1^*$. The following diagram commutes:

$$\begin{tikzcd}
(Y_{K_2}, \psi_{K_2}) \arrow[r, "\widehat{i}"] \arrow[d, "h_{K_2}"'] & (Y_{K_1}, \psi_{K_1}) \arrow[d, "h_{K_1}"'] \\
(\mathbb{Z}^d/K_2^*, \varphi_{K_2^*}) \arrow[r, "q"] & (\mathbb{Z}^d/K_1^*, \varphi_{K_1^*})
\end{tikzcd}$$

The obvious immediate consequence is the following.

Theorem 2.5. *Let H be a group with $\mathbb{Z}^d \subseteq H \subset \mathbb{Q}^d$. If $H_1, H_2, \ldots$ is any increasing sequence of finite-index extensions of $\mathbb{Z}^d$ with union H, then $\mathcal{G} = \{H_1^*, H_2^*, \ldots\}$ is a decreasing sequence of finite-index subgroups of $\mathbb{Z}^d$ and the $\mathbb{Z}^d$-systems (Y_H, ψ_H) and $(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ are conjugate.*

We also note the following, which establishes that the correspondence is actually bijective.

Theorem 2.6. *Let $\mathcal{G} = \{G_1, G_2, \ldots\}$ be a decreasing sequence of finite-index subgroups of $\mathbb{Z}^d$. Then the group*

$$H = \cup_{n=1}^{\infty} G_n^*$$

satisfies $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. Moreover, the $\mathbb{Z}^d$-systems (Y_H, ψ_H) and $(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ are conjugate.

In short, we have an equivalent formulation for $\mathbb{Z}^d$-odometers parametrized by the group H instead of the sequence $\mathcal{G}$. We believe that the parametrization by the group H is more natural. At this point, it is slightly simpler, being given as a single group, rather than a sequence of groups. That is rather trivial; we hope to make the case more convincing in subsequent sections.

Next, we list two relatively simple results which are worth noting.

Proposition 2.7. *Let $d_1, d_2 \geq 1$ and $\mathbb{Z}^{d_i} \subseteq H^i \subseteq \mathbb{Q}^{d_i}$ be groups for $i = 1, 2$. Then $(Y_{H^1 \oplus H^2}, \psi_{H^1 \oplus H^2})$ is conjugate to the product system $(Y_{H^1} \times Y_{H^2}, \psi_{H^1} \times \psi_{H^2})$ as $\mathbb{Z}^{d_1+d_2}$-systems.*

One other nice feature of our parameterization of odometers by $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ is that it makes the difference between conjugacy and isomorphism relatively easy to describe, as follows.

Proposition 2.8. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ be a group and let α be in $GL_d(\mathbb{Z})$. Then the $\mathbb{Z}^d$-odometers (Y_H, ψ_H) and $(Y_{\alpha H}, \psi_{\alpha H})$ are isomorphic via the automorphism of $\mathbb{Z}^d$ sending $n \in \mathbb{Z}^d$ to αn.*

Conversely, if $\mathbb{Z}^d \subseteq H, H' \subseteq \mathbb{Q}^d$ are two dense subgroups of $\mathbb{Q}^d$, such that the $\mathbb{Z}^d$-odometers, (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are isomorphic, then there is α in $GL_d(\mathbb{Z})$ such that $(Y_{\alpha H}, \psi_{\alpha H})$ and $(Y_{H'}, \psi_{H'})$ are conjugate.

Proof. It is clear that α induces automorphisms of both $\mathbb{Z}^d$ and $\mathbb{Q}^d$. It is then a simple matter to observe in our definition of (Y_H, ψ_H) that we have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}^d \arrow[r, "\hat{\rho}_H"] \arrow[d, "\alpha"'] & \widehat{H/\mathbb{Z}^d} \arrow[d, "\alpha"] \\ \mathbb{Z}^d \arrow[r, "\hat{\rho}_{\alpha H}"] & \widehat{\alpha H/\mathbb{Z}^d} \end{tikzcd}$$

In fact, the commutativity of this diagram is simply a re-phrasing of the desired isomorphism between the two systems, implemented by α.

For the second part, the isomorphism between the actions provides an automorphism of the group $\mathbb{Z}^d$. But such an automorphism is always implemented by a matrix α as above. From the first part, we know that (Y_H, ψ_H) and $(Y_{\alpha H}, \psi_{\alpha H})$ are isomorphic via α. It follows that the latter is conjugate to $(Y_{H'}, \psi_{H'})$. $\square$

The next result shows that the superindex can be computed from a given expression of H as a union of finite index extensions of $\mathbb{Z}^d$. This is useful, in view of Theorems 2.5 and 2.6. The proof is trivial and we omit it.

Proposition 2.9. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. If $H_n, n \geq 1$ is an increasing sequence of finite-index extensions of $\mathbb{Z}^d$ with union H, then*

$$[[H : \mathbb{Z}^d]] = \cup_{n=1}^{\infty} \{k \in \mathbb{N} \mid k | [H_n : \mathbb{Z}^d]\}.$$

3 Cohomology for $\mathbb{Z}^d$-actions

In this section, we provide some basic definitions and results regarding cohomology. We begin with a Cantor minimal $\mathbb{Z}^d$-system, (X, φ). We let $C(X, \mathbb{Z})$ denote the set of continuous integer-valued functions on X. Of course, each such function is simply the (finite) sum of integer multiples of characteristic functions of clopen subsets of X. We regard it as an abelian group with pointwise addition of functions. We note that the non-negative functions form a positive cone. It is also a $\mathbb{Z}^d$-module via $n \cdot f(x) = f(\varphi^n(x))$, for n in $\mathbb{Z}^d$, f in $C(X, \mathbb{Z})$ and x in X. We define $H^*(X, \varphi)$ to be the group cohomology of $\mathbb{Z}^d$ with coefficients in the module $C(X, \mathbb{Z})$. This was first considered by Forrest and Hunton [8].

We refer the reader to [2] for a more thorough treatment of cohomology. We remark that this may be described in the following fashion. For $n \geq 0$, let C^n be the group of integer-valued functions on $X \times \times_{i=1}^{n} \mathbb{Z}^d$ which are continuous in the product topology. We have a coboundary operator $d : C^n \to C^{n+1}$ defined by

$$\begin{aligned} d(\theta)(x, s_0, s_1, \ldots, s_n) &= \theta(\varphi^{s_0}(x), s_1, \ldots, s_n) \\ &+ \sum_{i=1}^{n} (-1)^i \theta(x, s_0, s_1, \ldots, s_{i-1} + s_i, \ldots, s_n) \\ &+ (-1)^{n+1} \theta(x, s_0, \ldots, s_{n-1}) \end{aligned}$$

for θ in C^n, x in X and $s_0, \ldots, s_n$ in $\mathbb{Z}^d$. We will let $Z^n(X, \varphi), B^n(X, \varphi)$ and $H^n(X, \varphi)$ denote the n-cocycles, n-coboundaries and n-cohomology groups, respectively, of this complex.

We will have particular interest in the group $H^1(X, \varphi)$. Notice that here we are looking at continuous functions $\theta : X \times \mathbb{Z}^d \to \mathbb{Z}$ and such a function is a 1-cocycle (i.e. $d(\theta) = 0$) if and only if

$$\theta(x, m + n) = \theta(x, m) + \theta(\varphi^m(x), n),$$

for all m, n in $\mathbb{Z}^d$ and x in X. A cocycle, θ, is a coboundary if there is h in $C(X, \mathbb{Z})$ such that $\theta(x, n) = h(\varphi^n(x)) - h(x)$, for all x in X and n in $\mathbb{Z}^d$.

Proposition 3.1. *The cohomology $H^*(X, \varphi)$ is an invariant of continuous orbit equivalence.*

We will not prove this. One rather long method of proof is by direct computation. Another is to observe that the cohomology is actually the groupoid cohomology of the étale groupoid $X \times \mathbb{Z}^d$ (see [18]) and that continuous orbit equivalence implies (or is actually equivalent to) isomorphism between the étale groupoids. It is worth noting that the cohomology is *not* an invariant of orbit equivalence because the cocycles are required to be continuous.

The fact that our cohomology groups are coming from dynamical systems provides extra tools for their study. Specifically, our systems always have invariant measures (unique invariant measures for odometers) and these can be paired with cocycles.

Proposition 3.2. *Let μ be an invariant probability measure for the Cantor $\mathbb{Z}^d$-system (X, φ). For any 1-cocycle define $\tau^1_\mu(\theta) : \mathbb{Z}^d \to \mathbb{R}$ by*

$$\tau^1_\mu(\theta)(n) = \int_X \theta(x, n) d\mu(x),$$

for n in $\mathbb{Z}^d$. Then $\tau_\mu^1(\theta)$ is a group homomorphism. Moreover, it is zero if θ is a coboundary and hence passes to a well-defined group homomorphism

$$\tau_\mu^1 : H^1(X, \varphi) \to Hom(\mathbb{Z}^d, \mathbb{R}).$$

Proof. First, we check that $\tau_\mu^1(\theta)$ is a group homomorphism. Using the invariance of μ, for m, n in $\mathbb{Z}^d$, we have

$$\begin{aligned}
\tau_\mu^1(\theta)(m+n) &= \int_X \theta(x, m+n) d\mu(x) \\
&= \int_X \left(\theta(x, m) + \theta(\varphi^m(x), n)\right) d\mu(x) \\
&= \int_X \theta(x, m) d\mu(x) + \int_X \theta(\varphi^m(x), n) d\mu(x) \\
&= \int_X \theta(x, m) d\mu(x) + \int_X \theta(x, n) d\mu(x) \\
&= \tau_\mu^1(\theta)(m) + \tau_\mu^1(\theta)(n).
\end{aligned}$$

Next, we check that if $\theta = dh$, then $\tau_\mu^1(\theta) = 0$. Let n be in $\mathbb{Z}^d$. Again using the invariance of μ, we have

$$\begin{aligned}
\tau_\mu^1(\theta)(n) &= \int_X \theta(x, n) d\mu(x) \\
&= \int_X h(\varphi^n(x)) - h(x) d\mu(x) \\
&= \int_X h(x) d\mu(x) - \int_X h(x) d\mu(x) \\
&= 0.
\end{aligned}$$

The fact that τ_μ^1 is additive is obvious. $\square$

We want to make one simplification to this result and that concerns the group $Hom(\mathbb{Z}^d, \mathbb{R})$. There is an obvious isomorphism from this group to $\mathbb{R}^d$, taking α in $Hom(\mathbb{Z}^d, \mathbb{R})$ to $(\alpha(e_1), \alpha(e_2), \ldots, \alpha(e_d))$ in $\mathbb{R}^d$. We simply build this into our definition, without changing our notation.

Definition 3.3. *Let μ be an invariant probability measure for the Cantor $\mathbb{Z}^d$-system (X, φ). We define $\tau_\mu^1 : H^1(X, \varphi) \to \mathbb{R}^d$ by*

$$\tau_\mu^1([\theta]) = \left(\tau_\mu^1(\theta)(e_1), \ldots, \tau_\mu^1(\theta)(e_d)\right),$$

for any 1-cocycle θ.

It is worth noting that this final version of the invariant depends on the generators of $\mathbb{Z}^d$. In particular, isomorphic systems do *not* have the same map.

We also introduce the group of co-invariants; we let $B(X, \varphi)$ denote the subgroup of $C(X, \mathbb{Z})$ generated by all functions of the form $h - h \circ \varphi^n$, where h is any element of $C(X, \mathbb{Z})$ and n is in $\mathbb{Z}^d$. We let

$$D(X, \varphi) = C(X, \mathbb{Z})/B(X, \varphi).$$

We let $[f]$ denote the coset of $f \in C(X, \mathbb{Z})$. We also endow it with the positive cone

$$D(X, \varphi)^+ = \{[f] \mid f \geq 0\}$$

and order unit $[1]$, where 1 denotes the constant function.

Once again, if μ is an invariant probability measure for the system (X, φ) then the formula

$$\tau_\mu([f]) = \int_X f(x) d\mu(x),$$

defines a positive group homomorphism from $D(X, \varphi)$ to $\mathbb{R}$.

We also define $B_m(X, \varphi)$ to be the set of all f in $C(X, \mathbb{Z})$ such that $\int_X f d\mu = 0$, for all φ-invariant measures on X. It evidently contains $B(X, \varphi)$ and we let $D_m(X, \varphi)$ denote the quotient with order structure analogous to the before. This is a quotient of $D(X, \varphi)$.

The importance of the ordered group $D_m(X, \varphi)$ is that, for minimal free actions of $\mathbb{Z}^d$ on the Cantor set, it is a complete invariant for orbit equivalence [11].

We remark here that for minimal, free Cantor $\mathbb{Z}^d$-systems, $D(X, \varphi)$ is actually isomorphic to $H^d(X, \varphi)$, although the latter has no natural order structure. The isomorphism is induced by taking an d-cocycle θ to the function $f(x) = \theta(x, e_1, \ldots, e_d)$ in $C(X, \mathbb{Z})$, where $e_1, \ldots, e_d$ is the standard basis for $\mathbb{R}^d$. We refer the reader to [8] although we will not use this fact. We also refer the reader to [13].

4 Cohomology for $\mathbb{Z}^d$-odometers

The main results of this section describe the cohomology of a free, minimal $\mathbb{Z}^d$-odometer and are based on two relatively simple results on cohomology.

Lemma 4.1. *Let*

$$(X_1, \varphi_1) \xleftarrow{\pi_1} (X_2, \varphi_2) \xleftarrow{\pi_2} \cdots$$

be a system of $\mathbb{Z}^d$-actions and let (X, φ) be their inverse limit. Then, for all $i \geq 0$, we have

$$H^i(X, \varphi) = \lim_{n \to \infty} H^i(X_1, \varphi_1) \xrightarrow{\pi_1^*} H^i(X_2, \varphi_2) \xrightarrow{\pi_2^*} \cdots$$

In addition, we have

$$D(X, \varphi) = \lim_{n \to \infty} D(X_1, \varphi_1) \xrightarrow{\pi_1^*} D(X_2, \varphi_2) \xrightarrow{\pi_2^*} \cdots$$

We will not provide a proof, but we refer the reader to [2]. In fact, the reader can easily construct a proof himself or herself by starting with the fact that $C(X \times \mathbb{Z}^d \times \ldots \times \mathbb{Z}^d, \mathbb{Z})$ is the inductive limit of

$$C(X_1 \times \mathbb{Z}^d \times \ldots \times \mathbb{Z}^d, \mathbb{Z}) \stackrel{\pi_1^*}{\to} C(X_2 \times \mathbb{Z}^d \times \ldots \times \mathbb{Z}^d, \mathbb{Z}) \stackrel{\pi_2^*}{\to} \cdots$$

The second basic result is the following, which is a very simple case of Shapiro's Lemma [2]. We will sketch a proof, in part for completeness and in part because we will need to use some aspects of the proof in the next computation.

Lemma 4.2. *Let $d \geq 1$ and let G be a finite index subgroup of $\mathbb{Z}^d$. For each θ in $Z^1(\mathbb{Z}^d/G, \varphi_G)$, we define $\alpha(\theta) : G \to \mathbb{Z}$ by*

$$\alpha(\theta)(g) = \theta(G, g), g \in G.$$

Then $\alpha(\theta)$ is a group homomorphism and α induces an isomorphism from $H^1(X_G, \varphi_G)$ to $Hom(G, \mathbb{Z})$.

Proof. The fact that $\alpha(\theta)$ is a group homomorphism is a trivial consequence of the cocycle condition on θ, when restricted to $\{G\} \times G$.

Second, it is a trivial computation to see that, if f is in $C(\mathbb{Z}^d/G, \mathbb{Z})$, then $\alpha(d(f)) = 0$. This implies that α descends to a well-defined map on cohomology.

Third, it is an easy matter to see that if θ and η are cocycles, then $\alpha(\theta + \eta) = \alpha(\theta) + \alpha(\eta)$.

Fourth, suppose that θ and η are cocycles and $\alpha(\theta) = \alpha(\eta)$. Select $k_i, 1 \leq i \leq [\mathbb{Z}^d : G]$ in $\mathbb{Z}^d$, one from each coset of G in $\mathbb{Z}^d$. Define f in $C(\mathbb{Z}^d/G, \mathbb{Z})$ by

$$f(k_i + G) = \theta(G, k_i) - \eta(G, k_i), 1 \leq i \leq [\mathbb{Z}^d : G].$$

It is a simple computation (using the cocycle condition) to prove that $\theta - \eta - d(f) = 0$. This shows that the map induced by α at the level of cohomology is injective.

Finally, let $\gamma : G \to \mathbb{Z}$ be a homomorphism. Let $k_i, 1 \leq i \leq [\mathbb{Z}^d : G]$ in $\mathbb{Z}^d$ be as above. To define a cocycle θ, it suffices to pick $1 \leq i, j \leq [\mathbb{Z}^d : G]$ and g in G and define $\theta(k_i + G, k_j + g)$. Given i, j, g, there is a unique $1 \leq l \leq [\mathbb{Z}^d : G]$ and g' in G such that $k_i + k_j + g = k_l + g'$ and we set

$$\theta(k_i + G, k_j + g) = \gamma(g').$$

It is a fairly simple matter to check that θ is a cocycle and it is obvious that $\alpha(\theta) = \gamma$. □

Theorem 4.3. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. Then we have $H^1(Y_H, \psi_H) \cong H$.*

Proof. Select an increasing sequence of finite-index extensions of $\mathbb{Z}^d$, $H_1, H_2, \ldots$ with union H. In view of Theorem 2.5, it suffices for us to prove that $H \cong H^1(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$, where $\mathcal{G} = \{H_1^*, H_2^*, \ldots\}$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
H^1(\mathbb{Z}^d/H_1^*, \varphi_{H_1^*}) & \longrightarrow & H^1(\mathbb{Z}^d/H_2^*, \varphi_{H_2^*}) & \longrightarrow & H^1(\mathbb{Z}^d/H_3^*, \varphi_{H_3^*}) & \longrightarrow & \cdots \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
Hom(H_1^*, \mathbb{Z}) & \longrightarrow & Hom(H_2^*, \mathbb{Z}) & \longrightarrow & Hom(H_3^*, \mathbb{Z}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H_1 & \longrightarrow & H_2 & \longrightarrow & H_3 & \longrightarrow & \cdots
\end{array}$$

.

Each of the upper vertical maps is an isomorphism by Lemma 4.2. Each of the lower vertical maps is an isomorphism by simple duality. The limit of the first line is $H^1(X_{\mathcal{G}}, \varphi_{\mathcal{G}})$ by Lemma 4.1, while the limit of the last line is H. □

While the next result undoubtedly holds for all d, the proof is geometric and rather simpler if we restrict to $d = 1, 2$. In fact, we will give the proof only for $d = 2$ sure that, having seen this, readers can easily supply the proof for $d = 1$.

Theorem 4.4. *Let $d = 1$ or $d = 2$ and let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. Let μ be the unique invariant probability measure for the system (Y_H, ψ_H). Then the map*

$$\tau^1_\mu : H^1(Y_H, \psi_H) \to H$$

is an isomorphism.

Proof. Fix a sequence $H_1, H_2 \ldots$ of finite-index extensions of $\mathbb{Z}^2$ with union H. Let $n \geq 1$ and h be in H_n. Let $G_n = H_n^*$ so that $H_n = Hom(G_n, \mathbb{Z})$ via the inner product. Referring to the commutative diagram in the proof of Theorem 4.3, h determines the homomorphism $< \cdot, h >$ in $Hom(G_n, \mathbb{Z})$ which in turn determines a cocycle θ (unique up to coboundary) in $Z^1(\mathbb{Z}^d/G_n, \varphi_{G_n})$ with $\alpha(\theta) =< \cdot, h >$. This means that

$$\theta(G_n, g) = \alpha(\theta)(g) =< g, h >,$$

for all g in G_n. Our first task is to use the proof of Lemma 4.2 to write θ explicitly.

This begins with the choice of $k_i, 1 \leq i \leq [\mathbb{Z}^2 : G_n]$, which represent the cosets of G_n. We may choose generators $(a, b), (c, d)$ of G_n such that these lie in the first quadrant and the line through (a, b) is below the line through (c, d); that is, $a, d > 0$, $b, c \geq 0$ and $ad - bc > 0$. Let k_i be the points in the integer lattice which are also in the parallelogram determined by (a, b) and (c, d); more precisely, let

$$F = \{s(a, b) + t(c, d) \mid 0 \leq s, t < 1\} \cap \mathbb{Z}^2.$$

We then define θ as in the proof of 4.2: for k_1, k_2 in F and g in G_n, we set $\theta(k_1 + G_n, k_2 + g) =< g', h >$, where k' in F and g' in G_n are such that $k_1 + k_2 + g = k' + g'$.

With a slight abuse of notation, we consider the cocycle, again denoted θ, in $Z^1(X_\mathcal{G}, \varphi_\mathcal{G})$ given by $\theta(x, k) = \theta(\pi_n(x), k)$, for x in $X_\mathcal{G}$ and k in $\mathbb{Z}^2$. It is the class of this θ that is mapped to h under the isomorphism of Theorem 4.3.

It follows from the definition of $\tau^1_{\mu_\mathcal{G}}$ and the formula for $\mu_\mathcal{G}$ given in Theorem 2.2 that

$$\tau^1_{\mu_\mathcal{G}}(\theta) = \sum_{k \in F} [\mathbb{Z}^d : G_n]^{-1}(\theta(k + G_n, (1, 0)), \theta(k + G_n, (0, 1)))$$

First, we note that $[\mathbb{Z}^d : G_n] = ad - bc$. We compute the first entry of $\tau^1_{\mu_{\mathcal{G}}}(\theta)$. Let k be in F. If $k+(1,0) = k'$ is also in F, then writing $k+(1,0) = k'+(0,0)$ means that $\theta(k + G_n, (1,0)) =< (0,0), h >= 0$.

Now let us write $k = (i,j)$. If we fix $0 \leq j < b+d$, the values of i for which (i,j) are in F form an interval, $i = i_0, \ldots, i_1$. If $i < i_1$, then $(i,j) + (1,0)$ is again in F and $\theta((i,j) + G_n, (1,0)) = 0$. Let us now consider $i = i_1$. If $0 \leq j < b$, then the point $(i_1, j) + (1,0)$ has moved out of F through its lower boundary, the line joining the origin and (a,b). In this case we write $(i_1, j) + (1,0) = (i_1 + 1 + c, j + d) - (c,d)$, where $(i_1 + 1 + c, j + d)$ is in F and $-(c,d)$ is in G_n. Hence, we have

$$\theta((i_1, j) + G_n, (1,0)) =< -(c,d), h > .$$

We note that there are exactly b such values of (i_1, j). (This conclusion also holds in the case $b = 0$, which we leave to the reader.)

If $b \leq j < b + d$, then $(i_1, j) + (1,0)$ has moved out of F through its right boundary and we write $(i_1, j) + (1,0) = (i_1 + 1 - a, j - b) + (a,b)$, with $(i_1 + 1 - a, j - b)$ in F and (a,b) in G_n, and we have

$$\theta((i_1, j) + G_n, (1,0)) =< (a,b), h > .$$

There are exactly d such values of (i_1, j). Altogether, we find the first entry of $\tau^1_{\mu_{\mathcal{G}}}(\theta)$ is

$$(ad - bc)^{-1} \left(b < -(c,d), h > +d < (a,b), h >\right) =< (1,0), h > .$$

In a similar way, the second entry is $< (0,1), h >$ and so we have

$$\tau^1_{\mu_{\mathcal{G}}}(\theta) = (< (1,0), h >, < (0,1), h >) = h.$$

This completes the proof. □

5 Classification of $\mathbb{Z}^d$-odometers

Corollary 5.1. *Let $\mathbb{Z}^d \subseteq H, H' \subseteq \mathbb{Q}^d$ be dense subgroups of $\mathbb{Q}^d$..*

1. *If $d = 1$ or $d = 2$, then the systems (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are conjugate if and only if $H = H'$.*
2. *The systems (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are isomorphic if and only if there exists α in $GL_2(\mathbb{Z})$ such that $\alpha H = H'$.*

Proof. The first part is an immediate consequence of Theorem 2.2. The second part is an immediate consequence of Theorem 2.8 and the first part. □

Example 5.2. *We remark that the condition that α be in $GL_2(\mathbb{Z})$ cannot be replaced by α in $SL_2(\mathbb{Z})$. Consider $H = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3] + \mathbb{Z}(1/5, 1/5)$ and $H' = \alpha H$, where $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We claim that if β is any matrix in $GL_2(\mathbb{Z})$ such that $\beta H = \alpha H$, then $\beta = \pm\alpha$ and hence there is no such β in $SL_2(\mathbb{Z})$. To see the claim, let K be the subgroup of H consisting of all elements h such that, for every $k \geq 1$, there is h' in G such that $2^k h' = h$. It is easy to see that $K = \mathbb{Z}[1/2] \oplus 0$. Similarly, we let L be an analogous group, replacing 2 by 3, so that $L = 0 \oplus \mathbb{Z}[1/3]$. It is then clear that the only β in $GL_2(\mathbb{Z})$ such that $\beta K = \alpha K$ and $\beta L = \alpha L$ are $\beta = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$. As we also require, $\beta(1/5, 1/5)$ to be in αH, this leaves only $\beta = \pm\alpha$.*

We now turn our attention to orbit equivalence for $\mathbb{Z}^d$-odometers. The following result is also trivial; we state it simply for emphasis.

Lemma 5.3. *Let G be a finite index subgroup of $\mathbb{Z}^d$. Let μ be the normalized counting measure on $\mathbb{Z}^d/G$. Then $B(\mathbb{Z}^d/G, \varphi_G) = B_m(\mathbb{Z}^d/G, \varphi_G)$ and*

$$\tau_\mu : D(\mathbb{Z}^d/G, \varphi_G) = D_m(\mathbb{Z}^d/G, \varphi_G) \to [\mathbb{Z}^d : G]^{-1}\mathbb{Z}$$

is an isomorphism of ordered abelian groups with order units.

Theorem 5.4. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$. Then $B(Y_H, \psi_H) = B_m(Y_H, \psi_H)$ and the map*

$$\tau_\mu^d : D(Y_H, \psi_H) \to \cup_{m \in [[H:\mathbb{Z}^d]]} m^{-1}\mathbb{Z}$$

is an isomorphism of ordered abelian groups with order units.

Proof. The proof is exactly the same as that of Theorem 4.3, with Lemma 5.3 replacing Lemma 4.2. We omit the details. □

Corollary 5.5. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ and $\mathbb{Z}^{d'} \subseteq H' \subseteq \mathbb{Q}^{d'}$ be dense subgroups. The systems (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are orbit equivalent if and only if $[[H : \mathbb{Z}^d]] = [[H' : \mathbb{Z}^{d'}]]$.*

Notice that the condition above does not require $d = d'$.

We finally turn our attention to the issue of continuous orbit equivalence. Here, our main result is Theorem 5.7 below. In fact, it is not difficult to obtain Theorem 5.7 from the results of [4], but we give an independent direct proof. Even if the terminology is different, the proofs share many features. We also direct the reader's attention to Theorem 1.2 of [18] where several other characterizations of continuous orbit equivalence are given.

The following preliminary result will be useful in the proof and possibly of some interest on its own.

Proposition 5.6. *Assume that K, H are groups with $\mathbb{Z}^d \subseteq K \subseteq H \subseteq \mathbb{Q}^d$, $[K : \mathbb{Z}^d]$ is finite and that H is dense in $\mathbb{Q}^d$. Let π denote the quotient map from $\widehat{H/\mathbb{Z}^d} = Y_H$ to $\widehat{K/\mathbb{Z}^d} = Y_K$, $\mathcal{P}$ be the partition of Y_H into clopen sets formed by the pre-images of the points of Y_K under π and Z be the element of $\mathcal{P}$ containing the identity element of Y_H.*

1. *If $a_1, a_2, \ldots, a_k$ are representatives of the cosets of K^* in $\mathbb{Z}^d$ (so that $k = [K : \mathbb{Z}^d]$), then $\mathcal{P}$ consists of the sets $\psi_H^{a_i}(Z), i = 1, \ldots, k$.*

2. *For any z in Z and n in $\mathbb{Z}^d$, $\psi_H^n(z)$ is in Z if and only if n is in K^*. Moreover, $\{\psi_H^n(z) \mid n \in K^*\}$ is dense in Z.*

3. *If we choose an integer matrix α with $\alpha^T \mathbb{Z}^d = K^*$, then the system (Z, ψ_H, K^*) is isomorphic to $(Y_{\alpha H}, \psi_{\alpha H}, \mathbb{Z}^d)$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K/\mathbb{Z}^d & \xrightarrow{i} & H/\mathbb{Z}^d & \xrightarrow{p} & H/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K/\mathbb{Z}^d & \longrightarrow & \mathbb{R}^d/\mathbb{Z}^d & \xrightarrow{q} & \mathbb{R}^d/K & \longrightarrow & 0 \end{array}$$

If we take Pontryagin duals throughout, the first line gives an exact sequence with $\hat{i} = \pi$. It is straightforward to calculate that, after identifying the dual of $\mathbb{R}^d/\mathbb{Z}^d$ with $\mathbb{Z}^d$, the image of $\hat{q}$ is exactly K^*. The first part follows at once, as does the first sentence of part 2. For the second sentence of part 2, we know that the orbit of z is dense in Y_H. On the other hand, points of the form $\psi_H^n(z)$ with $z \notin K^*$ do not lie in Z and as Z is clopen, such points cannot limit on points in Z.

For the last statement, we first note that $\alpha^T \mathbb{Z}^d = K^*$ implies that $K = \alpha^{-1}\mathbb{Z}^d$. We consider the following commutative diagram

$$\begin{tikzcd}
H/K \arrow[r, "\alpha"] \arrow[d] & \alpha H/\mathbb{Z}^d \arrow[d] \\
\mathbb{R}^d/K \arrow[r, "\alpha"] & \mathbb{R}^d/\mathbb{Z}^d
\end{tikzcd}$$

Of course, the horizontal maps are isomorphisms. Taking Pontryagin duals yields the last part. □

Theorem 5.7. *Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ and $\mathbb{Z}^{d'} \subseteq H' \subseteq \mathbb{Q}^{d'}$ be dense subgroups. If $d = d'$ and there is α in $GL_d(\mathbb{Q})$ with $det(\alpha) = \pm 1$ such that $\alpha H = H'$, then the odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are continuously orbit equivalent. The converse holds in the case $d = 1, 2$.*

Proof. First, let us assume that the systems are continuously orbit equivalent. We have noted in Proposition 3.1 that continuous orbit equivalence implies isomorphism of cohomology groups, that $H^d(Y_H, \psi_H) \cong D(Y_H, \psi_H)$ is non-trivial and that $H^k(X, \varphi) = 0$, for any $k > d$ and any $\mathbb{Z}^d$-action. We conclude from this that $d = d'$, being the largest integer with non-trivial cohomology in that degree.

Now let $h : Y_H \to Y_{H'}$ be the homeomorphism and $\alpha : H \times \mathbb{Z}^d \to \mathbb{Z}^d$ be the associated cocycle as in the definition of continuous orbit equivalence. By following h by rotation by $-h(0)$, we may assume that $h(0) = 0$ in the group $Y_{H'}$. Also let $H_n, n \geq 1$ be an increasing sequence of finite index extensions of $\mathbb{Z}^d$ with union H. This means that Y_H is (up to isomorphism) an inverse limit of finite spaces, $\mathbb{Z}^d/H_n^*$. Recall that we let π_n denote the map from Y_H to $\mathbb{Z}^d/H_n^*$. For each $i = 1, 2, \ldots, d$, the function $\alpha(\cdot, e_i)$ is continuous and takes values in $\mathbb{Z}^d$. Hence it is locally constant and we may find n such that each function $\alpha(\cdot, e_i), i = 1, \ldots, d$ is constant on the partition induced by π_n.

For convenience, we let $K = H_n$, $\pi : Y_H \to Y_K$ be the quotient map and the partition $\mathcal{P}$ be as in Proposition 5.6. It follows from the cocycle property of α and the fact that ψ_H simply permutes the elements of $\mathcal{P}$ that $\alpha(\cdot, m)$ is constant on each element of the partition for every m in $\mathbb{Z}^d$. Define $\alpha : K^* \to \mathbb{Z}^d$ by $\alpha(n) = \alpha(0_H, n), n \in K^*$. It is then clear from the cocycle condition and our choice of Z in Proposition 5.6 that $\alpha(K^*)$ is a subgroup of $\mathbb{Z}^d$.

It follows from Proposition 5.6 that

$$K^* = \{k \in \mathbb{Z}^d \mid \psi_H^k(0_H) \in Z\}.$$

Then as h is an orbit equivalence, we have

$$\alpha(K^*) = \{l \in \mathbb{Z}^d \mid \psi_{H'}^l(0_{H'}) \in h(Z)\}.$$

We claim that $K' = (\alpha(K^*)^*$ is actually a subgroup of H'. Now let H'_m be an increasing sequence of finite index extensions of $\mathbb{Z}^d$ with union H' and let π'_m be the natural map from $Y_{H'}$ to $Y_{H'_m}$. The clopen sets $(\pi'_m)^{-1}\{0_{H'_m}\}, m \geq 1$ form a decreasing system of sets which intersect to the identity in $Y_{H'}$. It follows that there exists some m such that $\pi'_m(H'_m)$ is contained in $h(Z)$. With another application of Proposition 5.6, we have

$$\begin{aligned} H'^*_m &= \left\{l \in \mathbb{Z}^d \mid \psi^l_{H'}(0_{H'}) \in \pi'^{-1}\{0_{H'_m}\}\right\} \\ &\subseteq \left\{l \in \mathbb{Z}^d \mid \psi^l_{H'}(0_{H'}) \in h(\pi^{-1}\{0_{H'_m}\})\right\} \\ &= \alpha(K^*). \end{aligned}$$

It follows that

$$H' \supseteq H'_m \supseteq (\alpha K^*)^* = K'.$$

as desired. For convenience, let $\pi' : Y_{H'} \to Y_{K'}$. Let Z' be the pre-image in $Y_{H'}$ of the identity of $Y_{K'}$ under π'.

It follows from two applications of the second part of Proposition 5.6 that $h(Z) = Z'$ and that h with α are an isomorphism between the systems (Z, K^*, ψ_H) and $(Z', K'^*, \psi_{H'})$. In addition, h bijectively maps the partition of Y_H into the elements of the partition induced by π' of $Y_{H'}$. From this we conclude that each partition has the same number of elements and so $[\mathbb{Z}^d : (K')^*] = [\mathbb{Z}^d : K^*]$.

Now choose integer matrices α and α' such that $K^* = \alpha^T\mathbb{Z}^d$ and $(K')^* = (\alpha')^T\mathbb{Z}^d$. We have $|det(\alpha)| = [K : \mathbb{Z}^d] = [K' : \mathbb{Z}^d] = |det(\alpha')|$. It follows from what we have above and the last part of 5.6 that $(Y_{\alpha H}, \psi_{\alpha H})$ and $(Y_{\alpha' H'}, \psi_{\alpha' H'})$ are isomorphic. Therefore, by 5.1 that there is β in $GL_d(\mathbb{Z})$ with $\beta\alpha H = \alpha' H'$. Hence, $(\alpha')^{-1}\beta\alpha H = H'$ is a matrix with rational entries and

$$|det((\alpha')^{-1}\beta\alpha)| = |det(\alpha')^{-1}||det(\beta)||det(\alpha)| = 1.$$

For the converse, suppose that there is a matrix α with rational entries and determinant ± 1 such that $\alpha H = H'$. The group H is the union of subgroups K with $[K : \mathbb{Z}^d]$ finite. For each element of H', we may find a K such that αK contains that element. As $\mathbb{Z}^d$ is finitely generated, we may find $\mathbb{Z}^d \subseteq K \subseteq H$ with $[K : \mathbb{Z}^d]$ finite and $\mathbb{Z}^d \subseteq \alpha K$. Find a positive integer

n such that $n\alpha$ has only integer entries. Then we have

$$\begin{aligned}
[K:\mathbb{Z}^d] &= [\mathbb{Z}^d : K^*] \\
&= [\mathbb{Z}^d : nK^*][K^* : nK^*]^{-1} \\
&= [\mathbb{Z}^d : n\alpha^T K'^*]n^{-d} \\
&= |det(n\alpha)|[\mathbb{Z}^d : K'^*]n^{-d} \\
&= [\mathbb{Z}^d : K'^*] \\
&= [K' : \mathbb{Z}^d]
\end{aligned}$$

Using the fact above, we let $a_1, a_2, \dots, a_m$ and $a'_1, a'_2, \dots, a'_m$ be distinct representatives of the cosets of K^* and K', respectively, in $\mathbb{Z}^d$. The map α evidently induces a homeomorphism which we denote by h between $Z = \widehat{H/K}$ and $\widehat{H'/K'}$, which together with $(\alpha^T)^{-1}$, provide an isomorphism between (Z, ψ_H, K^*) and $(Z', \psi_{H'}, K'^*)$. We extend h to all of Y_H by setting $h \mid \psi_H^{a_i}(Z) = \psi_{H'}^{a'_i} \circ h \circ \psi_H^{-a_i}$. It follows from Corollary 5.1 that h is an orbit equivalence between (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ and it is an easy matter to see that the associated cocycles are continuous. We omit the details. □

Remark 5.8. *We leave it as an exercise to show that, for H and H' as in Example 5.2, the only β in $GL_2(\mathbb{Q})$ with such that $\beta H = H'$ and* $\det(\beta) = \pm 1$ *is $\beta = \pm\alpha$. This implies that we cannot change the condition* $\det(\alpha) = \pm 1$ *to* $\det(\alpha) = 1$ *above.*

These results on classification lead to a surprising dichotomy between the cases $d = 1$ and $d = 2$, which essentially stems from the fact that if a subgroup of the rationals is a finite index extension of the integers, then the index uniquely determines the group.

Corollary 5.9. *Let $\mathbb{Z} \subseteq H, H' \subseteq \mathbb{Q}$ be dense. The following are equivalent.*

1. *The $\mathbb{Z}$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are conjugate.*
2. *The $\mathbb{Z}$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are isomorphic.*
3. *The $\mathbb{Z}$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are continuously orbit equivalent.*
4. *The $\mathbb{Z}$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are orbit equivalent.*

Proof. This is well-known, but we will give a proof here since it is quite short. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are all trivial. We must prove (4) $\Rightarrow$ (1). From 5.1, we know that $\tau^1(H^1(Y_H, \psi_H))$ is a complete invariant for conjugacy, while from [12] and Theorem 5.4, $\tau^d(D(Y_h, \psi))$ is a complete invariant for orbit equivalence. As we stated earlier, $D(Y_H, \psi_H) = H^1(Y_H, \psi_H)$. This completes the proof. $\square$

Let us take this opportunity to make some vague comments prompted by the last corollary. The first, very briefly, is that for $\mathbb{Z}^d$-odometers, if $H_n, n \geq 1$, is an increasing sequence of finite-index extensions of $\mathbb{Z}^d$ with union H, the invariant $H^1(Y_H, \psi_H)$ 'remembers' the *groups* $H_n/\mathbb{Z}^d$, whereas the invariant $H^d(Y_H, \psi_H)$ 'remembers' the *sets* $H_n/\mathbb{Z}^d$. Secondly, it is well-known that for general minimal, free $\mathbb{Z}^d$-actions on Cantor sets, the invariant $H^d(X, \varphi)$ provides the invariant for orbit equivalence. On the other hand, the role of $H^1(X, \varphi)$ is not well-understood. The fact that they coincide when $d = 1$ tends to confuse the issue, which is nicely illustrated in the last corollary.

The situation is remarkably different for $d = 2$. We list these examples for completeness, but leave the reader the easy task of verifying them. The first example is already in the work of Cortez [3]. The second and third already appear in Li [18]. The fourth example is already well-known since [11].

Example 5.10. *1. Let $H = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3]$ and $H' = \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/2]$. Then the $\mathbb{Z}^2$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are isomorphic (using $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$), but not conjugate.*

2. Let $H = \mathbb{Z}[1/2] \oplus 5^{-1}\mathbb{Z}[1/3]$ and $H' = 5^{-1}\mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3]$. Then the $\mathbb{Z}^2$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are continuously orbit equivalent (using $\alpha = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 5 \end{bmatrix}$), but not isomorphic.

3. Let $H = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/15]$ and $H' = \mathbb{Z}[1/10] \oplus \mathbb{Z}[1/3]$. Then the $\mathbb{Z}^2$-odometers (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are orbit equivalent, but not continuously orbit equivalent.

4. Let $\mathbb{Z}^d \subseteq H \subseteq \mathbb{Q}^d$ be any dense subgroup with $d > 1$. Let

$$H' = \cup_{m \in [[H:\mathbb{Z}^d]]} m^{-1}\mathbb{Z},$$

so that $\mathbb{Z} \subseteq H' \subseteq \mathbb{Q}$ is dense. Then (Y_H, ψ_H) and $(Y_{H'}, \psi_{H'})$ are orbit equivalent, but not continuously orbit equivalent.

One slightly unfortunate consequence of these examples is that it may leave the reader with the impression that dense subgroups of $\mathbb{Q}^2$ are mainly obtained by taking direct sums. To allay that, let us first mention an example due to Fuchs [10] of a group (already appearing in 5.2) $\mathbb{Z}^2 \subseteq H \subseteq \mathbb{Q}^2$ which cannot be written as an internal direct product in a non-trivial way; that is, the group is indecomposable:

$$H = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3] + \mathbb{Z}(1/5, 1/5).$$

Unfortunately, this may still leave the reader with the impression that dense subgroups of $\mathbb{Q}^2$ are direct sums, up to a finite-index subgroup. We present an example of a class which seem to be substantially further from direct sums. We have not been able to find in the literature an example of this type.

Proposition 5.11. *There exists a group $\mathbb{Z}^2 \subseteq H \subseteq \mathbb{Q}^2$ such that, H is dense in $\mathbb{Q}^2$ and, for every x in $\mathbb{Z}^2$, the group $\mathbb{Q}x \cap H$ is cyclic.*

Proof. We will consider a pair of positive integers $a, b \geq 2$ and the associated matrix

$$\alpha = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}.$$

Observe that the determinant of α is $|\alpha| = ab - 1 \neq 0$. First, we observe that, given any positive integer $K \geq 3$, there are $a, b \geq K$ such that $|\alpha|$ is prime. If we simply let $a = K$, the arithmetic progression $bK - 1$ is prime for infinitely many b since K and 1 are relatively prime [6].

Writing elements of $\mathbb{Z}^2$ as row vectors, observe the following, which we leave as exercises:

1. $\mathbb{Z}^2\alpha \subseteq \mathbb{Z}^2$ and hence $\mathbb{Z}^2 \subseteq \mathbb{Z}^2\alpha^{-1}$,

2. $\pm|\alpha|^{-1}(a, -1), \pm|\alpha|^{-1}(-1, b)$ are in $\mathbb{Z}^2\alpha^{-1}$.

3. $(\mathbb{Q} \oplus 0) \cap \mathbb{Z}^2\alpha^{-1} = \mathbb{Z} \oplus 0$ and $(0 \oplus \mathbb{Q}) \cap \mathbb{Z}^2\alpha^{-1} = 0 \oplus \mathbb{Z}$.

4. for $1 \leq k \leq a - 1$, $b \leq kb \leq (ab - 1) - b + 1$ and for $1 \leq k \leq b - 1$, $a \leq ka \leq (ab - 1) - a + 1$,

5. for $x \neq 0$ in $\mathbb{Z}^2\alpha^{-1}$, we have $|x|_1 \geq |\alpha|^{-1}\min\{a,b\}$, where $|\cdot|_1$ is the ℓ^1-norm.

We claim that α has the following property: if x is in $\mathbb{Z}^2\alpha^{-1} - \mathbb{Z}^2$ and $m > 1$ is such that mx is in $\mathbb{Z}^2$, then $|mx|_1 \geq K$. We know that

$$[\mathbb{Z}^2\alpha^{-1} : \mathbb{Z}^2] = [\mathbb{Z}^2 : \mathbb{Z}^2\alpha] = |\alpha|$$

is prime and so the element $x + \mathbb{Z}^2$ must have order $|\alpha|$ in the quotient group $\mathbb{Z}^2\alpha^{-1}/\mathbb{Z}^2$. It follows that $m \geq |\alpha|$. Combining this with the last item above, we have

$$|mx|_1 \geq |\alpha||x|_1 \geq \min\{a,b\} \geq K.$$

We now construct a sequence of matrices as above inductively. Choose α_1 using the constant $K = 1$. For $n = 2, 3, 4, \ldots$, we choose α_n for the constant $K_n = n\|\alpha_1\|\|\alpha_2\|\ldots\|\alpha_{n-1}\|$, where $\|\alpha\|$ is the operator norm, regarding α as an operator on $\ell^1(\mathbb{R}^2)$.

We define $H_n = \mathbb{Z}^2\alpha_n^{-1}\alpha_{n-1}^{-1}\cdots\alpha_1^{-1} \subseteq \mathbb{Z}^2\alpha_{n-1}^{-1}\cdots\alpha_1^{-1} = H_{n-1}$ and $H = \cup_n H_n$.

Now suppose that x is in $H_n - H_{n-1}$ and mx is in $\mathbb{Z}^2$, for some $m > 1$.It follows that $z = x\alpha_1\cdots\alpha_{n-1}$ is in $\mathbb{Z}^2\alpha_n^{-1}$ and mz is in $\mathbb{Z}^2\alpha_1\cdots\alpha_{n-1} \subseteq \mathbb{Z}^2$. It follows that $|mz|_1 \geq n\|\alpha_1\|\|\alpha_2\|\ldots\|\alpha_{n-1}\|$. On the other hand, we have

$$|mz|_1 = |mx\alpha_1\cdots\alpha_{n-1}|_1 \leq |mx|_1\|\alpha_1\|\|\alpha_2\|\ldots\|\alpha_{n-1}\|.$$

Combining these two estimates yields

$$|mx|_1 \geq n.$$

Put another way, we have proved that if y is in $\mathbb{Z}^2$ and $mx = y$ for some $m > 1$ and x in $H_n - H_{n-1}$, then $n \geq |y|_1$. From this it follows that

$$H_n \cap \mathbb{Q}y = H_{|y|_1} \cap \mathbb{Q}y,$$

if $n \geq |y|_1$. From this and the fact that $H_{|y|_1}$ is discrete, the desired conclusion follows.

To see that H is dense, we observe that its closure (in $\mathbb{R}^2$) is $(H^*)^*$. If we observe that , for any matrix α as above and any vector (x,y) in $\mathbb{R}^2$, we have

$$\begin{aligned}
\|(x,y)\alpha\|_1 &= |ax+y| + |x+by| \\
&\geq (a-1)|x| + (b-1)|y| \\
&\geq (K-1)\|(x,y)\|_1.
\end{aligned}$$

As $K_n \geq n$, it follows that the intersection of the ℓ^1-ball of radius n with

$$H_n^* = (\mathbb{Z}^2\alpha_n^{-1}\cdots\alpha_1^{-1})^* = \mathbb{Z}^2(\alpha_1\alpha_2\cdot\alpha_n)^T.$$

is trivial. It follows that

$$H^* = (\cup_n H_n)^* = \cap_n H_n^* = \{0\}.$$

Hence, we have $\overline{H} = (H^*)^* = \mathbb{R}^2$ and this completes the proof. $\square$

6 Rational subgroups and odometer factors

In this section, we turn to examine a general minimal, free action, φ, of $\mathbb{Z}^d$ on a Cantor set, X. Our interest will be in factor maps from (X, φ) to other systems, particularly odometers.

Theorem 6.1. *Let (X, φ) be a minimal, free $\mathbb{Z}^d$-Cantor system. The group $H^1(X, \varphi)$ is torsion-free.*

Proof. Suppose that $n \geq 1$ and we have an element of $H^1(X, \varphi)$ of order n. This means that there exists a cocycle θ in $Z^1(X, \varphi)$ such that $n\theta$ is in $B^1(X, \varphi)$. So we may find f in $C(X, \mathbb{Z})$ such that

$$n\theta(x, k) = d(f)(x, k) = f(x) - f(\varphi^k(x)), x \in X, k \in \mathbb{Z}^d.$$

For $i = 0, 1, 2, \ldots, n-1$, define

$$X_i = \{x \in X \mid f(x) \equiv i (mod\, n)\}.$$

In the formula above relating f and θ, it is clear that the left hand side is a multiple of n, which immediately implies that $\varphi^k(X_i) = X_i$, for all k in $\mathbb{Z}^d$. By minimality, all X_i are empty, except one, say $X_j = X$. It then follows that $n^{-1}(f - j)$ is in $C(X, \mathbb{Z})$ and $d(n^{-1}(f - j)) = \theta$. Thus, θ is zero in $H^1(X, \varphi)$. $\square$

Definition 6.2. *Let (X, φ) be a minimal, free $\mathbb{Z}^d$-Cantor system. We define the* rational subgroup *of $H^1(X, \varphi)$, denoted $\mathbb{Q}(H^1(X, \varphi))$, by*

$$\begin{aligned} \mathbb{Q}(H^1(X, \varphi)) \quad = \quad & \{a \in H^1(X, \varphi) \mid \text{ there exist} \\ & j \in \mathbb{Z}^d, k \geq 1, ka = \sum_i j_i[\varepsilon_i]\}. \end{aligned}$$

It is an easy matter to check that $\mathbb{Q}(H^1(X,\varphi))$ is a subgroup of $H^1(X,\varphi)$ and that it contains $[\varepsilon_i], 1 \leq j \leq d$.

Lemma 6.3. *Let (X,φ) be a minimal, free $\mathbb{Z}^d$-Cantor system. Let $f : X \to \mathbb{R}$ be continuous and let r be in $\mathbb{R}^d$. Assume that 0 is in the range of f. The following are equivalent.*

1. *r is in $\mathbb{Q}^d$ and*
$$e^{2\pi i f} \circ \varphi^l = e^{2\pi i(<r,l>+f)},$$
for all l in $\mathbb{Z}^d$.

2. *There is a positive integer k such that kf is in $C(X,\mathbb{Z})$ and kr is in $\mathbb{Z}^d$. In addition, the function*
$$\theta(x,l) = d(f)(x,l) - < r, l >, x \in X, l \in \mathbb{Z}^d,$$
is in $Z^1(X,\varphi)$. Moreover, we have $k[\theta] = \sum_{i=1}^d -kr_i[\varepsilon_i]$ and so $[\theta]$ is in $\mathbb{Q}(H^1(X,\varphi))$ and $\tau^1_\mu[\theta] = r$.

Proof. First, assume that condition 1 holds. Let F be the subgroup of $\mathbb{T}$ generated by $e^{2\pi i<r,l>}, l \in \mathbb{Z}^d$, which is evidently finite. Suppose that $f(x_0) = 0$. It follows from our hypothesis that, for any l in $\mathbb{Z}^d$, $e^{2\pi i f}(\varphi^l(x_0))$ is in F. As f is continuous and the orbit of x_0 is dense, we see that the range of $e^{2\pi i f}$ is contained in F. Let k be the least positive integer such that kr is in $\mathbb{Z}^d$. It follows that the range of $e^{2\pi i k f}$ is simply 1, so kf is in $C(X,\mathbb{Z})$.

It is clear that the function θ is continuous and
$$e^{2\pi i\theta(x,l)} = e^{2\pi i f} \circ \varphi e^{-2\pi i f} e^{-2\pi 1<r,l>} = 1,$$
so it is integer-valued. Finally, θ satisfies the cocycle condition because it is the sum of a coboundary and a homomorphism (even if neither are integer-valued).

For the next statement, we observe that $\sum_{i=1}^d kr_i\varepsilon_i(l) = \sum_{i=1}^d kr_il_i =< k < r, l >$ and that $k\theta = d(kf) - \sum_{i=1}^d kr_i\varepsilon_i$.

For the last part, we compute
$$\tau^1_\mu[\theta]_i = \int_X \theta(x,e_i) \int_X (f(\varphi^{e_i}(x) - f(x) - r_i)\, d\mu(x) = r_i.$$
Thus the second statement holds.

Now let us assume that the second statement holds and prove the first does also. First, it is clear that r is in $\mathbb{Q}^d$. Secondly, using the fact that θ takes integer values, exponentiating $2\pi i$ times the equation in part 2 yields the equation of part 1. $\square$

Theorem 6.4. *Let (X, φ) be a minimal, free $\mathbb{Z}^d$-Cantor system, x_0 be in X and let μ be a φ-invariant measure.*

1. *An element $r \in \mathbb{Q}^d$ is in $\tau^1_\mu(\mathbb{Q}(H^1(X, \varphi)))$ if and only if there is a continuous function $\xi : X \to \mathbb{T}$ such that $\xi(x_0) = 1$ and*

$$\xi \circ \varphi^l = e^{2\pi i < r,l >} \xi, l \in \mathbb{Z}^d.$$

   That is, $(e^{2\pi i r_1}, \dots, e^{2\pi i r_d})$ is a rational eigenvalue of φ with continuous eigenfunction.

2. *If $\xi : X \to \mathbb{T}$ is a continuous function with $\xi(x_0) = 1$ and $r \in \mathbb{Q}^d$ satisfy*

$$\xi \circ \varphi^l = e^{2\pi i < r,l >} \xi, l \in \mathbb{Z}^d.$$

   then $\xi(X)$ is a finite subgroup of $\mathbb{T}$.

3. *Then $\tau^1_\mu(\mathbb{Q}(H^1(X, \varphi))$ is a subgroup of $\mathbb{Q}^d$ and is independent of μ.*

4. *$\tau^1_\mu : \mathbb{Q}(H^1(X, \varphi)) \to \mathbb{Q}^d$ is an isomorphism to its image.*

Proof. For the first part, begin with r in $\tau^1_\mu(\mathbb{Q}(H^1(X, \varphi)))$. This means that we can find θ in $Z^1(X, \varphi)$, a positive integer k, j in $\mathbb{Z}^d$ and g in $C(X, \mathbb{Z})$ such that

$$k\theta = \sum_{i=1}^{d} j_i \varepsilon_i + d(g),$$

with $\tau^1_\mu[\theta] = r$. The last statement means that $\int_X \theta(x, e_i) d\mu(x) = r_i, 1 \leq i \leq d$. Using $l = e_i$ and integrating in the equation above gives $kr_i = j_i$, for all i. Then θ, r and $f = k^{-1}g$ satisfies the second condition of the last lemma. We conclude that $\xi = e^{2\pi i f}$ is a continuous eigenfunction with eigenvalue $(e^{2\pi i r_1}, \dots, e^{2\pi i r_d})$ as claimed.

Conversely, if $(e^{2\pi i r_1}, \dots, e^{2\pi i r_d})$ is a rational eigenvalue with continuous eigenfunction ξ, then as X is totally disconnected, it is a simple matter to find a continuous real-valued function f with $e^{2\pi i f} = \xi$. Then for any x_0 in

X, the function $f - f(x_0)$ and vector r satisfy the hypotheses of part 1 of the last lemma. The desired conclusion holds from part 2 of the lemma.

For the second part, it is clear that if $\xi(x) = 1$, for some x, then the orbit of x is mapped into the finite subgroup $\{e^{2\pi i <r,l>} \mid l \in \mathbb{Z}^d\}$. The conclusion follows from the continuity of ξ and the minimality of φ.

The third part is clear. For the last part, suppose that θ and η are in $Z^1(X, \varphi)$ and $\tau^1_\mu[\theta] = \tau^1_\mu[\eta]$. Find positive integers k, l and vectors j, m in $\mathbb{Z}^d$ such that $k[\theta] = \sum_i j_i[\varepsilon_i], l[\eta] = \sum_i m_i[\varepsilon_i]$. Evaluating τ^1_μ on each yields

$$r_i = \tau^1_\mu[\theta] = k^{-1} j_i = \tau^1_\mu[\eta] = l^{-1} m_i$$

It follows that

$$kl[\theta] = l \sum_i j_i[\varepsilon_i], k \sum_i m_i[\varepsilon_i] = kl[\eta].$$

It follows from Theorem 6.1 that $[\theta] = [\eta]$. □

For the moment, we will let G be a countable abelian group. In fact, all of our groups will be abelian, so we will use additive notation. An action (X, φ) of G is a *group rotation* if X is a (compact) abelian group and there is a group homomorphism, also denoted $\varphi : G \to X$ such that $\varphi^g(x) = x + \varphi(g)$, for all x in X, g in G. The system is minimal if and only if $\varphi(G)$ is dense in X.

Recall that if $\pi : (X, \varphi) \to (Y, \psi)$ is a factor map between actions of an abelian group G, we may define

$$E_\pi = \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}.$$

It is clear that E_π is an equivalence relation, is a closed subset of $X \times X$ and is invariant under $\varphi^g \times \varphi^g$, for every g in G. Conversely, if E satisfies these three conditions, then the quotient space X/E is Hausdorff and obtains an action of G in an obvious way and the quotient map, $\pi_E : X \to X/E$ becomes a factor map.

Recall also ([1]), that an action, (Y, ψ), of G is *equicontinuous* if, for any open set, U in $Y \times Y$ containing the diagonal, $\Delta_Y = \{(y, y) \mid y \in Y\}$, there is another open set V, also containing Δ_Y such that $(\psi \times \psi)^g(V) \subseteq U$, for every g in G. It is easy to see that every group rotation is equicontinuous and that any factor of an equicontinuous system is also equicontinuous. If $\pi : (X, \varphi) \to (Y, \psi)$ is a factor map and (Y, ψ) is equicontinuous, we also say that π and E_π are equicontinuous.

Given the system (X, φ), one may consider the family of all closed, $\varphi^g \times \varphi^g$-invariant, for all g in G, equivalence relations which are equicontinuous. Noting that $X \times X$ is in this family and that the given properties are preserved under intersections, one obtains a minimal closed, $\varphi \times \varphi$-invariant, equicontinuous equivalence relation, which we denote by E_{eq}. The minimality of E_{eq} means that its associated factor, which we denote $\pi_{eq} : (X, \varphi) \to (X_{eq}, \varphi_{eq})$, is the *maximal equicontinuous factor* of (X, φ).

Next, we note that every continuous eigenfunction of (X, φ), $\xi : X \to \mathbb{T}$ with eigenvalue γ in $\hat{G}$, factors through (X_{eq}, φ_{eq}) as follows. We can simply regard γ as a group homomorphism from G into $\mathbb{T}$ and so ξ is a factor map from (X, φ) to the group rotation $(\mathbb{T}, \gamma)$. As the latter is equicontinuous and (X_{eq}, φ_{eq}) is maximal, we have a factor map $\chi : (X_{eq}, \varphi_{eq}) \to (\mathbb{T}, \gamma)$ such that $\chi \circ \pi_{eq} = \xi$.

Now, we use the fact that every minimal equicontinuous action of a countable, abelian group G, (X, φ), is a group rotation. We refer the reader to [17] or [20] for a proof of this fact (although the latter deals only with the case $G = \mathbb{Z}$). The idea is to pick any x_0 in Y and define $\alpha(g) = \varphi^g(x_0)$, so that the orbit of x_0 obtains the structure of a group. Then one shows that the equicontinuity of the action implies this group structure can be extended to the closure, which is X.

We now observe that, for a minimal group rotation (Y, ψ), a continuous function $\chi : Y \to \mathbb{T}$ is a character of Y if and only if $\chi(0) = 1$ and it is a continuous eigenfunction for ψ. The 'only if' direction is trivial. For the 'if' direction, we observe that we have

$$\begin{aligned}
\chi(y + \psi(g)) &= << g, \gamma >> \chi(y) \\
&= << g, \gamma >> \chi(0)\chi(y) \\
&= \chi(0 + \psi(g))\chi(y) \\
&= \chi(y)\chi(\psi(g))
\end{aligned}$$

holds for every g in G and y in Y. The fact that $\psi(G)$ is dense in Y and χ is continuous means that it holds if we replace $\psi(g)$ with any y' in Y; i.e. χ is a character.

Next, we consider the connected subgroup of the identity in X_{eq}, which we denote by X_{eq}^0 and the quotient X_{eq}/X_{eq}^0, which is compact and totally disconnected. The easiest way to understand this is to examine that dual map to the quotient map: $q : X_{eq} \to X_{eq}/X_{eq}^0$. Any character on X_{eq}/X_{eq}^0 is induced by a character on X_{eq} and it is a simple matter to check that a character, χ, of X_{eq} passes to the quotient if and only if it satisfies the following equivalent conditions:

1. $\chi(X^0_{eq}) = \{1\}$,
2. $\chi(X_{eq})$ is a finite subgroup of $\mathbb{T}$,
3. $\chi(X_{eq})$ is a subgroup of the roots of unity.

We now restrict our attention to $G = \mathbb{Z}^d$ and consider (X, φ), a minimal $\mathbb{Z}^d$-action action on the Cantor set. Let μ be an invariant measure for φ and let $H = \tau^1_\mu(\mathbb{Q}(H^1(X, \varphi))$. We claim that $H/\mathbb{Z}^d$ is isomorphic to the dual of X_{eq}/X^0_{eq}. Let r be any element of H. we know from part 1 of Theorem 6.4 that there is a continuous function $\xi : X \to \mathbb{T}$ with $\xi(x_0) = 1$ and

$$\xi \circ \varphi^l = e^{2\pi i <r,l>} \xi,$$

for all l in $\mathbb{Z}^d$. It follows from part 2 of the same result that ξ has finite range. Then from the discussion above, we know that $\xi = \chi \circ q \circ \pi_{eq}$, for some character χ on X_{eq}/X^0_{eq}. The map sending r to χ is obviously trivial on $\mathbb{Z}^d$. It is a simple matter to check that this is an isomorphism as claimed and that the dual of this map is a conjugacy between the $\mathbb{Z}^d$-actions. We have proved the following.

Theorem 6.5. *Let (X, φ) be a minimal, free $\mathbb{Z}^d$-Cantor system. Choose μ to be a φ-invariant measure and let $H = \tau^1_\mu(\mathbb{Q}(H^1(X, \varphi))$. There is a factor map*

$$\pi : (X, \varphi) \to (Y_H, \psi_H).$$

Moreover, (Y_H, ψ_H) is the maximal totally disconnected, equicontinuous factor of (X, φ).

The factor map is called almost one-to-one if there is a point y in Y_H such that $\pi^{-1}\{y\}$ is a single point. The system (X, φ) is a so-called Toeplitz system if and only if it is expansive and the map π is almost one-to-one - see [3].

Acknowledgements

The authors would like to thank R. Grigorchuk and J. Hunton for helpful conversations and the Centre de Recerca Matemática in Barcelona for their hospitality and stimulating environment during the program 'Operator Algebras: Dynamics and Interactions', where much of this work was done.

References

[1] J. Auslander, *Minimal Flows and their Extensions*, Mathematical Sudies 153, Notas de Matemáca, North Holland, 1988.

[2] K.S. Brown, *Cohomology of Groups*, Graduate Texts in Math. 87, Springer, 1982.

[3] I.M. Cortez, *$\mathbb{Z}^d$-Toeplitz arrays*, Discrete and Contin. Dyn. Sys. 15(2006), 859–881.

[4] I.M. Cortez and K. Medynets, *On Virtual Conjugacy of Generalized Odometers*, J. London Math. Soc. (2) 94(2016), 545–556.

[5] I.M. Cortez and S. Petite, *G-odometers and their almost 1-1 extensions*, J. London Math. Soc. (2) 78(2008), 1–20.

[6] G. L. Dirichlet, *G. Lejeune Dirichlets Werke*, L, Kronecker and L. Fuchs, eds., American Mathematical Society, Providence, 1969.

[7] T. Downarowicz, *Survey of odometers and Toeplitz flows*, Algebraic and Topological Dynamics (Kolyada, Manin, Ward eds), Contemporary Math. 385(2005), 7-38.

[8] A.H. Forrest and J.R. Hunton, *The cohomology and K-theory of commuting homeomorphisms of the Cantor set*, Ergod. Th. and Dynam. Sys. 19(1999) 611-625.

[9] A.H. Forrest, J.R. Hunton and J. Kellendonk, *Topological invariants for projection method patterns*, Mem. Amer. Math. Soc. 758 (2002).

[10] L. Fuchs, *Abelian Groups*, Monographs in Mathematics, Springer, 2015.

[11] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, Invent. Math. 179(2010), 119-158.

[12] T. Giordano, I.F. Putnam and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469(1995), 51-111.

[13] T. Giordano, I.F. Putnam and C.F. Skau, *Cocycles for Cantor minimal $\mathbb{Z}^d$-systems*, Internat. J. Math., 20(2009), 1107-1135.

[14] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs, 101, American Mathematical Society, 2003.

[15] R. I. Grigorchuk, V. V. Nekrashevich and V.I Suschanskii, *Automata, dynamical systems, and groups*, Proc. Steklov Inst. Math. 231 (2000) no. 4 128203.

[16] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, Grundl. Math. Wiss. 115, Springer-Verlag, 1963.

[17] P. Kurka, *Topological and Symbolic Dynamics*, Cours Spécialités 11, Société Mathétique de France, 2003.

[18] X. Li, *Continuous orbit equivalence rigidity*, to appear in Ergod. Th. Dyn. Sys.

[19] V. Nekrashevych, *Self-similar Groups*, Mathematical Survey Monograph 117, American Mathematical Society, Providence, RI, 2005.

[20] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Math. 79, Springer, 1982.